STABILITY OF THE CAFFARELLI-KOHN-NIRENBERG INEQUALITY: THE EXISTENCE OF MINIMIZERS

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Abstract. In this paper, we consider the following variational problem:

$$\inf_{u\in D^{1,2}_a(\mathbb{R}^N)\backslash\mathcal{Z}}\frac{\|u\|^2_{D^{1,2}_a(\mathbb{R}^N)}-C^{-1}_{a,b,N}\|u\|^2_{L^{p+1}(|x|^{-b(p+1)},\mathbb{R}^N)}}{dist^2_{D^{1,2}_a}(u,\mathcal{Z})}:=c_{BE},$$

where $N\geq 2$, $b_{FS}(a)<b<a+1$ for $a<0$ and $a\leq b<a+1$ for $0\leq a<a_c:=\frac{N-2}{2}$ and $a+b>0$ with $b_{FS}(a)$ being the Felli-Schneider curve, $p=\frac{N+2(1+a-b)}{N-2(1+a-b)}$, $\mathcal{Z}=\{c\tau^{a_c-a}W(\tau x)\mid c\in\mathbb{R}\backslash\{0\},\tau>0\}$ and up to dilations and scalar multiplications, $W(x)$, which is positive and radially symmetric, is the unique extremal function of the following classical Caffarelli-Kohn-Nirenberg (CKN for short) inequality

$$\left(\int_{\mathbb{R}^N}|x|^{-b(p+1)}|u|^{p+1}dx\right)^{\frac{2}{p+1}}\leq C_{a,b,N}\int_{\mathbb{R}^N}|x|^{-2a}|\nabla u|^2dx$$

with $C_{a,b,N}$ being the optimal constant. It is known in [32] that $c_{BE}>0$. In this paper, we prove that the above variational problem has a minimizer for $N\geq 2$ under the following two assumptions:

(i) $a_c^*\leq a<a_c$ and $a\leq b<a+1$,

(ii) $a<a_c^*$ and $b^*_{FS}(a)\leq b<a+1$,

where $a_c^*=\left(1-\sqrt{\frac{N-1}{2N}}\right)a_c$ and

$$b^*_{FS}(a)=\frac{(a_c-a)N}{a_c-a+\sqrt{(a_c-a)^2+N-1}}+a-a_c.$$

Our results extend that of Konig in [24] for the Sobolev inequality to the CKN inequality. Moreover, we believe that our assumptions (i) and (ii) are optimal for the existence of minimizers of the above variational problem.

Keywords: Caffarelli-Kohn-Nirenberg inequality; Bianchi-Egnell type stability; Existence of minimizers.

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1. Introduction

Let $D^{1,2}_a(\mathbb{R}^N)$ be the Hilbert space given by

$$D^{1,2}_a(\mathbb{R}^N)=\{u\in D^{1,2}(\mathbb{R}^N)\mid\int_{\mathbb{R}^N}|x|^{-2a}|\nabla u|^2dx<+\infty\}\tag{1.1}$$ eqn886

with the inner product

$$\langle u,v\rangle_{D^{1,2}_a(\mathbb{R}^N)}=\int_{\mathbb{R}^N}|x|^{-2a}\nabla u\nabla v dx$$

and $D^{1,2}(\mathbb{R}^N)=\dot{W}^{1,2}(\mathbb{R}^N)$ being the usual homogeneous Sobolev space (cf. [16, Definition 2.1]). Then the classical Caffarelli-Kohn-Nirenberg (CKN for short) inequality, established by Caffarelli, Kohn and Nirenberg in the celebrated paper [4] in a more general version, states that

$$\left(\int_{\mathbb{R}^N}|x|^{-b(p+1)}|u|^{p+1}dx\right)^{\frac{2}{p+1}}\leq C_{a,b,N}\int_{\mathbb{R}^N}|x|^{-2a}|\nabla u|^2dx, \tag{1.2}$$

eq0001

for all $u\in D^{1,2}_a(\mathbb{R}^N)$, where $-\infty<a<a_c:=\frac{N-2}{2}$, $p=\frac{N+2(1+a-b)}{N-2(1+a-b)}$ and

$$\begin{cases} a+\dfrac{1}{2}<b<a+1, & N=1,\\ a<b<a+1, & N=2,\\ a\leq b<a+1, & N\geq 3.\end{cases}$$

Here, for the sake of simplicity, we denote $a_c=\frac{N-2}{2}$, as that in [10–12].

As that of many famous functional inequalities such as the Sobolev inequality, the Hardy-Littlewood-Sobolev inequality, the Gagliardo-Nirenberg-Sobolev inequality, the Euclidean logarithmic Sobolev inequality and so on, as a generalization of the Sobolev and Hardy-Sobolev inequalities, the CKN inequality (1.2) is also very helpful in understanding various problems in lots of mathematical fields, such as nonlinear partial differential equations, calculus of variations, geometric analysis, the theory of probability to mathematical physics and so on. For this purpose, a fundamental task in understanding the CKN inequality (1.2) is to study the optimal constant, the classification of extremal functions, as well as their qualitative properties for parameters in the full region. Under the above conditions, it is well known (cf. [1, 5, 7, 27, 31]) that the CKN inequality (1.2) has extremal functions if and only if under the following assumptions:

(1) $a<b<a+1$ and $a<0$ for $N\geq 2$,
(2) $a+\frac{1}{2}<b<a+1$ and $a<0$ for $N=1$,
(3) $a\leq b<a+1$ and $0\leq a<a_c$ for $N\geq 3$.

Moreover, let

$$b_{FS}(a)=\frac{N(a_c-a)}{2\sqrt{(a_c-a)^2+(N-1)}}+a-a_c>a \tag{1.3}$$

eqn991

be the Felli-Schneider curve found in [17], then it is also well known (cf. [1, 7, 10–13, 17, 27, 31]) that up to dilations $\tau^{a_c-a}u(\tau x)$ and scalar multiplications $Cu(x)$ (also up to translations $u(x+y)$ in the special case $a=b=0$), the CKN inequality (1.2) has a unique extremal function

$$W(x)=(2(p+1)(a_c-a)^2)^{\frac{1}{(p-1)}}\left(1+|x|^{(a_c-a)(p-1)}\right)^{-\frac{2}{p-1}} \tag{1.4}$$

eqq0097

either for $b_{FS}(a)\leq b<a+1$ with $a<0$ or for $a\leq b<a+1$ with $0\leq a<a_c$ in the cases of $N\geq 2$ while, extremal functions of (1.2) must be non-radial either for the full region of a and b in the case of $N=1$ or for $a<b<b_{FS}(a)$ with $a<0$ in the cases of $N\geq 2$. Moreover, it has been proved in [5, 28] that there are exactly two extremal functions of (1.2) in the case of $N=1$ up to dilations and scalar multiplications while in the cases of $N\geq 2$, extremal functions of (1.2) must have $\mathcal{O}(N-1)$ symmetry for $a<b<b_{FS}(a)$ with $a<0$, that is, extremal functions of (1.2) for $N \geq 2$ must depend on the radius r and the angle θ_N between the positive x_N-axis and $\overrightarrow{Ox}$ for $a < b < b_{FS}(a)$ with $a < 0$ up to rotations. To our best knowledge, whether the extremal function of (1.2) is unique or not for $a < b < b_{FS}(a)$ with $a < 0$ in the cases of $N \geq 2$ remains open.

Besides the existence and classification of extremal functions and the computation of the optimal constant, a more interesting and challenging problem in understanding functional inequalities is its quantitative stability, whose basic question one wants to address in this aspect is the following (cf. [15]):

(Q) Suppose we are given a functional inequality for which minimizers are known. Can we prove, in some quantitative way, that if a function "almost attains the equality" then it is close (in some suitable sense) to one of the minimizers?

The studies on the quantitative stability of functional inequalities were initialed by Brezis and Lieb in [3] by raising an open question for the classical Sobolev inequality ($a = b = 0$ in the CKN inequality (1.2)),

$$S_N\left(\int_{\mathbb{R}^N}|u|^{\frac{2N}{N-2}}dx\right)^{\frac{N-2}{N}} \leq \int_{\mathbb{R}^N}|\nabla u|^2dx \tag{1.5}$$ eqq0093

for $N \geq 3$, which was settled by Bianchi and Egnell in [2] by proving that

$$dist^2_{D^{1,2}}(u,\mathcal{U}) \lesssim \|u\|^2_{D^{1,2}(\mathbb{R}^N)} - S_N\|u\|^2_{L^{\frac{2N}{N-2}}(\mathbb{R}^N)}, \tag{1.6}$$ eqq0090

where S_N is the optimal constant of (1.5) and

$$\mathcal{U} = \{cU_{y,\lambda} \mid c \in \mathbb{R}\backslash\{0\}, \lambda > 0 \text{ and } y \in \mathbb{R}^N\}$$

with $U(x)$ being the Aubin-Talanti bubble (cf. [1,31]) and $U_{y,\lambda}(x) = \lambda^{\frac{N-2}{2}}U(\lambda(x - y))$. Since then, the stability of functional inequalities, which is similar to (1.6), is called the Bianchi-Egnell type stability. In the very recent paper [32], we prove the following Bianchi-Egnell type stability of the CKN inequality (1.2):

⟨thm0001⟩ **Theorem 1.1.** *Let $N \geq 3$, $a < a_c$ and $b_{FS}(a)$ be the Felli-Schneider curve given by (1.3) and assume that either*

(1) $b_{FS}(a) < b < a + 1$ *with* $a < 0$ *or*
(2) $a \leq b < a + 1$ *with* $a \geq 0$ *and* $a + b > 0$.

Then

$$dist^2_{D^{1,2}_a}(u,\mathcal{Z}) \lesssim \|u\|^2_{D^{1,2}_a(\mathbb{R}^N)} - C^{-1}_{a,b,N}\|u\|^2_{L^{p+1}(|x|^{-b(p+1)},\mathbb{R}^N)} \tag{1.7}$$ eqq0091

for all $u \in D^{1,2}_a(\mathbb{R}^N)$, *where*

$$\mathcal{Z} = \{cW_\tau(x) \mid c \in \mathbb{R}\backslash\{0\} \text{ and } \tau > 0\}$$

with $W_\tau(x) = \tau^{a_c - a}W(\tau x)$, $L^{p+1}(|x|^{-b(p+1)}, \mathbb{R}^N)$ *is the usual weighted Lebesgue space with its usual norm given by*

$$\|u\|_{L^{p+1}(|x|^{-b(p+1)},\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N}|x|^{-b(p+1)}|u|^{p+1}dx\right)^{\frac{1}{p+1}}.$$

Moreover, by the same argument as that used in [32] for proving Theorem 1.1, it is not difficult to obtain the following one.

⟨thm0002⟩ **Theorem 1.2.** *Let $N = 2$ and assume that $b_{FS}(a) < b < a + 1$ with $a < 0$ and $b_{FS}(a)$ being the Felli-Schneider curve given by (1.3). Then*

$$dist^2_{D^{1,2}_a}(u, \mathcal{Z}) \lesssim \|u\|^2_{D^{1,2}_a(\mathbb{R}^2)} - C^{-1}_{a,b,2}\|u\|^2_{L^{p+1}(|x|^{-b(p+1)},\mathbb{R}^2)}$$

for all $u \in D^{1,2}_a(\mathbb{R}^2)$.

