SHARP QUANTITATIVE STABILITY ESTIMATES FOR CRITICAL POINTS OF FRACTIONAL SOBOLEV INEQUALITIES

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ABSTRACT. By developing a unified approach based on integral representations, we establish sharp quantitative stability estimates of the fractional and higher-order Sobolev inequalities, induced by the embedding $\dot{H}^s(\mathbb{R}^n) \hookrightarrow L^{\frac{2n}{n-2s}}(\mathbb{R}^n)$ for any $s \in (0, \frac{n}{2})$, in the critical point setting.

1. INTRODUCTION

Given $n \in \mathbb{N}$ and $s \in \mathbb{R}$, let $\dot{H}^s(\mathbb{R}^n)$ be the homogeneous Sobolev space of fractional order s defined as

$$\dot{H}^s(\mathbb{R}^n) := \left\{ u \in \mathcal{S}'(\mathbb{R}^n) : \mathcal{F}u \in L^1_{\text{loc}}(\mathbb{R}^n),\ \|u\|_{\dot{H}^s(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n} |\xi|^{2s} |\mathcal{F}u(\xi)|^2 \mathrm{d}\xi \right)^{\frac{1}{2}} < \infty \right\}$$

where $\mathcal{F}u$ is the Fourier transform of u, and $\mathcal{S}'(\mathbb{R}^n)$ is the space of tempered distributions, i.e., the continuous dual space of the Schwartz space $\mathcal{S}(\mathbb{R}^n)$. As shown in [4], $\dot{H}^s(\mathbb{R}^n)$ is a Hilbert space if and only if $s < \frac{n}{2}$. Moreover, if $u \in \mathcal{S}(\mathbb{R}^n)$, then

$$\|u\|^2_{\dot{H}^s(\mathbb{R}^n)} = \left\| (-\Delta)^{\frac{s}{2}} u \right\|^2_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} u(-\Delta)^s u \quad \text{where } \mathcal{F}\left((-\Delta)^s u \right)(\xi) := |\xi|^{2s} \hat{u}(\xi).$$

The space $\dot{H}^s(\mathbb{R}^n)$ with $s < \frac{n}{2}$ is realized as the completion of $\mathcal{S}(\mathbb{R}^n)$ under the norm $\|\cdot\|_{\dot{H}^s(\mathbb{R}^n)}$.

For any $s \in (0, \frac{n}{2})$, there is an optimal constant $S_{n,s} > 0$ depending only on n and s such that

$$S_{n,s}\|u\|_{L^{p+1}(\mathbb{R}^n)} \le \|u\|_{\dot{H}^s(\mathbb{R}^n)} \quad \text{for all } u \in \dot{H}^s(\mathbb{R}^n) \quad \text{where } p := \frac{n+2s}{n-2s}, \tag{1.1}$$

referred to as the fractional Sobolev inequality. Lieb [46] proved that the set of the extremizers of (1.1) consists of non-zero constant multiples of the functions (often called the bubbles)

$$U[z,\lambda](x) = \alpha_{n,s} \left(\frac{\lambda}{1+\lambda^2|x-z|^2} \right)^{\frac{n-2s}{2}} \quad \text{for } x \in \mathbb{R}^n \tag{1.2}$$

where $\alpha_{n,s} := 2^{\frac{n-2s}{2}} [\Gamma(\frac{n+2s}{2})/\Gamma(\frac{n-2s}{2})]^{\frac{n-2s}{4s}}$.

According to the standard theory of calculus of variations, an extremizer of (1.1) always solves

$$(-\Delta)^s u = \mu |u|^{p-1} u \quad \text{in } \mathbb{R}^n, \quad u \in \dot{H}^s(\mathbb{R}^n) \tag{1.3}$$

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where $\mu \in \mathbb{R}$ is a Lagrange multiplier. Chen et al. [16] classified all positive solutions to (1.3), showing that they must assume the form in (1.2) up to a constant multiple. Furthermore, Dávila et al. [23] deduced that if $s \in (0,1)$, then the solution space of a linearized equation of (1.3)

$$(-\Delta)^s Z - p\,U[z,\lambda]^{p-1} Z = 0 \quad \text{in } \mathbb{R}^n, \quad Z \in L^\infty(\mathbb{R}^n).\ ^1 \tag{1.4}$$

is spanned by

$$Z^a[z,\lambda] = \frac{1}{\lambda} \frac{\partial U[\bar{z},\lambda]}{\partial \bar{z}^a}\bigg|_{\bar{z}=z} \quad \text{for } a = 1,\dots,n \quad \text{and} \quad Z^{n+1}[z,\lambda] = \lambda\, \frac{\partial U[z,\bar{\lambda}]}{\partial \bar{\lambda}}\bigg|_{\bar{\lambda}=\lambda},$$

where $\bar{z} = (\bar{z}_1,\dots,\bar{z}_n) \in \mathbb{R}^n$. In [45, Lemma 5.1], Li and Xiong extended this non-degeneracy theorem to all $s \in (0,\frac{n}{2})$. The condition $Z \in L^\infty(\mathbb{R}^n)$ in [45, Lemma 5.1] can be replaced with $Z \in \dot{H}^s(\mathbb{R}^n)$, as shown in Lemma A.1.

For a further understanding of (1.1), one can naturally consider its quantitative stability, as proposed by Brezis and Lieb [8]. Bianchi and Egnell [7] proved the existence of a constant $C_{\mathrm{BE}} > 0$ depending only on n such that

$$\inf_{z\in\mathbb{R}^n,\,\lambda>0,\,c\in\mathbb{R}} \|u - cU[z,\lambda]\|^2_{\dot{H}^1(\mathbb{R}^n)} \le C_{\mathrm{BE}} \left(\|u\|^2_{\dot{H}^1(\mathbb{R}^n)} - S^2_{n,s}\|u\|^2_{L^{p+1}(\mathbb{R}^n)}\right) \tag{1.5}$$

for any $u \in \dot{H}^1(\mathbb{R}^n)$. Later, their result was generalized by Chen et al. [15], who found a constant $C_{\mathrm{CFW}} > 0$ depending only on n and s such that

$$\inf_{z\in\mathbb{R}^n,\,\lambda>0,\,c\in\mathbb{R}} \|u - cU[z,\lambda]\|^2_{\dot{H}^s(\mathbb{R}^n)} \le C_{\mathrm{CFW}} \left(\|u\|^2_{\dot{H}^s(\mathbb{R}^n)} - S^2_{n,s}\|u\|^2_{L^{p+1}(\mathbb{R}^n)}\right) \tag{1.6}$$

for any $u \in \dot{H}^s(\mathbb{R}^n)$.

Another way to address the stability issue on (1.1) is to consider the qualitative stability for solutions to equation (1.3), which is the main objective of this paper. This problem is difficult because it requires controlling the quantitative behavior of approximate solutions with arbitrarily high energy. The starting point is the following Struwe-type profile decompositions for (1.1) derived by Gérard [38, Théorème 1.1]. Refer also to Palatucci and Pisante [49, Theorem 1.1] and Fang and González [31, Theorem 1.3].

Theorem A. *Suppose that $n \in \mathbb{N}$, $\nu \in \mathbb{N}$, $s \in (0,\frac{n}{2})$, $p = \frac{n+2s}{n-2s}$, and $S_{n,s} > 0$ is the constant in (1.1). Let $\{u_m\}_{m\in\mathbb{N}}$ be a sequence of non-negative functions in $\dot{H}^s(\mathbb{R}^n)$ such that $\left(\nu - \frac{1}{2}\right) S^n_{n,s} \le \|u_m\|^2_{\dot{H}^s(\mathbb{R}^n)} \le \left(\nu + \frac{1}{2}\right) S^n_{n,s}$. If it satisfies*

$$\|(-\Delta)^s u_m - u_m^p\|_{\dot{H}^{-s}(\mathbb{R}^n)} \to 0 \quad \text{as } k \to \infty,$$

then there exist a sequence $\{(z_{1,m},\dots,z_{\nu,m})\}_{m\in\mathbb{N}}$ of ν-tuples of points in $\mathbb{R}^n$ and a sequence $\{(\lambda_{1,m},\dots,\lambda_{\nu,m})\}_{m\in\mathbb{N}}$ of ν-tuples of positive numbers such that

$$\left\| u_m - \sum_{i=1}^{\nu} U[z_{i,m},\lambda_{i,m}] \right\|_{\dot{H}^s(\mathbb{R}^n)} \to 0 \quad \text{as } m \to \infty.$$

In addition, let $U_{i,m} := U[z_{i,m},\lambda_{i,m}]$ for $i = 1,\dots,\nu$. Then there exists $m_0 \in \mathbb{N}$ such that the sequence $\{(U_{1,m},\dots,U_{\nu,m})\}_{m\ge m_0}$ of ν-tuples of bubbles is δ-interacting in the following sense: If we define the quantity

$$q_{ij} = q(z_i,z_j,\lambda_i,\lambda_j) = \left(\frac{\lambda_i}{\lambda_j} + \frac{\lambda_i}{\lambda_j} + \lambda_i\lambda_j|z_i - z_j|^2\right)^{-\frac{n-2s}{2}} \quad \text{for } i,j = 1,\dots,\nu, \tag{1.7}$$

[1] More precisely, (1.4) is understood as the corresponding integral equation $Z = \Phi_{n,s} * \left(p\,U[z,\lambda]^{p-1}Z\right)$ in $\mathbb{R}^n$ where $\Phi_{n,s}$ is the Riesz potential in (1.10).

then

$$\max_{\substack{i,j=1,\dots,\nu,\\ i\neq j}} q\left(z_{i,m}, z_{j,m}, \lambda_{i,m}, \lambda_{j,m}\right) \le \delta \quad \text{for all } m \ge m_0.$$

If $s = 1$, the above theorem is reduced to one obtained by Struwe [56]. Also, the corresponding pointwise theory was established by Druet, Hebey, and Robert [29].

In this paper, we establish sharp quantitative stability estimates of the above decomposition provided any $n \in \mathbb{N}$ and $s \in (0, \frac{n}{2})$.

Theorem 1.1. *Let $n \in \mathbb{N}$, $\nu \in \mathbb{N}$, $s \in (0, \frac{n}{2})$, and $p = \frac{n+2s}{n-2s}$. There exist a small constant $\delta > 0$ and a large constant $C > 0$ depending only on n, s, and ν such that the following statement holds: If $u \in \dot{H}^s(\mathbb{R}^n)$ satisfies*

$$\left\| u - \sum_{i=1}^{\nu} U[\tilde{z}_i, \tilde{\lambda}_i] \right\|_{\dot{H}^s(\mathbb{R}^n)} \le \delta \tag{1.8}$$

for some δ-interacting family $\{U[\tilde{z}_i, \tilde{\lambda}_i]\}_{i=1}^{\nu}$, then there is a family $\{U[z_i, \lambda_i]\}_{i=1}^{\nu}$ of bubbles such that

$$\left\| u - \sum_{i=1}^{\nu} U[z_i, \lambda_i] \right\|_{\dot{H}^s(\mathbb{R}^n)} \le C \begin{cases} \Gamma(u) & \text{for } \nu = 1, \\ \Gamma(u) & \text{for } 2s < n < 6s \text{ and } \nu \ge 2, \\ \Gamma(u)|\log \Gamma(u)|^{\frac{1}{2}} & \text{for } n = 6s \text{ and } \nu \ge 2, \\ \Gamma(u)^{\frac{p}{2}} & \text{for } n > 6s \text{ and } \nu \ge 2 \end{cases} \tag{1.9}$$

where $\Gamma(u) := \|(-\Delta)^s u - |u|^{p-1}u\|_{\dot{H}^{-s}(\mathbb{R}^n)}$.

Furthermore, estimate (1.9) is sharp for $n > 2s$ and $\nu \ge 2$ in the sense that the power of $\Gamma(u)$ ($|\log \Gamma(u)|$, respectively) cannot be substituted with a larger (smaller, resp.) one.

As in the proof of [32, Corollary 3.4], we can combine Theorems A and 1.1 to find

Corollary 1.2. *Let $n \in \mathbb{N}$, $\nu \in \mathbb{N}$, and $s \in (0, \frac{n}{2})$. For any non-negative function $u \in \dot{H}^s(\mathbb{R}^n)$ such that $\left(\nu - \frac{1}{2}\right) S_{n,s}^n \le \|u\|_{\dot{H}^s(\mathbb{R}^n)} \le \left(\nu + \frac{1}{2}\right) S_{n,s}^n$, there exist ν bubbles $\{U[z_i, \lambda_i]\}_{i=1}^{\nu}$ such that (1.9) holds.*

The quantitative stability for functional and geometric inequalities is a fascinating subject that has captivated researchers for decades. Brezis and Nirenberg [9] and Brezis and Lieb [8] began this research direction, examining the Sobolev embeddings $H^1(\Omega) \hookrightarrow L^{\frac{2n}{n-2}}(\Omega)$ for bounded domains Ω in $\mathbb{R}^n$. Later, Bianchi and Egnell [7] obtained the optimal solution for the embedding $\dot{H}^1(\mathbb{R}^n) \hookrightarrow L^{\frac{2n}{n-2}}(\mathbb{R}^n)$. After these seminal works, numerous results of a similar nature appeared in the literature, and the following represents only a fraction of them; for the Sobolev inequalities $\dot{W}^{1,p}(\mathbb{R}^n) \hookrightarrow L^{\frac{np}{n-p}}(\mathbb{R}^n)$ in the non-Hilbert setting (for $p \neq 2$) [18, 34, 35], the fractional Sobolev inequalities and the Hardy-Littlewood-Sobolev (HLS) inequalities [15, 26], the conformally invariant Sobolev inequalities on Riemannian manifolds [30, 36], the isoperimetric inequalities [37, 33, 19, 17, 20], and so on. Besides, the smallest possible constants $C_{\text{BE}}, C_{\text{CFW}} > 0$ in (1.5)–(1.6) were estimated in [27, 42, 43, 13, 14].

In contrast, the quantitative stability of *almost* solutions (specifically, functions u with $\Gamma(u)$ small in our setting) to the Euler-Lagrange equations of functional and geometric inequalities has been less explored. However, recent advancements in [21, 32, 24] fully addressed when the Sobolev inequality $\dot{H}^1(\mathbb{R}^n) \hookrightarrow L^{\frac{2n}{n-2}}(\mathbb{R}^n)$ was considered: Ciraolo et al. [21] studied the one-bubble case ($\nu = 1$) for $n \ge 3$, Figalli and Glaudo [32] did the multi-bubble case ($\nu \ge 2$) for $n = 3, 4, 5$, and Deng et al. [24] did the multi-bubble case for $n \ge 6$. In related research, de Nitti and König [25] estimated the smallest possible constant $C > 0$ in (1.9) for $n \in \mathbb{N}$, $s \in (0, \frac{n}{2})$, and $\nu = 1$. Additionally, Aryan [3] deduced the stability result for the Euler-Lagrange equations of the fractional Sobolev inequalities $\dot{H}^s(\mathbb{R}^n) \hookrightarrow L^{\frac{2n}{n-2s}}(\mathbb{R}^n)$ with $s \in (0,1)$. Analogous results for other inequalities can be found in, e.g., [57, 58, 6, 47, 52, 12]. In this paper, we treat the fractional and higher-order Sobolev inequalities for all $n \in \mathbb{N}$, $s \in (0, \frac{n}{2})$, and $\nu \in \mathbb{N}$, thereby fully extending all the previous results [21, 32, 24, 3].

While numerous results in the literature investigated the existence and qualitative behavior of solutions to fractional elliptic problems $(-\Delta)^s u = f(u)$ when $s \in (0,1)$ and $f : \mathbb{R} \to \mathbb{R}$ is a certain function, studying the case $s \in (1, \frac{n}{2})$ is still at the beginning stage. Refer to a few works such as [16, 40, 41, 10, 2, 25, 42, 43, 48]. In fact, a study for the operator $(-\Delta)^s$ for $s > 1$ is an interesting research topic per se; refer to the extension results in [59, 11, 22], a recent survey paper of Abatangelo [1], and references therein. We believe that our results may facilitate further researches on higher-order local and non-local elliptic problems.

Novelty of the proof. Here, we outline the new features of our proof of Theorem 1.1. They mainly originate from the fact that we allow $s > 1$.

(1) Our method, primarily based on [24], offers a *unified* approach for any choice of $n \in \mathbb{N}$ and $s \in (0, \frac{n}{2})$. The choice of the norms with which we work depends on n: In what follows, we say that the dimension n is high if $n \geq 6s$ and low if $2s < n < 6s$. For the high-dimensional case, we use weighted $L^\infty(\mathbb{R}^n)$-type norms; refer to Definition 3.1. For the low-dimensional case, we utilize the standard $\dot{H}^s(\mathbb{R}^n)$-norm and $L^{\frac{2n}{n+2s}}(\mathbb{R}^n)$-norm; see Definition 5.1.

(2) In Proposition 2.2, we derive a spectral inequality that holds for all $s \in (0, \frac{n}{2})$ and δ-interacting families with $\delta > 0$ small. We do not use bump functions that appeared in the proof of Figalli and Glaudo [32] and Aryan [3] for the case $s \in (0, 1]$, resulting in a simpler proof.[2] Some key ingredients are the fractional Leibniz rule [39] and Li's Kenig-Ponce-Vega estimate [44].

(3) Proving Proposition 3.3 (linear theory) is one of the most delicate parts of the paper.

- We need to first ensure that the $*$-norm of f is finite when the $**$-norm of h is finite, because our domain $\mathbb{R}^n$ is unbounded. We will deduce the result for every $s \in (0, \frac{n}{2})$ simultaneously by repeatedly applying the integral representation of f along with the HLS inequality.
- When $s \in (0, 1]$, one can employ the barrier argument based on the maximum principle for narrow domains to control f in the neck region, as described in [24, 3]. However, extending this approach to large $s > 1$ is extremely challenging. In this study, we introduce a totally different method based on potential analysis, which simplifies the overall argument and allows us to handle the case $s \in (0, \frac{n}{2})$ at the same time.
- To estimate f in the core region, we require a Hölder continuity for the rescaled function $\hat{f}$, as mentioned in Lemma 4.6 and (4.37). While the standard theory of elliptic regularity is applicable for $s \in (0, 1]$ or $s \in (1, \frac{n}{2}) \cap \mathbb{N}$, it is not yet available for $s \in (1, \frac{n}{2}) \setminus \mathbb{N}$. We will directly analyze the representation of $\hat{f}$ to ensure that it has Hölder continuity or even higher-order differentiability.
- To deduce the limit equation (4.40), we have to analyze integrals on $\mathbb{R}^n$. We need to divide $\mathbb{R}^n$ into three distinct parts: the singular part, the uniformly convergent part, and the exterior part. Although this approach is relatively standard, the interaction between different bubbles necessitates a more refined analysis; see Appendix B.2 for further details. The same strategy can be applied in the derivation of (4.41); see Appendix B.3.

[2] Applying the idea developed in this paper, the first two authors obtained an analogous inequality in the setting of the Yamabe problem on compact Riemannian manifolds in [12].

