ON THE 3-D BREZIS-NIRENBERG PROBLEM WITH SMALL PARAMETER

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To the memory of Haïm Brezis with admiration

ABSTRACT. In 2023, H. Brezis [2] published a list of his "favorite open problems", which he described as challenges he had "raised throughout his career and has resisted so far". In this paper, we shall provide a partial answer to this question by presenting the existence of sign-changing solutions to the equation whenever the parameter is small enough. Our construction is based on the building blocks of Del Pino-Musso-Pacard-Pistoia sign-changing solutions to Yamabe problem.

1. INTRODUCTION

In 1983, Brezis and Nirenberg [3] proposed the problem

$$\begin{cases} \Delta u + \lambda u + |u|^{\frac{4}{n-2}} u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a smooth, bounded domain in $\mathbb{R}^n$. In their landmark study [3], they established the existence of at least one positive solution under the conditions: $0 < \lambda < \lambda_1$ for $n \geq 4$, and $0 < \lambda_* < \lambda < \lambda_1$ for $n = 3$, with λ_1 as the first eigenvalue of the Laplacian and λ_* a domain-dependent constant, later quantified by Druet [16] via Robin functions. When Ω is the unit ball in $\mathbb{R}^n$, they showed $\lambda_* = \frac{\lambda_1}{4}$, and positive solutions exist exactly when $\lambda \in (\frac{\lambda_1}{4}, \lambda_1)$. Additionally, using Pohozaev's identity, they proved that no solutions exist if $\lambda \leq 0$ and Ω is star-shaped. Since its debut, this problem has drawn significant interest and stands as a pivotal work in the study of nonlinear elliptic partial differential equations (PDEs) with critical Sobolev exponents; see [4, 8, 6, 5, 7, 14, 22] and references therein for the developments of this problem. Notably, the three-dimensional case remains largely unresolved, particularly the question of whether nontrivial solutions to (1.1) exist for $\lambda \in (0, \lambda_*)$. For the unit ball case, H. Brezis [2] posed it as Open Problem 1.1. Consider $\Omega = \mathbb{B}$, the unit ball in $\mathbb{R}^3$, with the problem:

$$\begin{cases} \Delta u + \lambda u + u^5 = 0 & \text{in } \mathbb{B}, \\ u = 0 & \text{on } \partial\mathbb{B}. \end{cases} \tag{1.2}$$

H. Brezis' first open problem is as follows:

Open Problem 1.1 (Implicit in [3]) Assume that

$$0 < \lambda < \frac{\lambda_1}{4}. \tag{1.3}$$

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Does there exist a non-trivial solution $u \not\equiv 0$ to (1.2)?

In a concurrent study, the authors of this paper provide a complete answer to this question. Specifically, the following conclusion is obtained:

Theorem A. [23, Theorem 1.1] *Assume that* $0 < \lambda < +\infty$. *Then there are infinitely many (sign-changing) solutions to* (1.2).

In the proof of Theorem A, two key factors play important role. First, we rely on a nodal solution to the Yamabe problem. Unlike most of the existing literature, which typically utilizes the standard positive Talenti bubble, we instead choose *sign-changing* solutions to the Yamabe equation:

$$\Delta u + |u|^{\frac{4}{n-2}} u = 0 \quad \text{in} \quad \mathbb{R}^n, \tag{1.4}$$

as constructed by Del Pino-Musso-Pacard-Pistoia [11]. Regarding sign-changing solutions to (1.4), it is remarkable to mention that Ding [15] pioneered the use of variational methods to derive an infinite family of conformally distinct sign-changing solutions with finite energy. This breakthrough spurred ongoing research into the existence of such solutions across $\mathbb{R}^n$. In [11], Del Pino-Musso-Pacard-Pistoia introduced an innovative construction method, yielding sign-changing solutions characterized by high energy. Their solution resembles a positive bubble adorned with m negative spikes arranged in a regular polygon of radius 1, earning it the designation "crown solution" due to its distinctive geometry. Notably, this crown-type solution exhibits invariance under both rotational and Kelvin transformations. We define Σ as the set of nonzero finite-energy solutions to (1.4):

$$\Sigma := \left\{ Q \in \mathcal{D}^{1,2}(\mathbb{R}^n \setminus \{0\}) : \ \Delta Q + |Q|^{\frac{4}{n-2}} Q = 0 \right\}. \tag{1.5}$$

It is straightforward to verify that equation (1.4) is invariant under four transformations: translation, dilation, orthogonal transformation, and Kelvin transformation; explicit definitions are provided in Section 2. Using the crown solution $q(z)$ and this invariance, we propose the following ansatz:

$$\frac{\varepsilon^{\frac{1}{2}}}{|z-y|} q\left(\frac{\varepsilon R_\beta (z-y)}{|z-y|^2} + \xi \right),$$

where $R_\beta \in SO(3)$ and ξ is a zero of $q(z)$. A novel aspect of this construction is that the solution now exhibits a decay of $1/|z|^2$, a significant improvement over the decay of the ansatz based on the standard bubble. Second, we need the crown solution is non-degenerate. Once a suitable building block has been identified, our next step is to seek a perturbation that, when added to the ansatz, yields an exact solution to the Brezis-Nirenberg problem. The process of incorporating this perturbation is intricate and hinges on a critical requirement: the linearized operator must be non-degenerate in the sense of Duyckaerts-Kenig-Merle [17]. This non-degeneracy condition stipulates that the kernel space of the linearized operator precisely coincides with the space spanned by the four transformations—translation, dilation, orthogonal transformation, and Kelvin transformation—previously discussed. Consider this issue, Musso-Wei [19] established that the crown solution satisfies this non-degeneracy property through rigorous analysis. This key result unlocks significant opportunities for further study. For instance, using the non-degeneracy of the crown solution, Musso-Wei [20] constructed a sign-changing solution to the Bahri-Coron problem. Beyond this, the non-degeneracy of crown solution has proven instrumental in a broader range of investigations into critical exponent problems, we refer the readers to [12, 13].

Once the building block is determined, the subsequent task is to decide the placement of the configurations and formulate an appropriate approximate solution. In [23], by using the symmetry of the unit ball we separate it into K subdomains, with K being a sufficiently large even integer that acts as the perturbation parameter in our analysis. Within each sector, we position a crown bubble in an alternating fashion, creating a necklace-like pattern of Del Pino-Musso-Pacard-Pistoia bubbles near the boundary. The interactions between these bubbles, as well as their mirror images relative to the boundary (arising from the Dirichlet boundary condition) are notably intense. This intensity results in a complex interplay among the parameters defining the approximate solution. By meticulously computing the energy of solution and pinpointing its precise leading-order terms, we can locate the critical point via minimization. This approach, combined with the classical reduction lemma, enables us to establish the existence of a solution to (1.2).

In contrast to the studies of [23] that employed K as the perturbation parameter, this study takes a different approach by selecting λ as the perturbation parameter. When λ is assumed to be sufficiently small, we establish the space for the six parameters defined in the approximate solution. Different from the strategy in [23], we place the crown bubble close to the center of the unit ball. Given that the center of crown bubble is nearly equidistant from the boundary of $\mathbb{B}$, we do not need an inner-outer gluing method to study the associated linearized problem. Instead, we can tackle the linearized problem directly, as the influence from the boundary remains negligible. Following this, by evaluating the corresponding energy and determining the critical points of the reduced energy functional, we show that the Lagrange multipliers vanish. This allows us to confirm the existence of a solution to (1.2) for sufficiently small λ. Moreover, we extend this result further.

To state the main result, let us first introduce some notations. Suppose that $G(z,p)$ is the Green's function for Ω in $\mathbb{R}^3$. We denote by $H_\Omega(z,p)$ the regular part of the Green's function, namely

$$H_\Omega(z,p) = 4\pi\left(\frac{1}{4\pi|z-p|} - G(z,p)\right). \tag{1.6}$$

Recall that the Robin function is defined by

$$\tau_\Omega(z) = H_\Omega(z,z).$$

We will assume the following conditions for Ω. Assume Ω is symmetric with respect to the plane $z_3 = 0$. After some translation if needed, assume $0 \in \Omega$ and

$$\nabla\tau_\Omega(0) = 0. \tag{C-1}$$

For any $w \in \mathbb{S}^1 \subset \mathbb{R}^3$, i.e. $w = (w_1, w_2, 0) \in \mathbb{R}^3$ with $w_1^2 + w_2^2 = 1$,

$$w^T \nabla^2_{z,p} H_\Omega(0,0) w > 0 \tag{C-2}$$

There exists $\theta_0 \geq 2$, $r_0 > 0$ and $C > 0$ such that for any $w \in \mathbb{S}^1 \subset \mathbb{R}^3$ and $|b| < r_0$,

$$w^T(\nabla^2_{z,p} H_\Omega(b,b) - \nabla^2_{z,p} H_\Omega(0,0))w \geq C|b|^{\theta_0}. \tag{C-3}$$

This finding is encapsulated in the following theorem:

Theorem 1.1. *Suppose that $\Omega \subset \mathbb{R}^3$ is a bounded smooth domain which is symmetric with respect to the plane $z_3 = 0$. Assume that (C-1), (C-2), and (C-3) hold for its H_Ω. Then there exists $\lambda_0 = \lambda_0(\Omega) > 0$ small such that for any $\lambda \in (0, \lambda_0)$ there are infinitely many sign-changing solutions to (1.1).*

The necessity of condition (C-1) is very natural. The solution we construct concentrates at the origin, it is reasonable to expect the origin is a critical point of the Robin function τ_Ω. Moreover, (C-2) and (C-3) imply that $w^T \nabla^2_{z,p} H_\Omega(b,b) w$ has a positive and strict local minimum at $b = 0$ for any fixed $w \in \mathbb{S}^1 \subset \mathbb{R}^3$. The local minimum is non-degenerate if and only if $\theta_0 = 2$, and in this case, (C-3) is equivalent to $\nabla^2_b (w^T \nabla^2_{z,p} H_\Omega(b,b)w)|_{b=0}$ being positive definite for any $w \in \mathbb{S}^1 \subset \mathbb{R}^3$.

It is well-known that in the unit ball case

$$H_{\mathbb{B}}(z,p) = \left| |z|p - \frac{z}{|z|} \right|^{-1} = \frac{1}{\sqrt{1 - 2z \cdot p + |z|^2 |p|^2}}. \tag{1.7}$$

Consequently,

$$\tau_{\mathbb{B}}(z) = \frac{1}{1-|z|^2}, \quad \partial_{z_i p_j} H_{\mathbb{B}}(b,b) = \frac{\delta_{ij} + b_i b_j}{(1-|b|^2)^3}, \quad w^T \nabla^2_{z,p} H_{\mathbb{B}}(b,b) w = \frac{|w|^2 + (b \cdot w)^2}{(1-|b|^2)^3}.$$

It is easy to see that $w^T \nabla^2_{z,p} H_{\mathbb{B}}(b,b) w$ takes a positive and non-degenerate global minimum at $b = 0$ for any given $w \in \mathbb{S}^2 \subset \mathbb{R}^3$. More precisely, one has the following

$$w^T [\nabla^2_{z,p} H_{\mathbb{B}}(b,b) - \nabla^2_{z,p} H_{\mathbb{B}}(0,0)] w = \frac{1 + (b \cdot w)^2}{(1-|b|^2)^3} - 1 \geq C|b|^2.$$

Thus $\mathbb{B}$ satisfies (C-1), (C-2), and (C-3). Therefore Theorem 1.1 implies that (1.2) has infinitely many sign-changing solutions when λ is small.

Corollary 1.2. *There exists $\lambda_0 > 0$ small such that for any $\lambda \in (0, \lambda_0)$ there are infinitely many sign-changing solutions to (1.2).*

Furthermore, Theorem 1.1 can be applied to ensure the existence of sign-changing solutions to problem (1.1) when Ω is a cuboid and λ is any positive number.

Theorem 1.3. *Let Ω be a cuboid, there are infinitely many sign-changing solutions to (1.1) for any positive λ.*

Considering the problem (1.1) in domains with less symmetry, we can also establish the existence of sign-changing solutions when the domain is sufficiently close to a ball in an appropriate sense. Suppose $m \geq 0$ and $0 < \alpha < 1$. Recall that $C^{m,\alpha}(\Omega)$ consists of u such that $\|u\|_{C^{m,\alpha}(\Omega)} < \infty$ where

$$\|u\|_{C^{m,\alpha}(\Omega)} := \sum_{k=0}^{m} \|D^k u\|_{C^0(\Omega)} + \sup_{x,y \in \Omega, x \neq y} \frac{|D^m u(x) - D^m u(y)|}{|x-y|^\alpha}.$$

Definition 1.4. *Assume Ω_1 and Ω_2 are smooth bounded domains in $\mathbb{R}^n$ which are $C^{m,\alpha}$ diffeomorphic to each other up to the boundary, that is, there exists a $C^{m,\alpha}$-diffeomorphism $G : \bar{\Omega}_1 \to \bar{\Omega}_2$ such that $G\left(\bar{\Omega}_1\right) = \bar{\Omega}_2$. If*

$$\|G - id\|_{C^{m,\alpha}(\Omega_1)} \leq \varepsilon$$

then we say that Ω_2 is ε-close to Ω_1 in $C^{m,\alpha}$-sense.

It has been proved by [1] that the dependence of H_Ω on the variation of domain Ω is at least C^1. More precisely, it implies that

$$\|H_\Omega(x,y) - H_{G(\Omega)}(G(x),G(y))\|_{C^{m,\alpha}(\Omega)} \leq C\|G - id\|_{C^{m,\alpha}(\Omega)} \tag{1.8}$$

provided $\|G - id\|_{C^{m,\alpha}(\Omega)} < \rho(\Omega)$ is small. Here C depends on Ω, m and α.