Remark 1.1. *In [32, Theorem 1.1], we have claimed that Theorem 1.1 holds true for $b_{FS}(a) \le b < a + 1$ with $a < 0$. However, it is incorrect for $b = b_{FS}(a)$ since it has been proved in [17] that W is degenerate for $b = b_{FS}(a)$. It follows that the spectral gap inequality,*

$$\|\rho\|^2_{D^{1,2}_a(\mathbb{R}^N)} \ge (p + \varepsilon) \int_{\mathbb{R}^N} |x|^{-b(p+1)} W^{p-1} \rho^2 dx$$

for all $\rho \in \mathcal{N}^\perp$ which plays the key role in proving the Bianchi-Egnell type stability of the CKN inequality in [32, Theorem 1.1], does not hold true any more for $b = b_{FS}(a)$, where $\varepsilon > 0$ is a fixed small constant and

$$\mathcal{N}^\perp = \{\rho \in D^{1,2}_a(\mathbb{R}^N) \mid \langle \rho, W\rangle_{D^{1,2}_a(\mathbb{R}^N)} = \langle \rho, W'\rangle_{D^{1,2}_a(\mathbb{R}^N)} = 0\}.$$

This has already been observed in [8, 19]. We now correct [32, Theorem 1.1] here to Theorem 1.1. Moreover, it is worth pointing out that in this degenerate situation, the Bianchi-Egnell stability of the CKN inequality still holds true for higher order of the distance functional, as proved in [19] whose ideas can be traced back to [20].

The power two of the distance in the left side of the Bianchi-Engell inequality (1.6) is well known to be optimal, which is also the case of the Bianchi-Egnell type stability of the CKN inequality (1.2) given by (1.7). Thus, we can define the following two variational problems:

$$\inf_{u \in D^{1,2}(\mathbb{R}^N)\setminus\mathcal{U}} \frac{\|u\|^2_{D^{1,2}(\mathbb{R}^N)} - S_N\|u\|^2_{L^{\frac{2N}{N-2}}(\mathbb{R}^N)}}{dist^2_{D^{1,2}}(u, \mathcal{U})} := s_{BE} \tag{1.8}$$ eqq0092

and

$$\inf_{u \in D^{1,2}_a(\mathbb{R}^N)\setminus\mathcal{Z}} \frac{\|u\|^2_{D^{1,2}_a(\mathbb{R}^N)} - C^{-1}_{a,b,N}\|u\|^2_{L^{p+1}(|x|^{-b(p+1)},\mathbb{R}^N)}}{dist^2_{D^{1,2}_a}(u, \mathcal{Z})} := c_{BE}, \tag{1.9}$$ eqq1001

By (1.6), we know that $s_{BE} > 0$, and by Theorems 1.1 and 1.2, we know that $c_{BE} > 0$ either for

(1) $b_{FS}(a) < b < a + 1$ with $a < 0$ or for
(2) $a \le b < a + 1$ with $a \ge 0$ and $a + b > 0$.

in the cases of $N \ge 2$. Moreover, as pointed out by Konig in [24], it is a long-standing open question that is to determine the best constant s_{BE} in (1.8). It has been proved by Konig in [24] that the variational problem (1.8) has a minimizer which makes the key step in determining the best constant s_{BE} and gives a positive answer to the open question proposed in [9]. Konig's proof in [24] is a concentration-compactness type argument on the distance functional $dist^2_{D^{1,2}}(u, \mathcal{U})$. By establishing two crucial energy estimates of s_{BE} (cf. [23, 24]), Konig excluded the dichotomy case and the vanishing case of $dist^2_{D^{1,2}}(u, \mathcal{U})$ and proved that the variational problem (1.8) has a minimizer. In this argument, a key integrant is the well understanding of the spectrum of the Laplacian operator $-\Delta$ in the weighted

Lebesgue space $L^2(U^{\frac{4}{N-2}}, \mathbb{R}^N)$, which is crucial in establishing one of the two important energy estimates of s_{BE}. Furthermore, as pointed out by Konig in [24], this strategy also works for the fractional Sobolev inequality of $N \geq 2$.

In this paper, by making a well understanding of the spectrum of the operator $-div(|x|^{-a}\nabla\cdot)$ in the weighted Lebesgue space $L^2(|x|^{-b(p+1)}W^{p-1}, \mathbb{R}^N)$, we adapt Konig's strategy in [24] to prove the following theorem.

⟨thmq0001⟩ **Theorem 1.3.** *Let $N \geq 2$ and assume that either*

(1) $b_{FS}(a) < b < a+1$ *with* $a < 0$ *or*
(2) $a \leq b < a+1$ *with* $0 \leq a < a_c$ *and* $a+b > 0$.

Then the variational problem (1.9) has a minimizer, provided

(i) $a_c^* < a < a_c$ *and* $a \leq b < a+1$,
(ii) $a \leq a_c^*$ *and* $b_{FS}^*(a) \leq b < a+1$,

where

$$a_c^* = \left(1 - \sqrt{\frac{N-1}{2N}}\right)a_c \quad \textit{and} \quad b_{FS}^*(a) = \frac{(a_c - a)N}{a_c - a + \sqrt{(a_c-a)^2 + N - 1}} + a - a_c.$$

Theorem 1.3 is the generalization of Konig's reuslt in [24] for the Sobolev inequality (1.5) to the CKN inequality (1.2). However, Konig has proved in [24] that s_{BE} is attained for all $N \geq 3$ (even in the fractional setting for all $N \geq 2$), while in Theorem 1.3, we only prove that c_{BE} is attained for $N \geq 2$ under the assumptions (i) and (ii) which do not cover the full region of the parameters a, b under the conditions (1) and (2). The main reason is that under the assumptions (i) and (ii) for $N \geq 2$, the spectral gap inequality of the operator $-div(|x|^{-a}\nabla\cdot)$ in the weighted Lebesgue space $L^2(|x|^{-b(p+1)}W^{p-1}, \mathbb{R}^N)$ is attained by a unique (up to scalar multiplications) function which is related to spherical harmonics on $\mathbb{S}^{N-1}$ of degree 0, while in the remaining case, that is, $N \geq 2$ with $a < a_c^*$ and $b_{FS}(a) < b < b_{FS}^*(a)$, the spectral gap inequality of the operator $-div(|x|^{-a}\nabla\cdot)$ in the weighted Lebesgue space $L^2(|x|^{-b(p+1)}W^{p-1}, \mathbb{R}^N)$ is attained by the functions which are related to spherical harmonics on $\mathbb{S}^{N-1}$ of degree 1. Thus, Konig's strategy in [24], that is, expansing the related functional at the possible best choice of test functions up to the third order term to derive a crucial energy estimate, works for c_{BE} under the assumptions (i) and (ii) since the expansion has a negative third order term (see the proof of Proposition 4.1) and is invalid in the remaining case since the expansion has a varnishing third order term (see the appendix). To go further, we expand the functional of (1.9) at the possible best choice of test functions up to the fourth order term. After tedious computations, we find that the possible best choice of test functions can not derive the desired energy estimate of c_{BE} in this situation any more, since this expansion has a varnishing third order term and a positive fourth order term (see the appendix for more details). Taking into account the fact that the test functions are possible to be optimal, we believe that c_{BE} will be not attained in this remaining case. We remark that a similar situation is also faced in proving the existence of minimizers of s_{BE} in the fractional setting for $N = 1$, see, for example, the very recent paper [25].

In the final of the introduction, we would like to point out that the studies on the stability of functional inequalities are growingly interested in recent years in the community of nonlinear analysis by its deep connections to many nonlinear partial differential equations, such as the fast diffusion equation, the Keller-Segel equation and so on. We refer the readers to the survey [14] and the Lecture notes [21] for their detailed introductions and references about the studies on lots of famous functional inequalities and their stability, such as the Sobolev inequality, the Hardy-Littlewood-Sobolev inequality, the Gagliardo-Nirenberg-Sobolev inequality, the Caffarelli-Kohn-Nirenberg inequality, the Euclidean logarithmic Sobolev inequality and so on. We also would like to refer the readers to the note [15] for the related studies on the stability of many geometric inequalities.

Notations. Throughout this paper, $a \sim b$ means that $C'b \leq a \leq Cb$ and $a \lesssim b$ means that $a \leq Cb$. Moreover, $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$.

2. Preliminaries

Let $D^{1,2}_a(\mathbb{R}^N)$ be the Hilbert space given by (1.1) with the norm $\|\cdot\|_{D^{1,2}_a(\mathbb{R}^N)}$, then by [5, Proposition 2.2], $D^{1,2}_a(\mathbb{R}^N)$ is isomorphic to the Hilbert space $H^1(\mathcal{C})$ through the Emden-Fowler transformation

$$u(x) = |x|^{-(a_c-a)}v(-\ln|x|, \frac{x}{|x|}), \tag{2.1}$$ eq0007

where $\mathcal{C} = \mathbb{R} \times \mathbb{S}^{N-1}$ is the standard cylinder, the inner product in $H^1(\mathcal{C})$ is given by

$$\begin{aligned} \langle w, v\rangle_{H^1(\mathcal{C})} &= \int_{\mathcal{C}} \partial_t w \partial_t v + \nabla_{\mathbb{S}^{N-1}} w \nabla_{\mathbb{S}^{N-1}} v + (a_c-a)^2 wv d\mu \\ &:= \int_{\mathcal{C}} \nabla w \nabla v + (a_c-a)^2 wv d\mu \end{aligned}$$

with $d\mu$ being the volume element on $\mathcal{C}$ and $w, v \in H^1(\mathcal{C})$.

The Euler-Lagrange equation of the CKN inequality (1.2) is given by

$$-div(|x|^{-a}\nabla u) = |x|^{-b(p+1)}|u|^{p-1}u, \quad \text{in } \mathbb{R}^N. \tag{2.2}$$ eq0018

It has been proved in [7, 12] that $W(x)$, given by (1.4), is the unique nonnegative solution of (2.2) in $D^{1,2}_a(\mathbb{R}^N)$ either for $b_{FS}(a) \leq b < a+1$ with $a < 0$ or for $a \leq b < a+1$ with $a \geq 0$ and $a+b > 0$ up to dilations $W_\tau = \tau^{a_c-a}W(\tau x)$ in the cases of $N \geq 2$. By the transformation (2.1), $W(x)$ and (2.2) are transformed into

$$\Psi(t) = \left(\frac{(p+1)(a_c-a)^2}{2}\right)^{\frac{1}{p-1}} \left(cosh(\frac{(a_c-a)(p-1)}{2}t)\right)^{-\frac{2}{p-1}} \tag{2.3}$$ eq0026

and

$$-\Delta_{\mathbb{S}^{N-1}} v - \partial_t^2 v + (a_c-a)^2 v = |v|^{p-1}v, \quad \text{in } \mathcal{C}, \tag{2.4}$$ eq0006

respectively, where $t = -\ln|x|$ and $\theta = \frac{x}{|x|}$ for $x \in \mathbb{R}^N\backslash\{0\}$ and $\Delta_{\mathbb{S}^{N-1}}$ is the Laplace-Beltrami operator on $\mathbb{S}^{N-1}$. Moreover, the CKN inequality (1.2) and the variational problem (1.9) is transformed into

$$C^{-1}_{a,b,N} = \inf_{v \in H^1(\mathcal{C})\backslash\{0\}} \frac{\|v\|^2_{H^1(\mathcal{C})}}{\|v\|^2_{L^{p+1}(\mathcal{C})}}, \tag{2.5}$$ eq0009

and

$$\inf_{v\in H^1(\mathcal{C})\backslash\mathcal{Y}}\frac{\|v\|^2_{H^1(\mathcal{C})}-C^{-1}_{a,b,N}\|v\|^2_{L^{p+1}(\mathcal{C})}}{dist^2_{H^1(\mathcal{C})}(v,\mathcal{Y})}:=c_{BE}, \tag{2.6}$$ eqq0001

respectively, where $L^{p+1}(\mathcal{C})$ is the usual Lebesgue space with its usual norm given by $\|u\|_{L^{p+1}(\mathcal{C})}=\left(\int_{\mathcal{C}}|u|^{p+1}d\mu\right)^{\frac{1}{p+1}}$ and

$$\mathcal{Y}=\{c\Psi_s(t)\mid c\in\mathbb{R}\backslash\{0\}\text{ and }s\in\mathbb{R}\} \tag{2.7}$$ eqq1030

with $\Psi_s(t)=\Psi(t-s)$.