(4) In Appendix A, we prove the non-degeneracy of the bubble in $\dot{H}^s(\mathbb{R}^n)$ and the removability of singularities for nonlocal equations. They hold for all $s \in (0, \frac{n}{2})$, are of independent interest, and may also be helpful in other contexts.

Organization of the paper. In Section 2, we derive a spectral inequality for δ-interacting families. In Sections 3–5, we prove Theorem 1.1: The cases $n \ge 6s$ and $2s < n < 6s$ are treated in Sections 3–4 and 5, respectively. In Appendix A, we obtain the auxiliary results described in (4) above. In Appendix B, we carry out technical computations needed in the proof.

We are mainly concerned with the multi-bubble case $\nu \ge 2$, as the single-bubble case $\nu = 1$ and $s \in (0, \frac{n}{2})$ has already treated in [25]. For the sake of brevity, we often omit proofs if there is a suitable reference to quote. In particular, we borrow several estimates obtained in [24] for $s = 1$, whenever similar estimates hold for all $s \in (0, \frac{n}{2})$.

Notations. We collect some notations used in the paper.

- Let $\lfloor s \rfloor$ be the greatest integer that does not exceed s.

- Let (A) be a condition. We set $\mathbf{1}_{(\mathrm{A})} = 1$ if (A) holds and 0 otherwise.

- For $x \in \mathbb{R}^n$ and $r > 0$, we write $B(x,r) = \{\omega \in \mathbb{R}^n : |\omega - x| < r\}$ and $B(x,r)^c = \{\omega \in \mathbb{R}^n : |\omega - x| \ge r\}$.

- Given $n \in \mathbb{N}$ and $s \in (0, \frac{n}{2})$, let $\Phi_{n,s}$ be the Riesz potential

$$\Phi_{n,s}(x) = \frac{\gamma_{n,s}}{|x|^{n-2s}} \quad \text{for } x \in \mathbb{R}^n \setminus \{0\} \quad \text{where } \gamma_{n,s} := \frac{\Gamma\left(\frac{n-2s}{2}\right)}{\pi^{n/2} 2^{2s} \Gamma(s)}. \tag{1.10}$$

- Given a function $u \in \dot{H}^s(\mathbb{R}^n)$, let $\mathcal{F}u$ be the Fourier transform of u. The notation $\hat{u}$ is reserved for other use, e.g., a suitable rescaling of u.

- We use the Japanese bracket notation $\langle x \rangle = \sqrt{1 + |x|^2}$ for $x \in \mathbb{R}^n$.

- Unless otherwise stated, $C > 0$ is a universal constant that may vary from line to line and even in the same line. We write $a_1 \lesssim a_2$ if $a_1 \le C a_2$, $a_1 \gtrsim a_2$ if $a_1 \ge C a_2$, and $a_1 \simeq a_2$ if $a_1 \lesssim a_2$ and $a_1 \gtrsim a_2$.

2. The spectral inequality

As a preparation step for the proof of Theorem 1.1, we derive a spectral inequality (2.4) which will be employed in the proof of Propositions 3.5 and 5.5. It was deduced in [5, Proposition 3.1] and [32, Proposition 3.10] when $s = 1$, and in [3, Lemma 2.5] when $s \in (0,1)$. Here, we present a proof based on a blow-up argument.

Definition 2.1. We write $U_i = U[z_i, \lambda_i]$ for $i = 1, \dots, \nu$. For $\nu \ge 2$, let q_{ij} be the quantity in (1.7) and

$$\mathscr{Q} = \max\{q_{ij} : i, j = 1, \dots, \nu,\ i \ne j\} \tag{2.1}$$

so that the ν-tuple $(U_1, \dots, U_\nu)$ of bubbles is δ-interacting if and only if $\mathscr{Q} \le \delta$. We also set

$$\mathscr{R}_{ij} = \max\left\{\sqrt{\frac{\lambda_i}{\lambda_j}},\ \sqrt{\frac{\lambda_j}{\lambda_i}},\ \sqrt{\lambda_i \lambda_j}|z_i - z_j|\right\} \simeq q_{ij}^{-\frac{1}{n-2s}} \quad \text{for } i, j = 1, \dots, \nu,\ i \ne j \tag{2.2}$$

and

$$\mathscr{R} = \frac{1}{2} \min\{\mathscr{R}_{ij} : i, j = 1, \dots, \nu,\ i \ne j\} \simeq \mathscr{Q}^{-\frac{1}{n-2s}}. \tag{2.3}$$

Proposition 2.2. *Let $n \in \mathbb{N}$, $\nu \in \mathbb{N}$, $s \in (0, \frac{n}{2})$, and $\delta_0 > 0$ is sufficiently small. Suppose that the ν-tuple $(U_1, \dots, U_\nu)$ of bubbles is δ'-interacting for some $\delta' \in (0, \delta_0)$. If $\varrho = \varrho(x)$ is a function in $\dot{H}^s(\mathbb{R}^n)$ that satisfies*

$$\int_{\mathbb{R}^n} (-\Delta)^{\frac{s}{2}} \varrho \, (-\Delta)^{\frac{s}{2}} U_i \mathrm{d}x = \int_{\mathbb{R}^n} (-\Delta)^{\frac{s}{2}} \varrho \, (-\Delta)^{\frac{s}{2}} Z_i^a \mathrm{d}x = 0$$

for all $i = 1, \dots, \nu$ and $a = 1, \dots, n+1$, then there exists a constant $c_0 \in (0,1)$ such that

$$\int_{\mathbb{R}^n} \sigma^{p-1} \varrho^2 \mathrm{d}x \le \frac{c_0}{p} \|\varrho\|_{\dot{H}^s(\mathbb{R}^n)}^2 \tag{2.4}$$

where $\sigma = \sum_{i=1}^{\nu} U_i$.

Proof. The case $\nu = 1$ is clear. In the sequel, we assume that $\nu \ge 2$.

To the contrary, suppose that there exist sequences of small positive numbers $\{\delta'_m\}_{m \in \mathbb{N}}$, δ'_m-interacting ν-tuples of bubbles $\{(U_{1,m}, \dots, U_{\nu,m})\}_{m \in \mathbb{N}}$, functions $\{\varrho_m\}_{m \in \mathbb{N}}$ in $\dot{H}^s(\mathbb{R}^n)$, and numbers $\{c_m\}_{m \in \mathbb{N}}$ in $(0,1]$ such that $\delta'_m \to 0$ and $c_m \to 1$ as $m \to \infty$,

$$\|\varrho_m\|_{\dot{H}^s(\mathbb{R}^n)} = 1, \quad \int_{\mathbb{R}^n} \sigma_m^{p-1} \varrho_m^2 \mathrm{d}x = \sup \left\{ \int_{\mathbb{R}^n} \sigma_m^{p-1} \varrho^2 \mathrm{d}x : \|\varrho\|_{\dot{H}^s(\mathbb{R}^n)} = 1 \right\} \ge \frac{c_m}{p}, \tag{2.5}$$

and

$$\int_{\mathbb{R}^n} U_{i,m}^p \varrho_m \mathrm{d}x = \int_{\mathbb{R}^n} U_{i,m}^{p-1} Z_{i,m}^a \varrho_m \mathrm{d}x = 0 \tag{2.6}$$

for $m \in \mathbb{N}$, $i = 1, \dots, \nu$, and $a = 1, \dots, n+1$. Here, $\sigma_m := \sum_{i=1}^{\nu} U_{i,m}$ and $Z_{i,m}^a := Z^a[z_{i,m}, \lambda_{i,m}]$. In view of (2.5)–(2.6), we know that

$$(-\Delta)^s \varrho_m - \mu_m \sigma_m^{p-1} \varrho_m = \sum_{i=1}^{\nu} \mu_{i,m} U_{i,m}^p + \sum_{i=1}^{\nu} \sum_{a=1}^{n+1} \mu_{i,m}^a U_{i,m}^{p-1} Z_{i,m}^a \quad \text{in } \mathbb{R}^n \tag{2.7}$$

where $\mu_m, \mu_{i,m}, \mu_{i,m}^a \in \mathbb{R}$ are Lagrange multipliers. Testing (2.7) with ρ_m and using (2.6) yield

$$\mu_m = \left(\int_{\mathbb{R}^n} \sigma_m^{p-1} \varrho_m^2 \mathrm{d}x \right)^{-1} \in \left[c(n,s,\nu), c_m^{-1} p \right] \tag{2.8}$$

where the lower bound $c(n,s,\nu)$ is positive and dependent only on n, s, and ν. Hence, we may assume that $\mu_m \to \mu_\infty \in [c(n,s,\nu), p]$ as $m \to \infty$.

Let $q_{ij,m}$, $\mathscr{Q}_m$, $\mathscr{R}_{ij,m}$, and $\mathscr{R}_m$ be the quantities introduced in (1.7), (2.1), (2.2), and (2.3), respectively, where $(z_i, z_j, \lambda_i, \lambda_j)$ is replaced with $(z_{i,m}, z_{j,m}, \lambda_{i,m}, \lambda_{j,m})$. We present the rest of the proof by dividing it into three steps.

STEP 1. We claim that

$$\sum_{i=1}^{\nu} |\mu_{i,m}| + \sum_{i=1}^{\nu} \sum_{a=1}^{n+1} |\mu_{i,m}^a| \to 0 \quad \text{as } m \to \infty. \tag{2.9}$$

Testing (2.7) with $U_{j,m}$ for $j = 1, \dots, \nu$ and employing (2.6), we obtain

$$-\mu_m \int_{\mathbb{R}^n} \sigma_m^{p-1} U_{j,m} \varrho_m \mathrm{d}x = \sum_{i=1}^{\nu} \mu_{i,m} \int_{\mathbb{R}^n} U_{i,m}^p U_{j,m} \mathrm{d}x + \sum_{i=1}^{\nu} \sum_{a=1}^{n+1} \mu_{i,m}^a \int_{\mathbb{R}^n} U_{i,m}^{p-1} Z_{i,m}^a U_{j,m} \mathrm{d}x.$$

On the other hand, [32, Proposition B.2] tells us that

$$\int_{\mathbb{R}^n} U_i^\alpha U_j^\beta \mathrm{d}x \simeq q_{ij}^{\min\{\alpha,\beta\}} \quad \text{for any } i,j = 1, \dots, \nu,\ i \ne j \tag{2.10}$$

provided $\alpha,\beta\ge 0$, $\alpha\ne\beta$, and $\alpha+\beta=p+1$. By (2.6) and (2.10), we have

$$\int_{\mathbb{R}^n} U_{i,m}^p U_{j,m}\mathrm{d}x=\begin{cases}\int_{\mathbb{R}^n} U_{1,0}^{p+1}\mathrm{d}x & \text{if } i=j,\\ O(q_{ij,m}) & \text{if } i\ne j,\end{cases}$$

$$\int_{\mathbb{R}^n} U_{i,m}^{p-1} Z_{i,m}^a U_{j,m}\mathrm{d}x=\begin{cases}0 & \text{if } i=j,\\ O(q_{ij,m}) & \text{if } i\ne j,\end{cases}$$

and

$$\begin{aligned}\int_{\mathbb{R}^n}\sigma_m^{p-1}U_{j,m}\varrho_m\mathrm{d}x&=\int_{\mathbb{R}^n}\left(\sigma_m^{p-1}-U_{j,m}^{p-1}\right)U_{j,m}\varrho_m\mathrm{d}x\\ &=O\left(\sum_{\substack{i=1,\dots,\nu,\\ i\ne j}}\left[\left\|U_{i,m}^{p-1}U_{j,m}\right\|_{L^{\frac{p+1}{p}}(\mathbb{R}^n)}+\left\|U_{i,m}^{p-2}U_{j,m}^2\right\|_{L^{\frac{p+1}{p}}(\mathbb{R}^n)}\mathbf{1}_{n<6s}\right]\right)\\ &=O\left(q_{ij,m}^{p-1}+q_{ij,m}^{\min\{p-2,1\}}\mathbf{1}_{n<6s}\right).\end{aligned}$$

Thus

$$\begin{aligned}O\left(\mathscr{Q}_m^{p-1}+\mathscr{Q}_m^{\min\{p-2,1\}}\mathbf{1}_{n<6s}\right)&=\left[\int_{\mathbb{R}^n}U_{1,0}^{p+1}\mathrm{d}x\right]\mu_j\\ &\quad+\sum_{\substack{i=1,\dots,\nu,\\ i\ne j}}O(q_{ij,m})\mu_i+\sum_{\substack{i=1,\dots,\nu,\\ i\ne j}}\sum_{a=1}^{n+1}O(q_{ij,m})\mu_i^a.\end{aligned}\tag{2.11}$$

Similarly, by testing (2.7) with $Z_{j,m}^b$, we get

$$\begin{aligned}O\left(\mathscr{Q}_m^{p-1}+\mathscr{Q}_m^{\min\{p-2,1\}}\mathbf{1}_{n<6s}\right)&=\sum_{\substack{i=1,\dots,\nu,\\ i\ne j}}O(q_{ij,m})\mu_i\\ &\quad+\sum_{\substack{i=1,\dots,\nu,\\ i\ne j}}\sum_{a=1}^{n+1}O(q_{ij,m})\mu_i^a+\left[\int_{\mathbb{R}^n}U_{1,0}^{p-1}\left(Z_{1,0}^b\right)^2\mathrm{d}x\right]\mu_j^b.\end{aligned}\tag{2.12}$$

Claim (2.9) follows from (2.11), (2.12), and the fact that $\mathscr{Q}_m\to 0$ as $m\to\infty$.

STEP 2. We verify

$$\lim_{m\to\infty}\int_{B_{i,m}}U_{i,m}^{p-1}\varrho_m^2\mathrm{d}x=0\quad\text{for each } i=1,\dots,\nu.\tag{2.13}$$

Let $\chi:\mathbb{R}^n\to[0,1]$ be an arbitrary smooth radial function such that $\chi=1$ in $B(0,1)$ and 0 on $B(0,2)^c$. Also, fixing $i=1,\dots,\nu$ and a sequence $\{r_m\}_{m\in\mathbb{N}}\subset(0,\infty)$ of positive numbers such that $r_m\to\infty$ as $m\to\infty$, we set

$$\hat{\varrho}_{i,m}(y)=\chi_m(y)\lambda_{i,m}^{-\frac{n-2s}{2}}\varrho_m\big(\lambda_{i,m}^{-1}y+z_{i,m}\big)\quad\text{for } y\in\mathbb{R}^n\quad\text{where } \chi_m(y):=\chi\left(\frac{2y}{r_m}\right).$$

By (2.7),

$$(-\Delta)^s\hat{\varrho}_{i,m}-\mu_m\left\{\lambda_{i,m}^{-\frac{n-2s}{2}}\sigma_m\big(\lambda_{i,m}^{-1}\cdot+z_{i,m}\big)\right\}^{p-1}\hat{\varrho}_{i,m}$$

$$= \chi_m \lambda_{i,m}^{-\frac{n+2s}{2}} \left[\sum_{j=1}^{\nu} \mu_{j,m} U_{j,m}^p + \sum_{j=1}^{\nu} \sum_{a=1}^{n+1} \mu_{j,m}^a U_{j,m}^{p-1} Z_{j,m}^a \right] \left(\lambda_{i,m}^{-1} \cdot + z_{i,m}\right) + \mathcal{R}_{i,m} \quad \text{in } \mathbb{R}^n \tag{2.14}$$

where

$$\mathcal{R}_{i,m}(y) := \left[(-\Delta)^s \hat{\varrho}_{i,m}\right](y) - \chi_m(y) \lambda_{i,m}^{-\frac{n+2s}{2}} \left[(-\Delta)^s \varrho_m\right] \left(\lambda_{i,m}^{-1} y + z_{i,m}\right) \quad \text{for } y \in \mathbb{R}^n.$$

In addition,

$$\begin{aligned} \|\hat{\varrho}_{i,m}\|_{\dot{H}^s(\mathbb{R}^n)} &\lesssim \left\|(-\Delta)^{\frac{s}{2}} \chi_m\right\|_{L^{\frac{n}{s}}(\mathbb{R}^n)} \|\varrho_m\|_{L^{\frac{2n}{n-2s}}(\mathbb{R}^n)} + \|\chi_m\|_{L^\infty(\mathbb{R}^n)} \|\varrho_m\|_{\dot{H}^s(\mathbb{R}^n)} \\ &\lesssim \left\|(-\Delta)^{\frac{s}{2}} \chi\right\|_{L^{\frac{n}{s}}(\mathbb{R}^n)} + 1 \lesssim 1. \end{aligned} \tag{2.15}$$

Here, we applied a fractional Leibniz rule (see e.g. [39, Theorem 1]) for the first inequality, and (1.1) and (2.5) for the second inequality. Also, we employed the Hausdorff-Young inequality, the assumption that $n > 2s$, and $\mathcal{F}\chi \in \mathcal{S}(\mathbb{R}^n)$ for the last inequality. Therefore, we may assume that

$$\hat{\varrho}_{i,m} \rightharpoonup \hat{\varrho}_{i,\infty} \quad \text{weakly in } \dot{H}^s(\mathbb{R}^n) \quad \text{and} \quad \hat{\varrho}_{i,m} \to \hat{\varrho}_{i,\infty} \quad \text{a.e.} \quad \text{as } m \to \infty$$

for some $\hat{\varrho}_{i,\infty} \in \dot{H}^s(\mathbb{R}^n)$. By (2.6),

$$\int_{\mathbb{R}^n} U[0,1]^p \hat{\varrho}_{i,\infty} \mathrm{d}y = \int_{\mathbb{R}^n} U[0,1]^{p-1} Z^a[0,1] \hat{\varrho}_{i,\infty} \mathrm{d}y = 0 \quad \text{for all } a = 1, \ldots, n+1. \tag{2.16}$$