Since $\tau_{\mathbb{B}}(z)$ and $w^T\nabla^2_{z,p}H_{\mathbb{B}}(b,b)w$ takes a positive and non-degenerate global minimum at near 0 for any given $w \in \mathbb{S}^2$, these properties will be preserved by the small variation of the unit ball considering (1.8).

Corollary 1.5. *If $\Omega \subset \mathbb{R}^3$ is a smooth bounded domain. If Ω is $C^{4,\alpha}$-close to a ball, then* (C-1)*,* (C-2)*, and* (C-3) *hold for H_Ω. Consequently, if, Ω is also symmetric with respect to the plane $z_3 = 0$, there exists $\lambda_0(\Omega) > 0$ such that for any $\lambda \in (0, \lambda_0)$ there are infinitely many sign-changing solutions to* (1.1).

The paper is organized as follows: In Section 2, we present some preliminary results on the nodal solution to the Yamabe problem and its non-degeneracy. In Sections 3 and 4 we define the approximate solution with correction terms and compute the energy for the approximate solution. In Section 5 we study the linear and nonlinear problems. While in Section 6 we reduce the infinite dimensional problem to a finite one and resolve the reduced problem by identifying the local minimum of the reduced energy and provide the proof of Theorem 1.1. In last section, we provide the proof of Theorem 1.3.

2. Preliminary results

In this section, we will outline the preliminary results utilized in this paper. To begin with, as highlighted in the introduction, it is essential to confirm that the range of ξ forms a smooth manifold. As demonstrated in [23], we have provided an in-depth analysis of the nodal set for the crown solution, which was introduced by Del Pino-Musso-Pacard-Pistoia in [11]. This step guarantees the robustness for the reduction process of the problem. Additionally, we require the crown solution to be non-degenerate in the sense of Duyckaerts-Kenig-Merle (see [17]), and we will interpret it within this section.

Let m_0 be a sufficiently large positive integer, for any $m > m_0$, the crown solution established by Del Pino-Musso-Pacard-Pistoia takes the following form:

$$q_m(z) = U_*(z) + \phi(z), \tag{2.1}$$

where

$$U_* = U(z) - \sum_{j=1}^m U_j(z) = U(z) - \sum_{j=1}^m \mu_m^{-\frac{1}{2}} U\left(\frac{z-\xi_j}{\mu_m}\right)$$
$$= 3^{\frac{1}{4}}\left(\frac{1}{1+|z|^2}\right)^{\frac{1}{2}} - 3^{\frac{1}{4}}\sum_{j=1}^m \mu_m^{-\frac{1}{2}}\left(\frac{1}{1+\mu_m^{-2}|z-\xi_j|^2}\right)^{\frac{1}{2}},$$

and

$$\phi(z) = \sum_{j=1}^m \tilde{\phi}_j(z) + \psi(z).$$

For U_*, the parameters μ_m and $\xi_j,\ j=1,\cdots,m$ are given as following:

$$\xi_j=\sqrt{1-\mu_m^2}\left(\cos\frac{2(j-1)\pi}{m},\sin\frac{2(j-1)\pi}{m},0\right),\tag{2.2}$$

and (for the choice of d_m, see [11, Page 2590])

$$\mu_m=\frac{d_m^2}{m^2(\log m)^2}\quad\text{and}\quad d_m=\sqrt{2}\frac{m\log m}{\sum_{j=1}^{m-1}\csc\frac{j\pi}{m}}+O\left(\frac{1}{m\log m}\right).\tag{2.3}$$

The error terms $\tilde{\phi}_j,\ j=1,\cdots,m$, and ψ satisfy

$$|\nabla^\ell\tilde{\phi}_j(z)|\le\frac{C}{1+\mu_m^{-1}|z-\xi_j|^{\ell+1}},\quad \ell=0,1,2,\quad j=1,\cdots,m,\tag{2.4}$$

and

$$|\psi(z)|\le\frac{C}{\log m}\quad\text{and}\quad|\nabla\psi(z)|+|\nabla^2\psi(z)|\le C.\tag{2.5}$$

For the derivation of (2.4)-(2.5), we refer the readers to [23] and the references therein. Consider the solution q_m we have the following theorem, see [23, Theorem 2.1]

Theorem 2.1. [23, Theorem 2.1.] *When m is large enough, q_m has a smooth embedding and compact nodal set $\mathcal{N}(q_m)$ such that $q_m(z)=0$ and $\nabla q_m(z)\neq 0$ for any $z\in\mathcal{N}(q_m)$. Near the center of each bump ξ_j, $\mathcal{N}(q_m)\cap\{z_3=0\}\sim\{|z-\xi_j|\sim 1/m\}$.* [1]

We prove Theorem 2.1 via the following steps. First, we must derive a precise expansion of ψ, which represents the outer error of the solution q_m. As established in [11], ψ is of the order $\frac{1}{\log m}$, however, for our purposes, we need to determine its explicit value near the center of each negative bubble at this order. Second, we must identify the locations where q_m equals zero. According to [23, Lemma 2.2], the distance from any zero of q_m to the set $\{\xi_1,\dots,\xi_m\}$ is of the order of $O\left(\frac{1}{m}\right)$. Moreover, between two adjacent points ξ_j and ξ_{j+1}, we define the midpoint as

$$\xi_{j+\frac12}=\sqrt{1-\mu_m^2}\left(\cos\frac{(2j-1)\pi}{m},\sin\frac{(2j-1)\pi}{m},0\right),$$

and there exists a small positive constant c_0 such that q_m has no zeros within the ball $B_{c_0/m}(\xi_{j+\frac12})$ for $j=1,\dots,m$. Here, $\xi_{m+\frac12}$ denotes the midpoint between ξ_0 and ξ_m, given by

$$\sqrt{1-\mu_m^2}\left(\cos\frac{(2m-1)\pi}{m},\sin\frac{(2m-1)\pi}{m},0\right).$$

This information implies that, topologically, the nodal set of q_m in the z_1z_2-plane forms a circle around each ξ_j for $j=1,\dots,m$. Finally, we must prove that for any point p in the nodal set of q_m, the gradient $\nabla q_m(p)\neq 0$. This is achieved by choosing a suitable direction and verifying that the derivative of q_m along this direction is non-zero, as shown in [23, Lemma 2.3]. For more details, we refer the readers to [23, Sections 2.1–2.2].

[1] Indeed, we can prove that near the center of each bump ξ_j, the distance between the zero point z of q_m and ξ_j is on the order of $1/m$. However, it is neither possible nor necessary for our proof to show that the nodal set resembles a ball in topology; it suffices to confirm that it is a smooth Riemannian manifold.

Another important ingredient of our proof is the non-degeneracy of solution. We denote by Σ the solution set of Yamabe equation

$$\Delta q + q^5 = 0 \quad \text{in} \quad \mathbb{R}^3. \tag{2.6}$$

We choose m large enough and fix a function $q = q_m$ and assume that its nodal set is a smooth Riemannian manifold. It is known that the solution set Σ is invariant under the following transformation:

(1) the translation $T_y : z \to z + y$ where $y \in \mathbb{R}^3$,
(2) the dilation $D_\varepsilon : z \to \varepsilon z$ for $\varepsilon > 0$,
(3) the rotation $R_\beta : z \to R_\beta z$ for $R_\beta \in SO(3)$,
(4) the inversion $J : z \to \frac{z}{|z|^2}$,
(5) the translation under inversion $\psi_\xi = J \circ T_\xi \circ J$.

For any set of parameters $A = (y, R_\beta, \varepsilon, \xi) \in \mathbb{R}^3 \times SO(3) \times \mathbb{R}_+ \times \mathbb{R}^3$, we define the transformation $\mathscr{T}_A = T_{-y} \circ R_\beta \circ D_\varepsilon \circ \psi_\xi$. Then $\Theta_A(z) = |\det(\mathscr{T}_A'(z))|^{\frac{1}{6}} q(\mathscr{T}_A(z))$ is also a solution to the Yamabe equation (2.6). Using $Jq = q$, we have

$$\Theta_A(z) = \frac{\varepsilon^{\frac{1}{2}}}{|z-y|} q\left(\frac{\varepsilon R_\theta(z-y)}{|z-y|^2} + \xi\right).$$

Choosing A near to $(0, Id, 1, 0)$, it generates a family of solutions near q. Taking the derivatives on each parameter in A, we obtain 10 functions

$$\begin{aligned} & -z_j q + |z|^2 \partial_{z_j} q - 2 z_j z \cdot \nabla q, \quad \partial_{z_j} q, \quad 1 \le j \le 3, \\ & \left(z_j \partial_{z_\ell} - z_\ell \partial_{z_j}\right) q, \quad 1 \le j < \ell \le 3, \quad \frac{1}{2} q + z \cdot \nabla q. \end{aligned} \tag{2.7}$$

Obviously, we have $L_q f = 0$ for any f in the above, where L_q is linearized operator near q

$$L_q = -\Delta - 5q^4.$$

We denote by $\mathcal{Z}_q$ the function space spanned by the 10 functions in (2.7) and define the kernel space of L_q by

$$\widetilde{\mathcal{Z}}_q = \{f \in D^{1,2}(\mathbb{R}^3) : L_q f = 0\}.$$

Musso-Wei [19] proved that $\widetilde{\mathcal{Z}}_q = \mathcal{Z}_q$. Together with the fact that

$$-z_3 q + |z|^2 \partial_{z_3} q - 2 z_3 z \cdot \nabla q = \partial_{z_3} q,$$

we can re-state their result as the following form:

Proposition 2.2 (Non-degeneracy)**.** *When m is large enough, q_m is non-degenerate and*

$$\dim \mathcal{Z}_{q_m} = 9,$$

where q_m is the one constructed in [11] in $\mathbb{R}^3$. The nine functions in (2.7) *(removing $\partial_{z_3} q$) are linearly independent.*

It follows from Proposition 2.2 that Q_A is non-degenerate for any A and the kernel of $\widetilde{\mathcal{Z}}_A := \widetilde{\mathcal{Z}}_{Q_A}$ has 9 dimensions. Indeed, $v \in \widetilde{\mathcal{Z}}_q$ if and only if $|\det(\mathscr{T}_A'(z))|^{\frac{1}{6}} v(\mathscr{T}_A(z) \in \widetilde{\mathcal{Z}}_A$ where $\mathscr{T}_A = T_{-y} \circ R_\beta \circ D_\varepsilon \circ \psi_\xi$.

By assuming the symmetry of Ω, we can reduce the complexity of the problem and do not need to use the full parameters of invariance. We shall construct a solution which is even on z_3. To that end, we define

$$\Gamma = \mathcal{N}(q) \cap \{x \in \mathbb{R}^3 : (z_1, z_2, 0)\} \tag{2.8}$$

the intersection of the nodal set of q_m with $z_1 z_2$-plane.

We redefine the set of parameters $A = (\varepsilon, \xi, a, b, \beta) \in \mathbb{R} \times \Gamma \times \mathbb{R} \times \mathbb{R}^2 \times \mathbb{S}$, and

$$Q_A(z) = \frac{\varepsilon^{\frac{1}{2}}}{|z-b|} q\left(\frac{\varepsilon R_\beta (z-b)}{|z-b|^2} + \xi + a\nu(\xi)\right) \tag{2.9}$$

where $\nu(\xi) = \frac{\nabla q(\xi)}{|\nabla q(\xi)|}$ is the unit normal to $\mathcal{N}(q)$, $R_\beta \in SO(2) \cong \mathbb{S}$ is the 2D rotation group. Using the symmetry of q, we have $\nu(\xi) \in \mathbb{R}^2$, i.e., $\nu_3(\xi) = 0$. The parameter space A is six dimensional, where $\varepsilon \in \mathbb{R}$ is the dilation, $b \in \mathbb{R}^2$ is the translation, $\xi + a\nu(\xi) \in \mathbb{R}^2$ is the translation under inversion, $R_\beta \in SO(2)$ is the rotation.

Due to the selection of the base point ξ, when the remaining parameters are fixed, we observe that $Q_A(z) = O(|z|^{-2})$ as $|z| \to \infty$, uniformly for $\xi \in \Gamma$ and $\beta \in \mathbb{S}$ when $a = 0$. This property is critical in the construction of solutions to the Brezis-Nirenberg problem.

In Section 5 and 6, we need a basis of the kernel $\widetilde{\mathcal{Z}}_{Q_A}$. One way is using (2.7) and making appropriate translations as mentioned above. The other way is differentiating Q_A on the parameters A directly. We follow the second approach and list the relevant expression of them.

$$Z_0(z) = \frac{\partial}{\partial \varepsilon} Q_A|_{\varepsilon=1,b=0,a=0} = \frac{1}{2|z|} q\left(R_\beta \frac{z}{|z|^2} + \xi\right) + \frac{1}{|z|^3} R_\beta^T \nabla q\left(R_\beta \frac{z}{|z|^2} + \xi\right) \cdot \begin{pmatrix} z_1 \\ z_2 \\ 0 \end{pmatrix},$$

$$Z_1(z) = \frac{\partial}{\partial a} Q_A|_{\varepsilon=1,b=0,a=0} = \frac{1}{|z|} q_{\nu(\xi)}\left(R_\beta \frac{z}{|z|^2} + \xi\right),$$

$$Z_2(z) = \frac{\partial}{\partial \xi} Q_A|_{\varepsilon=1,b=0,a=0} = \frac{1}{|z|} q_\xi\left(R_\beta \frac{z}{|z|^2} + \xi\right),$$

$$Z_3(z) = \frac{\partial}{\partial b_1} Q_A|_{\varepsilon=1,b=0,a=0} = \frac{2z_1}{|z|^2}\frac{\partial Q_A}{\partial \varepsilon} - \frac{1}{|z|^2}\frac{\partial Q_A}{\partial a} = \frac{2z_1}{|z|^2} Z_0 - \frac{1}{|z|^2} Z_1,$$

$$Z_4(z) = \frac{\partial}{\partial b_2} Q_A|_{\varepsilon=1,b=0,a=0,\beta=\theta_*} = \frac{2z_2}{|z|^2}\frac{\partial Q_A}{\partial \varepsilon} - \frac{1}{|z|^2}\frac{\partial Q_A}{\partial \xi} = \frac{2z_2}{|z|^2} Z_0 - \frac{1}{|z|^2} Z_2,$$

$$Z_5(x) = \frac{\partial}{\partial \beta} Q_A|_{\varepsilon=1,b=0,a=0,\beta=\theta_*} = \frac{z_1}{|z|^2}\frac{\partial Q_A}{\partial \xi} - \frac{z_2}{|z|^2}\frac{\partial Q_A}{\partial a} = \frac{z_1}{|z|^2} Z_2 - \frac{z_2}{|z|^2} Z_1.$$

There are three functions we did not list here, which come from the differentiation of translation in z_3, rotation in $z_1 z_3$-plane, and rotation in $z_2 z_3$-plane. They are all odd in z_3 and play no role in our proof.