On the other hand, it has been proved in [17] that $W(x)$ is nondegenerate in $D^{1,2}_a(\mathbb{R}^N)$ either for $b_{FS}(a)<b<a+1$ with $a<0$ or for $a\leq b<a+1$ with $a\geq 0$ and $a+b>0$ in the cases of $N\geq 2$. That is, up to scalar multiplications,

$$V(x):=\nabla W(x)\cdot x+(a_c-a)W(x)=\frac{\partial}{\partial\lambda}(\lambda^{(a_c-a)}W(\lambda x))|_{\lambda=1} \tag{2.8}$$ eq0010

is the only nonzero solution of the linearization of (2.2) around W in $D^{1,2}_a(\mathbb{R}^N)$ which is given by

$$-div(|x|^{-a}\nabla u)=p|x|^{-b(p+1)}W^{p-1}u,\quad\text{in }\mathbb{R}^N. \tag{2.9}$$ eq0017

By the transformation (2.1), the linear equation (2.9) can be rewritten as follows:

$$-\Delta_{\mathbb{S}^{N-1}}v-\partial_t^2v+(a_c-a)^2v=p\Psi^{p-1}v,\quad\text{in }\mathcal{C}. \tag{2.10}$$ eq0016

By applying the transformation (2.1) on (2.8), we know that

$$\Psi_s'(t)=\Psi'(t-s)=\frac{\partial}{\partial t}\Psi(t-s)=-\frac{\partial}{\partial s}\Psi(t-s)$$

is the only nonzero solution of (2.10) in $H^1(\mathcal{C})$.

3. Spectral gap inequality

We denote by $\mathcal{M}=\mathbb{R}\Psi\bigoplus\mathbb{R}\Psi'$. Since Ψ is Morse index 1, by the nondegeneracy of Ψ under the conditions (1) and (2) in the cases of $N\geq 2$, we have the following spectral gap inequality:

$$\|\rho\|^2_{H^1(\mathcal{C})}\geq(p+\varepsilon)\int_{\mathcal{C}}\Psi^{p-1}\rho^2d\mu \tag{3.1}$$ eqq10018

for all $\rho\in\mathcal{M}^\perp$ where $\varepsilon>0$ is a fixed small constant and

$$\mathcal{M}^\perp=\{\rho\in H^1(\mathcal{C})\mid\langle\rho,\Psi\rangle_{H^1(\mathcal{C})}=\langle\rho,\Psi'\rangle_{H^1(\mathcal{C})}=0\}.$$

In this section, we shall improve the spectral gap inequality (3.1) by proving the following result.

⟨propq0001⟩ **Proposition 3.1.** *Let $N\geq 2$ and assume that either*

(1) $b_{FS}(a)<b<a+1$ *with* $a<0$ *or*
(2) $a\leq b<a+1$ *with* $0\leq a<a_c$ *and* $a+b>0$.

Then for every $\rho\in\mathcal{M}^\perp$, we have

$$\|\rho\|^2_{H^1(\mathcal{C})}-\beta\int_{\mathcal{C}}\cosh^{-2}(\gamma t)\rho^2d\mu\geq\lambda_*\|\rho\|^2_{H^1(\mathcal{C})}, \tag{3.2}$$ eqq0021

where

$$\lambda_*=\begin{cases}\dfrac{2(p-1)}{3p-1}, & a_c^*<a<a_c \text{ and } a\le b<a+1,\\ \dfrac{2(p-1)}{3p-1}, & a\le a_c^* \text{ and } b_{FS}^*(a)\le b<a+1,\\ \dfrac{2q(a)-(p-2)(p+1)+(p-1)(1+q(a))^{\frac12}}{2+2q(a)+(p-1)(1+q(a))^{\frac12}}, & a\le a_c^* \text{ and } b_{FS}(a)<b<b_{FS}^*(a)\end{cases}$$

with

$$\beta=\frac{p(p+1)(a_c-a)^2}{2},\quad \gamma=\frac{(p-1)(a_c-a)}{2},\quad a_c^*=\left(1-\sqrt{\frac{N-1}{2N}}\right)a_c,$$

and

$$b_{FS}^*(a)=\frac{(a_c-a)N}{a_c-a+\sqrt{(a_c-a)^2+N-1}}+a-a_c,\quad q(a)=\frac{N-1}{(a_c-a)^2}.$$

Moreover, the equality of (3.2) *holds if and only if for*

$$\rho_{0,2}(t)=\mathcal{P}(\tanh(\gamma t))(\cosh(\gamma t))^{-\frac{2}{p-1}} \tag{3.3}$$ eqq1022

with

$$\mathcal{P}(z)=\frac18(1-z^2)^{-\frac{2}{p-1}}\frac{d^2}{dz^2}((1-z^2)^{2+\frac{2}{p-1}})$$

being a Jacobi polynomial under the assumptions

(i) $a_c^*\le a<a_c$ *and* $a\le b<a+1$,
(ii) $a<a_c^*$ *and* $b_{FS}^*(a)<b<a+1$,

the equality of (3.2) *holds if and only if for*

$$\rho_{1,0,l}(t,\theta)=(\cosh(\gamma t))^{-\frac{\sqrt{(a_c-a)^2+N-1}}{\gamma}}\Theta_{1,l},\quad l=1,2,\cdots,N, \tag{3.4}$$ eqq3022

under the assumption $a<a_c^*$ *and* $b_{FS}(a)<b<b_{FS}^*(a)$*, and the equality of* (3.2) *holds if and only if either for* $\rho_{0,2}(t)$ *or for* $\rho_{1,0,l}(t,\theta)$*,* $l=1,2,\cdots,N$*, under the assumption* $a<a_c^*$ *and* $b=b_{FS}^*(a)$*, where* $\Theta_{1,l}$ *are spherical harmonics of degree one.*

Proof. Since by (2.3), $\Psi(t)\to 0$ as $|t|\to+\infty$, it is well known that the operator $-\Delta_{\mathbb{S}^{N-1}}-\partial_t^2+(a_c-a)^2$ is compact in $L^2(p\Psi^{p-1},\mathcal{C})$. Thus, it is also well known that $\sigma(-\Delta_{\mathbb{S}^{N-1}}-\partial_t^2+(a_c-a)^2)=\{\lambda_l\}_{l\in\mathbb{N}}$ with $0<\lambda_1\le\lambda_2\le\cdots\le\lambda_l\to+\infty$ as $l\to\infty$, where $\sigma(-\Delta_{\mathbb{S}^{N-1}}-\partial_t^2+(a_c-a)^2)$ is the spectrum of the operator $-\Delta_{\mathbb{S}^{N-1}}-\partial_t^2+(a_c-a)^2$ in $L^2(p\Psi^{p-1},\mathcal{C})$.

Let us now consider the following eigenvalue problem

$$-\Delta_{\mathbb{S}^{N-1}}v-\partial_t^2v+(a_c-a)^2v=\lambda p\Psi^{p-1}v,\quad \text{in } \mathcal{C}. \tag{3.5}$$ eqq0016

where $\lambda>0$. As usual, since the spherical harmonics $\{\Theta_{i,l}\}_{i\in\mathbb{N}_0,1\le l\le l_{i,N}}$, which satisfies the following equation

$$-\Delta_{\mathbb{S}^{N-1}}\Theta_{i,l}=i(N-2+i)\Theta_{i,l}\quad \text{in } \mathbb{S}^{N-1}, \tag{3.6}$$ eqq10016

form a orthogonal basic of $L^2(\mathbb{S}^{N-1})$, we shall use $\{\Theta_{i,l}\}_{i\in\mathbb{N}_0,1\leq l\leq l_{i,N}}$ as the Fourier modes to expand the eigenvalue problem (2.10), where $l_{i,N}\in\mathbb{N}$. Since we have

$$v=\sum_{i=0}^{\infty}\sum_{l=1}^{l_{i,N}}\phi_{i,l}\Theta_{i,l}$$

for every $v\in L^2(\mathcal{C})$ with $\phi_{i,l}=\int_{\mathcal{C}}v\Theta_{i,l}d\theta$, by (3.5) and (3.6), v is a solution of the eigenvalue problem (3.5) if and only if $\phi_{i,l}$ satisfies the following ordinary differential equation

$$-\partial_t^2\phi_{i,l}-\lambda\beta\cosh^{-2}(\gamma t)\phi_{i,l}=-\tau_{a,i}\phi_{i,l},\quad \text{in }\mathbb{R} \qquad (3.7)\ \boxed{\texttt{eqq0017}}$$

for all $l=1,2,\cdots,l_{i,N}$ and $i\in\mathbb{N}_0$, where

$$\tau_{a,i}=(a_c-a)^2+i(N-2+i). \qquad (3.8)\ \boxed{\texttt{eqq5699}}$$

By [26, p. 74] (see also [18, 4.2.2. Example: Poschl-Teller potentials] or [17, p. 130]), the negative eigenvalues of the opreator $-\partial_t^2-\lambda\beta\cosh^{-2}(\gamma t)$ in $L^2(\mathbb{R})$ is given by

$$\sigma_j=-\frac{\gamma^2}{4}\bigg(-(2j+1)+\sqrt{1+4\lambda\beta\gamma^{-2}}\bigg)^2$$

where $j=0,1,2,\cdots,j_0$ with $j_0\in\mathbb{N}_0$ and $j_0\leq\frac{1}{2}\bigg(\sqrt{1+4\lambda\beta\gamma^{-2}}-1\bigg)$. It follows that the ordinary differential equation (3.7) is solvable if and only if

$$\frac{\gamma^2}{4}\bigg(-(2j+1)+\sqrt{1+4\lambda\beta\gamma^{-2}}\bigg)^2=\tau_{a,i}. \qquad (3.9)\ \boxed{\texttt{eqq0018}}$$

For every i and j, we denote the unique number of $\lambda>0$ which satisfies (3.9) by $\lambda_{i,j}$. Thus, all eigenvalues of (3.5) are $\{\lambda_{i,j}\}_{i,j\in\mathbb{N}_0}$. Since (3.5) with $\lambda=1$ is just (2.10), it has been proved in [17] that $\lambda_{0,1}=1$. Note that by (3.9), $\lambda_{j,i}<\lambda_{j,i+1}$ and $\lambda_{j,i}<\lambda_{j+1,i}$ for all i and j, thus, by $\lambda_{0,1}=1$, we have $\lambda_{0,0}<1$ and $1<\lambda_{i,j}$ for all other i and j except $\lambda_{1,0}$. Moreover, since Ψ has Morse index 1 under the conditions (1) and (2) in the cases of $N\geq2$, we must have $1<\lambda_{1,0}$. Thus, $\min\{\lambda_{0,2},\lambda_{1,1},\lambda_{1,0}\}$ is the smallest eigenvalue of (3.5) which is larger than 1.