Let $\psi \in C_c^\infty(\mathbb{R}^n)$ be a test function. Then

$$\begin{aligned} &\int_{\mathbb{R}^n} \left\{ \lambda_{i,m}^{-\frac{n-2s}{2}} \sigma_m \left(\lambda_{i,m}^{-1} y + z_{i,m}\right) \right\}^{p-1} \hat{\varrho}_{i,m} \psi \, \mathrm{d}y \\ &= \int_{\mathbb{R}^n} U[0,1]^{p-1} \hat{\varrho}_{i,m} \psi \, \mathrm{d}y + O\left(\max_{\alpha \in \left\{\frac{p+1}{p}, \frac{p^2-1}{p}\right\}} \sum_{\substack{k=1,\ldots,\nu, \\ k \neq i}} \left\| \lambda_{i,m}^{-\frac{n-2s}{2}} U_{k,m}\left(\lambda_{i,m}^{-1} \cdot + z_{i,m}\right) \right\|_{L^\alpha(\operatorname{supp}\psi \cap B(0,r_m))}^{\frac{\alpha p}{p+1}} \right) \\ &\to \int_{\mathbb{R}^n} U[0,1]^{p-1} \hat{\varrho}_{i,\infty} \psi \, \mathrm{d}y \quad \text{as } m \to \infty, \end{aligned}$$

because if we set $z_{ki,m} = \lambda_{k,m}(z_{i,m} - z_{k,m})$, then

$$\begin{aligned} &\left(\frac{\lambda_{k,m}}{\lambda_{i,m}}\right)^{\frac{\alpha(n-2s)}{2}} \int_{\operatorname{supp}\psi \cap B(0,r_m)} \frac{\mathrm{d}y}{\langle (\lambda_{k,m}/\lambda_{i,m}) y + z_{ki,m} \rangle^{\alpha(n-2s)}} \\ &= \left(\frac{\lambda_{k,m}}{\lambda_{i,m}}\right)^{\frac{\alpha(n-2s)}{2} - n} \int_{B(z_{ki,m}, \frac{\lambda_{k,m}}{\lambda_{i,m}} r_0)} \frac{\mathrm{d}Y}{\langle Y \rangle^{\alpha(n-2s)}} \\ &\lesssim \begin{cases} \left(\frac{\lambda_{k,m}}{\lambda_{i,m}}\right)^{\frac{\alpha(n-2s)}{2}} = o(1) & \text{if } \lim\limits_{m\to\infty} \dfrac{\lambda_{k,m}}{\lambda_{i,m}} = 0, \\ \left(\frac{\lambda_{k,m}}{\lambda_{i,m}}\right)^{\frac{\alpha(n-2s)}{2} - n} + \left(\frac{\lambda_{k,m}}{\lambda_{i,m}}\right)^{-\frac{\alpha(n-2s)}{2}} = o(1) & \text{if } \lim\limits_{m\to\infty} \dfrac{\lambda_{k,m}}{\lambda_{i,m}} = \infty, \\ \dfrac{1}{\mathscr{R}_{ki,m}^{\alpha(n-2s)}} = o(1) & \text{if } \lim\limits_{m\to\infty} \dfrac{\lambda_{k,m}}{\lambda_{i,m}} \in (0,\infty) \text{ (so that } \mathscr{R}_{ki,m} \simeq |z_{ki,m}|) \end{cases} \end{aligned}$$

for $\alpha \in \{\frac{p+1}{p}, \frac{p^2-1}{p}\}$, provided $\operatorname{supp}\psi \subset B(0, r_0)$ for some $r_0 > 0$ and $r_m \geq r_0$. Furthermore, Hölder's inequality and (2.9) give

$$\left|\int_{\mathbb{R}^n} \chi_m \lambda_{i,m}^{-\frac{n+2s}{2}} \left[\sum_{j=1}^{\nu} \mu_{j,m} U_{j,m}^p + \sum_{j=1}^{\nu}\sum_{a=1}^{n+1} \mu_{j,m}^a U_{j,m}^{p-1} Z_{j,m}^a\right] (\lambda_{i,m}^{-1} y + z_{i,m}) \psi \, \mathrm{d}y\right|$$
$$\le \left(\sum_{j=1}^{\nu} |\mu_{j,m}| + \sum_{j=1}^{\nu}\sum_{a=1}^{n+1} \left|\mu_{j,m}^a\right|\right) \|U[0,1]\|_{L^{\frac{2n}{n-2s}}(\mathbb{R}^n)}^p \|\psi\|_{L^{\frac{2n}{n-2s}}(\mathbb{R}^n)} = o(1).$$

Writing

$$\tilde{\varrho}_{i,m}(y) = \lambda_{i,m}^{-\frac{n-2s}{2}} \varrho_m\big(\lambda_{i,m}^{-1} y + z_{i,m}\big) \quad \text{and} \quad \mathcal{F}(D^{s,\eta}\psi)(\xi) = i^{-|\eta|} \partial_\xi^\eta (|\xi|^s)(\mathcal{F}\psi)(\xi)$$

where η is an n-dimensional multi-index, and invoking the generalized Kenig-Ponce-Vega estimate due to Li [44, Theorem 1.2], we deduce

$$\begin{aligned}\left|\int_{\mathbb{R}^n} \mathcal{R}_{i,m}\psi \, \mathrm{d}y\right| &\le \int_{\mathbb{R}^n} |\tilde{\varrho}_{i,m}(y)| \left|[(-\Delta)^s(\chi_m\psi)](y) - \chi_m(y)[(-\Delta)^s\psi](y)\right| \mathrm{d}y \\ &\le \|\varrho_m\|_{L^{\frac{2n}{n-2s}}(\mathbb{R}^n)} \left\|(-\Delta)^s(\chi_m\psi) - \chi_m[(-\Delta)^s\psi]\right\|_{L^{\frac{n}{s}}(\mathbb{R}^n)} \\ &\lesssim \|(-\Delta)^s\chi_m\|_{L^{\frac{2n}{s}}(\mathbb{R}^n)} \|\psi\|_{L^{\frac{2n}{s}}(\mathbb{R}^n)} + \sum_{1\le|\eta|\le 2s} \left\|\partial^\eta\chi_m D^{2s,\eta}\psi\right\|_{L^{\frac{n}{s}}(\mathbb{R}^n)} \\ &\lesssim r_m^{-\frac{3s}{2}} + r_m^{-1} = o(1).\end{aligned}$$

Here, the empty summation $\sum_{1\le|\eta|\le 2s}$ is understood as zero for $s \in (0, \frac{1}{2})$.

Accordingly, by taking the limit $m \to \infty$ on (2.14), we find

$$(-\Delta)^s \hat{\varrho}_{i,\infty} - \mu_\infty U[0,1]^{p-1} \hat{\varrho}_{i,\infty} = 0 \quad \text{in } \mathbb{R}^n \tag{2.17}$$

where $\mu_\infty \in [c(n,s,\nu), p]$. From (2.16), (2.17), the fact that $U[0,1]$ is an extremizer of (1.1), and Lemma A.1 (b), we conclude that $\hat{\varrho}_{i,\infty} = 0$ in $\mathbb{R}^n$. This and (2.15) imply

$$\int_{B_{i,m}} U_{i,m}^{p-1} \varrho_m^2 \mathrm{d}x \le \int_{\mathbb{R}^n} U[0,1]^{p-1} \hat{\varrho}_{i,m}^2 \mathrm{d}y \lesssim \left(\int_{\mathbb{R}^n} U[0,1]^p \hat{\varrho}_{i,m} \mathrm{d}y\right)^{\frac{p}{p-1}} \to 0 \quad \text{as } m \to \infty,$$

which reads (2.13).

STEP 3. Finally, we prove that

$$\lim_{m\to\infty} \int_{\mathbb{R}^n} \sigma_m^{p-1} \varrho_m^2 \mathrm{d}x = 0. \tag{2.18}$$

Its validity will imply that (2.4) holds, because it contradicts (2.8).

Given any number $L > 0$, let $B_{i,m} = B(z_{i,m}, \frac{L}{\lambda_{i,m}})$ and $B_{i,m}^c$ be its complement. It holds that

$$\int_{B_{i,m}^c} U_{i,m}^{p-1} \varrho_m^2 \mathrm{d}x \le \|U_{i,m}\|_{L^{p+1}(B_{i,m}^c)}^{p-1} \lesssim L^{-2s} \tag{2.19}$$

for $i = 1, \dots, \nu$. It follows from (2.13) and (2.19) that

$$\begin{aligned}\int_{\mathbb{R}^n} \sigma_m^{p-1} \varrho_m^2 \mathrm{d}x &\lesssim \sum_{i=1}^{\nu} \int_{\mathbb{R}^n} U_{i,m}^{p-1} \varrho_m^2 \mathrm{d}x \le \sum_{i=1}^{\nu} \left[\int_{B_{i,m}} U_{i,m}^{p-1} \varrho_m^2 \mathrm{d}x + \int_{B_{i,m}^c} U_{i,m}^{p-1} \varrho_m^2 \mathrm{d}x\right] \\ &\lesssim o(1) + L^{-2s},\end{aligned}$$

which yields (2.18), because $L > 0$ can be taken arbitrarily large. □

3. Quantitative stability estimate for dimension $n \ge 6s$ (1)

In this section, we establish Theorem 1.1 assuming that $n \ge 6s$. From now on, we always assume that $\nu \ge 2$.

The following two weighted $L^\infty(\mathbb{R}^n)$-norms were devised in [24] (for the case $s = 1$) to capture the precise pointwise behavior of the bubbles, which is crucial on determining the optimal exponents of $\Gamma(u)$ in the right-hand side of (1.9).

Definition 3.1. Recall the number $\mathscr{R} > 0$ in (2.3) and write $y_i = \lambda_i(x - z_i) \in \mathbb{R}^n$. We define

$$\|h\|_{**} = \sup_{x\in\mathbb{R}^n} |h(x)|\mathcal{V}^{-1}(x) \quad \text{and} \quad \|\rho\|_{*} = \sup_{x\in\mathbb{R}^n} |\rho(x)|\mathcal{W}^{-1}(x)$$

with

$$\mathcal{V}(x) := \sum_{i=1}^{\nu} \left(v_i^{\text{in}} + v_i^{\text{out}}\right)(x) \quad \text{and} \quad \mathcal{W}(x) := \sum_{i=1}^{\nu} \left(w_i^{\text{in}} + w_i^{\text{out}}\right)(x) \tag{3.1}$$

where

$$\begin{cases} v_i^{\text{in}}(x) := \lambda_i^{\frac{n+2s}{2}} \dfrac{\mathscr{R}^{2s-n}}{\langle y_i\rangle^{4s}} \mathbf{1}_{\{|y_i|<\mathscr{R}\}}(x), \quad v_i^{\text{out}}(x) := \lambda_i^{\frac{n+2s}{2}} \dfrac{\mathscr{R}^{-4s}}{|y_i|^{n-2s}} \mathbf{1}_{\{|y_i|\ge\mathscr{R}\}}(x), \\ w_i^{\text{in}}(x) := \lambda_i^{\frac{n-2s}{2}} \dfrac{\mathscr{R}^{2s-n}}{\langle y_i\rangle^{2s}} \mathbf{1}_{\{|y_i|<\mathscr{R}\}}(x), \quad w_i^{\text{out}}(x) := \lambda_i^{\frac{n-2s}{2}} \dfrac{\mathscr{R}^{-4s}}{|y_i|^{n-4s}} \mathbf{1}_{\{|y_i|\ge\mathscr{R}\}}(x) \end{cases} \tag{3.2}$$

for $n > 6s$ and $i = 1, \dots, \nu$, and

$$\begin{cases} v_i^{\text{in}}(x) := \lambda_i^{4s} \dfrac{\mathscr{R}^{-4s}}{\langle y_i\rangle^{4s}} \mathbf{1}_{\{|y_i|<\mathscr{R}^2\}}(x), \quad v_i^{\text{out}}(x) := \lambda_i^{4s} \dfrac{\mathscr{R}^{-2s}}{|y_i|^{5s}} \mathbf{1}_{\{|y_i|\ge\mathscr{R}^2\}}(x), \\ w_i^{\text{in}}(x) := \lambda_i^{2s} \dfrac{\mathscr{R}^{-4s}}{\langle y_i\rangle^{2s}} \mathbf{1}_{\{|y_i|<\mathscr{R}^2\}}(x), \quad w_i^{\text{out}}(x) := \lambda_i^{2s} \dfrac{\mathscr{R}^{-2s}}{|y_i|^{3s}} \mathbf{1}_{\{|y_i|\ge\mathscr{R}^2\}}(x) \end{cases} \tag{3.3}$$

for $n = 6s$ and $i = 1, \dots, \nu$.

Clearly, the norms $\|\cdot\|_*$ and $\|\cdot\|_{**}$ depend on the choice of $z_i \in \mathbb{R}^n$ and $\lambda_i \in (0, \infty)$. If we keep using v_i^{out} and w_i^{out} in (3.2) for $n = 6s$, their slow decay, specifically $|y_i|^{-4s}$ and $|y_i|^{-2s}$, causes additional technical complexity that does not arise when using (3.3).

By utilizing the above norms, we will derive (1.9) for all $n \ge 6s$ and small $\delta > 0$. The derivation is split into three steps.

Step 1. Let $\sigma = \sum_{i=1}^{\nu} U[z_i, \lambda_i] = \sum_{i=1}^{\nu} U_i$ be such that

$$\|u - \sigma\|_{\dot{H}^s(\mathbb{R}^n)} = \inf_{(\tilde{z}_1,\dots,\tilde{z}_\nu,\tilde{\lambda}_1,\dots,\tilde{\lambda}_\nu)\in\mathbb{R}^{n\nu}\times(0,\infty)^\nu} \left\| u - \sum_{i=1}^{\nu} U[\tilde{z}_i, \tilde{\lambda}_i] \right\|_{\dot{H}^s(\mathbb{R}^n)} \le \delta.$$

We also set $\rho = u - \sigma \in \dot{H}^s(\mathbb{R}^n)$ and $Z_i^a = Z^a[z_i, \lambda_i]$ for $a = 1, \dots, n+1$. Because of (1.8), the family $\{U_i\}_{i=1,\dots,\nu}$ is δ'-interacting for some $\delta' > 0$ where $\delta' \to 0$ as $\delta \to 0$. The function ρ satisfies

$$(-\Delta)^s\rho - \left[|\sigma+\rho|^{p-1}(\sigma+\rho) - \sigma^p\right] = \left(\sigma^p - \sum_{i=1}^{\nu} U_i^p\right) + \left[(-\Delta)^s u - |u|^{p-1}u\right] \tag{3.4}$$

in $\mathbb{R}^n$ and

$$\int_{\mathbb{R}^n} (-\Delta)^{\frac{s}{2}}\rho\,(-\Delta)^{\frac{s}{2}} Z_i^a \mathrm{d}x = \int_{\mathbb{R}^n} \rho U_i^{p-1} Z_i^a \mathrm{d}x = 0 \quad \text{for all } i = 1, \dots, \nu \text{ and } a = 1, \dots, n+1. \tag{3.5}$$

Consider the equation

$$\begin{cases} (-\Delta)^s\rho_0 - \left[|\sigma+\rho_0|^{p-1}(\sigma+\rho_0)-\sigma^p\right] = \left(\sigma^p - \sum_{i=1}^{\nu} U_i^p\right) + \sum_{i=1}^{\nu}\sum_{a=1}^{n+1} c_i^a U_i^{p-1} Z_i^a \quad \text{in } \mathbb{R}^n, \\ \rho_0 \in \dot{H}^s(\mathbb{R}^n),\ c_1^1,\dots,c_\nu^{n+1}\in\mathbb{R}, \\ \int_{\mathbb{R}^n} \rho_0 U_i^{p-1} Z_i^a \mathrm{d}x = 0 \quad \text{for all } i=1,\dots,\nu \text{ and } a=1,\dots,n+1. \end{cases} \tag{3.6}$$

To solve (3.6), we will use a pointwise estimate on the error term $\sigma^p - \sum_{i=1}^{\nu} U_i^p$.

Lemma 3.2. *There exists a constant $C>0$ depending only on n, s, and ν such that*

$$\left\| \sigma^p - \sum_{i=1}^{\nu} U_i^p \right\|_{**} \le C \tag{3.7}$$

provided $\delta>0$ small.

Proof. The proof is essentially the same as that of [24, Proposition 3.4], which we omit. □

In addition, we analyze an associated inhomogeneous equation

$$\begin{cases} (-\Delta)^s f - p\,\sigma^{p-1} f = h + \sum_{i=1}^{\nu}\sum_{a=1}^{n+1} c_i^a U_i^{p-1} Z_i^a \quad \text{in } \mathbb{R}^n, \\ f\in \dot{H}^s(\mathbb{R}^n),\ c_1^1,\dots,c_\nu^{n+1}\in\mathbb{R}, \\ \int_{\mathbb{R}^n} f U_i^{p-1} Z_i^a \mathrm{d}x = 0 \quad \text{for all } i=1,\dots,\nu \text{ and } a=1,\dots,n+1. \end{cases} \tag{3.8}$$

Proposition 3.3. *If $\|h\|_{**}<\infty$ and f satisfies (3.8), then $\|f\|_* < \infty$. Moreover, there exists a constant $C>0$ depending only on n, s, and ν such that*

$$\|f\|_* \le C\|h\|_{**} \tag{3.9}$$

provided $\delta>0$ small.

Deducing the above proposition is the most challenging part of the entire proof. Because of its complexity and length, we will put it off until Section 4.

From Lemma 3.2 and Proposition 3.3, we establish the unique existence of a solution to (3.6).

Proposition 3.4. *Assume that $\delta>0$ is small enough. Equation (3.6) has a solution ρ_0 and a family $\{c_i^a\}_{i=1,\dots,\nu,\ a=1,\dots,n+1}$ of numbers such that*

$$\|\rho_0\|_* \le C \tag{3.10}$$

where $C>0$ depends only on n, s, and ν. Besides,

$$\|\rho_0\|_{\dot{H}^s(\mathbb{R}^n)} \le C \begin{cases} \mathscr{Q}^{\frac{p}{2}} & \text{for } n>6s, \\ \mathscr{Q}|\log\mathscr{Q}|^{\frac12} & \text{for } n=6s \end{cases} \tag{3.11}$$

where $\mathscr{Q}>0$ is the value in (2.1).

Proof. A priori estimate (3.9) and the Fredholm alternative imply that a solution f to (3.8) uniquely exists for a given h with $\|h\|_{**}<\infty$. Therefore, relying on Lemma 3.2 and the fact that the main order term of $|\sigma+\rho_0|^{p-1}(\sigma+\rho_0)-\sigma^p$ is $p\,\sigma^{p-1}\rho_0$, we can apply a fixed point argument to yield the unique existence of ρ_0 and $\{c_i^a\}$ satisfying (3.6) and (3.10). By testing (3.4) with ρ_0 and employing (3.10), we also discover (3.11). For details, refer to the proof of Lemma 5.2 and Propositions 5.3, 5.4, and 6.1 in [24] in which the case $s=1$ is treated. □

STEP 2. Set $\rho_1 = \rho - \rho_0$. In light of (3.4), (3.5), and (3.6), we have

$$\begin{cases} (-\Delta)^s \rho_1 - \left[|\sigma+\rho_0+\rho_1|^{p-1}(\sigma+\rho_0+\rho_1) - |\sigma+\rho_0|^{p-1}(\sigma+\rho_0)\right] \\ \qquad\qquad = \left[(-\Delta)^s u - |u|^{p-1}u\right] - \sum_{i=1}^{\nu}\sum_{a=1}^{n+1} c_i^a U_i^{p-1} Z_i^a \quad \text{in } \mathbb{R}^n, \\ \rho_1 \in \dot{H}^s(\mathbb{R}^n),\ c_1^1, \ldots, c_\nu^{n+1} \in \mathbb{R}, \\ \int_{\mathbb{R}^n} \rho_1 U_i^{p-1} Z_i^a \mathrm{d}x = 0 \quad \text{for all } i = 1, \ldots, \nu \text{ and } a = 1, \ldots, n+1. \end{cases} \tag{3.12}$$

Proposition 3.5. *Assume that $\delta > 0$ is small enough. There exists a constant $C > 0$ depending only on n, s, and ν that*

$$\|\rho_1\|_{\dot{H}^s(\mathbb{R}^n)} \le C\left(\Gamma(u) + \mathscr{Q}^2\right) \tag{3.13}$$

where $\Gamma(u) = \|(-\Delta)^s u - |u|^{p-1}u\|_{\dot{H}^{-s}(\mathbb{R}^n)}$.