3. Approximate Solutions

In this section, we shall modify the family of bubbles Q_A, defined at (2.9), to satisfy a similar equation to the Brezis-Nirenberg problem in Ω and Dirichlet boundary conditions. This is the first step in the gluing process. More precisely, we define the approximate solution PQ_A (or the projection of Q_A) to be

$$\begin{cases} \Delta PQ_A + \lambda PQ_A + Q_A^5 = 0 & \text{in } \Omega, \\ PQ_A = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

At the end of this section, we will prove PQ_A is Q_A summing other terms with good control.

We shall use the following family of bubbles described in (2.9) in the Section 2,

$$Q_A(z) = \frac{\varepsilon^{\frac{1}{2}}}{|z-b|} q\left(\frac{\varepsilon R_\beta(z-b)}{|z-b|^2} + \xi + a\nu(\xi)\right),$$

where $A = (\varepsilon, \xi, a, b, \beta)$.

We will make some constraints on the parameters of A. The rationale for selecting these constraints is to make sure some functional (see $\Psi(A)$ in (4.12)) has an infimum achieved inside it. It will become clear throughout the computations in sections 3 and 4, culminating in the proof of Theorem 6.2. Denote $b = |b|e^{i\alpha_b}$. We make the following constraint.

$$\varepsilon \in [\delta\lambda, \delta^{-1}\lambda], \quad |a| \le \delta^{-1}\varepsilon^{3/2}, \quad |b|^{\theta_0} \le \delta^{-4}\varepsilon, \quad \xi \in \Gamma, \quad \beta \in \mathbb{S} \tag{3.2}$$

where $\delta \in (0,1)$ is some fixed small number that will be determined later. We always assume that $\lambda \ll \delta$ such that the above constraints imply that $\varepsilon \ll 1$, $|a| \ll 1$ and $|b| < r_0$. All the constants C are independent of λ and δ.

Under these constraints, we have

$$|q(\hat{\xi})| \le C|a| \le C\lambda, \quad \text{dist}(b, \partial\Omega) \ge \frac{1}{2}.$$

Hereafter, we adopt the following notations for the remainder of this paper.

$$\hat{\xi} = \xi + a\nu(\xi), \quad w = R_\beta^T \nabla q(\hat{\xi}), \quad W = R_\beta^T \nabla^2 q(\hat{\xi}) R_\beta. \tag{3.3}$$

Note that $C^{-1} \le |w| = |\nabla q(\hat{\xi})| \le C$ and $|W| = |\nabla^2 q(\hat{\xi})| \le C$ for a constant C just depend on q.

To get PQ_A in (3.1), we first solve φ_A from

$$\begin{cases} \Delta \varphi_A = 0 & \text{in} \quad \Omega, \\ \varphi_A = Q_A & \text{on} \quad \partial\Omega, \end{cases} \tag{3.4}$$

Lemma 3.1. *Under the constraint (3.2), for any $z \in \Omega$, one has*

$$\varphi_A(z) = \varepsilon^{\frac{1}{2}} q(\hat{\xi}) H_\Omega(z,b) + \varepsilon^{\frac{3}{2}} w \cdot \nabla_p H_\Omega(z,b) + \frac{1}{6}\varepsilon^{\frac{5}{2}} W_{ij} \partial^2_{p_i p_j} H_\Omega(z,b) + O(\varepsilon^{\frac{7}{2}}), \tag{3.5}$$

where $H_\Omega(z,p)$ is defined in (1.6). In particular, one has $|\varphi_A(z)| = O(\varepsilon^{\frac{3}{2}})$.

Proof. When $z \in \partial\Omega$, one has $\varepsilon/|z-b| \le C\varepsilon \ll 1$. Using Taylor expansion of q near $\hat{\xi}$, we have the expansion of $Q(z)$ for any z satisfying $|z-b| \ge \varepsilon$

$$Q_A(z) = \frac{\varepsilon^{\frac{1}{2}}}{|z-b|}\left[q(\hat{\xi}) + \frac{\varepsilon w\cdot(z-b)}{|z-b|^2} + \frac{\varepsilon^2(z-b)^T W(z-b)}{2|z-b|^4} + O\left(\frac{\varepsilon^3}{|z-b|^3}\right)\right]. \tag{3.6}$$

Recall that $H_\Omega(z,p) = |z-p|^{-1}$ when $z \in \partial\Omega$ and $p \in \Omega$. It is easy to verify the following for $z \in \partial\Omega$,

$$\nabla_{p_j} H_\Omega(z,b) = \frac{(z-b)_j}{|z-b|^3}, \quad \nabla^2_{p_j p_\ell} H_\Omega(z,b) = \frac{-\delta_{j\ell}}{|z-b|^3} + \frac{3(z-b)_j(z-b)_\ell}{|z-b|^5}.$$

Comparing the expansion of $Q_A(z)$ in (3.6), we found that Q_A on $\partial\Omega$ can be approximated by $H_\Omega(z,b)$ and its derivatives. Thus if we let

$$\varphi_A = \varepsilon^{\frac{1}{2}} q(\hat{\xi}) H_\Omega(z,b) + \varepsilon^{\frac{3}{2}} w\cdot\nabla_p H_\Omega(z,b) + \frac{1}{6}\varepsilon^{\frac{5}{2}} W_{ij}\partial^2_{p_i p_j} H_\Omega(z,b) + f_A(z),$$

then f_A satisfies $\Delta f_A(z) = 0$ in Ω and

$$f_A(z) = Q_A(z) - \frac{\varepsilon^{\frac{1}{2}} q(\hat{\xi})}{|z-b|} - \frac{\varepsilon^{\frac{3}{2}} w\cdot(z-b)}{|z-b|^3} - \frac{1}{6}\frac{\varepsilon^{\frac{5}{2}}(z-b)^T W(z-b)}{|z-b|^5} - \frac{1}{6}\frac{\varepsilon^{\frac{5}{2}}\Delta q(\hat{\xi})}{|z-b|^3}$$

for $z \in \partial\Omega$. Note that on $\partial\Omega$,

$$|f_A(z)| \le C\frac{\varepsilon^{\frac{5}{2}}|\Delta q(\hat{\xi})|}{|z-b|^3} + \frac{\varepsilon^{\frac{7}{2}}}{|z-b|^4} \le C\varepsilon^{\frac{5}{2}}|q(\hat{\xi})|^5 + C\varepsilon^{\frac{7}{2}} \le C\varepsilon^{\frac{7}{2}}.$$

Thus by maximum principle, we have $|f_A(z)| = O(\varepsilon^{\frac{7}{2}})$ for $z \in \Omega$. □

Let $PQ_A = Q_A - \varphi_A - \psi_A$. Using the above equation and the one of PQ_A in (3.1), ψ_A must satisfy

$$\begin{cases} \Delta\psi_A + \lambda\psi_A = \lambda(Q_A - \varphi_A) & \text{in } \Omega, \\ \psi_A = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.7}$$

We want to get the estimates of ψ_A.

Lemma 3.2. *Under the constraint* (3.2)*, for any $z \in \Omega$, one has $|\psi_A(z)| = O(\lambda\varepsilon^{\frac{3}{2}}|\log\varepsilon|)$.*

Proof. Let $y = (z-b)/\varepsilon$. Consider $\psi_A(z) = \varepsilon^{\frac{3}{2}}\tilde{\psi}((z-b)/\varepsilon)$, then $\Delta_z\psi_A + \lambda\psi_A = \varepsilon^{-\frac{1}{2}}[\Delta_y\tilde{\psi} + \lambda\varepsilon^2\tilde{\psi}]$. Thus $\tilde{\psi}(y)$ satisfies the following equation in $\Omega_\varepsilon := \{y = (z-b)/\varepsilon : z \in \Omega\}$,

$$\begin{cases} \Delta_y\tilde{\psi} + \lambda\varepsilon^2\tilde{\psi} = \lambda\tilde{F}(y) & \text{in } \Omega_\varepsilon, \\ \tilde{\psi} = 0 & \text{on } \partial\Omega_\varepsilon, \end{cases}$$

where

$$\tilde{F}(y) = \frac{1}{|y|} q\left(\frac{y}{|y|^2} + \hat{\xi}\right) - \varepsilon^{\frac{1}{2}}\varphi_A(\varepsilon y + b, b).$$

By the constraint (3.2), for any y in Ω_ε,

$$\frac{1}{|y|}\left|q\Big(\frac{y}{|y|^2} + \hat{\xi}\Big)\right| \le C\chi + C(|q(\hat{\xi})||y|^{-1} + |y|^{-2})(1-\chi) \le C|a|\langle y\rangle^{-1} + C\langle y\rangle^{-2}$$

where $\chi(x)$ is a cut-off function support at $B_2(0)$ and $\langle y\rangle = \sqrt{1+|y|^2}$. Define $K := diam(\Omega)+1$. If $y \in \Omega_\varepsilon$, then $\langle y\rangle \le K\varepsilon^{-1}$. Thus

$$|\tilde{F}(y)| \le C\varepsilon\langle y\rangle^{-1} + C\langle y\rangle^{-2} + C\varepsilon^2 \le C\langle y\rangle^{-2}, \quad \text{for} \quad y \in \Omega_\varepsilon.$$

It is easy to verify that $\bar{\psi}(y) = \log(K^2\varepsilon^{-1}) - \log\langle y\rangle$ is a super-solution if λ is small. Namely

$$\begin{aligned} \Delta\bar{\psi} + \lambda\varepsilon^2\bar{\psi} &\le -\frac{1}{2}\langle y\rangle^{-2} \quad \text{in} \quad \Omega_\varepsilon \\ \bar{\psi}(y) &\ge \log K \quad \text{on} \quad \partial\Omega_\varepsilon \end{aligned} \tag{3.8}$$

Thus there exists some uniform constant C large enough such that $\Delta(C\lambda\bar{\psi}\pm\tilde{\psi}_A)+\lambda\varepsilon^2(C\lambda\bar{\psi}\pm\tilde{\psi}_A) \le 0$ on Ω_ε and $C\lambda\bar{\psi}\pm\tilde{\psi}_A > 0$ on $\partial\Omega_\varepsilon$. Thus we have $C\lambda\bar{\psi}$ as a barrier function for $\tilde{\psi}_A$.

Note that the first eigenvalue is $\lambda_1(\Omega_\varepsilon) = \varepsilon^2\lambda_1(\Omega)$. Thus $\Delta_y + \lambda\varepsilon^2$ satisfies the maximum principle on Ω_ε when λ is small enough. Therefore $|\tilde{\psi}_A(y)| \le C\lambda|\bar{\psi}| = O(\lambda|\log\varepsilon|)$. The proof is complete. □

To summarize Lemma 3.1 and Lemma 3.2, we have the following proposition.

Proposition 3.3. *Assume that A satisfies the constraint (3.2). The solution of (3.1) can be written as $PQ_A = Q_A - \varphi_A - \psi_A$ where*

$$|\varphi_A + \psi_A| \le C\varepsilon^{\frac{3}{2}}. \tag{3.9}$$

Remark 3.4. *By elliptic theory, it is not hard to show that the dependence of $\varphi_A + \psi_A$ on the parameters in A is at least C^1.*

4. Energy expansion

In this section, we shall compute the energy of the approximate solution and find its leading-order term with respect to the parameters.

Define the energy of PQ_A as

$$\begin{aligned} J(PQ_A) &= \frac{1}{2}\int_\Omega (|\nabla PQ_A|^2 - \lambda|PQ_A|^2)dz - \frac{1}{6}\int_\Omega (PQ_A)^6 dz \\ &= \frac{1}{2}\int_\Omega Q_A^5 PQ_A dz - \frac{1}{6}\int_\Omega (PQ_A)^6 dz. \end{aligned}$$

For the first term on the right-hand side

$$\int_\Omega Q_A^5 PQ_A dz = \int_\Omega Q_A^6 dz - \int_\Omega Q_A^5(\varphi_A + \psi_A)dz,$$

For the second term

$$\int_\Omega (PQ_A)^6 dz = \int_\Omega (Q_A - \varphi_A - \psi_A)^6 dz = \int_\Omega \left(Q_A^6 - 6Q_A^5(\varphi_A+\psi_A)\right) dz + I,$$

where

$$\begin{aligned} I = \int_\Omega & 15Q_A^4(\varphi_A+\psi_A)^2 - 20Q_A^3(\varphi_A+\psi_A)^3 + 15Q_A^2(\varphi_A+\psi_A)^4 \\ & - 6Q_A(\varphi_A+\psi_A)^5 + (\varphi_A+\psi_A)^6 dz. \end{aligned}$$

To estimate each term, we need the following lemma.