Let us first compare $\lambda_{0,2}$ and $\lambda_{1,1}$. We define

$$f(\lambda)=\sqrt{1+4\lambda\beta\gamma^{-2}}.$$

Then by (3.9),

$$f(\lambda_{0,2})-f(\lambda_{1,1})=2(1-g_N(a)h_{N,a}^{-1}(b)),$$

where $g_N(a)=\sqrt{\frac{1}{4}+\frac{N-1}{4(a_c-a)^2}}-\frac{1}{2}$ and $h_{N,a}(b)=\frac{1+a-b}{N-2(1+a-b)}$. By direct calculations, we find that $h_{N,a}(b)$ is decreasing for b with

$$\min_{a\leq b\leq a+1}h_{N,a}(b)=h_{N,a}(a+1)=0,\qquad \max_{a\leq b\leq a+1}h_{N,a}(b)=h_{N,a}(a)=\bigg(\frac{1}{N-2}\bigg)_+$$

and $g_N(a)$ is increasing for a with

$$\min_{-\infty<a\leq\frac{N-2}{2}}g_N(a)=g_N(-\infty)=0,\qquad \max_{-\infty<a\leq\frac{N-2}{2}}g_N(a)=g_N(\frac{N-2}{2})=+\infty.$$

Here, $\left(\frac{1}{N-2}\right)_+ = \frac{1}{N-2}$ for $N \geq 3$ and $\left(\frac{1}{N-2}\right)_+ = +\infty$ for $N = 2$. Note that $g_N(0) = \left(\frac{1}{N-2}\right)_+$, thus, for $0 \leq a < \frac{N-2}{2}$ which implies $N \geq 3$, we always have $g_N(a)h^{-1}_{N,a}(b) > 1$ for all $0 \leq a < \frac{N-2}{2}$ and $a \leq b < a+1$ with $a+b > 0$, which implies that $f(\lambda_{0,2}) < f(\lambda_{1,1})$ for all $0 \leq a < \frac{N-2}{2}$ and $a \leq b < a+1$ with $a+b > 0$. Moreover, for every $a < 0$, there exists a unique $a < b_{N,a} < a+1$ such that $g_N(a)h^{-1}_{N,a}(b_{N,a}) = 1$, which implies that $b_{N,a} = b_{FS}(a)$ given by (1.3). Thus, by the monotone property of $h_{N,a}(b)$, we see that $g_N(a)h^{-1}_{N,a}(b) > 1$ for all $a < 0$ and $b_{FS}(a) < b < a+1$, which implies that $f(\lambda_{0,2}) < f(\lambda_{1,1})$ for all $a < 0$ and $b_{FS}(a) < b < a+1$. It follows that we always have $\lambda_{0,2} < \lambda_{1,1}$ under the conditions (1) and (2) in the cases of $N \geq 2$.

It is sufficiently to compare $\lambda_{0,2}$ and $\lambda_{1,0}$ to determine the smallest eigenvalue of (3.5) which is larger than 1 under the conditions (1) and (2) in the cases of $N \geq 2$. As above, we have

$$f(\lambda_{0,2}) - f(\lambda_{1,0}) = 2(2 - g_N(a)h^{-1}_{N,a}(b)).$$

Now, using the monotone properties of $g_N(a)$ and $h_{N,a}(b)$, we can compute as above to find that $g_N(a)h^{-1}_{N,a}(b) > 2$ for $a^*_c < a < a_c$ with all $a \leq b < a+1$ in the cases of $N \geq 2$, while for $a \leq a^*_c$ in the cases of $N \geq 2$, $g_N(a)h^{-1}_{N,a}(b) > 2$ for $b^*_{FS}(a) < b < a+1$, $g_N(a)h^{-1}_{N,a}(b) = 2$ for $b = b^*_{FS}(a)$ and $g_N(a)h^{-1}_{N,a}(b) < 2$ for $b_{FS}(a) < b < b^*_{FS}(a)$. If follows that $\lambda_{0,2} < \lambda_{1,0}$ either for $a \leq a^*_c$ with $b^*_{FS}(a) < b < a+1$ or for $a^*_c < a < a_c$ with all $a \leq b < a+1$, $\lambda_{0,2} = \lambda_{1,0}$ for $a \leq a^*_c$ with $b = b^*_{FS}(a)$ and $\lambda_{1,0} < \lambda_{0,2}$ for $a < a^*_c$ with $b_{FS}(a) < b < b^*_{FS}(a)$, which, together with the fact that $\lambda_{0,2} < \lambda_{1,1}$ under the conditions (1) and (2) in the cases $N \geq 2$, implies that $\lambda_{0,2}$ is the smallest eigenvalue of (3.5) which is larger than 1 either for $a \leq a^*_c$ with $b^*_{FS}(a) < b < a+1$ or for $a^*_c < a < a_c$ with all $a \leq b < a+1$, $\lambda_{0,2}$ and $\lambda_{1,0}$ are both the smallest eigenvalue of (3.5) which is larger than 1 for $a \leq a^*_c$ with $b = b^*_{FS}(a)$, and $\lambda_{1,0}$ is the smallest eigenvalue of (3.5) which is larger than 1 for $a < a^*_c$ with $b_{FS}(a) < b < b^*_{FS}(a)$.

Since $\lambda_{0,1} = 1$, $\lambda_{0,0} < 1$ and $1 < \lambda_{i,j}$ for all other i and j, we have

$$\|\rho\|^2_{H^1(\mathcal{C})} - \beta \int_{\mathcal{C}} \cosh^{-2}(\gamma t)\rho^2 d\mu \geq \lambda_* \|\rho\|^2_{H^1(\mathcal{C})} \tag{3.10}$$ eqq1021

for every $\rho \in \mathcal{M}^\perp$, where

$$\lambda_* = \begin{cases} \dfrac{\lambda_{0,2}-1}{\lambda_{0,2}}, & a^*_c < a < a_c \text{ with } a \leq b < a+1, \\ \dfrac{\lambda_{0,2}-1}{\lambda_{0,2}}, & a \leq a^*_c \text{ with } b^*_{FS}(a) < b < a+1, \\ \dfrac{\lambda_{0,2}-1}{\lambda_{0,2}} = \dfrac{\lambda_{1,0}-1}{\lambda_{1,0}}, & a \leq a^*_c \text{ with } b = b^*_{FS}(a), \\ \dfrac{\lambda_{1,0}-1}{\lambda_{1,0}}, & a \leq a^*_c \text{ with } b_{FS}(a) < b < b^*_{FS}(a). \end{cases}$$

By (3.9), we can compute

$$\frac{\lambda_{0,2}-1}{\lambda_{0,2}}=\frac{(2(a_c-a)+4\gamma)\gamma}{(a_c-a+3\gamma)(a_c-a+2\gamma)}=\frac{2(p-1)}{3p-1} \tag{3.11}$$ eqq1029

and

$$\frac{\lambda_{1,0}-1}{\lambda_{1,0}}=\frac{\sqrt{(a_c-a)^2+N-1}(\sqrt{(a_c-a)^2+N-1}+\gamma)-\beta}{\sqrt{(a_c-a)^2+N-1}(\sqrt{(a_c-a)^2+N-1}+\gamma)} \tag{3.12}$$ eqq3029

which, together with (3.10), implies that (3.2) holds true for every $\rho\in\mathcal{M}^\perp$ under the conditions (1) and (2) in the cases of $N\geq 2$.

It remains to prove that the equality of (3.2) holds if and only if for the functions given by (3.3) and (3.4). By [22, p. 129, Case 1], we have

$$\phi_{i,j}(t)=\chi_k(\sinh(\gamma t))(\cosh(\gamma t))^{-\frac{j\gamma+\sqrt{\tau_{a,i}}}{\gamma}}, \tag{3.13}$$ eqq0020

where $\chi_k(z)$ is a polynomial of degree at most k which depends on i,j. Moreover, by [22, Theorem 7], $\chi_k(z)$ satisfies the following equation

$$-\gamma^2(z^2+1)D_z^2\chi_k(z)-((1-2j)\gamma^2-2\gamma\sqrt{\tau_{a,i}})D_z\chi_k(z)+R\chi_k(z)=0 \tag{3.14}$$ eqq1090

where $R\in\mathbb{R}$ can be taken arbitrary values. As that in [26, p. 74] (see also [30, p. 529]), we introduce the function

$$\varphi_{j,k}(z)=(1+z^2)^{-\frac{j}{2}}\chi_k(z)\quad\text{and}\quad\widetilde{\varphi}_{j,k}(y)=\varphi_{j,k}\left(\frac{y}{\sqrt{1-y^2}}\right)$$

with $z=\frac{y}{\sqrt{1-y^2}}$. Then by direct calculations, $1+z^2=\frac{1}{1-y^2}$. Moreover,

$$D_y\widetilde{\varphi}_{j,k}(y)=\frac{D_z\varphi_{j,k}(z)}{(1-y^2)^{\frac{3}{2}}}$$

and

$$D_y^2\widetilde{\varphi}_{j,k}(y)=\frac{D_z^2\varphi_{j,k}(z)}{(1-y^2)^3}+\frac{3yD_z\varphi_{j,k}(z)}{(1-y^2)^{\frac{5}{2}}}$$

with

$$D_z\varphi_{j,k}(z)=-j(1+z^2)^{-\frac{j+2}{2}}z\chi_k(z)+(1+z^2)^{-\frac{j}{2}}D_z\chi_k(z)$$

and

$$\begin{aligned}D_z^2\varphi_{j,k}(z)&=(1+z^2)^{-\frac{j}{2}}D_z^2\chi_k(z)-2j(1+z^2)^{-\frac{j+2}{2}}z\chi_k(z)\\&\quad+j\left(\frac{(j+2)z^2}{1+z^2}-1\right)(1+z^2)^{-\frac{j+2}{2}}\chi_k(z).\end{aligned}$$

It follows that for every $\widetilde{\tau},\widetilde{\mu}$ and q,

$$\begin{aligned}&(1-y^2)D_y^2\widetilde{\varphi}_{j,k}(y)+(\widetilde{\tau}-\widetilde{\mu}-(\widetilde{\tau}+\widetilde{\mu}+2))D_y\widetilde{\varphi}_{j,k}(y)+q(\widetilde{\tau}+\widetilde{\mu}+q+1)\widetilde{\varphi}_{j,k}(y)\\=\;&(1+z^2)^{\frac{-j+4}{2}}D_z^2\chi_k(z)+((\widetilde{\tau}-\widetilde{\mu})(1+z^2)^{\frac{-j+3}{2}}+(1-2j-\widetilde{\tau}-\widetilde{\mu})(1+z^2)^{\frac{-j+2}{2}}z)D_z\chi_k(z)\\&+(-j(\widetilde{\tau}-\widetilde{\mu})(1+z^2)^{\frac{1}{2}}+(j^2+j(\widetilde{\tau}+\widetilde{\mu}))z^2+q(\widetilde{\tau}+\widetilde{\mu}+q+1)-j)(1+z^2)^{\frac{-j}{2}}\chi_k(z).\end{aligned}$$

By (3.14), we find that $\widetilde{\varphi}_{j,k}(y)$ satisfies the following Jacobi equation

$$(1-y^2)D_y^2\widetilde{\varphi}_{j,k}(y)-2\left(\frac{\sqrt{\tau_{a,i}}}{\gamma}+1\right)yD_y\widetilde{\varphi}_{j,k}(y)+j\left(\frac{2\sqrt{\tau_{a,i}}}{\gamma}+j+1\right)\widetilde{\varphi}_{j,k}(y)=0.$$

It follows from [29, p. 22] that $\widetilde{\varphi}_{j,k}(y)$ is the Jacobi polynomial given by

$$\widetilde{\varphi}_{j,k}(y) = \frac{(-1)^j}{2^j j!}(1-y^2)^{-\frac{\sqrt{\tau_{a,i}}}{\gamma}} \frac{d^j}{dy^j}\left((1-y^2)^{j+\frac{\sqrt{\tau_{a,i}}}{\gamma}}\right). \tag{3.15}$$ eqq0027

Thus, by (3.2), we know that the equality of (3.2) holds if and only if for the functions given by (3.3) and (3.4). $\square$

⟨rmkq0001⟩ **Remark 3.1.** *Since* $\tanh(\gamma t)$ *is bounded in* $\mathbb{R}$, *by* (3.13) *and* (3.15),

$$|\phi_{i,j}| \lesssim \Psi \quad \text{for all } i \text{ and } l.$$

In particular, by (3.3) *and* (3.4), *we have* $|\rho_{0,2}| \lesssim \Psi$ *and* $|\rho_{1,0,l}| \lesssim \Psi$ *for all* $l = 1, 2, \cdots, N$.