Proof. Applying the spectral inequality (2.4), one can adapt the argument in the proof of Lemmas 6.2, 6.3, and Proposition 6.4 in [24]. The details are omitted. ☐

Putting (3.11) and (3.13) together leads

$$\|\rho\|_{\dot{H}^s(\mathbb{R}^n)} \le C \begin{cases} \Gamma(u) + \mathscr{Q}^{\frac{p}{2}} & \text{for } n > 6s, \\ \Gamma(u) + \mathscr{Q}|\log \mathscr{Q}|^{\frac{1}{2}} & \text{for } n = 6s. \end{cases} \tag{3.14}$$

STEP 3. Thanks to (3.14), we only need to check that $\mathscr{Q} \lesssim \Gamma(u)$ to establish (1.9).

Since $\sigma > 0$, $\rho \in \mathbb{R}$, and $1 < p \le 2$ for $n \ge 6s$, it holds that

$$\left||\sigma+\rho|^{p-1}(\sigma+\rho) - \sigma^p - p\,\sigma^{p-1}\rho\right| \lesssim \min\left\{\sigma^{p-2}\rho^2, |\rho|^p\right\} \lesssim \sigma^{p-2}\rho^2. \tag{3.15}$$

For any $j = 1, \ldots, \nu$ and $a = 1, \ldots, n+1$,

$$\left(\sigma^{p-1} - U_j^{p-1}\right)\left|Z_j^a\right| \lesssim \left(\sigma^{p-1} - U_j^{p-1}\right) U_j \lesssim \sum_{i=1}^{\nu}\left(\sigma^{p-1} - U_i^{p-1}\right) U_i = \sigma^p - \sum_{i=1}^{\nu} U_i^p, \tag{3.16}$$

so

$$\begin{aligned} \int_{\mathbb{R}^n}\left(\sigma^{p-1} - U_j^{p-1}\right)|\rho_0|\left|Z_j^{n+1}\right|\mathrm{d}x &\lesssim \int_{\mathbb{R}^n}\left(\sigma^p - \sum_{i=1}^{\nu} U_i^p\right)|\rho_0|\mathrm{d}x \\ &\lesssim \int_{\mathbb{R}^n} \mathcal{V}\mathcal{W}\mathrm{d}x \le \|\mathcal{V}\|_{L^{\frac{p+1}{p}}(\mathbb{R}^n)} \|\mathcal{W}\|_{L^{p+1}(\mathbb{R}^n)} \\ &\lesssim \begin{cases} \mathscr{R}^{-\frac{n+2s}{2}} \cdot \mathscr{R}^{-\frac{n+2s}{2}} \simeq \mathscr{Q}^p & \text{for } n > 6s, \\ \mathscr{R}^{-8s}\log \mathscr{R} \simeq \mathscr{Q}^2|\log \mathscr{Q}| & \text{for } n = 6s. \end{cases} \end{aligned} \tag{3.17}$$

Here, the second inequality in (3.17) is a consequence of (3.7) and (3.10), and the fourth inequality can be achieved through straightforward computations; refer to [24, Lemma 3.7] for $s = 1$. We also used (2.3) in the last line.

By testing (3.4) with Z_j^{n+1} for any fixed $j = 1, \ldots, \nu$, and applying (3.15), (3.17), Hölder's inequality, (1.1), (3.13), and (3.14), we observe

$$\begin{aligned} &\left|\int_{\mathbb{R}^n}\left(\sigma^p - \sum_{i=1}^{\nu} U_i^p\right) Z_j^{n+1}\mathrm{d}x\right| \\ &\lesssim \int_{\mathbb{R}^n}\left(\sigma^{p-1} - U_j^{p-1}\right)|\rho_0|\left|Z_j^{n+1}\right|\mathrm{d}x + \int_{\mathbb{R}^n}\sigma^{p-1}|\rho_1|\left|Z_j^{n+1}\right|\mathrm{d}x + \int_{\mathbb{R}^n}\sigma^{p-2}\rho^2\left|Z_j^{n+1}\right|\mathrm{d}x + \Gamma(u) \end{aligned}$$

$$\begin{aligned}
&\lesssim \int_{\mathbb{R}^n}\left(\sigma^p-\sum_{u=1}^{\nu}U_i^p\right)|\rho_0|\mathrm{d}x+\int_{\mathbb{R}^n}\sigma^p|\rho_1|\mathrm{d}x+\int_{\mathbb{R}^n}\sigma^{p-1}\rho^2\mathrm{d}x+\Gamma(u) \qquad (3.18)\\
&\lesssim \left\{\begin{matrix}\mathscr{Q}^p & \text{for } n>6s,\\ \mathscr{Q}^2|\log\mathscr{Q}| & \text{for } n=6s\end{matrix}\right\}+\|\rho_1\|_{\dot{H}^s(\mathbb{R}^n)}+\|\rho\|^2_{\dot{H}^s(\mathbb{R}^n)}+\Gamma(u)\\
&\lesssim \Gamma(u)+\begin{cases}\mathscr{Q}^p & \text{for } n>6s,\\ \mathscr{Q}^2|\log\mathscr{Q}| & \text{for } n=6s.\end{cases}
\end{aligned}$$

Furthermore, the proof of [24, Lemma 2.1] shows

$$\int_{\mathbb{R}^n}\left(\sigma^p-\sum_{i=1}^{\nu}U_i^p\right)Z_j^{n+1}\mathrm{d}x=\sum_{\substack{i=1,\dots,\nu,\\ i\neq j}}\int_{\mathbb{R}^n}U_i^p\lambda_j\partial_{\lambda_j}U_j\mathrm{d}x+o(\mathscr{Q})\quad\text{for all } j=1,\dots,\nu, \tag{3.19}$$

which together with (3.18) yields

$$\left|\sum_{\substack{i=1,\dots,\nu,\\ i\neq j}}\int_{\mathbb{R}^n}U_i^p\lambda_j\partial_{\lambda_j}U_j\mathrm{d}x\right|\lesssim\Gamma(u)+o(\mathscr{Q})\quad\text{for all } j=1,\dots,\nu \tag{3.20}$$

where $o(\mathscr{Q})$ is a term such that $o(\mathscr{Q})/\mathscr{Q}\to 0$ as $\mathscr{Q}\to 0$. As can be seen in the proof of [24, Lemma 2.3], one can draw the desired inequality $\mathscr{Q}\lesssim\Gamma(u)$ from (3.20). This completes the proof of (1.9) for $n\geq 6s$ under the validity of Proposition 3.3.

The sharpness of (1.9) can be proven as in [24, Section 7], which we omit.

4. Quantitative stability estimate for dimension $n \geq 6s$ (2)

This section is devoted to the proof of Proposition 3.3 for $n\geq 6s$. We divide it into three substeps.

Substep 1. We verify the first claim in the statement of Proposition 3.3.

Lemma 4.1. *If $\|h\|_{**}<\infty$ and f satisfies (3.8), then $\|f\|_*<\infty$.*

Proof. Suppose first that $n>6s$. It suffices to confirm that

$$f\in L^{\frac{2n}{n-2s}}(\mathbb{R}^n)\text{ and }\left\|\langle\,\cdot\,\rangle^{n-2s}h\right\|_{L^\infty(\mathbb{R}^n)}<\infty\Rightarrow\left\|\langle\,\cdot\,\rangle^{n-4s}f\right\|_{L^\infty(\mathbb{R}^n)}<\infty. \tag{4.1}$$

Following the proof of [16, Theorem 4.5] and exploiting $n>4s$ to control the term h, we get the integral representation of f from (3.8):

$$f=\Phi_{n,s}*\left(p\,\sigma^{p-1}f+h+\sum_{i=1}^{\nu}\sum_{a=1}^{n+1}c_i^aU_i^{p-1}Z_i^a\right)\quad\text{in }\mathbb{R}^n \tag{4.2}$$

where $\Phi_{n,s}$ is the Riesz potential in (1.10). By virtue of the hypothesis on h in (4.1), there exists a large constant $c>0$ depending only on n, s, ν, h, z_i, λ_i, and c_i^a such that

$$|f(x)|\leq\int_{\mathbb{R}^n}\frac{c}{|x-\omega|^{n-2s}}\left[\frac{|f(\omega)|}{\langle\omega\rangle^{4s}}+\frac{1}{\langle\omega\rangle^{n-2s}}\right]\mathrm{d}\omega\quad\text{for } x\in\mathbb{R}^n.$$

Since

$$\int_{\mathbb{R}^n}\frac{1}{|x-\omega|^{n-2s}}\frac{\mathrm{d}\omega}{\langle\omega\rangle^{n-2s}}\lesssim\frac{1}{|x|^{n-4s}}\quad\text{for all } x\in\mathbb{R}^n\text{ with } d:=\frac{|x|}{2}\geq 1, \tag{4.3}$$

we have

$$|f(x)| \le c\left[\int_{\mathbb{R}^n} \frac{1}{|x-\omega|^{n-2s}} \frac{|f(\omega)|}{\langle\omega\rangle^{4s}} \mathrm{d}\omega + \frac{1}{\langle x\rangle^{n-4s}}\right] \quad \text{for } x \in \mathbb{R}^n. \tag{4.4}$$

Hence, by the HLS inequality,

$$\|f\|_{L^{t^*}(\mathbb{R}^n)} \lesssim c\left(\left\||f| * \frac{1}{|\cdot|^{n-2s}}\right\|_{L^{t^*}(\mathbb{R}^n)} + \left\|\frac{1}{\langle\cdot\rangle^{n-4s}}\right\|_{L^{t^*}(\mathbb{R}^n)}\right) \lesssim c\left(\|f\|_{L^t(\mathbb{R}^n)} + 1\right) \tag{4.5}$$

for any $t \in [\frac{2n}{n-2s}, \frac{n}{2s})$ (which is a non-empty interval for $n > 6s$) and $t^* = \frac{nt}{n-2st}$. By employing (4.5) and arguing as follows, we can show that $f \in L^{\tilde{t}}(\mathbb{R}^n)$ for all $\tilde{t} \ge \frac{2n}{n-2s}$:

- We take $t = t_1 := \frac{2n}{n-2s}$ in (4.5). Then $f \in L^{t_2}(\mathbb{R}^n)$ for $t^* = t_2 := \frac{nt_1}{n-2st_1}$, and so $f \in L^{\tilde{t}}(\mathbb{R}^n)$ for all $\tilde{t} \in [t_1, t_2]$. We check whether $t_2 \ge \frac{n}{2s}$ or not.
- If $t_2 \ge \frac{n}{2s}$, we put $t = \frac{n}{2s} - \epsilon$ for any small $\epsilon > 0$ into (4.5). It implies that $f \in L^{\tilde{t}}(\mathbb{R}^n)$ for all $\tilde{t} \ge \frac{2n}{n-2s}$.
- If $t_2 < \frac{n}{2s}$, we plug $t = t_2$ into (4.5). It gives that $f \in L^{\tilde{t}}(\mathbb{R}^n)$ for all $\tilde{t} \in [t_2, t_3]$ where $t_3 := \frac{nt_2}{n-2st_2}$. We check whether $t_3 \ge \frac{n}{2s}$ or not.
- We iterate the above process. It terminates in a finite step because $t_{m+1} \ge (1 + \frac{4s}{n-6s})t_m$ for all m.

Let us fix some $t \gg 1$ large enough. Computing as in (4.3) and writing the Hölder conjugate of t as t', we find

$$\begin{aligned}\int_{\mathbb{R}^n} \frac{1}{|x-\omega|^{n-2s}} \frac{|f(\omega)|}{\langle\omega\rangle^{4s}} \mathrm{d}\omega &\lesssim \left(\int_{\mathbb{R}^n} \frac{1}{|x-\omega|^{(n-2s)t'}} \frac{\mathrm{d}\omega}{\langle\omega\rangle^{4st'}}\right)^{\frac{1}{t'}} \|f\|_{L^t(\mathbb{R}^n)} \\ &\lesssim \frac{1}{\langle x\rangle^{\frac{n(t'-1)}{t'}+2s}} \lesssim 1 \quad \text{for } x \in \mathbb{R}^n.\end{aligned} \tag{4.6}$$

From this and (4.4), we deduce that $f \in L^\infty(\mathbb{R}^n)$. Putting this fact into (4.4) and working as in (4.3) produce

$$|f(x)| \lesssim c\left[\int_{\mathbb{R}^n} \frac{1}{|x-\omega|^{n-2s}} \frac{\mathrm{d}\omega}{\langle\omega\rangle^{4s}} + \frac{1}{\langle x\rangle^{n-4s}}\right] \lesssim \frac{c}{\langle x\rangle^{\min\{2s,n-4s\}}} \quad \text{for } x \in \mathbb{R}^n.$$

Feeding this back to (4.4), we further obtain that $\|\langle\cdot\rangle^{\min\{4s,n-4s\}} f\|_{L^\infty(\mathbb{R}^n)} < \infty$. Repeating this process finitely many times, we conclude that the estimate for f in (4.1) is true.

If $n = 6s$, we still have (4.4) with $\langle x\rangle^{n-4s}$ replaced by $\langle x\rangle^{3s}$. However, we cannot proceed as in (4.5), because $[\frac{2n}{n-2s}, \frac{n}{2s}) = [3, 3) = \emptyset$. Fortunately, thanks to $f \in \dot{H}^s(\mathbb{R}^n) \subset L^3(\mathbb{R}^n)$, the HLS inequality, and Hölder's inequality, we see

$$\begin{aligned}\|f\|_{L^{t^*}(\mathbb{R}^n)} &\lesssim c\left(\left\|\frac{|f|}{\langle\cdot\rangle^{4s}} * \frac{1}{|\cdot|^{4s}}\right\|_{L^{t^*}(\mathbb{R}^n)} + \left\|\frac{1}{\langle\cdot\rangle^{3s}}\right\|_{L^{t^*}(\mathbb{R}^n)}\right) \\ &\lesssim c\left(\left\|\frac{|f|}{\langle\cdot\rangle^{4s}}\right\|_{L^{\zeta_2}(\mathbb{R}^n)} + 1\right) \lesssim c\left(\|f\|_{L^3(\mathbb{R}^n)} \left\|\frac{1}{\langle\cdot\rangle^{4s}}\right\|_{L^{\zeta_1}(\mathbb{R}^n)} + 1\right)\end{aligned} \tag{4.7}$$

for $\zeta_1 \in (3,\infty)$, $\zeta_2 = \frac{3\zeta_1}{\zeta_1+3} \in (\frac{3}{2},3)$, and $t^* = \frac{3\zeta_2}{3-\zeta_2} \in (3,\infty)$. This means that $f \in L^{\tilde{t}}(\mathbb{R}^n)$ for all $\tilde{t} \geq 3$.[3] As in the case $n > 6s$, we conclude that $\| \langle \cdot \rangle^{3s} f\|_{L^\infty(\mathbb{R}^n)} < \infty$. ☐

SUBSTEP 2. We estimate the coefficients c_i^a's in (3.8).

Lemma 4.2. *There is a constant $C > 0$ depending only on n, s, and ν such that*

$$\sum_{i=1}^{\nu}\sum_{a=1}^{n+1}|c_i^a| \leq C\left(\|h\|_{**}\mathscr{R}^{2s-n} + \|f\|_* \times \begin{cases}\mathscr{R}^{-(n+2s)} & \text{for } n > 6s, \\ \mathscr{R}^{-8s}\log\mathscr{R} & \text{for } n = 6s\end{cases}\right) \tag{4.8}$$

provided $\delta > 0$ small.

Proof. The proof of (4.8) is essentially the same as that of [24, Lemma 5.2], so we skip it. ☐

SUBSTEP 3. We prove that (3.9) holds for $\delta > 0$ sufficiently small.

Suppose that (3.9) is false. By virtue of Lemma 4.1, there exist sequences of small positive numbers $\{\delta'_m\}_{m\in\mathbb{N}}$, δ'_m-interacting families $\{\{U_{i,m} = U[z_{i,m},\lambda_{i,m}]\}_{i=1,\dots,\nu}\}_{m\in\mathbb{N}}$, functions $\{f_m\}_{m\in\mathbb{N}} \subset \dot{H}^s(\mathbb{R}^n)$ and $\{h_m\}_{m\in\mathbb{N}}$, and numbers $\{c_{i,m}^a\}_{i=1,\dots,\nu,\, a=1,\dots,n+1,\, m\in\mathbb{N}}$ such that

$$\delta'_m \to 0 \quad \text{and} \quad \|h_m\|_{**} \to 0 \quad \text{as } m \to \infty, \quad \|f_m\|_* = 1 \quad \text{for all } m \in \mathbb{N}, \tag{4.9}$$

and

$$\begin{cases}(-\Delta)^s f_m - p\,\sigma_m^{p-1} f_m = h_m + \displaystyle\sum_{i=1}^{\nu}\sum_{a=1}^{n+1} c_{i,m}^a U_{i,m}^{p-1} Z_{i,m}^a & \text{in } \mathbb{R}^n, \\ \displaystyle\int_{\mathbb{R}^n} U_{i,m}^{p-1} Z_{i,m}^a f_m dx = 0 & \text{for all } i = 1,\dots,\nu \text{ and } a = 1,\dots,n+1.\end{cases} \tag{4.10}$$

Here, $\sigma_m = \sum_{i=1}^{\nu} U_{i,m}$ and $Z_{i,m}^a = Z^a[z_{i,m},\lambda_{i,m}]$. By reordering the indices $i, j = 1,\dots,\nu$ and taking a subsequence, one can assume that

$$\begin{cases}\lambda_{1,m} \leq \lambda_{2,m} \leq \cdots \leq \lambda_{\nu,m} \text{ for all } m \in \mathbb{N}, \\ \text{either } \lim\limits_{m\to\infty} z_{ij,m} = z_{ij,\infty} \in \mathbb{R}^n \text{ or } \lim\limits_{m\to\infty} |z_{ij,m}| \to \infty\end{cases} \tag{4.11}$$

where $z_{ij,m} := \lambda_{i,m}(z_{j,m} - z_{i,m}) \in \mathbb{R}^n$.

Let $\mathcal{V}_m = \sum_{i=1}^{\nu}(v_{i,m}^{\text{in}} + v_{i,m}^{\text{out}})$ and $\mathcal{W}_m = \sum_{i=1}^{\nu}(w_{i,m}^{\text{in}} + w_{i,m}^{\text{out}})$ be the functions $\mathcal{V}$ and $\mathcal{W}$ in (3.1) with $(z_i,\lambda_i) = (z_{i,m},\lambda_{i,m})$, respectively. To reach a contradiction, we will establish that

$$\left(|f_m|\mathcal{W}_m^{-1}\right)(x) \leq \tfrac{1}{2} \quad \text{for all } x \in \mathbb{R}^n \tag{4.12}$$

provided $m \in \mathbb{N}$ large. Clearly, (4.12) implies $\|f_m\|_* \leq \frac{1}{2}$, which is absurd.