Lemma 4.1. *Assuming the constraints (3.2), we have*

$$\int_\Omega |Q_A|^\ell dz \le C \begin{cases} \varepsilon^{\frac{3}{2}} + \varepsilon^{\frac{1}{2}}|a| & \text{if } \ell = 1, \\ \varepsilon^{3-\frac{\ell}{2}} & \text{if } \ell = 2,3,4,5, \end{cases} \tag{4.1}$$

$$\int_\Omega Q_A^6 dz = \int_{\mathbb{R}^3} q(z)^6 dz + O(\varepsilon^9), \tag{4.2}$$

$$\int_\Omega Q_A^5 dz = \varepsilon^{\frac{1}{2}} 4\pi q(\hat{\xi}) + O(\varepsilon^{\frac{15}{2}}), \tag{4.3}$$

$$\int_\Omega Q_A^5(z)(z-b) dz = \varepsilon^{\frac{3}{2}} 4\pi R_\beta^T \nabla q(\hat{\xi}) + O(\varepsilon^{\frac{15}{2}}), \tag{4.4}$$

$$\int_\Omega |Q_A|^5 |z-b|^2 dz = O(\varepsilon^{\frac{5}{2}}). \tag{4.5}$$

Proof. To prove (4.1). Since q is Kelvin invariant, then $q(z) = O(|z|^{-1})$ as $|z| \to \infty$. When $|z-b| < \varepsilon$, one has

$$\left| q\left(\frac{\varepsilon(z-b)}{|z-b|^2} + \hat{\xi} \right) \right| \le C \frac{|z-b|}{\varepsilon}$$

thus $|Q_A(z)| \le \varepsilon^{-\frac{1}{2}}$ and

$$\int_{|z-b|<\varepsilon} |Q_A|^\ell dz \le C\varepsilon^{3-\frac{\ell}{2}}.$$

When $\varepsilon < |z-b|$, (3.6) leads to

$$|Q_A(z)| \le \frac{\varepsilon^{\frac{1}{2}}|a|}{|z-b|} + C\frac{\varepsilon^{\frac{3}{2}}}{|z-b|^2}, \quad \text{for any } |z-b| \ge \varepsilon. \tag{4.6}$$

One can integrate its right-hand side respectively.

$$\varepsilon^{\frac{3}{2}\ell} \int_{\{z\in\Omega : |z-b|>\varepsilon\}} \frac{dz}{|z-b|^{2\ell}} \le \begin{cases} C\varepsilon^{\frac{3}{2}} & \text{if } \ell = 1, \\ C\varepsilon^{3-\frac{\ell}{2}} & \text{if } \ell \ge 2. \end{cases}$$

$$\varepsilon^{\frac{\ell}{2}}|a|^\ell \int_{\{z\in\Omega : |z-b|>\varepsilon\}} \frac{dz}{|z-b|^{\ell}} \le \begin{cases} C\varepsilon^{\frac{1}{2}}|a| & \text{if } \ell = 1, \\ C\varepsilon|a|^2 & \text{if } \ell = 2, \\ C\varepsilon^{\frac{3}{2}}|a|^3|\log\varepsilon| & \text{if } \ell = 3, \\ C\varepsilon^{3-\frac{\ell}{2}}|a|^\ell & \text{if } \ell = 4,5. \end{cases}$$

The proof of (4.1) is complete by combining the above three equations.

To prove (4.2) and (4.3), we split the integral $\int_\Omega Q_A^\ell dz = \int_{\mathbb{R}^3} Q_A^\ell dz - \int_{\mathbb{R}^3\setminus\Omega} Q_A^\ell dz$ for $\ell = 5,6$. On the one hand, we use (4.6) to get

$$\int_{\mathbb{R}^3\setminus\Omega} Q_A^\ell dz \le C\varepsilon^{\frac{3\ell}{2}} + C\varepsilon^\ell |a|^\ell \le C\varepsilon^{\frac{3\ell}{2}}, \quad \ell = 5,6.$$

On the other hand, making a change of variables, $z = b + \varepsilon\frac{R_\beta^T x}{|x|^2}$, one obtains that

$$\int_{\mathbb{R}^3} Q_A^6 dz = \int_{\mathbb{R}^3} q(z+\hat{\xi})^6 dz = \int_{\mathbb{R}^3} q(z)^6 dz.$$

and

$$\int_{\mathbb{R}^3} Q_A^5 dz = \int_{\mathbb{R}^3} \frac{\varepsilon^{\frac{5}{2}}}{|z-b|^5}\left(q\left(\frac{\varepsilon(z-b)}{|z-b|^2}+\hat{\xi}\right)\right)^5 dz$$
$$= \varepsilon^{\frac{1}{2}} \int_{\mathbb{R}^3} \frac{1}{|z|}[q(z+\hat{\xi})]^5 dz = \varepsilon^{\frac{1}{2}} \int_{\mathbb{R}^3} \frac{1}{|z-\hat{\xi}|} q(z)^5 dz = \varepsilon^{\frac{1}{2}} 4\pi q(\hat{\xi}).$$

where we have used

$$\int_{\mathbb{R}^3} \frac{1}{|z-\xi|} q(z)^5 dz = 4\pi q(\xi)$$

for any ξ in the last step. This completes (4.2) and (4.3).

The proof of (4.4) and (4.5) is similar to the previous proofs. We omit it. □

It follows from (3.9) and (4.1) that $I = O(\varepsilon^4)$. Therefore

$$J(PQ_A) = \frac{1}{3}\int_\Omega Q_A^6 dz + \frac{1}{2}\int_\Omega Q_A^5(\varphi_A + \psi_A) dz + O(\varepsilon^4). \tag{4.7}$$

We will compute the first two terms on the right-hand side.

Lemma 4.2. *Under the constraint* (3.2)*, we have*

$$\frac{1}{4\pi}\int_\Omega Q_A^5 \varphi_A dz = \varepsilon[q(\hat{\xi})]^2 \tau_\Omega(b) + \varepsilon^2 q(\hat{\xi}) w \cdot \nabla \tau_\Omega(b) + \varepsilon^3 w^T \nabla^2_{z,p} H_\Omega(b,b) w + O(\varepsilon^4).$$

Proof. Recall the expansion of φ_A in (3.5). Applying Lemma 4.1 yields

$$\begin{aligned}\int_\Omega Q_A^5 \varphi_A dz &= \varepsilon^{\frac{1}{2}} q(\hat{\xi}) \int_\Omega Q_A^5 H_\Omega(z,b) dz + \varepsilon^{\frac{3}{2}} w \cdot \int_\Omega Q_A^5 \nabla_p H_\Omega(z,b) dz \\ &\quad + \frac{1}{6}\varepsilon^{\frac{5}{2}} W_{ij} \int_\Omega Q_A^5 \partial^2_{p_i p_j} H_\Omega(z,b) dz + O(\varepsilon^4).\end{aligned} \tag{4.8}$$

Let us compute each term on the right-hand side. Using (1.7), one has the Taylor expansion

$$H_\Omega(z,b) = H_\Omega(b,b) + \nabla_z H_\Omega(b,b) \cdot (z-b) + O(|z-b|^2).$$

Applying (4.3)-(4.5), we have

$$\begin{aligned}\int_\Omega Q_A^5 H_\Omega(z,b) dz &= H_\Omega(b,b)\int_\Omega Q_A^5 dz + \nabla_z H_\Omega(b,b) \cdot \int_\Omega Q_A^5 (z-b) dz + O(\varepsilon^{\frac{5}{2}}) \\ &= \varepsilon^{\frac{1}{2}} 4\pi q(\hat{\xi}) H_\Omega(b,b) + \varepsilon^{\frac{3}{2}} 4\pi w \cdot \nabla_z H_\Omega(b,b) + O(\varepsilon^{\frac{5}{2}}).\end{aligned}$$

Similarly, using $\nabla_p H_\Omega(z,b) = \nabla_p H_\Omega(b,b) + \nabla^2_{z,p} H_\Omega(b,b) \cdot (z-b) + O(|z-b|^2)$,

$$\begin{aligned}\int_\Omega Q_A^5 \nabla_p H_\Omega(z,b) dz &= \nabla_p H_\Omega(b,b)\int_\Omega Q_A^5 dz + \nabla^2_{z,p} H_\Omega(b,b) \cdot \int_\Omega Q_A^5 (z-b) dz + O(\varepsilon^{\frac{5}{2}}) \\ &= \varepsilon^{\frac{1}{2}} 4\pi q(\hat{\xi}) \nabla_p H_\Omega(b,b) + \varepsilon^{\frac{3}{2}} 4\pi \nabla^2_{z,p} H_\Omega(b,b) R_\beta^T \nabla q(\hat{\xi}) + O(\varepsilon^{\frac{5}{2}})\end{aligned}$$

and

$$\begin{aligned}\int_\Omega Q_A^5 \partial^2_{p_i p_j} H_\Omega(z,b) dz &= \int_\Omega Q_A^5 \partial^2_{p_i p_j} H_\Omega(b,b) dz + \nabla_z \partial^2_{p_i p_j} H_\Omega(b,b) \cdot \int_\Omega Q_A^5 (z-b) dz + O(\varepsilon^{\frac{5}{2}}) \\ &= O(\varepsilon^{\frac{1}{2}}|q(\hat{\xi})|) + O(\varepsilon^{\frac{3}{2}}) + O(\varepsilon^{\frac{5}{2}}) = O(\varepsilon^{\frac{3}{2}}).\end{aligned}$$

Inserting the above three estimates back to (4.8), and using $\nabla_z H_\Omega(b,b) = \nabla_p H_\Omega(b,b) = \frac{1}{2}\nabla\tau_\Omega(b)$, we can get the conclusion. □

Lemma 4.3. *Under the constraint* (3.2)*, we have*

$$\int_\Omega Q_A^5\psi_A dz = -\lambda\int_\Omega Q_A^2 dz + O(\lambda\varepsilon^3).$$

Proof. Using (3.1) and (3.7)

$$\begin{aligned}\int_\Omega Q_A^5\psi_A dz &= -\int_\Omega(\Delta PQ_A+\lambda PQ_A)\psi_A dz = -\int_\Omega PQ_A(\Delta\psi_A+\lambda\psi_A)dz\\ &= -\lambda\int_\Omega(Q_A-\varphi_A-\psi_A)(Q_A-\varphi_A)dz\\ &= -\lambda\int_\Omega Q_A^2dz+\lambda\int_\Omega Q_A(2\varphi_A+\psi_A)dz-\lambda\int_\Omega\varphi_A(\varphi_A+\psi_A)dz.\end{aligned}\tag{4.9}$$

Recall (3.9), Lemma 3.1 and (4.1),

$$\left|\int_\Omega Q_A(2\varphi_A+\psi_A)dz\right| \le C\varepsilon^{\frac{3}{2}}\int_\Omega|Q_A|dz\le C\varepsilon^3.$$

Similarly,

$$\left|\int_\Omega\varphi_A(\varphi_A+\psi_A)dz\right|\le C\varepsilon^3.$$

Therefore, plugging in the above estimates back to (4.9),

$$\int_\Omega Q_A^5\psi_A dz = -\lambda\int_\Omega Q_A^2 dz + O(\lambda\varepsilon^3).$$

□

Now, let us compute $\int_\Omega Q_A^2 dz$. We define

$$\rho(b) = \frac{1}{4\pi}\int_\Omega\frac{1}{|z-b|^2}dz.\tag{4.10}$$

Note that $\rho(b)$ is a smooth and bounded function for $b\in\Omega$. Taking the derivative to b on both sides, we get

$$\nabla\rho(b) = \frac{1}{4\pi}P.V.\int_\Omega\frac{2(z-b)}{|z-b|^4}dz = -\frac{1}{4\pi}P.V.\int_{\mathbb{R}^3\setminus\Omega}\frac{2(z-b)}{|z-b|^4}dz.$$

Lemma 4.4. *Assuming* (3.2)*, we have*

$$\frac{1}{4\pi}\int_\Omega Q_A^2dz = \varepsilon^2 P.V.\int_{\mathbb{R}^3}\frac{[q(z+\xi)]^2}{4\pi|z|^4}dz + O(\varepsilon^3+|a|^2\varepsilon+\varepsilon^2|a|).$$

Proof. Note that

$$\frac{1}{4\pi}\int_\Omega Q_A^2dz = \frac{1}{4\pi}\int_\Omega\frac{\varepsilon}{|z-b|^2}q\left(\frac{\varepsilon R_\beta(z-b)}{|z-b|^2}+\hat{\xi}\right)^2dz.$$

Notice that $\frac{1}{4\pi}\int_\Omega \frac{\varepsilon q(\hat{\xi})^2}{|z-b|^2}dz = \varepsilon[q(\hat{\xi})]^2\rho(b) = O(|a|^2\varepsilon)$. It suffices to estimate

$$I = \frac{1}{4\pi}\int_\Omega \frac{\varepsilon}{|z-b|^2}\left(q\left(\frac{\varepsilon R_\beta(z-b)}{|z-b|^2}+\hat{\xi}\right)^2 - q(\hat{\xi})^2\right)dz.$$

Again, we split the integral into two on $\mathbb{R}^3$ and $\mathbb{R}^3 \setminus \Omega$. First, using a change of variable,

$$\begin{aligned}
&\int_{\mathbb{R}^3} \frac{\varepsilon}{|z-b|^2}\left[q(\tfrac{\varepsilon R_\beta(z-b)}{|z-b|^2}+\hat{\xi})^2 - q(\hat{\xi})^2\right]dz = \varepsilon^2 P.V.\int_{\mathbb{R}^3}\frac{[q(z+\hat{\xi})]^2-[q(\hat{\xi})]^2}{|z|^4}dz\\
&= \varepsilon^2 P.V.\int_{\mathbb{R}^3}\frac{[q(z+\xi)]^2}{|z|^4}dz + \varepsilon^2 P.V.\int_{\mathbb{R}^3}\frac{[q(z+\hat{\xi})]^2-[q(\hat{\xi})]^2-[q(z+\xi)]^2}{|z|^4}dz\\
&= \varepsilon^2 P.V.\int_{\mathbb{R}^3}\frac{[q(z+\xi)]^2}{|z|^4}dz + O(\varepsilon^2|a|).
\end{aligned}$$

Second, since $\varepsilon/|z-b| \lesssim 1$ for $z \in \mathbb{R}^3 \setminus \Omega$, then

$$q\left(\frac{\varepsilon R_\beta(z-b)}{|z-b|^2}+\hat{\xi}\right)^2 - q(\hat{\xi})^2 = 2q(\hat{\xi})\nabla q(\hat{\xi})\cdot\frac{\varepsilon R_\beta(z-b)}{|z-b|^2} + O(\varepsilon^2|z-b|^{-2})$$

Therefore

$$\begin{aligned}
&\frac{1}{4\pi}\int_{\mathbb{R}^3\setminus\Omega}\frac{\varepsilon}{|z-b|^2}\left(q\left(\frac{\varepsilon R_\beta(z-b)}{|z-b|^2}+\hat{\xi}\right)^2 - q(\hat{\xi})^2\right)dz\\
&= \varepsilon^2 q(\hat{\xi})w\cdot\frac{1}{2\pi}P.V.\int_{\mathbb{R}^3\setminus\Omega}\frac{(z-b)}{|z-b|^4} + O(\varepsilon^3)\\
&= -\varepsilon^2 q(\hat{\xi})w\cdot\nabla\rho(b) + O(\varepsilon^3) = O(|a|\varepsilon^2+\varepsilon^3).
\end{aligned}$$

Combining the two results, the proof is complete.