Remark 3.2. *By* (3.4) *and* (3.12), *we find that* $\rho_{1,0,l}(t,\theta) = (\cosh(\gamma t))^{\frac{p+1}{p-1}}\theta_{1,l}$ *and* $\lambda_{1,0} = 1$ *for* $b = b_{FS}(a)$, *which coincides with the computations in [19, Lemma 7] (see also [17, (2.10)]).*

4. Energy estimates of c_{BE}

To prove c_{BE} is achieved, we shall follow the ideas of Konig in [24] to derive two crucial energy estimates of c_{BE}. For this purpose, we need first to establish the following expression of $dist^2_{H^1(\mathcal{C})}(v, \mathcal{Y})$, where $\mathcal{Y}$ is given by (2.7).

⟨lemq0001⟩ **Lemma 4.1.** *Let* $N \geq 2$ *and assume that either*

(1) $b_{FS}(a) < b < a+1$ *with* $a < 0$ *or*
(2) $a \leq b < a+1$ *with* $0 \leq a < a_c$ *and* $a+b > 0$.

Then for every $v \in H^1(\mathcal{C})$,

$$dist^2_{H^1(\mathcal{C})}(v, \mathcal{Y}) = \|v\|^2_{H^1(\mathcal{C})} - C^{-1}_{a,b,N} \sup_{h \in \mathcal{Y}_1} (\langle v, h^p \rangle_{L^2(\mathcal{C})})^2 \tag{4.1}$$ eqq0031

where

$$\mathcal{Y}_1 = \{v \in \mathcal{Y} \mid \|v\|_{L^{p+1}(\mathcal{C})} = 1\}.$$

Moreover, $\sup_{h \in \mathcal{Y}_1} (\langle v, h^p \rangle_{L^2(\mathcal{C})})^2$ *is attained for every* $v \in H^1(\mathcal{C})$.

Proof. The proof is a "completion of the square" argument which is the same as that of [9, Lemma 3] (see also the proof of [24, Lemma 2.2]), so we omit it here. $\square$

For the convenience of the readers, we provide here a standard computation of the integral $\int_{\mathbb{R}}(\cosh(s))^{-\alpha}(\cosh^2(s)-1)^{\beta}ds$ which is also used in the appendix:

$$\begin{aligned}
\int_{\mathbb{R}}(\cosh(s))^{-\alpha}(\cosh^2(s)-1)^{\beta}ds &= 2\int_0^{+\infty}(\cosh(s))^{-\alpha}(\cosh^2(s)-1)^{\beta}ds \\
&= 2\int_0^{+\infty}(1-\cosh^{-2}(s))^{\beta}(\cosh(s))^{2\beta-\alpha}ds \\
&= -\int_0^{+\infty}(1-\cosh^{-2}(s))^{\beta-\frac{1}{2}}(\cosh(s))^{2\beta-\alpha+2}d(\cosh^{-2}(s)) \\
&= \int_0^1 (1-x)^{\beta-\frac{1}{2}}x^{\frac{\alpha}{2}-\beta-1}dx \\
&= \mathbb{B}(\frac{\alpha}{2}-\beta, \beta+\frac{1}{2}),
\end{aligned} \tag{4.2}$$ eqq1041

for $\frac{\alpha}{2} > \beta$ and $\beta > -\frac{1}{2}$. Now, we have the following crucial energy estimates of c_{BE}.

⟨propq0002⟩ **Proposition 4.1.** *Let $N \geq 2$ and assume that either*

(1) $b_{FS}(a) < b < a+1$ *with* $a < 0$ *or*
(2) $a \leq b < a+1$ *with* $0 \leq a < a_c$ *and* $a+b > 0$.

Then $c_{BE} < 2 - 2^{\frac{1}{p+1}}$. Moreover, if either

(i) $a_c^* \leq a < a_c$ *and* $a \leq b < a+1$ *or*
(ii) $a < a_c^*$ *and* $b_{FS}^*(a) \leq b < a+1$.

Then $c_{BE} < \frac{2(p-1)}{3p-1}$, where a_c^* and $b_{FS}^*(a)$ are given in Proposition 3.1.

Proof. Let us first prove that

$$c_{BE} < \frac{2(p-1)}{3p-1}. \tag{4.3}$$ eqq1034

under the assumptions (i) and (ii). Testing c_{BE} by the function $u = \Psi + \varepsilon\rho_{0,2}$ with $\varepsilon \to 0$, then by the definition of c_{BE}, we have

$$c_{BE} \leq \frac{\|\Psi + \varepsilon\rho_{0,2}\|^2_{H^1(\mathcal{C})} - C^{-1}_{a,b,N}\|\Psi + \varepsilon\rho_{0,2}\|^2_{L^{p+1}(\mathcal{C})}}{dist^2_{H^1(\mathcal{C})}(\Psi + \varepsilon\rho_{0,2}, \mathcal{Y})}. \tag{4.4}$$ eqq0023

where $\rho_{0,2}$ is given by (3.3). By $\rho_{0,2} \in \mathcal{M}^\perp$ and Lemma 4.1,

$$dist^2_{H^1(\mathcal{C})}(\Psi + \varepsilon\rho_{0,2}, \mathcal{Y}) = \varepsilon^2\|\rho_{0,2}\|^2_{H^1(\mathcal{C})},$$

which, together with (4.4), implies that

$$c_{BE} \leq \frac{\|\Psi + \varepsilon\rho_{0,2}\|^2_{H^1(\mathcal{C})} - C^{-1}_{a,b,N}\|\Psi + \varepsilon\rho_{0,2}\|^2_{L^{p+1}(\mathcal{C})}}{\varepsilon^2\|\rho_{0,2}\|^2_{H^1(\mathcal{C})}}. \tag{4.5}$$ eqq1023

By Remark 3.1, we can expand $\|\Psi + \varepsilon\rho_{0,2}\|^{p+1}_{L^{p+1}(\mathcal{C})}$ by the Taylor expansion to arbitrary order terms. Thus,

$$\begin{aligned}\|\Psi + \varepsilon\rho_{0,2}\|^{p+1}_{L^{p+1}(\mathcal{C})} &= \|\Psi\|^{p+1}_{L^{p+1}(\mathcal{C})} + \varepsilon(p+1)\langle\Psi^p, \rho_{0,2}\rangle_{L^2(\mathcal{C})} + \varepsilon^2\frac{(p+1)p}{2}\langle\Psi^{p-1}, \rho_{0,2}^2\rangle_{L^2(\mathcal{C})}\\ &\quad + \varepsilon^3\frac{p(p^2-1)}{6}\langle\Psi^{p-2}, \rho_{0,2}^3\rangle_{L^2(\mathcal{C})} + o(\varepsilon^3).\end{aligned}$$

It follows from (2.4), $\rho_{0,2} \in \mathcal{M}^\perp$ and the Taylor expansion once more, that

$$\begin{aligned}C^{-1}_{a,b,N}\|\Psi + \varepsilon\rho_{0,2}\|^2_{L^{p+1}(\mathcal{C})} &= C^{-1}_{a,b,N}\|\Psi\|^2_{L^{p+1}(\mathcal{C})} + p\varepsilon^2\langle\Psi^{p-1}, \rho_{0,2}^2\rangle_{L^2(\mathcal{C})}\\ &\quad + \frac{p(p-1)\varepsilon^3}{3}\langle\Psi^{p-2}, \rho_{0,2}^3\rangle_{L^2(\mathcal{C})} + o(\varepsilon^3).\end{aligned} \tag{4.6}$$ eqq1033

By Proposition 3.1,

$$\rho_{0,2} = \frac{p(\cosh(\gamma s))^{-\frac{2}{p-1}}}{4(p-1)^2}(4(p+1) - (6p+2)(\cosh(\gamma s))^{-2})$$

under the assumptions (i) and (ii). It follows from (2.3) and (4.2) that

$$\begin{aligned}\langle\Psi^{p-2},\rho_{0,2}^3\rangle_{L^2(\mathcal{C})} &= \left(\frac{(p+1)(a_c-a)^2}{2}\right)^{\frac{p-2}{p-1}}|\mathbb{S}^{N-1}|\int_{\mathbb{R}}(\cosh(\gamma s))^{-\frac{2(p-2)}{p-1}}\rho_{0,2}^3ds\\ &= \frac{p^3}{8\gamma(p-1)^6}\left(\frac{(p+1)(a_c-a)^2}{2}\right)^{\frac{p-2}{p-1}}|\mathbb{S}^{N-1}|\bigg(8(p+1)^3\mathbb{B}(\frac{p+2}{p-1},\frac{1}{2})\\ &\quad -12(p+1)^2(3p+1)\mathbb{B}(\frac{p+2}{p-1}+1,\frac{1}{2})+6(p+1)(3p+1)^2\mathbb{B}(\frac{p+2}{p-1}+2,\frac{1}{2})\\ &\quad -(3p+1)^3\mathbb{B}(\frac{p+2}{p-1}+3,\frac{1}{2})\bigg),\end{aligned}$$

which, together with the well known fact that $\mathbb{B}(m,n)=\frac{m-1}{m-1+n}\mathbb{B}(m-1,n)$, implies that

$$\begin{aligned}\langle\Psi^{p-2},\rho_{0,2}^3\rangle_{L^2(\mathcal{C})} &= \frac{2(p+1)p^3}{\gamma(7p-3)(5p-1)(p-1)^6}\left(\frac{(p+1)(a_c-a)^2}{2}\right)^{\frac{p-2}{p-1}}|\mathbb{S}^{N-1}|\\ &\quad \times\mathbb{B}(\frac{p+2}{p-1},\frac{1}{2})(p^4-6p^2+8p-3)\\ &> 0\end{aligned}$$

since $p>1$. Thus, by (2.4), (2.5), (4.5), (4.6) and Propostion 3.1,

$$\begin{aligned}c_{BE} &\leq \frac{\|\rho_{0,2}\|_{H^1(\mathcal{C})}^2-p\langle\Psi^{p-1},\rho_{0,2}^2\rangle_{L^2(\mathcal{C})}}{\|\rho_{0,2}\|_{H^1(\mathcal{C})}^2}-\varepsilon\frac{p(p-1)\langle\Psi^{p-2},\rho_{0,2}^3\rangle_{L^2(\mathcal{C})}}{3\|\rho_{0,2}\|_{H^1(\mathcal{C})}^2}+o(\varepsilon)\\ &< \frac{\|\rho_{0,2}\|_{H^1(\mathcal{C})}^2-p\langle\Psi^{p-1},\rho_{0,2}^2\rangle_{L^2(\mathcal{C})}}{\|\rho_{0,2}\|_{H^1(\mathcal{C})}^2}\\ &= \frac{2(p-1)}{3p-1}\end{aligned}$$

for $\varepsilon>0$ sufficiently small, which implies that (4.3) holds true under the assumptions (i) and (ii).