TREE STRUCTURE: To prove (4.12), we will exploit the tree structure of δ-interacting ν-tuples of bubbles as $\delta \to 0$, as described in Lemma 4.4 below. The bubble-tree structure for $s = 1$ was investigated in e.g. [28, 53, 24]. The concept of bubble-trees dates back to the work of Parker and Wolfson [51] for pseudo-holomorphic maps, and those of Parker [50] and Qing and Tian [54] for harmonic maps on Riemann surfaces.

[3]Unlike (4.5), we cannot ignore the factor $\langle\omega\rangle^{-4s}$ to get meaningful information through (4.7). On the other hand, without appealing to the iteration process as in Substep 1 of the proof of Proposition 3.3, we can directly achieve $f \in L^{t^*}(\mathbb{R}^n)$ for any large t^* by choosing $\zeta_1 > 0$ large.

Definition 4.3. Let $\preceq$ be a partial order on a set $\mathcal{T}$, and $\prec$ the corresponding strict partial order on $\mathcal{T}$.

- A partially ordered set $(\mathcal{T}, \preceq)$ is called a directed tree if for each $t \in \mathcal{T}$, the set $\{s \in \mathcal{T} : s \preceq t\}$ is well-ordered by the relation $\preceq$.

- A root is the least element of the set $\{s \in \mathcal{T} : s \preceq t\}$ for some $t \in \mathcal{T}$.

- A rooted tree is a directed tree with roots; a rooted forest is a disjoint union of rooted trees.

- A descendant of $s \in \mathcal{T}$ is any element $t \in \mathcal{T}$ such that $s \prec t$. Let $\mathcal{D}(s)$ be the set of descendants of $s \in \mathcal{T}$, that is, $\mathcal{D}(s) = \{t \in \mathcal{T} : s \prec t\}$.

Lemma 4.4. *Recalling (4.11), we set a relation $\prec$ (and $\succ$) on I by*

$$i \prec j \ \Leftrightarrow\ j \succ i \ \Leftrightarrow\ \left[i < j \text{ and } \lim_{m\to\infty} z_{ij,m} \in \mathbb{R}^n\right].$$

Then $\prec$ is a strict partial order (that corresponds to a non-strict order $\preceq$) and there is a number $\nu^* \in \{1, \dots, \nu\}$ such that I can be expressed as a rooted forest.

Proof. A slight modification of the argument in [24, Subsection 4.2] works for any $s \in (0, \frac{n}{2})$. $\square$

<u>Decomposition of $\mathbb{R}^n$:</u> We set $y_{i,m} = \lambda_{i,m}(x - z_{i,m}) \in \mathbb{R}^n$ and recall $z_{ij,m} = \lambda_{i,m}(z_{j,m} - z_{i,m}) \in \mathbb{R}^n$. Given any $L > 1$ large and $\varepsilon \in (0, 1)$ small, we define

$$\Omega_{i,m;L,\varepsilon} = \{x \in \mathbb{R}^n : |y_{i,m}| \le L,\ |y_{i,m} - z_{ij,m}| \ge \varepsilon \text{ for all } j \in \mathcal{D}(i)\}$$

and

$$\mathcal{A}_{i,m;L,\varepsilon} = \bigcup_{j\in\mathcal{D}(i)} \left[\{x \in \mathbb{R}^n : |y_{i,m} - z_{ij,m}| < \varepsilon\} \setminus \bigcup_{k\in\mathcal{D}(i)} \{x \in \mathbb{R}^n : |y_{k,m}| \le L\}\right].$$

Then $\mathbb{R}^n$ is decomposed into three disjoint subsets:

$$\mathbb{R}^n = \Omega_{\text{Ext},m;L} \cup \Omega_{\text{Core},m;L,\varepsilon} \cup \Omega_{\text{Neck},m;L,\varepsilon}$$

where

$$\begin{cases} \text{Exterior region:} & \Omega_{\text{Ext},m;L} = \bigcap_{i=1}^{\nu} \{x \in \mathbb{R}^n : |y_{i,m}| > L\}, \\ \text{Core region:} & \Omega_{\text{Core},m;L,\varepsilon} = \bigcup_{i=1}^{\nu} \Omega_{i,m;L,\varepsilon}, \\ \text{Neck region:} & \Omega_{\text{Neck},m;L,\varepsilon} = \bigcup_{i=1}^{\nu} \mathcal{A}_{i,m;L,\varepsilon}. \end{cases}$$

<u>Preliminary estimates:</u> Let $\mathscr{R}_{ij,m}$ and $\mathscr{R}_m$ be the quantities in (2.2)–(2.3) where the parameter $(z_{i,m}, z_{j,m}, \lambda_{i,m}, \lambda_{j,m})$ is substituted for $(z_i, z_j, \lambda_i, \lambda_j)$ so that $\mathscr{R}_m \to \infty$ as $m \to \infty$. Then (4.8) gives

$$\sum_{i=1}^{\nu} \sum_{a=1}^{n+1} |c_{i,m}^a| = o\left(\mathscr{R}_m^{2s-n}\right) = o(\delta_m') \to 0 \quad \text{as } m \to \infty \tag{4.13}$$

where $o(\mathscr{R}_m^{2s-n})/\mathscr{R}_m^{2s-n} \to 0$ as $m \to \infty$. By (4.2), (4.9), and (4.13), we know

$$|f_m(x)| \lesssim \int_{\mathbb{R}^n} \frac{1}{|x-\omega|^{n-2s}} \left[\left(\sigma_m^{p-1}|f_m|\right)(\omega) + o(1)\mathcal{V}_m(\omega) + o\left(\mathscr{R}_m^{2s-n}\right) \sum_{i=1}^{\nu} U_{i,m}^p(\omega)\right] \mathrm{d}\omega \tag{4.14}$$

for $x \in \mathbb{R}^n$.

We readily observe that

$$\int_{\mathbb{R}^n} \frac{1}{|x-\omega|^{n-2s}} U_{i,m}^p(\omega)\mathrm{d}\omega = \gamma_{n,s}^{-1}\left(\Phi_{n,s} * U_{i,m}^p\right)(x) = \gamma_{n,s}^{-1} U_{i,m}(x) \lesssim \frac{\lambda_{i,m}^{\frac{n-2s}{2}}}{\langle y_{i,m}\rangle^{n-2s}}. \tag{4.15}$$

Moreover, the following estimates are true.

Lemma 4.5. *There exists a constant $C>0$ depending only on n, s, and ν such that*

$$\int_{\mathbb{R}^n} \frac{1}{|x-\omega|^{n-2s}} \left(v_{j,m}^{\mathrm{in}} + v_{j,m}^{\mathrm{out}}\right)(\omega)\mathrm{d}\omega \le C\left(w_{j,m}^{\mathrm{in}} + w_{j,m}^{\mathrm{out}}\right)(x) \tag{4.16}$$

for $j = 1, \dots, \nu$ and

$$\begin{aligned} &\frac{1}{\mathcal{W}_m(x)} \int_{\mathbb{R}^n} \frac{1}{|x-\omega|^{n-2s}} \left(\sigma_m^{p-1}\mathcal{W}_m\right)(\omega)\mathrm{d}\omega \\ &\le C \begin{cases} M^{3n}\sum_{i=1}^{\nu}\left(\frac{1}{\langle y_{i,m}\rangle^{2s}}\mathbf{1}_{\{|y_{i,m}|<\mathscr{R}_m\}} + \frac{\log|y_{i,m}|}{|y_{i,m}|^{2s}}\mathbf{1}_{\{|y_{i,m}|\ge\mathscr{R}_m\}}\right) + M^{4s}\mathscr{R}_m^{-2s} + M^{-2s} & \text{for } n>6s, \\ M^{3n}\sum_{i=1}^{\nu}\left(\frac{\log(2+|y_{i,m}|)}{\langle y_{i,m}\rangle^{s}}\mathbf{1}_{\{|y_{i,m}|<\mathscr{R}_m^2\}} + \frac{\log|y_{i,m}|}{|y_{i,m}|^{s}}\mathbf{1}_{\{|y_{i,m}|\ge\mathscr{R}_m^2\}}\right) + M^{4s}\mathscr{R}_m^{-2s} + M^{-2s} & \text{for } n=6s \end{cases} \end{aligned} \tag{4.17}$$

holds for any $x\in\mathbb{R}^n$, $M>1$, and $m\in\mathbb{N}$ large.

Proof. A minor variation of the proof of [24, Lemma 3.6] yields (4.16). Besides, if $n>6s$, we have the following interaction estimates as in [24, Lemma 4.1]: If $\lambda_{i,m}\le\lambda_{j,m}$, then

$$U_{j,m}^{p-1} w_{i,m}^{\mathrm{in}} \lesssim \mathscr{R}_{ij,m}^{-2s} v_{j,m}^{\mathrm{in}} + \mathscr{R}_m^{-2s} v_{j,m}^{\mathrm{out}} + \mathscr{R}_m^{-2s} v_{i,m}^{\mathrm{in}}, \tag{4.18}$$

$$U_{j,m}^{p-1} w_{i,m}^{\mathrm{out}} \lesssim \mathscr{R}_{ij,m}^{-2s} v_{j,m}^{\mathrm{in}} + \mathscr{R}_m^{-2s} v_{j,m}^{\mathrm{out}} + \mathscr{R}_m^{-2s} v_{i,m}^{\mathrm{out}}, \tag{4.19}$$

$$U_{i,m}^{p-1} w_{j,m}^{\mathrm{in}} \lesssim \mathscr{R}_{ij,m}^{-2s} v_{j,m}^{\mathrm{in}}, \tag{4.20}$$

$$U_{i,m}^{p-1} w_{j,m}^{\mathrm{out}} \lesssim \langle z_{ij,m}\rangle^{-2s}\left(v_{i,m}^{\mathrm{in}} + v_{i,m}^{\mathrm{out}} + v_{j,m}^{\mathrm{out}}\right), \tag{4.21}$$

and

$$U_{i,m}^{p-1} w_{j,m}^{\mathrm{out}} \lesssim \left[\left(\frac{\lambda_{i,m}}{\lambda_{j,m}}\right)^2 + \epsilon^2\right]^s v_{j,m}^{\mathrm{out}} \quad \text{if } |y_{i,m} - z_{ij,m}| \le \epsilon, \tag{4.22}$$

$$w_{j,m}^{\mathrm{out}} \lesssim \langle z_{ij,m}\rangle^{2n-10s}\epsilon^{4s-n}\left(w_{i,m}^{\mathrm{in}} + w_{i,m}^{\mathrm{out}}\right) \quad \text{if } |y_{i,m} - z_{ij,m}| \ge \epsilon \tag{4.23}$$

for any $\epsilon\in(0,1)$ and $m\in\mathbb{N}$ large. Taking these estimates into account, one can mimic the proof of [24, Proposition 4.3] to achieve (4.17).

If $n=6s$, we can obtain (4.17) by deriving analogous inequalities to (4.18)–(4.23) as in [24, Lemma 4.2]. We skip the details. □

In the remainder of this section, we restrict ourselves to the case $n>6s$. The proof for the case $n=6s$ goes through without serious modification, so we omit it for conciseness.

Suppose that for any given $\zeta\in(0,1)$, there exists a number $m_\zeta\in\mathbb{N}$ depending on ζ such that

$$m\ge m_\zeta \ \Rightarrow\ \int_{\mathbb{R}^n} \frac{1}{|x-\omega|^{n-2s}} \left(\sigma_m^{p-1}|f_m|\right)(\omega)\mathrm{d}\omega \le \zeta\mathcal{W}_m(x) \quad \text{for all } x\in\mathbb{R}^n. \tag{4.24}$$

Then, owing to (4.14)–(4.16), the desired inequality (4.12) will hold. We have

$$\begin{aligned}&\int_{\mathbb{R}^n} \frac{1}{|x-\omega|^{n-2s}} \left(\sigma_m^{p-1}|f_m|\right)(\omega)\mathrm{d}\omega \\ &= \left(\int_{\Omega_{\mathrm{Ext},m;L}} + \int_{\Omega_{\mathrm{Core},m;L,\varepsilon}} + \int_{\Omega_{\mathrm{Neck},m;L,\varepsilon}}\right) \frac{1}{|x-\omega|^{n-2s}} \left(\sigma_m^{p-1}|f_m|\right)(\omega)\mathrm{d}\omega \\ &=: \mathcal{I}_{\mathrm{Ext},m;L}(x) + \mathcal{I}_{\mathrm{Core},m;L,\varepsilon}(x) + \mathcal{I}_{\mathrm{Neck},m;L,\varepsilon}(x) \quad \text{for all } x \in \mathbb{R}^n.\end{aligned} \tag{4.25}$$

By choosing suitable L and ε, we shall estimate each terms $\mathcal{I}_{\mathrm{Ext},m;L}$, $\mathcal{I}_{\mathrm{Core},m;L,\varepsilon}$, and $\mathcal{I}_{\mathrm{Neck},m;L,\varepsilon}$ to deduce (4.24).

<u>Estimate of $\mathcal{I}_{\mathrm{Ext},m;L}$</u>: By (4.9) and (4.14)–(4.17), there is a constant $C_0 > 0$ depending only on n, s, and ν such that

$$\sup_{x\in\Omega_{\mathrm{Ext},m;L}} \left(|f_m|\mathcal{W}_m^{-1}\right)(x) \le C_0\left(M^{3n}L^{-2s}\log L + M^{4s}\mathscr{R}_m^{-2s} + M^{-2s}\right) + o(1). \tag{4.26}$$

In light of (4.26) and (4.17), there exists $C_1 > 0$ depending only on n, s, and ν such that

$$\begin{aligned}&\mathcal{I}_{\mathrm{Ext},m;L}(x) \\ &\le \left[C_0\left(M^{3n}L^{-2s}\log L + M^{4s}\mathscr{R}_m^{-2s} + M^{-2s}\right) + o(1)\right] \int_{\mathbb{R}^n} \frac{1}{|x-\omega|^{n-2s}} \left(\sigma_m^{p-1}\mathcal{W}_m\right)(\omega)\mathrm{d}\omega \\ &\le C_1\left[C_0\left(M^{3n}L^{-2s}\log L + M^{4s}\mathscr{R}_m^{-2s} + M^{-2s}\right) + o(1)\right]\mathcal{W}_m(x)\end{aligned}$$

for any $x \in \mathbb{R}^n$ and $m \in \mathbb{N}$ large. We pick numbers M_0 and L_0 so large that $C_1C_0M_0^{-2s} < \frac{\zeta}{12}$ and $C_1C_0M_0^{3n}L_0^{-2s}\log L_0 < \frac{\zeta}{12}$. Then

$$\mathcal{I}_{\mathrm{Ext},m;L_0}(x) \le \frac{\zeta}{3}\mathcal{W}_m(x) \tag{4.27}$$

for all $x \in \mathbb{R}^n$ and $m \in \mathbb{N}$ large.

By taking larger values for L_0 if required, we can assume that

$$L_0 > C^* := 1 + \max\{|z_{ij}| : i,j = 1,\dots,\nu,\ j \in \mathcal{D}(i)\}, \tag{4.28}$$

which will be frequently used later. Hereafter, we fix $L = L_0$ and omit the subscript L_0 for simplicity, writing e.g. $\Omega_{\mathrm{Ext},m} = \Omega_{\mathrm{Ext},m;L_0}$ or $\mathcal{I}_{\mathrm{Core},m;\varepsilon} = \mathcal{I}_{\mathrm{Core},m;L_0,\varepsilon}$.

<u>Estimate of $\mathcal{I}_{\mathrm{Core},m;\varepsilon}$</u>: Fix any $\varepsilon \in (0,1)$. By employing the blow-up argument, we will first show that

$$\sup_{x\in\Omega_{\mathrm{Core},m;\varepsilon}} \left(|f_m|\mathcal{W}_m^{-1}\right)(x) = o(1) \quad \text{as } m \to \infty. \tag{4.29}$$

If (4.29) is not true, we will have points $x_m \in \Omega_{\mathrm{Core},m;\varepsilon}$ for $m \in \mathbb{N}$ and a number $\theta_0 \in (0,1)$ such that $\theta_0 \le (|f_m|\mathcal{W}_m^{-1})(x_m) \le 1$ for all $m \in \mathbb{N}$. By passing to a subsequence, we may assume that $x_m \in \Omega_{i_0,m;\varepsilon}$ for some $i_0 = 1,\dots,\nu$ and all $m \in \mathbb{N}$. The following lemma is crucial.

Lemma 4.6. *Let $\hat{f}_m$ be a function in $\dot{H}^s(\mathbb{R}^n)$ defined as*

$$\hat{f}_m(y) = \mathcal{W}_m^{-1}(x_m)f_m\left(\lambda_{i_0,m}^{-1}y + z_{i_0,m}\right) \quad \textit{for } y \in \mathbb{R}^n$$

and $\widetilde{\mathcal{Z}}_\infty = \{z_{i_0j,\infty} : j \in \mathcal{D}(i_0)\}$. Then, up to a subsequence,

$$\hat{f}_m \to 0 \quad \textit{in } C^0_{\mathrm{loc}}\left(\mathbb{R}^n \setminus \widetilde{\mathcal{Z}}_\infty\right) \quad \textit{as } m \to \infty. \tag{4.30}$$

Proof. There are several lengthy technical calculations in this proof. To make the main strategy of this proof clearer, we will postpone their derivations to Appendix B.