□

Inserting Lemma 4.2, Lemma 4.3 and Lemma 4.4 to (4.7), we obtain

Proposition 4.5.

$$J(PQ_A) = \frac{1}{3}\int_{\mathbb{R}^3} q^6 dz + 2\pi\Psi(A) + O(\varepsilon^4+\lambda\varepsilon^3), \tag{4.11}$$

where

$$\Psi(A) = \varepsilon[q(\hat{\xi})]^2\tau_\Omega(b) + \varepsilon^2 q(\hat{\xi})w\cdot\nabla\tau_\Omega(b) + \varepsilon^3 w^T\nabla^2_{z,p}H_\Omega(b,b)w - \lambda\varepsilon^2 C_*(\xi). \tag{4.12}$$

and

$$C_*(\xi) = P.V.\int_{\mathbb{R}^3}\frac{[q(z+\xi)]^2}{4\pi|z|^4}dz. \tag{4.13}$$

Remark 4.6. *Clearly, $\Psi(A)$ depends smoothly on the parameters of A. By elliptic theory and Remark 3.4, it is not hard to show that the dependence of $J(PQ_A)$ on the parameters in A is at least C^1.*

5. THE LINEAR AND NONLINEAR PROBLEM

From the discussion in the previous Section, we will consider

$$PQ_A = Q_A - \varphi_A - \psi_A \tag{5.1}$$

as the approximate solution to our problem. Using PQ_A, we will present the results investigating the corresponding linearized and nonlinear problems. From which we obtain a solution, up to some Lagrange multipliers. In the next Section, we shall find parameters through the minimization method to make the coefficients of these Lagrange multipliers vanish, thereby finding a genuine solution to the Brezis-Nirenberg problem.

Based on PQ_A, we introduce the following form of PQ_A for convenience

$$\begin{aligned} PQ_{A'} &= \varepsilon^{\frac{1}{2}} Q_A - \varepsilon^{\frac{1}{2}} \varphi_A - \varepsilon^{\frac{1}{2}} \psi_A \\ &= Q_{A'} - \varepsilon^{\frac{1}{2}} \varphi_A - \varepsilon^{\frac{1}{2}} \psi_A \\ &= \frac{1}{|y-b_\varepsilon|} q\left(\frac{R_\beta(y-b_\varepsilon)}{|y-b_\varepsilon|^2} + \varepsilon^{-1}\xi + \varepsilon^{-1} a\nu(\xi)\right) - \varepsilon^{\frac{1}{2}} \varphi_A - \varepsilon^{\frac{1}{2}} \psi_A, \end{aligned}$$

where $y = \frac{z}{\varepsilon}$ and $b_\varepsilon = \frac{b}{\varepsilon}$. It is crucial to point out that the functions $PQ_{A'}$ and $Q_{A'}$ depend on the parameters $A' = (\Lambda, \varepsilon^{-1}a, \varepsilon^{-1}\xi, b_\varepsilon, \beta)$. Particularly, the dependence of $Q_{A'}$ on Λ can be understood as follows:

$$Q_{A'} = \frac{\Lambda^{\frac{1}{2}}}{|y-b_\varepsilon|} q\left(\frac{R_\beta \Lambda(y-b_\varepsilon)}{|y-b_\varepsilon|^2} + \varepsilon^{-1}\xi + \varepsilon^{-1} a\nu(\xi)\right)\Bigg|_{\Lambda=1}.$$

Since we scale Q_A by ε for the space variable, so the parameter Λ does not appear in $Q_{A'}$ and we will not carry it in the expression of $Q_{A'}$ in the argument.

The Brezis-Nirenberg problem is equivalent to finding ϕ such that

$$\Delta_y(PQ_{A'} + \phi) + \lambda\varepsilon^2(PQ_{A'} + \phi) + (PQ_{A'} + \phi)^5 = 0, \tag{5.2}$$

which can be rewritten as

$$\begin{cases} \Delta_y \phi + \lambda\varepsilon^2\phi + 5PQ_{A'}^4\phi = -E - N_\varepsilon(\phi) & \text{in} \quad \Omega_\varepsilon, \\ \phi = 0 & \text{on} \quad \partial\Omega_\varepsilon, \end{cases} \tag{5.3}$$

where

$$\begin{aligned} E &= \Delta PQ_{A'} + \lambda\varepsilon^2 PQ_{A'} + PQ_{A'}^5 = PQ_{A'}^5 - Q_{A'}^5 \\ &= -5\varepsilon^{\frac{1}{2}} Q_{A'}^4(\varphi_A + \psi_A) + O(\varepsilon^4)Q_{A'}^3 + O(\varepsilon^6)Q_{A'}^2 + O(\varepsilon^8)Q_{A'} + O(\varepsilon^{10}), \end{aligned}$$

and

$$N_\varepsilon(\phi) = (PQ_{A'} + \phi)^5 - PQ_{A'}^5 - 5PQ_{A'}^4\phi.$$

In order to study the problem (5.3), we will outline the findings concerning its linearized problem as well as the nonlinear problem. Given that this part is now well-established and follows standard procedures, we will present the result and refer the readers to [10, 20, 21, 23] and the references therein for details.

For the following linearized problem

$$\begin{cases} \Delta\phi + \lambda\varepsilon^2\phi + 5PQ_{A'}^4\phi = h + \sum_{j=0}^{5} c_j PQ_{A'}^4 \hat{Z}_j(y), & \text{in } \Omega_\varepsilon, \\ \phi = 0 & \text{on } \partial\Omega_\varepsilon, \\ \int_{B_{1/\varepsilon}(0)} \phi PQ_{A'}^4 \hat{Z}_j(y)dy = 0, \quad j = 0, 1, \cdots, 5, \end{cases} \tag{5.4}$$

where $A' = (\Lambda, \varepsilon^{-1}a, \varepsilon^{-1}\xi, b_\varepsilon, \beta)$ and $\hat{Z}_j(y) = \chi(y)Z_j(y)$. Here $Z_j(y)$ is the kernel function introduced in section 2.3 [2] and $\chi(y)$ is the characteristic function such that

$$\chi(y) = \begin{cases} 1, & y \in B_{1/4\varepsilon}(b_\varepsilon), \\ 0, & y \in B_{1/2\varepsilon}(b_\varepsilon)^c. \end{cases}$$

To state the main result concerning the linearized problem, we introduce the following weighted function space:

$$\|h\|_{**} = \sup_{y \in B_{1/\varepsilon}(0)} \left| \langle y - b_\varepsilon \rangle^{3+2\sigma} h(y) \right|, \tag{5.5}$$

$$\|\phi\|_* = \sup_{y \in \Omega_\varepsilon} |\langle y - b_\varepsilon \rangle \phi(y)| + \sup_{y \in \Omega_\varepsilon} |\langle y - b_\varepsilon \rangle^2 \nabla\phi(y)|, \tag{5.6}$$

where σ is a sufficiently small positive number and $\langle y \rangle = \sqrt{1 + |y|^2}$. The first result of this section is the following.

Proposition 5.1. [23, Proposition 6.1] *Suppose that the parameters A and ε satisfy the relation in (3.2). Then there exists λ_0 small enough such that for all $\lambda < \lambda_0$ and all $h \in C^\alpha(\Omega_\varepsilon)$ which is even in z_3, the problem (5.4) has a unique solution $\phi \equiv L_\varepsilon(h)$ which is even in z_3, and*

$$\|\phi\|_* \le C\|h\|_{**}, \quad |c_j| \le C\|h\|_{**},$$

and

$$\|\nabla_{A'}\phi\|_* \le C\|h\|_{**}.$$

Next, we shall solve the nonlinear problem

$$\begin{cases} \Delta\phi + \lambda\varepsilon^2\phi + 5PQ_{A'}^4\phi = -(N_\varepsilon(\phi) + E) + \sum_{j=0}^{5} c_j PQ_{A'}^4 \hat{Z}_j & \text{in } \Omega_\varepsilon(0), \\ \phi = 0 & \text{on } \partial\Omega_\varepsilon(0). \end{cases} \tag{5.7}$$

Consider (5.7) we have the following proposition

Proposition 5.2. [23, Proposition 6.2] *Suppose the constraints hold on the parameters A. Then there exists a small constant λ_0 such that for all $\lambda < \lambda_0$, there is a unique solution ϕ to problem (5.7) with*

$$\|\phi\|_* \le C\varepsilon^2, \quad \|\nabla_{A'}\phi\|_* \le C\varepsilon^2. \tag{5.8}$$

[2] The parameters $(1, \xi, 0, 0, \theta_*)$ are replaced by $(\Lambda, \varepsilon^{-1}\xi, \varepsilon^{-1}a, b_\varepsilon, \beta)$.

6. THE FINITE-DIMENSIONAL REDUCTION AND THE CRITICAL POINT

In this section, we will first set up the reduction that transforms the original infinite-dimensional problem into a finite-dimensional one. Then, we will determine the critical points of the energy with respect to the parameters, using these to establish the proof of Theorem 1.1.

Suppose that $\phi_{A'}$ is the solution of (5.7). Let $u_\varepsilon(y) = PQ_{A'}(y) + \phi_{A'}(y)$, then

$$\Delta_y u_\varepsilon(y) + \lambda\varepsilon^2 u_\varepsilon(y) + u_\varepsilon^5(y) = \sum_{j=0}^{5} c_j PQ_{A'}^4 \hat{Z}_j.$$

Note that u_ε will satisfy (5.2) if the Lagrange multiplier $c_j = 0$ for $j = 0, 1, \cdots, 5$. The following reduction Lemma says that this is equivalent to the criticality of A' in a finite-dimensional space. Returning to the original variable z and A (before scaling), denoting $PQ_A(z) = \varepsilon^{-1/2} PQ_{A'}(z/\varepsilon)$ and $\phi_A(z) = \varepsilon^{-1/2}\phi_{A'}(z/\varepsilon)$, then $PQ_A(z) + \phi_A(z)$ will be a solution to the Brezis-Nirenberg problem under the criticality of A.

Lemma 6.1. *$u_\varepsilon = PQ_{A'} + \phi_{A'}(y)$ is a solution of problem (5.2) if and only if A' is a critical point of the energy*

$$J_\varepsilon(u_\varepsilon) := \frac{1}{2}\int_{\Omega_\varepsilon} |\nabla_y u_\varepsilon|^2 dy - \frac{\lambda\varepsilon^2}{2}\int_{\Omega_\varepsilon} u_\varepsilon^2 dy - \frac{1}{6}\int_{\Omega_\varepsilon} u_\varepsilon^6 dy.$$

Equivalently, $PQ_A(z) + \phi_A(z)$ is a solution to (1.2) if and only if A is a critical point of the energy $J(PQ_A + \phi_A) = J_\varepsilon(u_\varepsilon)$ where

$$\begin{aligned} J(PQ_A + \phi_A) := & \frac{1}{2}\int_{\Omega} |\nabla_z (PQ_A + \phi_A)|^2 dz - \frac{\lambda}{2}\int_{\Omega} (PQ_A + \phi_A)^2 dz \\ & - \frac{1}{6}\int_{\Omega} (PQ_A + \phi_A)^6 dz. \end{aligned}$$

Proof. Let $A'_\ell,\ A_\ell,\ \ell = 0, \cdots, 5$ be the elements of A' and A respectively. Considering the derivative of $J(PQ_A + \phi_A)$ with respect to A, we see that $\frac{\partial J(PQ_A+\phi_A)}{\partial A_\ell} = 0$ is equivalent to say that $\frac{\partial J_\varepsilon(PQ_{A'}+\phi_{A'})}{\partial A'_\ell} = 0$. Next we compute

$$\frac{\partial}{\partial A'_\ell} J_\varepsilon(PQ_{A'} + \phi_{A'}) = DJ_\varepsilon(PQ_{A'} + \phi_{A'})\left[\frac{\partial}{\partial A'_\ell} PQ_{A'} + \frac{\partial}{\partial A'_\ell}\phi_{A'}\right].$$

On the other hand, one sees that

$$\frac{\partial PQ_{A'}}{\partial A'_\ell} = \hat{Z}_2(y) + o(1).$$

Using Proposition 5.2 one can show

$$\left\|\frac{\partial \phi_{A'}}{\partial A'_\ell}\right\|_{**} = O(\varepsilon^2) \quad \text{as} \quad \lambda \to 0,$$

then it implies that

$$DJ_\varepsilon(PQ_{A'} + \phi_{A'})\left[\hat{Z}_\ell + o(1)\right] = 0, \quad \forall \ell = 0, \cdots, 5. \tag{6.1}$$