It remains to prove that $c_{BE}<2-2^{\frac{2}{p+1}}$ for $N\geq 2$ under the conditions (1) and (2). For this purpose, we use $v_s=\Psi+\Psi_s$ as a test function of c_{BE}, where $s\to+\infty$. Then we have

$$c_{BE}\leq\frac{\|v_s\|_{H^1(\mathcal{C})}^2-C_{a,b,N}^{-1}\|v_s\|_{L^{p+1}(\mathcal{C})}^2}{dist_{H^1(\mathcal{C})}^2(v_s,\mathcal{Y})}. \tag{4.7}$$ eqq1035

By (2.3), (2.4) and direct calculations, we have

$$\begin{aligned}\|v_s\|_{H^1(\mathcal{C})}^2 &= 2\|\Psi\|_{H^1(\mathcal{C})}^2+2\int_{\mathcal{C}}\Psi^p\Psi_sd\mu\\ &= 2\|\Psi\|_{H^1(\mathcal{C})}^2+2\int_{\{t<\frac{s}{2}\}\times\mathbb{S}^{N-1}}(\Psi^p\Psi_s+\Psi_s^p\Psi)d\mu\\ &= 2\|\Psi\|_{H^1(\mathcal{C})}^2+2A_0e^{-\frac{2}{p-1}\gamma s}+\mathcal{O}(e^{-\frac{p+1}{p-1}\gamma s}),\end{aligned}\tag{4.8}$$ eqq0040

where

$$A_0=\left(\frac{(p+1)(a_c-a)^2}{2}\right)^{\frac{p+1}{p-1}}\int_{\mathcal{C}}(\cosh(\gamma t))^{-\frac{2p}{p-1}}e^{\frac{2}{p-1}\gamma t}dt.$$

Since $\Psi_s \leq \Psi$ in $(-\infty, \frac{s}{2})$ by (2.3), by (2.3) once more and the Taylor expansion,

$$\begin{aligned}
\|v_s\|^{p+1}_{L^{p+1}(\mathcal{C})} &= 2\int_{\{t<\frac{s}{2}\}\times\mathbb{S}^{N-1}}(\Psi_s+\Psi)^{p+1}d\mu \\
&= 2\int_{\{t<\frac{s}{2}\}\times\mathbb{S}^{N-1}}\Psi^{p+1}d\mu + 2(p+1)\int_{\{t<\frac{s}{2}\}\times\mathbb{S}^{N-1}}\Psi^p\Psi_s d\mu + \int_{\{t<\frac{s}{2}\}\times\mathbb{S}^{N-1}}\mathcal{O}(\Psi^{p-1}\Psi_s^2)d\mu \\
&= 2\|\Psi\|^{p+1}_{L^{p+1}(\mathcal{C})} + 2(p+1)A_0e^{-\frac{2}{p-1}\gamma s} + o(e^{-\frac{2}{p-1}\gamma s}).
\end{aligned}$$

Thus, by (2.4), (2.5) and the Taylor expasion, we have

$$\|v_s\|^2_{H^1(\mathcal{C})} - C^{-1}_{a,b,N}\|v_s\|^2_{L^{p+1}(\mathcal{C})} = (2-2^{\frac{2}{p+1}})\|\Psi\|^2_{H^1(\mathcal{C})} - 2A_0e^{-\frac{2}{p-1}\gamma s} + o(e^{-\frac{2}{p-1}\gamma s}). \tag{4.9}$$ eqq1036

On the other hand, by (2.4) and Lemma 4.1,

$$\begin{aligned}
dist^2_{H^1(\mathcal{C})}(v_s, \mathcal{Y}) &= \|v_s\|^2_{H^1(\mathcal{C})} - C^{-1}_{a,b,N}\sup_{h\in\mathcal{Y}_1}(\langle v_s, h^p\rangle_{L^2(\mathcal{C})})^2 \\
&= \|v_s\|^2_{H^1(\mathcal{C})} - C^{-1+\frac{2p}{p-1}}_{a,b,N}\sup_{\tau\in\mathbb{R}}(\langle \Psi+\Psi_s, \Psi^p_\tau\rangle_{L^2(\mathcal{C})})^2 \qquad (4.10)
\end{aligned}$$ eqq0035

We denote $H_s(\tau) = F(\tau) + G_s(\tau)$ with

$$F(\tau) = \langle \Psi, \Psi^p_\tau\rangle_{L^2(\mathcal{C})} \quad \text{and} \quad G_s(\tau) = \langle \Psi_s, \Psi^p_\tau\rangle_{L^2(\mathcal{C})}.$$

Clearly, by (2.3) and the symmetry of Ψ,

$$\sup_{\tau\in\mathbb{R}} H_s(\tau)^2 = \max_{0\leq\tau\leq\frac{s}{2}} H_s(\tau)^2 = (\max_{0\leq\tau\leq\frac{s}{2}} H_s(\tau))^2.$$

Moreover, $H_s(\tau)$ is strictly increasing in $(-\infty, 0)$ and strictly decreasing in $(s, +\infty)$. We denote

$$H_s(\tau(s)) = \max_{0\leq\tau\leq\frac{s}{2}} H_s(\tau).$$

Note that by (2.3) and the symmetry of Ψ, $F(\tau)$ is also strictly increasing in $(-\infty, 0)$ and strictly decreasing in $(0, +\infty)$. Thus, $F(0)$ is the unique strictly global maximum of $F(\tau)$. Since we also have

$$F(\tau) = \langle \Psi_{-\tau}, \Psi^p\rangle_{L^2(\mathcal{C})} \quad \text{and} \quad G_s(\tau) = \langle \Psi_{s-\tau}, \Psi^p\rangle_{L^2(\mathcal{C})},$$

by similar estimates of (4.8),

$$F(0) + o(1) = H_s(0) \leq H_s(\tau(s)) = F(\tau(s)) + o(1) \leq F(0) + o(1)$$

as $s \to +\infty$. It follows from the continuity and monotone property of $F(\tau)$ that $\tau(s) = o(1)$ as $s \to +\infty$. Again, by similar estimates of (4.8) and the Taylor expansion,

$$\begin{aligned}
H_s(\tau(s)) &= F(\tau(s)) + G_s(\tau(s)) \\
&= F(0) + \frac{F''(0)}{2}\tau(s)^2 + o(\tau(s)^2) + 2A_0e^{-\frac{2}{p-1}\gamma(s-\tau(s))} + \mathcal{O}(e^{-\frac{p+1}{p-1}\gamma s}),
\end{aligned}$$

which, together with $H_s(\tau(s)) \geq H(0) = F(0) + G_s(0)$ and $F''(0) < 0$, implies that

$$\begin{aligned}
\tau(s)^2 &\lesssim e^{-\frac{2}{p-1}\gamma(s-\tau(s))} - e^{-\frac{2}{p-1}\gamma s} + \mathcal{O}(e^{-\frac{p+1}{p-1}\gamma s}) \\
&\lesssim \tau(s)e^{-\frac{2}{p-1}\gamma s} + o(\tau(s)^2) + \mathcal{O}(e^{-\frac{p+1}{p-1}\gamma s}).
\end{aligned}$$

It follows from $p>1$ that $\tau(s)=o(e^{-\frac{1}{p-1}\gamma s})$. Thus, by the Taylor expansion and similar estimates of (4.8) once more, we have

$$
\begin{aligned}
H_s(\tau(s)) &= F(0)+G_s(0)+\mathcal{O}(\tau(s)^2)+G_s(\tau(s))-G(0)\\
&= F(0)+G_s(0)++\mathcal{O}(\tau(s)^2)+\mathcal{O}(\tau(s)e^{-\frac{2}{p-1}\gamma s}+e^{-\frac{p+1}{p-1}\gamma s})\\
&= F(0)+G_s(0)+o(e^{-\frac{2}{p-1}\gamma s}).
\end{aligned}
$$

By (2.4), (2.5), (4.8) and (4.10),

$$
\begin{aligned}
dist^2_{H^1(\mathcal{C})}(v_s,\mathcal{Y}) &= 2\|\Psi\|^2_{H^1(\mathcal{C})}+2\int_{\mathcal{C}}\Psi^p\Psi_s d\mu - C_{a,b,N}^{-1+\frac{2p}{p-1}}\left(\|\Psi\|^{p+1}_{L^{p+1}(\mathcal{C})}+\int_{\mathcal{C}}\Psi^p\Psi_s d\mu\right)^2\\
&\quad +o(e^{-\frac{2}{p-1}\gamma s})\\
&= \|\Psi\|^2_{H^1(\mathcal{C})}-C_{a,b,N}^{-1+\frac{2p}{p-1}}(\int_{\mathcal{C}}\Psi^p\Psi_s d\mu)^2+o(e^{-\frac{2}{p-1}\gamma s})\\
&= \|\Psi\|^2_{H^1(\mathcal{C})}+o(e^{-\frac{2}{p-1}\gamma s}). \qquad (4.11)
\end{aligned}
$$
eqq1037

By (4.7), (4.9) and (4.11), we have

$$
\begin{aligned}
c_{BE} &\leq \frac{\|v_s\|^2_{H^1(\mathcal{C})}-C_{a,b,N}^{-1}\|v_s\|^2_{L^{p+1}(\mathcal{C})}}{dist^2_{H^1(\mathcal{C})}(v_s,\mathcal{Y})}\\
&= 2-2^{\frac{2}{p+1}}-2A_0C_{a,b,N}^{\frac{2}{p-1}}e^{-\frac{2}{p-1}\gamma s}+o(e^{-\frac{2}{p-1}\gamma s})\\
&< 2-2^{\frac{2}{p+1}}
\end{aligned}
$$

for $s>0$ sufficiently large, which completes the proof. □

5. Proof of main results

We mainly follow the strategy of Konig in [24] to prove Theorem 1.3.