We set

$$\mathcal{H}_m(y) = \lambda_{i_0,m}^{-2s}\mathcal{W}_m^{-1}(x_m)\left[h_m + \sum_{j=1}^{\nu}\sum_{a=1}^{n+1} c_{j,m}^a U_{j,m}^{p-1} Z_{j,m}^a\right]\left(\lambda_{i_0,m}^{-1}y + z_{i_0,m}\right) \quad \text{for } y \in \mathbb{R}^n. \tag{4.31}$$

From (4.10), we see that

$$\begin{cases} \hat{f}_m = \Phi_{n,s} * \left[p\left\{\lambda_{i_0,m}^{-\frac{n-2s}{2}}\sigma_m\left(\lambda_{i_0,m}^{-1}\cdot + z_{i_0,m}\right)\right\}^{p-1}\hat{f}_m + \mathcal{H}_m\right] \quad \text{in } \mathbb{R}^n, \\ \displaystyle\int_{\mathbb{R}^n} U[0,1]^{p-1}Z^a[0,1]\hat{f}_m \mathrm{d}y = 0 \quad \text{for all } i = 1,\dots,\nu \text{ and } a = 1,\dots,n+1. \end{cases} \tag{4.32}$$

We notice from (3.1) that

$$\mathcal{W}_m(x_m) \ge \frac{\lambda_{i_0}^{\frac{n-2s}{2}}\mathscr{R}_m^{2s-n}}{\langle\lambda_{i_0,m}(x_m - z_{i_0,m})\rangle^{2s}} \gtrsim L_0^{-2s}\lambda_{i_0}^{\frac{n-2s}{2}}\mathscr{R}_m^{2s-n}. \tag{4.33}$$

Furthermore, given any $M > 0$, the proof of [24, Lemma 4.7] shows the existence of a sequence $\{\eta_{M,m}\}_{m\in\mathbb{N}} \subset (0,\infty)$ such that $\eta_{M,m} \to 0$ as $m \to \infty$ and

$$\left(w_{j,m}^{\text{in}} + w_{j,m}^{\text{out}}\right)\left(\lambda_{i_0,m}^{-1}y + z_{i_0,m}\right) = \eta_{M,m}w_{i_0,m}^{\text{in}}\left(\lambda_{i_0,m}^{-1}y + z_{i_0,m}\right) \tag{4.34}$$

for all $y \in B(0,M)$, $j \notin \mathcal{D}(i_0)$, and $m \in \mathbb{N}$ large.[4] Using the elementary inequality

$$\frac{\sum_{i=1}^{\nu}a_i}{\sum_{i=1}^{\nu}b_i} \le \max\left\{\frac{a_1}{b_1},\dots,\frac{a_\nu}{b_\nu}\right\} \le \sum_{i=1}^{\nu}\frac{a_i}{b_i} \quad \text{for } a_1,\dots,a_\nu \ge 0 \text{ and } b_1,\dots,b_\nu > 0,$$

(4.33), (4.34), and (4.28), we verify that

$$\begin{aligned} \frac{\mathcal{W}_m\left(\lambda_{i_0,m}^{-1}y + z_{i_0,m}\right)}{\mathcal{W}_m(x_m)} &\le \frac{\left[(1+\nu\eta_{M,m})w_{i_0,m}^{\text{in}} + \sum_{j\in\mathcal{D}(i_0)}\left(w_{j,m}^{\text{in}} + w_{j,m}^{\text{out}}\right)\right]\left(\lambda_{i_0,m}^{-1}y + z_{i_0,m}\right)}{\left[w_{i_0,m}^{\text{in}} + \sum_{j\in\mathcal{D}(i_0)}\left(w_{j,m}^{\text{in}} + w_{j,m}^{\text{out}}\right)\right](x_m)} \\ \lesssim (1+\nu\eta_{M,m})L_0^{2s} + \sum_{j\in\mathcal{D}(i_0)} &\left[\frac{L_0^{2s}}{|y - z_{i_0j,m}|^{2s}}\mathbf{1}_{\left\{\left|\frac{\lambda_{j,m}}{\lambda_{i_0,m}}(y - z_{i_0j,m})\right| < \mathscr{R}_m\right\}} + \frac{L_0^{n-4s}}{|y - z_{i_0j,m}|^{n-4s}}\mathbf{1}_{\left\{\left|\frac{\lambda_{j,m}}{\lambda_{i_0,m}}(y - z_{i_0j,m})\right| \ge \mathscr{R}_m\right\}}\right] \end{aligned} \tag{4.35}$$

for $y \in B(0,M) \setminus \{z_{i_0j,m} : j \in \mathcal{D}(i_0)\}$. Given any $l > 0$, let

$$\mathcal{K}_l := \left\{y \in \mathbb{R}^n : |y| \le l,\ |y - z_{i_0j,\infty}| \ge l^{-1} \text{ for all } j \in \mathcal{D}(i_0)\right\} \subset \mathbb{R}^n \setminus \widetilde{\mathcal{Z}}_\infty.$$

Then there exists a large number $m_l \in \mathbb{N}$ depending on l such that

$$y \in \mathcal{K}_l,\ j \in \mathcal{D}(i_0),\ m \ge m_l \ \Rightarrow\ \left|\frac{\lambda_{j,m}}{\lambda_{i_0,m}}(y - z_{i_0j,\infty})\right| \ge \frac{1}{2}l^{-1}\sqrt{\frac{\lambda_{j,m}}{\lambda_{i_0,m}}}\mathscr{R}_m \gg \mathscr{R}_m.$$

Thus (4.35) gives

$$\left|\hat{f}_m(y)\right| \lesssim (1+\nu\eta_{l,m})L_0^{2s} + \sum_{j\in\mathcal{D}(i_0)}\frac{L_0^{n-4s}}{|y - z_{i_0j,m}|^{n-4s}} \quad \text{uniformly in } \mathcal{K}_l \text{ for } m \ge m_l. \tag{4.36}$$

[4] In principle, we may have that $\liminf_{m\to\infty}\sup_{M>0}\eta_{M,m} = \infty$ because of the presence of bubbles that belong to a different bubble-tree than the one associated with the index i_0.

Let B and B' be a bounded open ball such that $\overline{B'} \subset B \subset \mathcal{K}_l$ for some large $l > 0$, where $\overline{B'}$ is the closure of B'. In Appendix B.1, we will directly use (4.32) to prove that if $s \in (\frac{1}{2}, \frac{n}{2})$, then

$$\|\hat{f}_m\|_{C^1(\overline{B'})} \lesssim L_0^{n-4s} \quad \text{for } m \in \mathbb{N} \text{ large.} \tag{4.37}$$

If $s \in (0, \frac{1}{2}]$, the standard elliptic regularity (see e.g. [55, Corollary 2.5]) combined with Remark B.2, $\|\hat{f}_m\|_{L^\infty(B)} \lesssim L_0^{n-4s}$, and $\mathcal{H}_m \to 0$ in $L^\infty(B)$ yields

$$\begin{aligned} \|\hat{f}_m\|_{C^{0,\alpha}(\overline{B'})} &\lesssim \left\| \langle \cdot \rangle^{-(n+2s)} \hat{f}_m \right\|_{L^1(\mathbb{R}^n)} + \|\hat{f}_m\|_{L^\infty(B)} \\ &\quad + \left\| p \left\{ \lambda_{i_0,m}^{-\frac{n-2s}{2}} \sigma_m\left(\lambda_{i_0,m}^{-1} \cdot + z_{i_0,m}\right) \right\}^{p-1} \hat{f}_m + \mathcal{H}_m \right\|_{L^\infty(B)} \\ &\lesssim L_0^{n-4s} \quad \text{for } m \in \mathbb{N} \text{ large} \end{aligned} \tag{4.38}$$

provided $\alpha \in (0, 2s)$. Employing (4.37)–(4.38), a standard covering argument (to obtain the $C^1(\mathcal{K}_l)$ or $C^{0,\alpha}(\mathcal{K}_l)$-estimate for $\hat{f}_m$), and the diagonal argument (to take $l \to \infty$), we obtain

$$\hat{f}_m \to \hat{f}_\infty \quad \text{in } C^0_{\text{loc}}\left(\mathbb{R}^n \setminus \widetilde{\mathcal{Z}}_\infty\right) \quad \text{as } m \to \infty \tag{4.39}$$

for some function $\hat{f}_\infty$.

From (4.36) and (4.39), we see that

$$\left|\hat{f}_\infty(y)\right| \lesssim L_0^{2s} + \sum_{j \in \mathcal{D}(i_0)} \frac{L_0^{n-4s}}{|y - z_{i_0 j,\infty}|^{n-4s}} \quad \text{in } \mathbb{R}^n \setminus \widetilde{\mathcal{Z}}_\infty.$$

In Appendix B.2 and B.3, we will confirm

$$\hat{f}_\infty = \Phi_{n,s} * \left(pU[0,1]^{p-1}\hat{f}_\infty\right) \quad \text{in } \mathbb{R}^n \setminus \widetilde{\mathcal{Z}}_\infty \tag{4.40}$$

and

$$\int_{\mathbb{R}^n} U[0,1]^{p-1} Z^a[0,1] \hat{f}_\infty \mathrm{d}y = 0 \quad \text{for all } i = 1, \ldots, \nu \text{ and } a = 1, \ldots, n+1. \tag{4.41}$$

Then, Lemma A.2 implies that each singularity $z_{i_0 j,\infty}$ of $\hat{f}_\infty$ is removable, namely, $\hat{f}_\infty$ extends to a function in $L^\infty(\mathbb{R}^n)$ satisfying (4.40) in $\mathbb{R}^n$. By Lemma A.1 (a), it follows that $\hat{f}_\infty = 0$ in $\mathbb{R}^n$. As a result, (4.30) must hold. □

Let $Y_m = \lambda_{i_0,m}(x_m - z_{i_0,m})$ so that $\sup_{m \in \mathbb{N}} |Y_m| \le L_0 < \infty$. One can assume that $Y_m \to Y_\infty$ as $m \to \infty$ for some $Y_\infty \in \mathbb{R}^n$ such that $|Y_\infty| \le L_0$ and $|Y_\infty - z_{i_0 j,\infty}| \ge \varepsilon$ for all $j \in \mathcal{D}(i_0)$. By (4.30), one concludes that $\hat{f}_m(Y_m) \to 0$ as $m \to \infty$, which is impossible because $|\hat{f}_m(Y_m)| = (|f_m|\mathcal{W}_m^{-1})(x_m) \ge \theta_0 > 0$. Therefore, (4.29) is true.

Now, from (4.29) and (4.17), we infer that

$$\mathcal{I}_{\text{Core},m;\varepsilon}(x) = o(1) \int_{\Omega_{\text{Core},m;\varepsilon}} \frac{1}{|x - \omega|^{n-2s}} \left(\sigma_m^{p-1}\mathcal{W}_m\right)(\omega) \mathrm{d}\omega = o(1)\mathcal{W}_m(x) \tag{4.42}$$

for all $x \in \mathbb{R}^n$ and $m \in \mathbb{N}$ large.

ESTIMATE OF $\mathcal{I}_{\mathrm{Neck},m;\varepsilon}$: By reasoning as in Case 3 of the proof of [24, Lemma 5.1], we find that for any fixed $\theta \in (0,1)$ and $i = 1,\dots,\nu$,

$$\begin{aligned}\sigma_m^{p-1}\mathcal{W}_m \lesssim \left[(C^*)^{\frac{2s(n-6s)}{n-2s}}\theta + L_0^{-2s} + o(1)\right] \sum_{j\in\mathcal{D}(i)} \left(v_{j,m}^{\mathrm{in}} + v_{j,m}^{\mathrm{out}}\right) \\ + \left[(C^*)^{\frac{2s(n-6s)}{n-2s}}\theta^{-\frac{n-4s}{2s}} + o(1)\right] v_{i,m}^{\mathrm{in}} \quad \text{in } \mathcal{A}_{i,m;\varepsilon},\end{aligned} \tag{4.43}$$

provided $L_0 > 3C^*$ (see (4.28) for the definition of C^*) and $m \in \mathbb{N}$ large. By (4.43),

$$\begin{aligned}&\mathcal{I}_{\mathrm{Neck},m;\varepsilon}(x)\\ &\le \sum_{i=1}^{\nu} \int_{\mathcal{A}_{i,m;\varepsilon}} \frac{1}{|x-\omega|^{n-2s}} \left(\sigma_m^{p-1}\mathcal{W}_m\right)(\omega)\mathrm{d}\omega\\ &\lesssim \sum_{i=1}^{\nu} \int_{\mathcal{A}_{i,m;\varepsilon}} \frac{1}{|x-\omega|^{n-2s}} \left[\left\{(C^*)^{2s}\theta + L_0^{-2s} + o(1)\right\}\mathcal{V}_m(\omega) + (C^*)^{2s}\theta^{-\frac{n-4s}{2s}} v_{i,m}^{\mathrm{in}}(\omega)\right]\mathrm{d}\omega\\ &\lesssim (C^*)^{2s}\theta^{-\frac{n-4s}{2s}} \sum_{i=1}^{\nu} \int_{\mathcal{A}_{i,m;\varepsilon}} \frac{1}{|x-\omega|^{n-2s}} v_{i,m}^{\mathrm{in}}(\omega)\mathrm{d}\omega\\ &\quad + \left\{(C^*)^{2s}\theta + L_0^{-2s} + o(1)\right\} \int_{\mathbb{R}^n} \frac{1}{|x-\omega|^{n-2s}} \mathcal{V}_m(\omega)\mathrm{d}\omega.\end{aligned}$$

Hence, by applying (4.16) and possibly increasing the value of L_0, we achieve

$$\mathcal{I}_{\mathrm{Neck},m;\varepsilon}(x) \le C(C^*)^{2s}\theta^{-\frac{n-4s}{2s}} \sum_{i=1}^{\nu} \int_{\mathcal{A}_{i,m;\varepsilon}} \frac{1}{|x-\omega|^{n-2s}} v_{i,m}^{\mathrm{in}}(\omega)\mathrm{d}\omega + \frac{\zeta}{6}\mathcal{W}_m(x) \tag{4.44}$$

for any $x \in \mathbb{R}^n$ and $\theta \in (0,1)$ small. Moreover,

$$\begin{aligned}&\int_{\mathcal{A}_{i,m;\varepsilon}} \frac{1}{|x-\omega|^{n-2s}} v_{i,m}^{\mathrm{in}}(\omega)\mathrm{d}\omega\\ &\le \sum_{j\in\mathcal{D}(i)} \underbrace{\int_{B(z_{j,m},\frac{\varepsilon}{\lambda_{i,m}})\setminus\cup_{k\in\mathcal{D}(i)} B(z_{k,m},\frac{L_0}{\lambda_{k,m}})} \frac{1}{|x-\omega|^{n-2s}} \frac{\lambda_{i,m}^{\frac{n+2s}{2}} \mathscr{R}_m^{2s-n}}{\langle\lambda_{i,m}(\omega - z_{i,m})\rangle^{4s}} \mathbf{1}_{B(z_{i,m},\frac{\mathscr{R}_m}{\lambda_{i,m}})}(\omega)\mathrm{d}\omega}_{=:\mathcal{J}_{i,m;\varepsilon}(x)}\end{aligned} \tag{4.45}$$

for $x \in \mathbb{R}^n$. We will estimate $\mathcal{J}_{i,m;\varepsilon}(x)$ by considering three separate cases.

Case 1: Fixing any $L' \ge 2C^*$, we assume that $|x - z_{j,m}| \ge \frac{L'}{\lambda_{i,m}}$ for some $j \in \mathcal{D}(i)$.

Letting $\tilde{y}_{ij,m} = \lambda_{i,m}(x - z_{j,m})$ and $\tilde{\omega}_{ij,m} = \lambda_{i,m}(\omega - z_{j,m})$, and recalling $z_{ij,m} = \lambda_{i,m}(z_{j,m} - z_{i,m})$, we evaluate

$$\begin{aligned}\mathcal{J}_{i,m;\varepsilon}(x) &\le \int_{B(z_{j,m},\frac{\varepsilon}{\lambda_{i,m}})} \frac{1}{|x-\omega|^{n-2s}} \frac{\lambda_{i,m}^{\frac{n+2s}{2}} \mathscr{R}_m^{2s-n}}{\langle\lambda_{i,m}(\omega - z_{i,m})\rangle^{4s}} \mathbf{1}_{B(z_{i,m},\frac{\mathscr{R}_m}{\lambda_{i,m}})}(\omega)\mathrm{d}\omega\\ &\le \lambda_{i,m}^{\frac{n-2s}{2}} \mathscr{R}_m^{2s-n} \int_{B(0,\varepsilon)} \frac{1}{|\tilde{y}_{ij,m} - \tilde{\omega}_{ij,m}|^{n-2s}} \frac{\mathrm{d}\tilde{\omega}_{ij,m}}{\langle\tilde{\omega}_{ij,m} + z_{ij,m}\rangle^{4s}}\\ &\lesssim \varepsilon^n \lambda_{i,m}^{\frac{n-2s}{2}} \mathscr{R}_m^{2s-n} \frac{1}{|\tilde{y}_{ij,m}|^{n-2s}}\end{aligned} \tag{4.46}$$

where we used $|\tilde{y}_{ij,m}-\tilde{\omega}_{ij,m}| \ge \frac{1}{2}|\tilde{y}_{ij,m}|$, which comes from $|\tilde{y}_{ij,m}| \ge L'$, to get the last inequality. Notice that $|z_{ij,m}| \le \frac{1}{2}|\tilde{y}_{ij,m}|$ and

$$|\tilde{y}_{ij,m}| - |z_{ij,m}| \le \lambda_{i,m}|x-z_{i,m}| \le |\tilde{y}_{ij,m}| + |z_{ij,m}|,$$

which imply

$$\frac{1}{2}|\tilde{y}_{ij,m}| \le |y_{i,m}| = \lambda_{i,m}|x-z_{i,m}| \le \frac{3}{2}|\tilde{y}_{ij,m}|.$$

Thus

$$\begin{aligned}
&\varepsilon^n \lambda_{i,m}^{\frac{n-2s}{2}} \mathscr{R}_m^{2s-n} \frac{1}{|\tilde{y}_{ij,m}|^{n-2s}} \\
&\lesssim \varepsilon^n \left[(L')^{4s-n} \frac{\lambda_{i,m}^{\frac{n-2s}{2}} \mathscr{R}_m^{2s-n}}{\langle y_{i,m}\rangle^{2s}} \mathbf{1}_{\{|y_{i,m}|<\mathscr{R}_m\}} + (L')^{-2s} \mathscr{R}_m^{6s-n} \frac{\lambda_{i,m}^{\frac{n-2s}{2}} \mathscr{R}_m^{-4s}}{|y_{i,m}|^{n-4s}} \mathbf{1}_{\{|y_{i,m}|\ge\mathscr{R}_m\}} \right] \\
&\le \varepsilon^n \mathcal{W}_m(x).
\end{aligned}$$

Case 2: We assume that $\frac{2\varepsilon}{\lambda_{i,m}} \le |x-z_{j,m}| \le \frac{L'}{\lambda_{i,m}}$ for some $j \in \mathcal{D}(i)$.

As in (4.46), we compute

$$\begin{aligned}
\mathcal{J}_{i,m;\varepsilon}(x) &\le \lambda_{i,m}^{\frac{n-2s}{2}} \mathscr{R}_m^{2s-n} \int_{B(0,\varepsilon)} \frac{1}{|\tilde{y}_{ij,m}-\tilde{\omega}_{ij,m}|^{n-2s}} \frac{\mathrm{d}\tilde{\omega}_{ij,m}}{\langle \tilde{\omega}_{ij,m}+z_{ij,m}\rangle^{4s}} \\
&\lesssim \varepsilon^{2s-n} \lambda_{i,m}^{\frac{n-2s}{2}} \mathscr{R}_m^{2s-n} \int_{B(0,\varepsilon)} \mathrm{d}\tilde{\omega}_{ij,m} \lesssim \varepsilon^{2s} \lambda_{i,m}^{\frac{n-2s}{2}} \mathscr{R}_m^{2s-n}
\end{aligned}$$

where we employed $|\tilde{y}_{ij,m}-\tilde{\omega}_{ij,m}| \ge |\tilde{y}_{ij,m}| - |\tilde{\omega}_{ij,m}| \ge 2\varepsilon-\varepsilon = \varepsilon$ to obtain the second inequality. Since

$$|y_{i,m}| = \lambda_{i,m}|x-z_{i,m}| \le \lambda_{i,m}(|x-z_{j,m}| + |z_{i,m}-z_{j,m}|) \le L' + C^* < \mathscr{R}_m$$

for $m \in \mathbb{N}$ large, we see

$$\varepsilon^{2s} \lambda_{i,m}^{\frac{n-2s}{2}} \mathscr{R}_m^{2s-n} \lesssim \varepsilon^{2s} \left[1 + (L'+C^*)^{2s}\right] \frac{\lambda_{i,m}^{\frac{n-2s}{2}} \mathscr{R}_m^{2s-n}}{\langle y_{i,m}\rangle^{2s}} \mathbf{1}_{\{|y_{i,m}|<\mathscr{R}_m\}} \lesssim \varepsilon^{2s}(L'+C^*)^{2s} \mathcal{W}_m(x).$$

Case 3: We assume that $|x-z_{j,m}| \le \frac{2\varepsilon}{\lambda_{i,m}}$ for some $j \in \mathcal{D}(i)$.