From the fact that $DJ_\varepsilon(PQ_{A'}+\phi_{A'})[g]=0$ for all functions such that $\int_{\Sigma_{K,\varepsilon}} PQ_{A'}^4 \hat{Z}_\ell g dy = 0$, we can see that (6.1) can be written as

$$DJ_\varepsilon(PQ_{A'}+\phi_{A'})\left[\hat{Z}_\ell + o(1)\Xi\right] = 0, \quad \forall \ell = 0, \cdots, 5, \tag{6.2}$$

where Ξ is a uniformly bounded function, that belongs to the vector space generated by the functions $\hat{Z}_\ell$. Thus,

$$DJ_\varepsilon(PQ_{A'}+\phi_{A'})[\hat{Z}_\ell] = 0, \quad \forall \ell = 0, \cdots, 5.$$

By definition of c_ℓ in (5.7), then we readily derive that this is equivalent to $c_\ell = 0$ for all ℓ and it finishes the proof. □

In the following calculation, we shall the see the major in the expansion of $J(PQ_A+\phi_A)$ is $J(PQ_A)$.

$$\begin{aligned} J(PQ_A+\phi_A) &= J_\varepsilon(PQ_{A'}+\phi_{A'}) \\ &= \frac{1}{2}\int_{\Omega_\varepsilon} |\nabla_y (PQ_{A'}+\phi_{A'})|^2 dy - \frac{\lambda\varepsilon^2}{2}\int_{\Omega_\varepsilon}(PQ_{A'}+\phi_{A'})^2 dy \\ &\quad - \frac{1}{6}\int_{\Omega_\varepsilon}(PQ_{A'}+\phi_{A'})^6 dy. \end{aligned}$$

By expanding all terms and grouping them, we get

$$\begin{aligned} J_\varepsilon(PQ_{A'}+\phi_{A'}) = J_\varepsilon(PQ_{A'}) &- \int_{\Omega_\varepsilon}(\Delta PQ_{A'} + \lambda\varepsilon^2 PQ_{A'} + PQ_{A'}^5)\phi_{A'} dy \\ &- \frac{1}{2}\int_{\Omega_\varepsilon}(\Delta\phi_{A'} + \lambda\varepsilon^2\phi_{A'} + 5PQ_{A'}^4\phi_{A'})\phi_{A'} dy \\ &- \frac{1}{6}\int_{\Omega_\varepsilon}(20PQ_{A'}^3\phi_{A'}^3 + 15PQ_{A'}^2\phi_{A'}^4 + 6PQ_{A'}\phi_{A'}^5 + \phi_{A'}^6) dy. \end{aligned} \tag{6.3}$$

It is known that $\|\phi_A\|_* \le C\varepsilon^2$, one can easily show that

$$\left|\int_{\Omega_\varepsilon}(20PQ_{A'}^3\phi_{A'}^3 + 15PQ_{A'}^2\phi_{A'}^4 + 6PQ_{A'}\phi_{A'}^5 + \phi_{A'}^6) dy\right| \le C\varepsilon^6.$$

Consider the second term on the right-hand side of (6.3), we have

$$\Delta PQ_{A'} + \lambda\varepsilon^2 PQ_{A'} + PQ_{A'}^5 = E(PQ_{A'}), \tag{6.4}$$

where

$$\begin{aligned} E(PQ_{A'}) = PQ_{A'}^5 - Q_{A'}^5 = &- 5\varepsilon^{\frac{1}{2}} Q_{A'}^4(\varphi_A+\psi_A) + O(\varepsilon^4)Q_{A'}^3 + O(\varepsilon^6)Q_{A'}^2 \\ &+ O(\varepsilon^8)Q_{A'} + O(\varepsilon^{10}). \end{aligned}$$

Then

$$\begin{aligned} \int_{\Omega_\varepsilon}(\Delta PQ_{A'} + \lambda\varepsilon^2 PQ_{A'} + PQ_{A'}^5)\phi_{A'} dy &= -5\int_{\Omega_\varepsilon}\varepsilon^{\frac{1}{2}} Q_{A'}^4(\varphi_A+\psi_A)\phi_{A'} dy + O(\varepsilon^6) \\ &= O(\varepsilon^4)\int_{\Omega_\varepsilon} Q_{A'}^4 dy + O(\varepsilon^6) = O(\varepsilon^4). \end{aligned} \tag{6.5}$$

While for the third term on the right hand side of (6.3),

$$\Delta\phi_{A'} + \lambda\varepsilon^2\phi_{A'} + 5PQ_{A'}^4\phi_{A'} = -E(PQ_{A'}) - N(\phi_{A'}) + \sum_{j=0}^{5} c_j PQ_{A'}^4\hat{Z}_j. \tag{6.6}$$

As the above computation,

$$\int_{\Omega_\varepsilon} E(PQ_{A'})\phi_{A'}dy = O(\varepsilon^4). \tag{6.7}$$

While for the higher order term $N(\phi_{A'})$, it is easy to see that

$$\int_{\Omega_\varepsilon} N(\phi_{A'})\phi_{A'}dy = O(\varepsilon^6). \tag{6.8}$$

The multiplicative of $c_j PQ_A^4\hat{Z}_j$ abd $\phi_{A'}$ is obvious zero due to the setting of $\phi_{A'}$. Therefore, we conclude that

$$\int_{\Omega_\varepsilon} (\Delta\phi_{A'} + \lambda\varepsilon^2\phi_{A'} + 5PQ_{A'}^4\phi_{A'})\phi_{A'}dy = O(\varepsilon^4). \tag{6.9}$$

Thus we conclude that

$$J(PQ_A + \phi_A) = J_\varepsilon(PQ_{A'} + \phi_{A'}) = J_\varepsilon(PQ_{A'}) + O(\varepsilon^4) = J(PQ_A) + O(\varepsilon^4). \tag{6.10}$$

Theorem 6.2. *There exists δ small such that for K large enough, the* $\inf_{A\in\mathscr{C}} J(PQ_A + \phi_A)$ *is achieved in the interior of the set $\mathscr{C}$ defined by* (3.2)*, i.e.*

$$\varepsilon \in [\delta\lambda, \delta^{-1}\lambda], \quad |a| \le \delta^{-1}\varepsilon^{3/2}, \quad |b|^{\theta_0} \le \delta^{-4}\varepsilon, \quad \xi \in \mathcal{N}(q), \quad \beta \in \mathbb{S}. \tag{6.11}$$

Proof. Since the constraint set $\mathscr{C}$ is closed, the infimum of $J(PQ_A + \phi_A)$ over $A \in \mathscr{C}$ is attained at some point $A = (\varepsilon, \xi, a, d, \alpha_b, \beta) \in \mathscr{C}$. Note that since we denote $b = |b|e^{i\alpha_b}$, then it is equivalent to write $A = (\varepsilon, \xi, a, b, \beta)$.

We will prove that for each point on the boundary $\partial\mathscr{C}$ there is another interior point whose value is strictly smaller than that. Thus, the infimum must be achieved in the interior. First, let us recall that

$$J(PQ_A + \phi_A) = J(PQ_A) + O(\varepsilon^4) = C_q + 2\pi\Psi(A) + O(\varepsilon^4 + \lambda\varepsilon^3). \tag{6.12}$$

Here we denote $C_q := \frac{1}{3}\int_{\mathbb{R}^3} q^6$ is a constant and does not depend on A. We need to study $\Psi(A)$ where

$$\Psi(A) = \varepsilon[q(\hat{\xi})]^2\tau_\Omega(b) + \varepsilon^2 q(\hat{\xi})w\cdot\nabla\tau_\Omega(b) + \varepsilon^3 w^T\nabla^2_{z,p}H_\Omega(b,b)w - \lambda\varepsilon^2 C_*(\xi). \tag{6.13}$$

First, we notice that ξ and β belong to some smooth compact manifold respectively. Thus the infimum of $J(PQ_A + \phi_A)$ is automatically achieved in the interior for these two parameters. We only need to consider ε, a, and b.

(1) Consider the variation of ε, and fix all the other variables in A. Thus one can think of $J(PQ_A + \phi_A) = \mathcal{J}_1(\varepsilon)$ as a function of ε on $[\delta\lambda, \delta^{-1}\lambda]$. If $\varepsilon = \delta\lambda$, then using (6.12), (6.13), and the definition of $\mathscr{C}$ in (6.11),

$$\mathcal{J}_1(\delta\lambda) \ge C_q - 2\pi\delta^2\lambda^3 C_*(\xi) + O(\delta^3\lambda^4).$$

If $\varepsilon = \delta^{-1}\lambda$, then the same estimates yields

$$\mathcal{J}_1(\delta^{-1}\lambda) \geq C_q + 2\pi \left(\delta^{-3}\lambda^3 w^T \nabla^2_{z,p} H_\Omega(b,b) w - \delta^{-2}\lambda^3 C_*(\xi)\right) + O(\delta^{-4}\lambda^4).$$

However, one can choose another $\varepsilon_* = \delta_*\lambda$ with $\delta_* = 2C_*(\xi)/(3w^T\nabla^2_{z,p}H_\Omega(b,b)w)$ and compute its energy. Notice that assumption (C-2) implies that δ_* is uniformly bounded when λ is small. Using the bounds in the constraint, we have $|\varepsilon_*[q(\hat{\xi})]^2\tau_\Omega(b)| \leq C\varepsilon_*|a|^2 \leq C\delta_*^{-1}\lambda^4$. Therefore

$$\begin{aligned}\mathcal{J}_1(\delta_* K^{-3}) &\leq C_q + 2\pi \left(\frac{2}{3}\lambda(\varepsilon_*)^2 C_*(\xi) - \lambda(\varepsilon_*)^2 C_*(\xi)\right) + O(\delta_*^{-1}\lambda^4)\\ &\leq C_q - \frac{\pi}{2}\lambda(\varepsilon_*)^2 C_*(\xi) = C_q - \frac{\pi}{2}\lambda^3\delta_*^2 C_*(\xi).\end{aligned}$$

Comparing the order λ^3 of the above three cases, one can choose a sufficiently small δ satisfying $C^{-1}\delta < \delta_* < C\delta^{-1}$ and sufficiently small λ such that

$$\mathcal{J}_1(\delta_*\lambda) < \min\{\mathcal{J}_1(\delta\lambda), \mathcal{J}_1(\delta^{-1}\lambda)\}.$$

(2) Consider the variation of a, and fix all the other variables in A. Thus one can think of $J(PQ_A + \phi_A) = \mathcal{J}_2(|a|)$ as a function of $|a|$ on $[0, \delta^{-1}\varepsilon^{3/2}]$. Notice that $q(\hat{\xi}) = a|\nabla q(\xi)| + O(a^2)$ and $w = R_\beta^T \nabla q(\xi) + O(|a|)$. Denote $w_0 := R_\beta^T\nabla q(\xi)$. If $|a| = \delta^{-1}\varepsilon^{3/2}$,

$$\begin{aligned}\mathcal{J}_2(\delta^{-1}\varepsilon^{3/2}) = C_q &+ 2\pi \left(\delta^{-2}\varepsilon^4 |\nabla q(\xi)|^2 \tau_\Omega(b) + \varepsilon^3 w_0^T \nabla^2_{z,p} H_\Omega(b,b) w_0 - \lambda\varepsilon^2 C_*(\xi)\right)\\ &+ O(\varepsilon^4 + \lambda\varepsilon^3).\end{aligned}$$

However, if we choose $a = 0$ then

$$\mathcal{J}_2(0) \leq C_q + 2\pi \left(\varepsilon^3 w_0^T \nabla^2_{z,p} H_\Omega(b,b) w_0 - \lambda\varepsilon^2 C_*(\xi)\right) + O(\varepsilon^4 + \lambda\varepsilon^3).$$

Taking λ small enough and δ small enough, we have

$$\mathcal{J}_2(0) < \mathcal{J}_2(\delta^{-1}\varepsilon^{3/2}).$$

(3) Consider the variation of b, and fix all the other variables in A. Thus one can think of $J(PQ_A + \phi_A) = \mathcal{J}_3(b)$ as a function of b with $|b|^{\theta_0} \leq \delta^{-4}\varepsilon$. We assume that $\lambda \ll \delta$ such that $\delta^{-4}\varepsilon < r_0^{\theta_0}$ where r_0 is defined in (C-3).

$$\begin{aligned}\frac{1}{2\pi}[\mathcal{J}_3(b) - \mathcal{J}_3(0)] &= \varepsilon[q(\hat{\xi})]^2(\tau_\Omega(b) - \tau_\Omega(0)) + \varepsilon^2 q(\hat{\xi}) w \cdot \nabla\tau_\Omega(b)\\ &\quad + \varepsilon^3\left[w^T\nabla^2_{z,p}H_\Omega(b,b)w - w^T\nabla^2_{z,p}H_\Omega(0,0)w\right] + O(\varepsilon^4 + \lambda\varepsilon^3)\end{aligned}$$

Now take b such that $|b|^{\theta_0} = \delta^{-4}\varepsilon$. It is easy to see that $|\varepsilon^2 q(\hat{\xi})w \cdot \nabla\tau_\Omega(b)| \leq C|a||b| \leq C\delta^{-3}\varepsilon^4$. Moreover, assumption (C-2) implies that

$$\varepsilon^3[w^T\nabla^2_{z,p}H_\Omega(b,b)w - w^T\nabla^2_{z,p}H_\Omega(0,0)w] \geq c_0\varepsilon^3|b|^{\theta_0}|w|^2 \geq \tilde{c}_0\delta^{-4}\varepsilon^4.$$

Therefore, for such b,

$$\frac{1}{2\pi}[\mathcal{J}_3(b) - \mathcal{J}_3(0)] \geq \tilde{c}_0\delta^{-4}\varepsilon^4 - C\delta^{-3}\varepsilon^4 + O(\varepsilon^4 + \lambda\varepsilon^3) > 0$$

provided δ is chosen small and λ is small enough. Consequently, $\mathcal{J}_3(0) < \inf_{\{b||b|^{\theta_0} = \delta^{-4}\varepsilon\}} \mathcal{J}_3(b)$.