Proof of Theorem 1.3: Let v_n be a minimizing suquence of (2.6). Then we have $\{v_n\}\subset H^1(\mathcal{C})\backslash\mathcal{Y}$ and

$$
\frac{\|v_n\|^2_{H^1(\mathcal{C})}-C_{a,b,N}^{-1}\|v_n\|^2_{L^{p+1}(\mathcal{C})}}{dist^2_{H^1(\mathcal{C})}(v_n,\mathcal{Y})}=c_{BE}+o_n(1). \tag{5.1}
$$
eqq0030

As that in [24], we normlize v_n by assuming $\|v_n\|^2_{L^{p+1}(\mathcal{C})}=1$. It follows from (5.1) that

$$
(c_{BE}+o_n(1))dist^2_{H^1(\mathcal{C})}(v_n,\mathcal{Y})+C_{a,b,N}^{-1}=\|v_n\|^2_{H^1(\mathcal{C})}. \tag{5.2}
$$
eq0002

Since by Proposition 4.1, (3.11) and (3.12), $0<c_{BE}<1$ under the conditions (1) and (2), by Lemma 4.1,

$$
(1-c_{BE}+o_n(1))\|v_n\|^2_{H^1(\mathcal{C})}=C_{a,b,N}^{-1}-(c_{BE}+o_n(1))C_{a,b,N}^{-1}\sup_{h\in\mathcal{Y}_1}(\langle v,h^p\rangle_{L^2(\mathcal{C})})^2.
$$

Thus, it is easy to see that $\{v_n\}$ is bounded in $H^1(\mathcal{C})$, which together with Lemma 4.1, also implies that $\{dist^2_{H^1(\mathcal{C})}(v_n,\mathcal{Y})\}$ is bounded. As that in [24], by the Lions lemma (cf. [5, Lemma 4.1]), up to translating the sequence $\{v_n\}$, we may assume that $v_n\rightharpoonup f$ weakly in $H^1(\mathcal{C})$ for some non-zero f. We decompose

$$
v_n=f+g_n \quad \text{in } H^1(\mathcal{C}) \text{ where } g_n\rightharpoonup 0 \text{ weakly in } H^1(\mathcal{C}).
$$

For the sake of clarity, we divide the following proof into three steps.

Step. 1 We prove that $g_n \to 0$ strongly in $H^1(\mathcal{C})$ as $n \to \infty$.

Suppose the contrary that $g_n \not\to 0$ strongly in $H^1(\mathcal{C})$, then by the Lions lemma (cf. [5, Lemma 4.1]) once more and the fact that $g_n \rightharpoonup 0$ weakly in $H^1(\mathcal{C})$, there exist $s_n \in \mathbb{R}$ such that $|s_n| \to +\infty$ and $g_n(\cdot - s_n) \rightharpoonup g_0 \neq 0$ weakly in $H^1(\mathcal{C})$. We denote $\mathbb{M}(v) = \sup_{h\in\mathcal{Y}_1}(\langle v, h^p\rangle_{L^2(\mathcal{C})})^2$. Then by Lemma 4.1,

$$\mathbb{M}(g_n) = (\langle g_n, h^p_{n,*}\rangle_{L^2(\mathcal{C})})^2 \quad \text{and} \quad \mathbb{M}(f) = (\langle f, h^p_f\rangle_{L^2(\mathcal{C})})^2.$$

Since $g_n(\cdot - s_n) \rightharpoonup g_0 \neq 0$ weakly in $H^1(\mathcal{C})$ with $|s_n| \to +\infty$, we must have $h_{n,*} = C^{\frac{1}{p-1}}_{a,b,N}\Psi(t - s'_{n,*})$ with $|s'_{n,*}| \to +\infty$. It follows that

$$\mathbb{M}(v_n) \geq (\langle v_n, h^p_{n,*}\rangle_{L^2(\mathcal{C})})^2 = \mathbb{M}(g_n) + o_n(1)$$

and

$$\mathbb{M}(v_n) \geq (\langle v_n, h^p_f\rangle_{L^2(\mathcal{C})})^2 = \mathbb{M}(f) + o_n(1).$$

Thus,

$$\mathbb{M}(v_n) \geq \max\left\{\mathbb{M}(g_n), \mathbb{M}(f)\right\} + o_n(1). \tag{5.3}$$ eqq2036

We denote $\mathcal{S}(v) = \frac{\|v\|^2_{H^1(\mathcal{C})}}{\|v\|^2_{L^{p+1}(\mathcal{C})}}$. Moreover, without loss of generality, we assume that $\|g_n\|^2_{L^{p+1}(\mathcal{C})} \leq \|f\|^2_{L^{p+1}(\mathcal{C})}$. Then by (2.6), (4.1), (5.2), (5.3) and the fact that $c_{BE} < 1$,

$$\begin{aligned}
o_n(1) &= o_n(1)dist^2_{H^1(\mathcal{C})}(v_n, \mathcal{Y}) \\
&= \|v_n\|^2_{H^1(\mathcal{C})} - C^{-1}_{a,b,N} - c_{BE}dist^2_{H^1(\mathcal{C})}(v_n, \mathcal{Y}) \\
&= (1 - c_{BE})\|v_n\|^2_{H^1(\mathcal{C})} - C^{-1}_{a,b,N} + c_{BE}C^{-1}_{a,b,N}\sup_{h\in\mathcal{Y}_1}(\langle v_n, h^p\rangle_{L^2(\mathcal{C})})^2 \\
&\geq \|f\|^2_{H^1(\mathcal{C})} - C^{-1}_{a,b,N}\|f\|^2_{L^{p+1}(\mathcal{C})} - c_{BE}dist^2_{H^1(\mathcal{C})}(f, \mathcal{Y}) + (1 - c_{BE})\|g_n\|^2_{H^1(\mathcal{C})} \\
&\quad -C^{-1}_{a,b,N}\left((\|f\|^{p+1}_{L^{p+1}(\mathcal{C})} + \|g_n\|^{p+1}_{L^{p+1}(\mathcal{C})})^{\frac{2}{p+1}} - \|f\|^2_{L^{p+1}(\mathcal{C})}\right) + o_n(1) \\
&\geq \left(1 - c_{BE} - \frac{C^{-1}_{a,b,N}}{\mathcal{S}(g_n)}\left(\frac{(q^{p+1}_n + 1)^{\frac{2}{p+1}} - 1}{q^2_n}\right)\right)\|g_n\|^2_{H^1(\mathcal{C})} + o(1),
\end{aligned} \tag{5.4}$$ eqq1040

where $q_n = \frac{\|g_n\|_{L^{p+1}(\mathcal{C})}}{\|f\|_{L^{p+1}(\mathcal{C})}} \leq 1$. By [24, Lemma 2.3], we have

$$\frac{(q^{p+1}_n + 1)^{\frac{2}{p+1}} - 1}{q^2_n} \leq 2^{\frac{2}{p+1}} - 1,$$

which, together with (2.5) and (5.4), implies that

$$c_{BE} \geq 2 - 2^{\frac{2}{p+1}}.$$

It contradicts Proposition 4.1. Thus, we must have $g_n \to 0$ strongly in $H^1(\mathcal{C})$.

Step. 2 We prove that $dist^2_{H^1(\mathcal{C})}(f, \mathcal{Y}) > 0$ under the assumptions (i) and (ii).

Again, we suppose the contrary that $dist_{H^1(\mathcal{C})}(f, \mathcal{Y}) = 0$, then by Step. 1, we have $g_n \to 0$ strongly in $H^1(\mathcal{C})$. It follows that $dist(v_n, \mathcal{Y}) \to 0$. Now, we are in the same situation as that in the proof of [24, Proposition 4.1]. Thanks to Proposition 3.1, we can use the same argument as that used for [6, Proposition 2] (see also the proof of [32, Proposotion 4.1]) to show that $c_{BE} \geq \frac{\lambda_{0,2}-1}{\lambda_{0,2}}$ under the assumptions (i) and (ii), which contradicts Proposition 4.1 under the assumptions (i) and (ii). Thus, we must have $dist^2_{H^1(\mathcal{C})}(f, \mathcal{Y}) > 0$ under the assumptions (i) and (ii).

Step. 3 We prove that the variational problem (2.6) has a minimizer f under the assumptions (i) and (ii).

Since by Step. 1, $g_n \to 0$ strongly in $H^1(\mathcal{C})$ and by Step. 2, $dist^2_{H^1(\mathcal{C})}(f, \mathcal{Y}) > 0$ under the assumptions (i) and (ii), the variational problem (2.6) has a minimizer f under the assumptions (i) and (ii). □

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7. Appendix: The remaining case

The remaining case, that is, $N \geq 2$ with $a < a_c^*$ and $b_{FS}(a) < b < b^*_{FS}(a)$, is very special for the variational problem (2.6). On one hand, by Proposition 4.1, the energy estimate $c_{BE} < 2 - 2^{\frac{2}{p+1}}$ still holds for this case. On the other hand, if we can establish the energy estimate $c_{BE} < \lambda_* = \frac{\lambda_{1,0}-1}{\lambda_{1,0}}$ for this case as that for the cases (i) and (ii) in Proposition 4.1, then by the same arguments as that used for Theorem 1.3, we can still prove that c_{BE} is attained in this case, where λ_* is given by Proposition 3.1. In what follows, we shall show that the test function $u = \Psi + \varepsilon\rho_{1,0,l}$ with $\varepsilon \to 0$, which seems to be the possiblely optimal test functions according to Proposition 3.1, is invalid in deriving the energy estimate $c_{BE} < \lambda_*$ in this case.

By Proposition 3.1, it is easy to see that $\langle \Psi^{p-2}, \rho^3_{1,0,l}\rangle_{L^2(\mathcal{C})} = 0$ for all $l = 1, 2, \cdots, N$. Now, as that in the proof of Proposition 4.1, we will have $c_{BE} \leq \lambda_* + o(\varepsilon)$ in the remaining case. Thus, to go further, we need to expand $\|\Psi + \varepsilon\rho_{0,2,l}\|^2_{L^{p+1}(\mathcal{C})}$ to higher oder terms. Since by Remark 3.1, we can expand $\|\Psi + \varepsilon\rho_{0,2,l}\|^2_{L^{p+1}(\mathcal{C})}$ to arbitrary order terms, by the Taylor expansion,

$$\begin{aligned} \|\Psi + \varepsilon\rho_{1,0,l}\|^{p+1}_{L^{p+1}(\mathcal{C})} &= \|\Psi\|^{p+1}_{L^{p+1}(\mathcal{C})} + \varepsilon(p+1)\langle \Psi^p, \rho_{1,0,l}\rangle_{L^2(\mathcal{C})} + \varepsilon^2\frac{(p+1)p}{2}\langle \Psi^{p-1}, \rho^2_{1,0,l}\rangle_{L^2(\mathcal{C})} \\ &\quad + \varepsilon^3\frac{p(p^2-1)}{6}\langle \Psi^{p-2}, \rho^3_{1,0,l}\rangle_{L^2(\mathcal{C})} + \varepsilon^4\frac{p(p^2-1)(p-2)}{12}\langle \Psi^{p-3}, \rho^4_{1,0,l}\rangle_{L^2(\mathcal{C})} \\ &\quad + o(\varepsilon^4). \end{aligned} \tag{7.1}$$

eqq0050

It follows from (2.4), $\rho_{1,0,l} \in \mathcal{M}^\perp$ and the Taylor expansion once more, that

$$\begin{aligned} C^{-1}_{a,b,N}\|\Psi + \varepsilon\rho_{1,0,l}\|^2_{L^{p+1}(\mathcal{C})} &= C^{-1}_{a,b,N}\|\Psi\|^2_{L^{p+1}(\mathcal{C})} + p\varepsilon^2\langle \Psi^{p-1}, \rho^2_{1,0,l}\rangle_{L^2(\mathcal{C})} \\ &\quad + \frac{p(p-1)\varepsilon^3}{3}\langle \Psi^{p-2}, \rho^3_{1,0,l}\rangle_{L^2(\mathcal{C})} + \frac{p(p-1)(p-2)\varepsilon^4}{12}\langle \Psi^{p-3}, \rho^4_{1,0,l}\rangle_{L^2(\mathcal{C})} \\ &\quad - C^{\frac{p+1}{p-1}}_{a,b,N}\frac{(p-1)p^2\varepsilon^4}{4}(\langle \Psi^{p-1}, \rho^2_{1,0,l}\rangle_{L^2(\mathcal{C})})^2 + o(\varepsilon^4). \end{aligned} \tag{7.2}$$

eqq2033

Then by Proposition 3.1, (7.1) and (7.2), we will have the following energy estimate:

$$c_{BE} \leq \lambda_* - \frac{\widehat{\mathbb{Z}}_{a,b,N,l}}{\|\rho_{1,0,l}\|^2_{H^1(\mathcal{C})}}\varepsilon^2 + o(\varepsilon^2), \tag{7.3}$$ eqq3051

where we denote

$$\widehat{\mathbb{Z}}_{a,b,N,l} = \frac{p(p-1)(p-2)}{12}\langle \Psi^{p-3}, \rho^4_{1,0,l}\rangle_{L^2(\mathcal{C})} - C^{\frac{p+1}{p-1}}_{a,b,N}\frac{(p-1)p^2}{4}(\langle \Psi^{p-1}, \rho^2_{1,0,l}\rangle_{L^2(\mathcal{C})})^2.$$