We calculate

$$\begin{aligned}
\mathcal{J}_{i,m;\varepsilon}(x) &\le \int_{B(x,\frac{3\varepsilon}{\lambda_{i,m}})} \frac{1}{|x-\omega|^{n-2s}} \frac{\lambda_{i,m}^{\frac{n+2s}{2}} \mathscr{R}_m^{2s-n}}{\langle \lambda_{i,m}(\omega-z_{i,m})\rangle^{4s}} \mathbf{1}_{B(z_{i,m},\frac{\mathscr{R}_m}{\lambda_{i,m}})}(\omega)\mathrm{d}\omega \\
&\le \lambda_{i,m}^{\frac{n+2s}{2}} \mathscr{R}_m^{2s-n} \int_{B(x,\frac{3\varepsilon}{\lambda_{i,m}})} \frac{1}{|x-\omega|^{n-2s}} \mathrm{d}\omega \lesssim \varepsilon^{2s} \lambda_{i,m}^{\frac{n-2s}{2}} \mathscr{R}_m^{2s-n}.
\end{aligned}$$

Because

$$|y_{i,m}| \le \lambda_{i,m}(|x-z_{j,m}| + |z_{i,m}-z_{j,m}|) \le 2\varepsilon + C^* < \mathscr{R}_m$$

for $m \in \mathbb{N}$ large, we observe

$$\varepsilon^{2s} \lambda_{i,m}^{\frac{n-2s}{2}} \mathscr{R}_m^{2s-n} \lesssim \varepsilon^{2s}(2\varepsilon + C^*)^{2s} \mathcal{W}_m(x).$$

From the above analysis for the three cases and (4.44)–(4.45), we find

$$\begin{aligned}\mathcal{I}_{\mathrm{Neck},m;\varepsilon}(x) &\le C(C^*)^{2s}\theta^{-\frac{n-4s}{2s}}\sum_{i=1}^{\nu}\sum_{j\in\mathcal{D}(i)}\mathcal{J}_{i,m;\varepsilon}(x)+\frac{\zeta}{6}\mathcal{W}_m(x)\\ &\le \left[C(C^*)^{2s}\theta^{-\frac{n-4s}{2s}}\varepsilon^{2s}\mathcal{W}_m(x)+\frac{\zeta}{6}\right]\mathcal{W}_m(x)\le\frac{\zeta}{3}\mathcal{W}_m(x)\end{aligned}\tag{4.47}$$

for all $x\in\mathbb{R}^n$, provided $\varepsilon\in(0,1)$ small and $m\in\mathbb{N}$ large.

Now, by inserting (4.27), (4.42), and (4.47) into (4.25), we obtain (4.24). Consequently, the contradictory inequality (4.12) holds for all $x\in\mathbb{R}^n$ and large $m\in\mathbb{N}$, implying the validity of (3.9). This completes the proof of Proposition 3.3.

5. Quantitative stability estimate for dimension $2s<n<6s$

Having the spectral inequality (2.4) in hand, one may attempt to argue as in [32] to derive (1.9) for $2s<n<6s$. Indeed, Aryan pursued this approach in [3, Section 2], getting the result provided $s\in(0,1)$.

Here, we present an alternative proof of (1.9) whose scheme is close to those in the previous sections. One can use standard integral norms at this time, and computations in the proof are more straightforward than the high-dimensional case $n\ge 6s$.

Definition 5.1. We redefine the $*$- and $**$-norms as

$$\|\rho\|_*=\|\rho\|_{\dot H^s(\mathbb{R}^n)}\quad\text{and}\quad\|h\|_{**}=\|h\|_{L^{\frac{2n}{n+2s}}(\mathbb{R}^n)}.$$

As before, the derivation of (1.9) is split into three steps.

Step 1. Assume that $2s<n<6s$. We set σ, ρ, and ρ_0 as in Step 1 of Section 3.

Lemma 5.2. *There exists a constant $C>0$ depending only on n, s, and ν such that*

$$\left\|\sigma^p-\sum_{i=1}^{\nu}U_i^p\right\|_{**}\le C\mathscr{Q}\tag{5.1}$$

where $\mathscr{Q}>0$ is the value in (2.1).

Proof. By elementary calculus, (2.10), and the condition $n<6s$, we have

$$\begin{aligned}\left\|\sigma^p-\sum_{i=1}^{\nu}U_i^p\right\|_{L^{\frac{2n}{n+2s}}(\mathbb{R}^n)} &\lesssim \sum_{\substack{i,j=1,\dots,\nu,\\ i\ne j}}\left\|U_i^{p-1}U_j\right\|_{L^{\frac{2n}{n+2s}}(\mathbb{R}^n)}=\sum_{\substack{i,j=1,\dots,\nu,\\ i\ne j}}\left(\int_{\mathbb{R}^n}U_i^{\frac{2(p-1)n}{n+2s}}U_j^{\frac{2n}{n+2s}}\right)^{\frac{n+2s}{2n}}\\ &\lesssim \mathscr{Q}^{\min\{p-1,1\}\frac{2n}{n+2s}\cdot\frac{n+2s}{2n}}=\mathscr{Q}.\end{aligned}$$

□

We next analyze an associated inhomogeneous equation (3.8).

Proposition 5.3. *If $h\in L^{\frac{2n}{n+2s}}(\mathbb{R}^n)$ and f satisfies (3.8), then there exists a constant $C>0$ depending only on n, s, and ν such that*

$$\|f\|_*\le C\|h\|_{**}.\tag{5.2}$$

Proof. Since the condition $f\in\dot H^s(\mathbb{R}^n)$ was assumed in (3.8), we clearly have that $\|f\|_*<\infty$. The proof consists of two substeps.

SUBSTEP 1. We claim that there is a constant $C > 0$ depending only on n, s, and ν such that

$$\sum_{i=1}^{\nu}\sum_{a=1}^{n+1}|c_i^a| \le C\left(\|h\|_{**} + \mathscr{Q}\|f\|_*\right). \tag{5.3}$$

To show it, we test (3.8) with Z_j^b for any fixed $j = 1, \dots, \nu$ and $b = 1, \dots, n+1$ and employ (2.10), (3.16), (3.7), and (1.1). Then we arrive at

$$\begin{aligned} &c_j^b \int_{\mathbb{R}^n} U[0,1]^{p-1}Z^b[0,1]^2 + \sum_{\substack{i=1,\dots,\nu,\\ i\neq j}} c_i^a O(q_{ij}) = p\int_{\mathbb{R}^n}\left(U_j^{p-1} - \sigma^{p-1}\right) fZ_j^b - \int_{\mathbb{R}^n} hZ_j^b \\ &= O\left(\left\|\sigma^p - \sum_{i=1}^{\nu} U_i^p\right\|_{L^{\frac{2n}{n+2s}}(\mathbb{R}^n)} \|f\|_{L^{\frac{2n}{n-2s}}(\mathbb{R}^n)}\right) + O\left(\|h\|_{L^{\frac{2n}{n+2s}}(\mathbb{R}^n)}\right) = O\left(\mathscr{Q}\|f\|_* + \|h\|_{**}\right), \end{aligned}$$

which implies (5.3).

SUBSTEP 2. We assert that (5.2) holds. Suppose not. There exist sequences of small positive numbers $\{\delta_m'\}_{m\in\mathbb{N}}$, δ_m'-interacting families $\{\{U_{i,m} = U[z_{i,m}, \lambda_{i,m}]\}_{i=1,\dots,\nu}\}_{m\in\mathbb{N}}$, functions $\{h_m\}_{m\in\mathbb{N}} \subset L^{\frac{2n}{n+2s}}(\mathbb{R}^n)$ and $\{f_m\}_{m\in\mathbb{N}} \subset \dot{H}^s(\mathbb{R}^n)$, and numbers $\{c_{i,m}^a\}_{i=1,\dots,\nu,\, a=1,\dots,n+1,\, m\in\mathbb{N}}$ satisfying (4.9)–(4.10). By (5.3),

$$\sum_{i=1}^{\nu}\sum_{a=1}^{n+1}\left|c_{i,m}^a\right| \to 0 \quad \text{as } m\to\infty. \tag{5.4}$$

Testing (4.10) with f_m and using Hölder's inequality, (1.1), (4.9) and (5.4), we obtain

$$p\int_{\mathbb{R}^n}\sigma_m^{p-1}f_m^2 = \|f_m\|_{\dot{H}^s(\mathbb{R}^n)}^2 + O\left(\|h_m\|_{L^{\frac{2n}{n+2s}}(\mathbb{R}^n)} + \sum_{i=1}^{\nu}\sum_{a=1}^{n+1}\left|c_{i,m}^a\right|\right) \to 1 \quad \text{as } m\to\infty.$$

On the other hand, the argument in the proof of (2.18) demonstrates a contradictory result

$$\lim_{m\to\infty}\int_{\mathbb{R}^n}\sigma_m^{p-1}f_m^2 = 0. \tag{5.5}$$

Indeed, the sequence $\{f_m\}_{m\in\mathbb{N}}$ shares crucial properties of $\{\varrho_m\}_{m\in\mathbb{N}}$ used in the proof of (2.18):

- Each f_m solves an inhomogeneous problem (4.10) whose right-hand side tends to 0 in $\dot{H}^{-s}(\mathbb{R}^n)$ as $m\to\infty$.
- $\|f_m\|_{\dot{H}^s(\mathbb{R}^n)} = 1$ and $f_m \perp Z_{i,m}^a$ in $\dot{H}^s(\mathbb{R}^n)$ for all $m\in\mathbb{N}$, $i = 1,\dots,\nu$, and $a = 1,\dots,n+1$.

These, combined with Lemma A.1 (b), yield (5.5). We omit the details. □

A fixed point argument with Lemma 5.2 and Proposition 5.3 leads to the next result; cf. Proposition 3.4.

Proposition 5.4. *Equation (3.6) has a solution ρ_0 and a family $\{c_i^a\}_{i=1,\dots,\nu,\ a=1,\dots,n+1}$ of numbers such that*

$$\|\rho_0\|_* \le C\mathscr{Q} \quad \text{and} \quad \sum_{i=1}^{\nu}\sum_{a=1}^{n+1}|c_i^a| \le C\mathscr{Q} \tag{5.6}$$

where $C > 0$ depends only on n, s, and ν and $\mathscr{Q} > 0$ is the value in (2.1).

The estimate for c_i^a's in (5.6) results from (5.1)–(5.3), and the fixed point argument.

STEP 2. Set $\rho_1 = \rho - \rho_0$. Then it satisfies (3.12).

Proposition 5.5. *There exists a constant $C > 0$ depending only on n, s, and ν that*

$$\|\rho_1\|_* \leq C\left(\Gamma(u) + \mathscr{Q}^2\right) \tag{5.7}$$

where $\Gamma(u) = \|(-\Delta)^s u - |u|^{p-1}u\|_{\dot{H}^{-s}(\mathbb{R}^n)}$.

Proof. As in the proof of Proposition 3.5, one can adapt the argument in the proof of Lemmas 6.2, 6.3, and Proposition 6.4 in [24], which uses the spectral inequality (2.4).

Compared to the high-dimensional case, we have more terms to treat here, because $n < 6s$ implies that $p > 2$ and so

$$\begin{cases} \left||\sigma+\rho_0+\rho_1|^{p-1}(\sigma+\rho_0+\rho_1) - |\sigma+\rho_0|^{p-1}(\sigma+\rho_0) - p|\sigma+\rho_0|^{p-1}\rho_1\right| \\ \qquad\qquad\qquad\qquad \lesssim |\rho_1|^p + |\sigma+\rho_0|^{p-2}|\rho_1|^2; \\ \left|(\sigma+\rho_0)^{p-1} - \sigma^{p-1}\right| \lesssim |\rho_0|^{p-1} + \sigma^{p-2}|\rho_0|; \\ \left|(\sigma+\rho_0)^{p-1} - U_k^{p-1}\right| \lesssim \displaystyle\sum_{\substack{i=1,\dots,\nu,\\ i\neq k}} U_i^{p-1} + |\rho_0|^{p-1} + U_k^{p-2}\Bigg(\sum_{\substack{i=1,\dots,\nu,\\ i\neq k}} U_i + |\rho_0|\Bigg). \end{cases} \tag{5.8}$$

Fortunately, the additional terms such as $|\sigma+\rho_0|^{p-2}|\rho_1|^2$ and $\sigma^{p-2}|\rho_0|$ in (5.8) can be controlled well, and the bound (3.13) for ρ_1 in Proposition 3.5 keeps unchanged. □

Putting (5.6) and (5.7) together leads

$$\|\rho\|_{\dot{H}^s(\mathbb{R}^n)} \leq \|\rho_0\|_* + \|\rho_1\|_* \leq C\left(\Gamma(u) + \mathscr{Q}\right). \tag{5.9}$$

STEP 3. Thanks to (5.9), we only need to check that $\mathscr{Q} \lesssim \Gamma(u)$ to establish (1.9).

Since $n < 6s$, it holds that $p > 2$. Hence

$$\left||\sigma+\rho|^{p-1}(\sigma+\rho) - \sigma^p - p\,\sigma^{p-1}\rho\right| \lesssim \sigma^{p-2}\rho^2 + |\rho|^p. \tag{5.10}$$

Testing (3.4) with Z_j^{n+1} for any fixed $j = 1,\dots,\nu$, and employing (5.10), (3.16), Hölder's inequality, (1.1), (5.1), (5.6), (5.7), (5.9), and $p > 2$, we observe

$$\begin{aligned} &\left|\int_{\mathbb{R}^n}\left(\sigma^p - \sum_{i=1}^{\nu} U_i^p\right) Z_j^{n+1}\right| \\ &\lesssim \int_{\mathbb{R}^n}\left(\sigma^p - \sum_{u=1}^{\nu} U_i^p\right)|\rho_0| + \int_{\mathbb{R}^n}\sigma^p|\rho_1| + \int_{\mathbb{R}^n}\sigma^{p-1}\rho^2 + \int_{\mathbb{R}^n}|\rho|^p\left|Z_j^{n+1}\right| + \Gamma(u) \\ &\lesssim \Gamma(u) + \mathscr{Q}^2 + \|\rho_1\|_{\dot{H}^s(\mathbb{R}^n)} + \|\rho\|^2_{\dot{H}^s(\mathbb{R}^n)} + \|\rho\|^p_{\dot{H}^s(\mathbb{R}^n)} \lesssim \Gamma(u) + \mathscr{Q}^2. \end{aligned} \tag{5.11}$$

Besides, a suitable modification of the proof of [24, Lemma 2.1] gives (3.19) provided $n < 6s$. During the derivation of (3.19), we need the estimate

$$\begin{aligned} \int_{\mathbb{R}^n} U_i^{p-1}U_jU_k &\lesssim \left(\int_{\mathbb{R}^n} U_i^{\frac{3}{2}(p-1)}U_j^{\frac{3}{2}}\right)^{\frac{1}{3}}\left(\int_{\mathbb{R}^n} U_i^{\frac{3}{2}(p-1)}U_k^{\frac{3}{2}}\right)^{\frac{1}{3}}\left(\int_{\mathbb{R}^n} U_j^{\frac{3}{2}}U_k^{\frac{3}{2}}\right)^{\frac{1}{3}} \\ &\lesssim \sqrt{q_{ij}q_{ik}q_{jk}}\,|\log q_{jk}|^{\frac{1}{3}} \lesssim \mathscr{Q}^{\frac{3}{2}}|\log \mathscr{Q}|^{\frac{1}{3}} = o(\mathscr{Q}), \end{aligned}$$

which holds for any $n < 6s$ and $i, j, k = 1,\dots,\nu$ such that $i \neq j$, $j \neq k$, and $i \neq k$.

As a consequence, we deduce (3.20) from (5.11) and (3.19). The desired inequality $\mathscr{Q} \lesssim \Gamma(u)$ follows from (3.20). This completes the proof of (1.9) for $2s < n < 6s$.

To derive the sharpness of (1.9), one can modify the argument in [12, Section 5.1]. We skip it.

Appendix A. Auxiliary results

A.1. **Non-degeneracy result.** We prove the non-degeneracy of the bubble $U[z,\lambda]$, which is a minor variation of ones in [23, 45]. We believe that it is of practical use elsewhere.

Lemma A.1. *Let $n \in \mathbb{N}$ and $s \in (0, \frac{n}{2})$. Assume that one of the followings hold:*

(a) *$Z \in L^\infty(\mathbb{R}^n)$ solves*
$$Z = \Phi_{n,s} * \left(p\, U[0,1]^{p-1} Z\right) \quad \text{in } \mathbb{R}^n. \tag{A.1}$$

(b) *If $Z \in \dot{H}^s(\mathbb{R}^n)$ solves*
$$(-\Delta)^s Z - p\, U[0,1]^{p-1} Z = 0 \quad \text{in } \mathbb{R}^n. \tag{A.2}$$

Then $Z \in \operatorname{span}\{Z^0[0,1], Z^1[0,1], \dots, Z^{n+1}[0,1]\}$.

Proof. Case (a) was treated in [45, Lemma 5.1]. In the rest of the proof, we will prove that Case (b) can be reduced to Case (a).

If $Z \in \dot{H}^s(\mathbb{R}^n)$ solves (A.2), then (A.1) holds similarly to (4.2). Additionally, one can verify that $Z \in L^\infty(\mathbb{R}^n)$ as follows:

- If $n \ge 6s$, we apply the iteration process in Substep 1 of the proof of Proposition 3.3 to (A.1).
- If $2s < n < 6s$, then the HLS inequality and Hölder's inequality imply
$$\|Z\|_{L^{t^*}(\mathbb{R}^n)} \lesssim \left\| \frac{|Z|}{\langle\,\cdot\,\rangle^{4s}} * \frac{1}{|\cdot|^{n-2s}} \right\|_{L^{t^*}(\mathbb{R}^n)} \lesssim \left\| \frac{|Z|}{\langle\,\cdot\,\rangle^{4s}} \right\|_{L^{\zeta_2}(\mathbb{R}^n)} \lesssim \|Z\|_{L^t(\mathbb{R}^n)} \left\| \frac{1}{\langle\,\cdot\,\rangle^{4s}} \right\|_{L^{\zeta_1}(\mathbb{R}^n)}$$
for $t = \frac{2n}{n-2s}$, $\zeta_1 \in (\frac{n}{2s}, \frac{2n}{6s-n})$, $\zeta_2 = \frac{t\zeta_1}{\zeta_1+t}$, and $t^* = \frac{n\zeta_2}{n-2s\zeta_2} \in (\frac{2n}{n-2s}, \infty)$. This means that $Z \in L^{\tilde{t}}(\mathbb{R}^n)$ for all $\tilde{t} \ge \frac{2n}{n-2s}$. From (4.6), we conclude that $Z \in L^\infty(\mathbb{R}^n)$. □

A.2. **Removability of singularity.** We derive a result on the removability of singularities of a solution to an integral equation, which will be used in the proof of Lemma 4.6.