Now, combining the previous (1)-(3) parts, we know that the infimum of $J(PQ_A+\phi_A)$ must be achieved when the parameters $\varepsilon, \xi, a, b, \beta$ are in the interior of the constraint set. Hence we finish the whole proof. □

Proof of Theorem 1.1. By Theorem 6.2, it follows that $\inf_{A\in\mathscr{C}} J(PQ_A+\phi_A)$ is attained in the interior of $\mathscr{C}$. By Remark 4.6 and Proposition 5.1, $J(PQ_A+\phi_A)$ is at least C^1 on the parameters of A. Then the partial derivatives of $J(PQ_A+\phi_A)$ with respect to the six parameters of A are zero at a minimum point in the interior of $\mathscr{C}$. Then using Lemma 6.1, we find a nontrivial solution to (1.1) and prove Theorem 1.1. □

7. The Green function of Cube and the proof of Theorem 1.3

In this section, we shall study the Green function of a cube and cuboid. In particular, we shall prove that its Green function satisfies (C-1)-(C-3). Let $G(z,p)$ be the Green function satisfying

$$\begin{cases}\Delta G(z,p)+\delta_p(z)=0 & \text{in } [-1,1]^3,\\ G(z,p)=0 & \text{on } \partial\left([-1,1]^3\right).\end{cases} \tag{7.1}$$

By the method of reflection, we can write the Green function and decompose it as

$$G(z,p)=\frac{1}{4\pi}\sum_{l,m,n=-\infty}^{\infty}(-1)^{l+m+n}\frac{1}{|z-p_{l,m,n}|}=\frac{1}{4\pi}\frac{1}{|z-p|}-\frac{1}{4\pi}H(z,p). \tag{7.2}$$

where $p_{l,m,n}=(2l+(-1)^lp_1, 2m+(-1)^mp_2, 2n+(-1^np_3))$ and $H(z,p)$ denotes the regular part. Given that the series convergence is at best conditional, we must first address the summation order. Following [9, Section 5.15], we define

$$\sigma_n\left(\frac{1}{|z-p_{l,m,n}|}\right)=\frac{1}{|z-p_{l,m,n+1}|}-\frac{1}{|z-p_{l,m,n}|}.$$

For fixed l,m, the inner sum over the index n in the series expression can be written as

$$N'(l,m)=\sum_{n\text{ odd}}\sigma_n\left(\frac{1}{|z-p_{l,m,n}|}\right)=-\sum_{n\text{ even}}\sigma_n\left(\frac{1}{|z-p_{l,m,n}|}\right). \tag{7.3}$$

Clearly, $N'(l,m)\to\infty$ as $l^2+m^2\to\infty$. Similarly, we define

$$N''(l)=\sum_{m\text{ odd}}\sigma_m\left(N'(l,m)\right)=-\sum_{n\text{ even}}\sigma_m\left(N'(l,m)\right),$$

and

$$G(z,p)=\sum_{l\text{ odd}}\sigma_l\left(N''(l)\right)=-\sum_{n\text{ even}}\sigma_l\left(N''(l)\right),$$

where

$$\sigma_m\left(N'(l,m)\right)=N'(l,m+1)-N'(l,m),\quad \sigma_l\left(N''(l)\right)=N''(l+1)-N''(l).$$

Thus, we can express

$$G(z,p)=\frac{1}{4\pi}\sum_{l\text{ odd}}\sum_{m\text{ odd}}\sum_{n\text{ odd}}\sigma_l\sigma_m\sigma_n\left(\frac{1}{|z-p_{l,m,n}|}\right). \tag{7.4}$$

For fixed z, p in cube and $l^2+m^2+n^2$ large, we have

$$\left|\sigma_l\sigma_m\sigma_n\left(\frac{1}{|z-p_{l,m,n}|}\right)\right| < \frac{c}{(l^2+m^2+n^2)^2}.$$

For fixed z, p within a cube and large $l^2+m^2+n^2$, we have

$$\left|\sigma_l\sigma_m\sigma_n\left(\frac{1}{|z-p_{l,m,n}|}\right)\right| < \frac{c}{(l^2+m^2+n^2)^2}.$$

As a result, the series in (7.4) is absolutely convergent, allowing term-by-term differentiation. Furthermore, differentiating the Green function at least three times results in each term decaying as $(l^2+m^2+n^2)^{-2}$, allowing the summation order to be disregarded due to absolute convergence. One can verify that this summation order ensures $G(z, p)$ is the intended function. Specifically, on the boundary of cube, the individual terms cancel pairwise, resulting in the Green function vanishing.

Now we consider the regular part, expressed as

$$H(z,p) = -\sum_{l,m,n\neq(0,0,0)}(-1)^{l+m+n}\frac{1}{|z-p_{l,m,n}|}, \tag{7.5}$$

with the summation adhering to the previously specified order. Since our focus is on the 2×2 submatrix, we compute the Hessian elements at $z = p = b$:

$$\begin{aligned}
\left.\frac{\partial^2 H(z,p)}{\partial z_1\partial p_1}\right|_{z=p=b} &= \sum_{(l,m,n)\neq(0,0,0)}\left(\frac{(-1)^{m+n+1}}{|b(l,m,n)|^3}+\frac{3(-1)^{m+n}(b_1-2l-(-1)^l b_1)^2}{|b(l,m,n)|^5}\right),\\
\left.\frac{\partial^2 H(z,p)}{\partial z_1\partial p_2}\right|_{z=p=b} &= \sum_{(l,m,n)\neq(0,0,0)}\frac{3(-1)^{l+n}(b_1-2l-(-1)^l b_1)(b_2-2m-(-1)^m b_2)}{|b(l,m,n)|^5},\\
\left.\frac{\partial^2 H(z,p)}{\partial z_2\partial p_1}\right|_{z=p=b} &= \sum_{(l,m,n)\neq(0,0,0)}\frac{3(-1)^{m+n}(b_1-2l-(-1)^l b_1)(b_2-2m-(-1)^m b_2)}{|b(l,m,n)|^5},\\
\left.\frac{\partial^2 H(z,p)}{\partial z_2\partial p_2}\right|_{z=p=b} &= \sum_{(l,m,n)\neq(0,0,0)}\left(\frac{(-1)^{l+n+1}}{|b(l,m,n)|^3}+\frac{3(-1)^{l+n}(b_2-2m-(-1)^m b_2)^2}{|b(l,m,n)|^5}\right),
\end{aligned}$$

where

$$|b(l,m,n)| = |(b_1-2l-(-1)^l b_1, b_2-2m-(-1)^m b_2, b_3-2n-(-1)^n b_3)|.$$

The Hessian matrix summation follows the previously defined order. To confirm the first two assumptions about the regular part of the Green function, we refer to [18, Theorem 1.1]. Using the symmetry of the cube, it is evident that 0 is a critical point of the Robin function $\tau(z) = H(z, z)$. Furthermore, Grossi showed that $\partial_z^2\tau(z)|_{z=0}$ is positive definite with zero off-diagonal terms. Using this result and straightforward calculations, we obtain

$$\partial^2_{z_j z_l}\tau(z) = 2\partial^2_{z_j z_l}H(z,z) + 2\partial^2_{z_j p_l}H(z,z). \tag{7.6}$$

At $z=0$, exploiting the symmetry and the fact that $H(z,p)$ is a harmonic function with respect to z, we find $\partial^2_{z_1p_1}H(z,p)\big|_{z=p=0}$ and $\partial^2_{z_2p_2}H(z,p)\big|_{z=p=0}$ are strictly positive. Symmetry also implies that the off-diagonal terms of the mixed Hessian matrix $\frac{\partial^2 H(z,p)}{\partial z_j \partial p_l}\Big|_{z=p=0}$ are zero. Thus, we establish that $\nabla^2_{z,p}H(0,0)$ is positive definite.

It remains to check the third assumption. We set

$$f_{jl}(b)=\frac{\partial^2 H(z,p)}{\partial z_j\partial p_l}\ |_{z=p=b},\quad j,l=1,2.$$

By direct computation, we have

$$\begin{aligned}
\frac{\partial f_{11}}{\partial b_1}\Big|_{b=0} &= \sum_{(l,m,n)\neq(0,0,0)}\frac{(-1)^{m+n+1}(1-(-1)^l)}{16}\left[\frac{9l}{(l^2+m^2+n^2)^{\frac52}}-\frac{15l^3}{(l^2+m^2+n^2)^{\frac72}}\right],\\
\frac{\partial f_{11}}{\partial b_2}\Big|_{b=0} &= \sum_{(l,m,n)\neq(0,0,0)}\frac{(-1)^{m+n+1}(1-(-1)^m)}{16}\left[\frac{3l}{(l^2+m^2+n^2)^{\frac52}}-\frac{15l^2m}{(l^2+m^2+n^2)^{\frac72}}\right],\\
\frac{\partial f_{12}}{\partial b_1}\Big|_{b=0} &= \sum_{(l,m,n)\neq(0,0,0)}\frac{(-1)^{l+n+1}(1-(-1)^l)}{16}\left[\frac{3m}{(l^2+m^2+n^2)^{\frac52}}-\frac{15l^2m}{(l^2+m^2+n^2)^{\frac72}}\right],\\
\frac{\partial f_{12}}{\partial b_2}\Big|_{b=0} &= \sum_{(l,m,n)\neq(0,0,0)}\frac{(-1)^{l+n+1}(1-(-1)^m)}{16}\left[\frac{3l}{(l^2+m^2+n^2)^{\frac52}}-\frac{15lm^2}{(l^2+m^2+n^2)^{\frac72}}\right],\\
\frac{\partial f_{21}}{\partial b_1}\Big|_{b=0} &= \sum_{(l,m,n)\neq(0,0,0)}\frac{(-1)^{m+n+1}(1-(-1)^l)}{16}\left[\frac{3m}{(l^2+m^2+n^2)^{\frac52}}-\frac{15l^2m}{(l^2+m^2+n^2)^{\frac72}}\right],\\
\frac{\partial f_{21}}{\partial b_2}\Big|_{b=0} &= \sum_{(l,m,n)\neq(0,0,0)}\frac{(-1)^{m+n+1}(1-(-1)^m)}{16}\left[\frac{3l}{(l^2+m^2+n^2)^{\frac52}}-\frac{15lm^2}{(l^2+m^2+n^2)^{\frac72}}\right],\\
\frac{\partial f_{22}}{\partial b_1}\Big|_{b=0} &= \sum_{(l,m,n)\neq(0,0,0)}\frac{(-1)^{l+n+1}(1-(-1)^l)}{16}\left[\frac{3m}{(l^2+m^2+n^2)^{\frac52}}-\frac{15m^2l}{(l^2+m^2+n^2)^{\frac72}}\right],\\
\frac{\partial f_{22}}{\partial b_2}\Big|_{b=0} &= \sum_{(l,m,n)\neq(0,0,0)}\frac{(-1)^{l+n+1}(1-(-1)^m)}{16}\left[\frac{9m}{(l^2+m^2+n^2)^{\frac52}}-\frac{15m^3}{(l^2+m^2+n^2)^{\frac72}}\right].
\end{aligned}$$

It is evident that each term in the series now decays as $\frac{1}{(l^2+m^2+n^2)^2}$, ensuring absolute convergence and allowing summation in any order. Using symmetry, all eight terms cancel out. Next, we compute the Hessian matrix for each term $f_{jl},\ j,l=1,2$,

$$\frac{\partial^2 f_{11}}{\partial b_j b_l}\Big|_{b=0}=\begin{cases}
\sum\limits_{(l,m,n)\neq(0,0,0)}\frac{(-1)^{m+n}(1-(-1)^l)^2}{32}\left[\frac{9}{(l^2+m^2+n^2)^{\frac52}}-\frac{90l^2}{128(l^2+m^2+n^2)^{\frac72}}+\frac{105l^4}{512(l^2+m^2+n^2)^{\frac92}}\right], & j=l=1,\\
\sum\limits_{(l,m,n)\neq(0,0,0)}\frac{(-1)^{m+n+1}(1-(-1)^l)(1-(-1)^m)}{32}\left[\frac{45lm}{(l^2+m^2+n^2)^{\frac72}}-\frac{105l^3m}{(l^2+m^2+n^2)^{\frac92}}\right], & j=1,l=2,\\
\sum\limits_{(l,m,n)\neq(0,0,0)}\frac{(-1)^{m+n+1}(1-(-1)^l)(1-(-1)^m)}{32}\left[\frac{45lm}{(l^2+m^2+n^2)^{\frac72}}-\frac{105l^3m}{(l^2+m^2+n^2)^{\frac92}}\right], & j=2,l=1,\\
\sum\limits_{(l,m,n)\neq(0,0,0)}\frac{(-1)^{m+n}(1-(-1)^m)^2}{32}\left[\frac{3}{(l^2+m^2+n^2)^{\frac52}}-\frac{15(l^2+m^2)}{(l^2+m^2+n^2)^{\frac72}}+\frac{105l^2m^2}{(l^2+m^2+n^2)^{\frac92}}\right], & j=l=2,
\end{cases}$$