Clearly, if we want to derive the desired energy estimate $c_{BE} < \lambda_*$, we need to show that $\mathbb{Z}_{a,b,N,l} > 0$ where $p > 2$ is necessary. Recall that $p = \frac{N+2(1+a-b)}{N-2(1+a-b)}$ with $a \leq b < a+1$, Thus, we must have $2 \leq N \leq 5$ for $p > 2$. It follows that $\widehat{\mathbb{Z}}_{a,b,N,l} < 0$ for $N \geq 6$, which implies that we can not derive the desired estimate $c_{BE} < \lambda_*$ for $N \geq 6$ and $a < a^*_c$ with $b_{FS}(a) < b < b^*_{FS}(a)$ any more by the possiblely optimal test functions $u = \Psi + \varepsilon\rho_{1,0,l}$ as $\varepsilon \to 0$.

For $2 \leq N \leq 5$, we know that $p > 2$ is equivalent to

$$b < b^{**}_{FS}(a) := a - a_c + \frac{N}{3}.$$

It follows from $a < a^*_c$ with $b_{FS}(a) < b < b^*_{FS}(a)$ that $a^{**}_c < a < a^*_c$ where

$$a^{**}_c = a_c - \frac{2}{\sqrt{5}}\sqrt{N-1}.$$

Moreover, $b^{**}_{FS}(a) < b^*_{FS}(a)$ for $a < a^{***}_c$ and $b^{**}_{FS}(a) > b^*_{FS}(a)$ for $a > a^{***}_c$ where

$$a^{***}_c = a_c - \frac{\sqrt{3}}{3}\sqrt{N-1}.$$

We remark that since $2 \leq N \leq 5$, we have $a^{***}_c < a^*_c$. Since p is decreasing for b, we have

$$q_*(a) \leq p \leq 2q_*(a) - 1 \quad \text{for } a^{***}_c \leq a < a^*_c \tag{7.4}$$ eqq9099

and

$$2 \leq p \leq 2q_*(a) - 1 \quad \text{for } a^{**}_c \leq a < a^{***}_c, \tag{7.5}$$ eqq9098

where

$$q_*(a) = \sqrt{1 + \frac{N-1}{(a_c - a)^2}}. \tag{7.6}$$ eqq9097

We also remark that for $a \in [a^{**}_c, a^*_c)$,

$$2 \leq q_*(a) \leq \left(\frac{N+2}{N-2}\right)_+ \quad \text{for } a^{***}_c \leq a < a^*_c \tag{7.7}$$ eqq9095

and

$$\frac{3}{2} \leq q_*(a) \leq 2 \quad \text{for } a^{**}_c \leq a < a^{***}_c. \tag{7.8}$$ eqq9094

By Proposition 3.1 and (2.3), we have

$$\langle \Psi^{p-3}, \rho^4_{1,0,l}\rangle_{L^2(\mathcal{C})} = \left(\frac{p+1}{2}(a_c - a)^2\right)^{\frac{p-3}{p-1}} \int_{\mathbb{S}^{N-1}} \theta^4_{1,l}d\theta \int_{\mathbb{R}} (\cosh(\gamma t))^{-2(\frac{p-3}{p-1}+\frac{2\sqrt{\tau_{a,1}}}{\gamma})}dt$$

and

$$\langle\Psi^{p-1},\rho_{1,0,l}^2\rangle_{L^2(\mathcal{C})}=\frac{p+1}{2}(a_c-a)^2\int_{\mathbb{S}^{N-1}}\theta_{1,l}^2d\theta\int_{\mathbb{R}}(\cosh(\gamma t))^{-2(1+\frac{\sqrt{\tau_{a,1}}}{\gamma})}dt$$

where $\tau_{a,1}$ is given by (3.8). Since by symmetry, we have

$$\int_{\mathbb{S}^{N-1}}\theta_{1,l}^2d\theta=\frac{1}{N}|\mathbb{S}^{N-1}|\quad\text{and}\quad\int_{\mathbb{S}^{N-1}}\theta_{1,l}^4d\theta=\frac{3}{N(N+2)}|\mathbb{S}^{N-1}|,$$

by (4.2) and the explicit formula of $C_{a,b,N}^{-1}$ given by [12, Corollary 1,3], that is,

$$C_{a,b,N}^{-1}=\frac{p+1}{2}(a_c-a)^{\frac{p+3}{p+1}}\left(\frac{2\sqrt{\pi}\Gamma\left(\frac{p+1}{p-1}\right)}{(p-1)\Gamma\left(\frac{3p+1}{2(p-1)}\right)}\right)^{\frac{p-1}{p+1}},$$

we have

$$\begin{aligned}\widehat{\mathbb{Z}}_{a,b,N,l}&=(a_c-a)^{\frac{p-5}{p-1}}\frac{p(p-2)|\mathbb{S}^{N-1}|}{2N(N+2)}\left(\frac{p+1}{2}\right)^{\frac{p-3}{p-1}}\\&\times\left(\mathbb{B}(\frac{p-3}{p-1}+2\frac{\sqrt{\tau_{a,1}}}{\gamma},\frac{1}{2})-\frac{pD_N\mathbb{B}^2(1+\frac{\sqrt{\tau_{a,1}}}{\gamma},\frac{1}{2})}{(p-2)\mathbb{B}(\frac{p+1}{p-1},\frac{1}{2})}\right),\end{aligned}\tag{7.9}$$

eqq9093

where

$$D_N=\frac{(N+2)|\mathbb{S}^{N-1}|}{N}.$$

Since it is well known that

$$|\mathbb{S}^{N-1}|=\begin{cases}\frac{2\pi^m}{(m-1)!}, & N=2m,\\ \frac{2(2\pi)^m}{(2m-1)!!}, & N=2m+1,\end{cases}$$

we have

$$D_N=\begin{cases}4\pi, & N=2,\\ \frac{20\pi}{3}, & N=3,\\ 3\pi^2, & N=4,\\ \frac{56\pi^2}{15}, & N=5.\end{cases}\tag{7.10}$$

eqq9096

Recall that by the definitions of γ and $\tau_{a,1}$ given by Proposition 3.1 and (3.8), respectively, we have

$$\frac{\sqrt{\tau_{a,1}}}{\gamma}-\frac{2}{p-1}=\frac{2}{p-1}(q_*(a)-1).$$

Thus, by (7.4) and (7.5),

$$1\leq\frac{\sqrt{\tau_{a,1}}}{\gamma}-\frac{2}{p-1}\leq 2,$$

where we have also used the monotone property of $q_*(a)$. As that in the computations for the cases (i) and (ii) in the proof of Proposition 4.1, by the monotone property of the beta function $\mathbb{B}(m,n)$ in terms of m and the equality $\mathbb{B}(m,n) = \frac{m-1}{m-1+n}\mathbb{B}(m-1,n)$,

$$\begin{aligned}
&\mathbb{B}(\frac{p-3}{p-1}+2\frac{\sqrt{\tau_{a,1}}}{\gamma},\frac{1}{2})-\frac{pD_N\mathbb{B}^2(1+2\frac{\sqrt{\tau_{a,1}}}{\gamma},\frac{1}{2})}{(p-2)\mathbb{B}(\frac{p+1}{p-1},\frac{1}{2})}\\
=&\frac{2\frac{\sqrt{\tau_{a,1}}}{\gamma}-\frac{2}{p-1}}{2\frac{\sqrt{\tau_{a,1}}}{\gamma}-\frac{2}{p-1}+\frac{1}{2}}\mathbb{B}(2\frac{\sqrt{\tau_{a,1}}}{\gamma}-\frac{2}{p-1},\frac{1}{2})-\frac{pD_N\mathbb{B}(1+2\frac{\sqrt{\tau_{a,1}}}{\gamma},\frac{1}{2})}{(p-2)\frac{\frac{p+1}{p-1}}{\frac{p+1}{p-1}+\frac{1}{2}}\times\frac{\frac{p+1}{p-1}+1}{\frac{p+1}{p-1}+\frac{3}{2}}\mathbb{B}(\frac{p+1}{p-1}+2,\frac{1}{2})}\mathbb{B}(1+2\frac{\sqrt{\tau_{a,1}}}{\gamma},\frac{1}{2})\\
\leq&4\left(\frac{2q_*(a)-1}{8q_*(a)+p-5}-\frac{2D_Np^2(p+1)}{(p-2)(2p+1)(5p-1)}\right)\mathbb{B}(1+2\frac{\sqrt{\tau_{a,1}}}{\gamma},\frac{1}{2})\\
=&\frac{4\mathbb{B}(1+2\frac{\sqrt{\tau_{a,1}}}{\gamma},\frac{1}{2})\overline{f}_{a,N}(p)}{(8q_*(a)+p-5)(p-2)(2p+1)(5p-1)},
\end{aligned}\tag{7.11}$$

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where

$$\begin{aligned}
\overline{f}_{a,N}(p)=&-2D_Np^4-2(4D_N-5)(2q_*(a)-1)p^3-(17(2q_*(a)-1)+2D_N(8q_*(a)-5))p^2\\
&-7(2q_*(a)-1)p+2(2q_*(a)-1)
\end{aligned}$$

with $q_*(a)$ given by (7.6). Since $q_*(a)\geq\frac{3}{2}$ and $D_N\geq\pi$ by (7.7)-(7.8) and (7.10), respectively, we have

$$\begin{aligned}
\overline{f}_{a,N}(p)\leq&-2D_Np^4-2(4D_N-5)(2q_*(a)-1)p^3-(17(2q_*(a)-1)+2D_N(8q_*(a)-5))p^2\\
\leq&-p^2(2D_Np^2+2(4D_N-5)(2q_*(a)-1)p)\\
<&\ 0
\end{aligned}$$

for $p>1$. It follows from (7.9) and (7.11) that $\widehat{\mathbb{Z}}_{a,b,N,l}<0$ for $2\leq N\leq 5$ and $a<a_c^*$ with $b_{FS}(a)<b<b^*_{FS}(a)$, which implies that we also can not derive the desired estimate $c_{BE}<\lambda_*$ from (7.3) for $2\leq N\leq 5$ and $a<a_c^*$ with $b_{FS}(a)<b<b^*_{FS}(a)$.

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