$$\frac{\partial^2 f_{12}}{\partial b_j b_l}\Big|_{b=0}=\begin{cases}
\sum\limits_{(l,m,n)\neq(0,0,0)}\frac{(-1)^{l+n}(1-(-1)^l)^2}{32}\left[-\frac{45lm}{(l^2+m^2+n^2)^{\frac72}}+\frac{105l^3m}{(l^2+m^2+n^2)^{\frac92}}\right], & j=l=1,\\
\sum\limits_{(l,m,n)\neq(0,0,0)}\frac{(-1)^{l+n}(1-(-1)^l)(1-(-1)^m)}{32}\left[\frac{3}{(l^2+m^2+n^2)^{\frac52}}-\frac{15(l^2+m^2)}{(l^2+m^2+n^2)^{\frac72}}+\frac{105l^2m^2}{(l^2+m^2+n^2)^{\frac92}}\right], & j=1,l=2,\\
\sum\limits_{(l,m,n)\neq(0,0,0)}\frac{(-1)^{l+n}(1-(-1)^l)(1-(-1)^m)}{32}\left[\frac{3}{(l^2+m^2+n^2)^{\frac52}}-\frac{15(l^2+m^2)}{(l^2+m^2+n^2)^{\frac72}}+\frac{105l^2m^2}{(l^2+m^2+n^2)^{\frac92}}\right], & j=2,l=1,\\
\sum\limits_{(l,m,n)\neq(0,0,0)}\frac{(-1)^{l+n}(1-(-1)^m)^2}{32}\left[-\frac{45lm}{(l^2+m^2+n^2)^{\frac72}}+\frac{105lm^3}{(l^2+m^2+n^2)^{\frac92}}\right], & j=l=2,
\end{cases}$$

$$\frac{\partial^2 f_{21}}{\partial b_j b_l}\Big|_{b=0} = \begin{cases} \sum\limits_{(l,m,n)\neq(0,0,0)} \frac{(-1)^{m+n}(1-(-1)^l)^2}{32}\left[-\frac{45lm}{(l^2+m^2+n^2)^{\frac{7}{2}}}+\frac{105l^3m}{(l^2+m^2+n^2)^{\frac{9}{2}}}\right], & j=l=1,\\ \sum\limits_{(l,m,n)\neq(0,0,0)} \frac{(-1)^{m+n}(1-(-1)^l)(1-(-1)^m)}{32}\left[\frac{3}{(l^2+m^2+n^2)^{\frac{5}{2}}}-\frac{15(l^2+m^2)}{(l^2+m^2+n^2)^{\frac{7}{2}}}+\frac{105l^2m^2}{(l^2+m^2+n^2)^{\frac{9}{2}}}\right], & j=1,l=2,\\ \sum\limits_{(l,m,n)\neq(0,0,0)} \frac{(-1)^{m+n}(1-(-1)^l)(1-(-1)^m)}{32}\left[\frac{3}{(l^2+m^2+n^2)^{\frac{5}{2}}}-\frac{15(l^2+m^2)}{(l^2+m^2+n^2)^{\frac{7}{2}}}+\frac{105l^2m^2}{(l^2+m^2+n^2)^{\frac{9}{2}}}\right], & j=2,l=1,\\ \sum\limits_{(l,m,n)\neq(0,0,0)} \frac{(-1)^{m+n}(1-(-1)^m)^2}{32}\left[-\frac{45lm}{(l^2+m^2+n^2)^{\frac{7}{2}}}+\frac{105lm^3}{(l^2+m^2+n^2)^{\frac{9}{2}}}\right], & j=l=2. \end{cases}$$

$$\frac{\partial^2 f_{22}}{\partial b_j b_l}\Big|_{b=0} = \begin{cases} \sum\limits_{(l,m,n)\neq(0,0,0)} \frac{(-1)^{l+n}(1-(-1)^l)^2}{32}\left[\frac{3}{(l^2+m^2+n^2)^{\frac{5}{2}}}-\frac{15(l^2+m^2)}{(l^2+m^2+n^2)^{\frac{7}{2}}}+\frac{105l^2m^2}{(l^2+m^2+n^2)^{\frac{9}{2}}}\right], & j=l=1,\\ \sum\limits_{(l,m,n)\neq(0,0,0)} \frac{(-1)^{l+n+1}(1-(-1)^l)(1-(-1)^m)}{32}\left[\frac{45lm}{(l^2+m^2+n^2)^{\frac{7}{2}}}-\frac{105lm^3}{(l^2+m^2+n^2)^{\frac{9}{2}}}\right], & j=1,l=2,\\ \sum\limits_{(l,m,n)\neq(0,0,0)} \frac{(-1)^{l+n+1}(1-(-1)^l)(1-(-1)^m)}{32}\left[\frac{45lm}{(l^2+m^2+n^2)^{\frac{7}{2}}}-\frac{105lm^3}{(l^2+m^2+n^2)^{\frac{9}{2}}}\right], & j=2,l=1,\\ \sum\limits_{(l,m,n)\neq(0,0,0)} \frac{(-1)^{l+n}(1-(-1)^m)^2}{32}\left[\frac{9}{(l^2+m^2+n^2)^{\frac{5}{2}}}-\frac{90m^2}{(l^2+m^2+n^2)^{\frac{7}{2}}}+\frac{105m^4}{(l^2+m^2+n^2)^{\frac{9}{2}}}\right], & j=l=2, \end{cases}$$

As the gradient term, each single term in the series now decays like $\frac{1}{(l^2+m^2+n^2)^{\frac{5}{2}}}$, confirming absolute convergence and permitting arbitrary summation order. Symmetry reveals that the off-diagonal terms of $\frac{\partial^2 f_{11}}{\partial b_j \partial b_l}$ and $\frac{\partial^2 f_{22}}{\partial b_j \partial b_l}$ are zero, as are the diagonal terms of $\frac{\partial^2 f_{12}}{\partial b_j \partial b_l}$ and $\frac{\partial^2 f_{21}}{\partial b_j \partial b_l}$. Using Mathematica or Matlab, we derive the value of the non-zero terms are

$$\sum_{(l,m,n)\neq(0,0,0)} \frac{(-1)^{m+n}(1-(-1)^l)^2}{32}\left[\frac{9}{(l^2+m^2+n^2)^{\frac{5}{2}}}-\frac{90l^2}{(l^2+m^2+n^2)^{\frac{7}{2}}}+\frac{105l^4}{(l^2+m^2+n^2)^{\frac{9}{2}}}\right]\approx 7.11,$$
$$\sum_{(l,m,n)\neq(0,0,0)} \frac{(-1)^{m+n}(1-(-1)^m)^2}{32}\left[\frac{3}{(l^2+m^2+n^2)^{\frac{5}{2}}}-\frac{15(l^2+m^2)}{(l^2+m^2+n^2)^{\frac{7}{2}}}+\frac{105l^2m^2}{(l^2+m^2+n^2)^{\frac{9}{2}}}\right]\approx 1.62,$$
$$\sum_{(l,m,n)\neq(0,0,0)} \frac{(-1)^{m+n}(1-(-1)^l)(1-(-1)^m)}{32}\left[\frac{3}{(l^2+m^2+n^2)^{\frac{5}{2}}}-\frac{15(l^2+m^2)}{(l^2+m^2+n^2)^{\frac{7}{2}}}+\frac{105l^2m^2}{(l^2+m^2+n^2)^{\frac{9}{2}}}\right]\approx -0.97.$$

As a consequence, we have

$$\frac{\partial^2 f_{11}}{\partial b_j b_l}\Big|_{b=0}\approx\begin{pmatrix}7.11 & 0\\ 0 & 1.62\end{pmatrix},\ \frac{\partial^2 f_{12}}{\partial b_j b_l}\Big|_{b=0}\approx\begin{pmatrix}0 & -0.97\\ -0.97 & 0\end{pmatrix},$$
$$\frac{\partial^2 f_{11}}{\partial b_j b_l}\Big|_{b=0}\approx\begin{pmatrix}0 & -0.97\\ -0.97 & 0\end{pmatrix},\ \frac{\partial^2 f_{11}}{\partial b_j b_l}\Big|_{b=0}\approx\begin{pmatrix}1.62 & 0\\ 0 & 7.11\end{pmatrix}.$$

In order to verify the assumption that $w^T \frac{\partial^2 H(z,p)}{\partial z_j \partial p_l}\Big|_{z=p=b} w$ has a local minimum at $b=0$ for any $w\in\mathbb{S}^1$, we compute:

$$\begin{aligned} w^T\frac{\partial^2 H(z,p)}{\partial z_j\partial p_l}\Big|_{z=p=b} w &= f_{11}(b)w_1^2+f_{22}(b)w_2^2+f_{12}(b)w_1w_2+f_{21}(b)w_1w_2\\ &\approx w^T\begin{pmatrix}7.11b_1^2+1.67b_2^2 & -1.94b_1b_2\\ -1.94b_1b_2 & 1.67b_1^2+7.11b_2^2\end{pmatrix}w+O(|b|^4)\\ &=(b_1,b_2)\begin{pmatrix}7.11w_1^2+1.62w_2^2 & -1.94w_1w_2\\ -1.94w_1w_2 & 1.62w_1^2+7.11w_2^2\end{pmatrix}\begin{pmatrix}b_1\\ b_2\end{pmatrix}\\ &\quad+O(|b|^4). \end{aligned} \tag{7.7}$$

Clearly,

$$\begin{pmatrix} 7.11w_1^2 + 1.62w_2^2 & -1.94w_1w_2 \\ -1.94w_1w_2 & 1.62w_1^2 + 7.11w_2^2 \end{pmatrix} \text{ is a positive definite matrix for any } w \in \mathbb{S}^1.$$

This implies that $w^T \frac{\partial^2 H(z,p)}{\partial z_j \partial p_l}\Big|_{z=p=b} w$ has a local minimum at $b = 0$. Thus, all assumptions are verified for the Green function of cube.

According to the above arguments, it is not difficult to see that if the cube is replaced by a cuboid with the length of three sides a, b, c satisfying that

$$1 \leq \max\left\{\frac{a}{b}, \frac{b}{a}, \frac{a}{c}, \frac{c}{a}, \frac{b}{c}, \frac{c}{b}\right\} < 1 + \epsilon, \tag{7.8}$$

we can still verify the assumptions (C-2)-(C-3) provided ϵ is sufficiently small. While the first assumption (C-1) is a direct consequence following [18] and the symmetry of cuboid.

With the above preparation, we are able to provide the proof of Theorem 1.3

Proof of Theorem 1.3. Let the side lengths of a cuboid be denoted by L_1, L_2, L_3. Suppose the ratios $\frac{L_1}{L_2}$, $\frac{L_2}{L_3}$, $\frac{L_3}{L_1}$ are rational. Then, we can express the side lengths as

$$L_1 : L_2 : L_3 = p : q : r, \tag{7.9}$$

where p, q, r are positive integers.

We divide the cuboid's sides into pN, qN, rN equal parts, respectively, for a sufficiently large integer N. This decomposes the cuboid into $pqrN^3$ smaller cubes, each with side length

$$\ell = \frac{L_1}{pN} = \frac{L_2}{qN} = \frac{L_3}{rN}.$$

In each small cube, we study the problem (1.1). By rescaling the spatial variable $x \to y = \frac{x}{\ell}$, this problem transforms into:

$$\begin{cases} \frac{1}{\ell^2}\Delta u + \lambda u + u^5 = 0 & \text{in } [-1,1]^3, \\ u = 0 & \text{on } \partial([-1,1]^3). \end{cases} \tag{7.10}$$

Introducing $v = \ell^{1/2}u$, the equation becomes:

$$\begin{cases} \Delta v + \lambda \ell^2 v + v^5 = 0 & \text{in } [-1,1]^3, \\ v = 0 & \text{on } \partial([-1,1]^3). \end{cases} \tag{7.11}$$

For sufficiently large N, the side length ℓ is small, ensuring $\lambda \ell^2 < \lambda_0$, where λ_0 is given by Theorem 1.1. Using Theorem 1.1 and the property of the Green function for the cuboid, we establish a sign-changing solution to the rescaled equation. Scaling back to the original variable and applying odd reflections across the boundaries of each small cube, we obtain a sign-changing solution to the original problem (1.1).

If the ratios $\frac{L_1}{L_2}$, $\frac{L_2}{L_3}$, $\frac{L_3}{L_1}$ are not all rational, then we may assume that $\frac{L_1}{L_2}$ and $\frac{L_2}{L_3}$ are irrational. We can find positive integers p_1, p_2, q_1, q_2 such that:

$$\frac{p_1}{q_1} < \frac{L_1}{L_2} < \frac{p_1+1}{q_1}, \quad \frac{p_2}{q_2} < \frac{L_2}{L_3} < \frac{p_2+1}{q_2}.$$

We divide the the sides of cuboid into p_1p_2, q_1p_2, q_1q_2 equal parts, resulting in $p_1p_2^2q_1^2q_2$ smaller cuboids. Consider one such cuboid, denoted by $\mathcal{C}_0$, with side lengths $a_1 = \frac{L_1}{p_1p_2}$, $a_2 = \frac{L_2}{q_1p_2}$, $a_3 = \frac{L_3}{q_1q_2}$. The ratios satisfy:

$$1 < \frac{a_1}{a_2} < \frac{p_1q_1+q_1}{p_1q_1}, \quad 1 < \frac{a_2}{a_3} < \frac{p_2q_2+q_2}{p_2q_2}.$$

By choosing sufficiently large p_1, p_2, q_1, q_2, we ensure that

$$\max\left\{ \frac{a_i}{a_j}, \frac{a_j}{a_i} \,\middle|\, i,j=1,2,3,\ i\neq j \right\} < 1+\epsilon,$$

where ϵ is specified in (7.8). As a consequence, we see that the Green function of $\mathcal{C}_0$ satisfies conditions (C-1)-(C-3). We further subdivide $\mathcal{C}_0$ into smaller cuboids by dividing each side into an equal number of parts, construct a solution to (BN) in one of these using Theorem 1.1, and apply odd reflections to extend the solution to the entire cuboid. This completes the proof of Theorem 1.3. □

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