Lemma A.2. *Suppose that $n \in \mathbb{N}$, $s \in (0, \frac{n}{2})$, $\alpha \in (0,n)$, and $\beta > 2s$. Given any $N \in \mathbb{N}$, let $\mathfrak{y}_1, \dots, \mathfrak{y}_N$ be distinct points in $\mathbb{R}^n$. If f and V are functions such that*
$$\begin{cases} f = \Phi_{n,s} * (Vf) & \text{in } \mathbb{R}^n \setminus \{\mathfrak{y}_1, \dots, \mathfrak{y}_N\}, \\ |f(y)| \le C\left(1 + \displaystyle\sum_{i=1}^N \frac{1}{|y-\mathfrak{y}_i|^\alpha}\right),\ |V(y)| \le \dfrac{C}{\langle y\rangle^\beta} & \text{for } y \in \mathbb{R}^n \setminus \{\mathfrak{y}_1, \dots, \mathfrak{y}_N\} \end{cases} \tag{A.3}$$
for some $C > 0$, then $f \in L^\infty(\mathbb{R}^n)$.

Proof. Let $C^{**} = 1 + \max_{i=1,\dots,N} |\mathfrak{y}_i|$. By (A.3), f is clearly bounded in the set $B(0, 4C^{**})^c$.

Suppose that $y \in B(0,4C^{**}) \setminus \{\mathfrak{y}_1, \dots, \mathfrak{y}_N\}$, it holds that
$$\begin{aligned} |f(y)| &\lesssim \int_{\mathbb{R}^n} \frac{1}{|y-\omega|^{n-2s}} \frac{d\omega}{\langle\omega\rangle^\beta} + \sum_{i=1}^N \int_{B(0,2C^{**})^c} \frac{1}{|y-\omega|^{n-2s}} \frac{d\omega}{|\omega|^{\alpha+\beta}} \\ &\quad + \sum_{i=1}^N \int_{B(0,2C^{**})} \frac{1}{|y-\omega|^{n-2s}} \frac{d\omega}{|\omega-\mathfrak{y}_i|^\alpha} \\ &\lesssim \frac{1 + \log(2+|y|)\mathbf{1}_{\{\beta=n\}}}{\langle y\rangle^{\min\{\beta,n\}-2s}} + \sum_{i=1}^N \int_{B(0,2C^{**})} \frac{1}{|y-\omega|^{n-2s}} \frac{d\omega}{|\omega-\mathfrak{y}_i|^\alpha} \end{aligned} \tag{A.4}$$
where the integrals on the first line were computed as in (4.3).

We shall estimate the rightmost integral in (A.4). Fix any non-negative $\zeta \in (\alpha-2s, \min\{\alpha, n-2s\}]$. By handling the cases $\{\omega \in \mathbb{R}^n : |\omega - y| < \frac{1}{2}|y-\mathfrak{y}_i|\}$, $\{\omega \in \mathbb{R}^n : |\omega - \mathfrak{y}_i| < \frac{1}{2}|y-\mathfrak{y}_i|\}$, and $\{\omega \in \mathbb{R}^n : \min\{|\omega - y|, |\omega-\mathfrak{y}_i|\} \ge \frac{1}{2}|y-\mathfrak{y}_i|\}$ separately, one can derive

$$\frac{1}{|y-\omega|^{n-2s}} \frac{1}{|\omega-\mathfrak{y}_i|^{\alpha}} \le \frac{c}{|y-\mathfrak{y}_i|^{\zeta}} \left(\frac{1}{|\omega-y|^{n-2s+\alpha-\zeta}} + \frac{1}{|\omega-\mathfrak{y}_i|^{n-2s+\alpha-\zeta}} \right) \tag{A.5}$$

where $c > 0$ is determined by n, s, α, and ζ. By (A.5) and the estimate

$$\begin{aligned}\int_{B(0,2C^{**})} \frac{d\omega}{|\omega-\mathfrak{y}_i|^{n-2s+\alpha-\zeta}} &\lesssim \int_{\left\{|\omega-\mathfrak{y}_i|<\frac{|\mathfrak{y}_i|}{2}\right\}} \frac{d\omega}{|\omega-\mathfrak{y}_i|^{n-2s+\alpha-\zeta}} + \frac{1}{|\mathfrak{y}_i|^{n-2s+\alpha-\zeta}} \int_{\left\{|\omega|<\frac{|\mathfrak{y}_i|}{2}\right\}} d\omega \\ &\quad + \int_{\left\{\frac{|\mathfrak{y}_i|}{2} \le |\omega| < 2C^{**}\right\} \cap \left\{|\omega-\mathfrak{y}_i| \ge \frac{|\mathfrak{y}_i|}{2}\right\}} \frac{d\omega}{|\omega|^{n-2s+\alpha-\zeta}} \\ &\lesssim |\mathfrak{y}_i|^{\zeta-(\alpha-2s)} + |\mathfrak{y}_i|^{\zeta-(\alpha-2s)} + (C^{**})^{\zeta-(\alpha-2s)} \lesssim (C^{**})^{\zeta-(\alpha-2s)},\end{aligned}$$

we deduce

$$\int_{B(0,2C^{**})} \frac{1}{|y-\omega|^{n-2s}} \frac{d\omega}{|\omega-\mathfrak{y}_i|^{\alpha}} \lesssim \frac{c(C^{**})^{\zeta-(\alpha-2s)}}{|y-\mathfrak{y}_i|^{\zeta}}$$

provided $|y| < 4C^{**}$.

Therefore,

$$|f(y)| \le C' \left(1 + \sum_{i=1}^{N} \frac{1}{|y-\mathfrak{y}_i|^{\zeta}}\right) \quad \text{for } y \in \mathbb{R}^n \setminus \{\mathfrak{y}_1, \ldots, \mathfrak{y}_N\}$$

where $C' > 0$ is determined by n, s, α, β, ζ, C^{**}, and C in (A.3). Feeding back this information into (A.4), we can iterate the above process until we get $f \in L^{\infty}(\mathbb{R}^n)$. □

Appendix B. Technical computations

Throughout this appendix, we assume that $n > 6s$.

B.1. **Derivation of (4.37).** We recall that

$$f_m(x) = \int_{\mathbb{R}^n} \frac{\gamma_{n,s}}{|x-\omega|^{n-2s}} \left(p\,\sigma_m^{p-1} f_m + h_m + \sum_{i=1}^{\nu} \sum_{a=1}^{n+1} c_{i,m}^a U_{i,m}^{p-1} Z_{i,m}^a \right)(\omega) d\omega \quad \text{for } x \in \mathbb{R}^n.$$

For $s \in (\frac{1}{2}, \frac{n}{2})$, we will prove that

$$\nabla f_m(x) = (2s-n)\gamma_{n,s} \int_{\mathbb{R}^n} \frac{(x-\omega)}{|x-\omega|^{n-2s+2}} \left(p\,\sigma_m^{p-1} f_m + h_m + \sum_{i=1}^{\nu} \sum_{a=1}^{n+1} c_{i,m}^a U_{i,m}^{p-1} Z_{i,m}^a \right)(\omega) d\omega \tag{B.1}$$

for $x \in \mathbb{R}^n$. It suffices to verify that the integral on the right-hand side of (B.1), denoted as $g_m(x)$, is well-behaved in order to apply the Lebesgue dominated convergence theorem.

To analyze the integral, we decompose $\mathbb{R}^n$ into two subsets $\{|y_{i,m}| \le \frac{3}{2}\mathscr{R}_m\}$ and $\{|y_{i,m}| > \frac{3}{2}\mathscr{R}_m\}$. Then one can examine

$$\begin{aligned}&\int_{\mathbb{R}^n} \frac{1}{|x-\omega|^{n-2s+1}} U_{i,m}^{p-1}(\omega) \left(w_{i,m}^{\text{in}} + w_{i,m}^{\text{out}}\right)(\omega) d\omega \\ &\qquad \lesssim \frac{\lambda_{i,m}\left(w_{i,m}^{\text{in}} + w_{i,m}^{\text{out}}\right)(x)}{\langle y_{i,m}\rangle} \left(\frac{1}{\langle y_{i,m}\rangle^{2s}} \mathbf{1}_{\{|y_{i,m}|<\mathscr{R}_m\}} + \frac{\log|y_{i,m}|}{|y_{i,m}|^{2s}} \mathbf{1}_{\{|y_{i,m}|\ge\mathscr{R}_m\}} \right)\end{aligned} \tag{B.2}$$

and
$$\int_{\mathbb{R}^n} \frac{1}{|x-\omega|^{n-2s+1}} \left(v_{i,m}^{\rm in} + v_{i,m}^{\rm out}\right)(\omega) d\omega \lesssim \frac{\lambda_{i,m}}{\langle y_{i,m}\rangle} \left(w_{i,m}^{\rm in} + w_{i,m}^{\rm out}\right)(x). \tag{B.3}$$
By applying (B.2)–(B.3) and (4.18)–(4.23), we observe that for any $M > 1$,
$$\begin{aligned}
&\int_{\mathbb{R}^n} \frac{1}{|x-\omega|^{n-2s+1}} \left(p\,\sigma_m^{p-1} f_m + h_m\right)(\omega) d\omega \\
&\lesssim \sum_{i=1}^{\nu} \frac{\lambda_{i,m} \left(w_{i,m}^{\rm in} + w_{i,m}^{\rm out}\right)(x)}{\langle y_{i,m}\rangle} \Bigg[M^{3n} \left(\frac{1}{\langle y_{i,m}\rangle^{2s}} \mathbf{1}_{\{|y_{i,m}|<\mathscr{R}_m\}} + \frac{\log |y_{i,m}|}{|y_{i,m}|^{2s}} \mathbf{1}_{\{|y_{i,m}|\ge\mathscr{R}_m\}}\right) \\
&\qquad + M^{4s}\mathscr{R}_m^{-2s} + M^{-2s} + \|h_m\|_{**}\Bigg] \\
&\lesssim \sum_{i=1}^{\nu} \frac{\lambda_{i,m}}{\langle y_{i,m}\rangle} \left(w_{i,m}^{\rm in} + w_{i,m}^{\rm out}\right)(x).
\end{aligned}$$
This gives rise to
$$|g_m(x)| \lesssim \sum_{i=1}^{\nu} \frac{\lambda_{i,m}}{\langle y_{i,m}\rangle} \left(w_{i,m}^{\rm in} + w_{i,m}^{\rm out}\right)(x) \quad \text{for } x \in \mathbb{R}^n,$$
since (4.8) implies
$$\begin{aligned}
&\int_{\mathbb{R}^n} \frac{1}{|x-\omega|^{n-2s+1}} \left|\sum_{i=1}^{\nu}\sum_{a=1}^{n+1} c_{i,m}^a (U_{i,m}^{p-1} Z_{i,m}^a)(\omega)\right| d\omega \\
&\qquad \lesssim \left(\|h_m\|_{**}\mathscr{R}_m^{2s-n} + \|f_m\|_{*}\mathscr{R}_m^{-(n+2s)}\right) \sum_{i=1}^{\nu} \frac{\lambda_{i,m}}{\langle y_{i,m}\rangle} U_{i,m}(x).
\end{aligned}$$
Therefore, (B.1) is valid.

Now, considering the relationship that
$$\left|\nabla \hat{f}_m(y)\right| = \lambda_{i_0,m}^{-1} \mathcal{W}_m(x_m)^{-1} |\nabla f_m(x)| \quad \text{for } x = \lambda_{i_0,m}^{-1} y + z_{i_0,m}$$
and $\lambda_{i_0,m}|x_m - z_{i_0,m}| \le L_0$, we find
$$\begin{aligned}
\left|\nabla \hat{f}_m(y)\right| &\lesssim \frac{1}{\lambda_{i_0,m}\mathcal{W}_m(x_m)} \sum_{j=1}^{\nu} \frac{\lambda_{j,m}}{\left\langle \lambda_{j,m}\lambda_{i_0,m}^{-1}(y - z_{i_0 j,m})\right\rangle} \left(w_{j,m}^{\rm in} + w_{j,m}^{\rm out}\right)\left(\lambda_{i_0,m}^{-1} y + z_{i_0,m}\right) \\
&\lesssim L_0^{2s} + L_0^{2s} \sum_{j \prec i_0} \left(\frac{\lambda_{j,m}}{\lambda_{i_0,m}}\right)^{\frac{n-2s}{2}+1} + \sum_{j \succ i_0} \Bigg[\frac{L_0^{2s}}{|y - z_{i_0 j,m}|^{2s+1}} \mathbf{1}_{\left\{\left|\frac{\lambda_{j,m}}{\lambda_{i_0,m}}(y - z_{i_0 j,m})\right| < \mathscr{R}_m\right\}} \\
&\qquad + \frac{L_0^{n-4s}}{|y - z_{i_0 j,m}|^{n-4s+1}} \mathbf{1}_{\left\{\left|\frac{\lambda_{j,m}}{\lambda_{i_0,m}}(y - z_{i_0 j,m})\right| \ge \mathscr{R}_m\right\}}\Bigg] \\
&\quad + \sum_{\{j:\, |z_{i_0 j,m}| \to \infty\}} \Bigg[\frac{\lambda_{j,m} \left\langle \lambda_{j,m}(x_m - z_{j,m})\right\rangle^{2s}}{\lambda_{i_0.m} \left\langle \frac{\lambda_{j,m}}{\lambda_{i_0.m}}(y - z_{i_0 j,m})\right\rangle^{2s+1}} \mathbf{1}_{\left\{\left|\frac{\lambda_{j,m}}{\lambda_{i_0,m}}(y - z_{i_0 j,m})\right| < \mathscr{R}_m\right\}} \\
&\qquad + \frac{\lambda_{j,m} |\lambda_{j,m}(x_m - z_{j,m})|^{n-4s}}{\lambda_{i_0,m} \left|\frac{\lambda_{j,m}}{\lambda_{i_0.m}}(y - z_{i_0 j,m})\right|^{n-4s+1}} \mathbf{1}_{\left\{\left|\frac{\lambda_{j,m}}{\lambda_{i_0,m}}(y - z_{i_0 j,m})\right| \ge \mathscr{R}_m\right\}}\Bigg] \\
&\lesssim L_0^{n-4s}
\end{aligned}$$

for $y \in \overline{B'}$ and $m \in \mathbb{N}$ large, where $\overline{B'}$ is any compact ball in $\mathbb{R}^n \setminus \widetilde{\mathcal{Z}}_\infty$. This concludes the proof of (4.37).

Remark B.1. Let γ be an integer such that $\gamma \in [1, \lfloor 2s \rfloor]$ for $s \in (\frac{1}{2}, \frac{n}{2}) \setminus \frac{1}{2}\mathbb{N}$ and $\gamma \in [1, 2s-1]$ for $s \in (\frac{1}{2}, \frac{n}{2}) \cap \frac{1}{2}\mathbb{N}$. The previous argument reveals that

$$\begin{aligned} \left|\nabla^\gamma f_m(x)\right| &\lesssim \left| \int_{\mathbb{R}^n} \frac{1}{|x-\omega|^{n-2s+\gamma}} \left(p\,\sigma_m^{p-1} f_m + h_m + \sum_{i=1}^{\nu} \sum_{a=1}^{n+1} c_{i,m}^a U_{i,m}^{p-1} Z_{i,m}^a \right)(\omega) \mathrm{d}\omega \right| \\ &\lesssim \sum_{i=1}^{\nu} \left(\frac{\lambda_{i,m}}{\langle y_{i,m} \rangle} \right)^{\gamma} \left(w_{i,m}^{\text{in}} + w_{i,m}^{\text{out}} \right)(x) \quad \text{for } x \in \mathbb{R}^n, \end{aligned}$$

since $n - 2s + \gamma < n$. Consequently, we have

$$\|\hat{f}_m\|_{C^\gamma(\overline{B'})} \lesssim L_0^{n-4s} \quad \text{for } m \in \mathbb{N} \text{ large.}$$

B.2. **Derivation of (4.40).** Let B' be a bounded open ball in $\mathbb{R}^n \setminus \widetilde{\mathcal{Z}}_\infty$. We choose any $y \in B'$, which will be fixed throughout this subsection. It follows directly from (4.35) that

$$\frac{\mathcal{W}_m\left(\lambda_{i_0,m}^{-1} y + z_{i_0,m}\right)}{\mathcal{W}_m(x_m)} \lesssim L_0^{n-4s} \tag{B.4}$$

for $m \in \mathbb{N}$ large. Arguing as in [24, Lemma 4.7], one has

$$U_{j,m}\left(\lambda_{i_0,m}^{-1} y + z_{i_0,m}\right) = o(1) U_{i_0,m}\left(\lambda_{i_0,m}^{-1} y + z_{i_0,m}\right) \quad \text{for } j \notin \mathcal{D}(i_0) \text{ and } y \in B'. \tag{B.5}$$

Recalling (4.31), we infer from (4.16), (B.4), (4.15), (4.8), (4.9), and (B.5) that

$$\begin{aligned} &\int_{\mathbb{R}^n} \frac{1}{|y-\omega|^{n-2s}} \mathcal{H}_m(\omega) \mathrm{d}\omega \\ &\quad \lesssim \|h_m\|_{**} \frac{\mathcal{W}_m\left(\lambda_{i_0,m}^{-1} y + z_{i_0,m}\right)}{\mathcal{W}_m(x_m)} + \frac{1}{\mathcal{W}_m(x_m)} \sum_{j=1}^{\nu} \sum_{a=1}^{n+1} |c_{j,m}^a| U_{j,m}\left(\lambda_{i_0,m}^{-1} y + z_{i_0,m}\right) \to 0 \end{aligned} \tag{B.6}$$

as $m \to \infty$.

In the following, we will justify the equality

$$\begin{aligned} &\lim_{m \to \infty} \int_{\mathbb{R}^n} \frac{1}{|y-\omega|^{n-2s}} \left\{ \lambda_{i_0,m}^{-\frac{n-2s}{2}} \sigma_m\left(\lambda_{i_0,m}^{-1} \omega + z_{i_0,m}\right) \right\}^{p-1} \hat{f}_m(\omega) \mathrm{d}\omega \\ &= \int_{\mathbb{R}^n} \frac{1}{|y-\omega|^{n-2s}} \left(U[0,1]^{p-1} \hat{f}_\infty \right)(\omega) \mathrm{d}\omega \end{aligned} \tag{B.7}$$

for each $y \in B'$. Indeed, if it is true, (4.40) will be an immediate consequence of (4.32), (4.39), (B.6), and (B.7).

Given any $M > 4C^*$ large and $\epsilon \in (0,1)$ small, we decompose $\mathbb{R}^n$ into

$$\begin{aligned} \mathbb{R}^n &= \left(\cup_{i \in \mathcal{D}(i_0)} B(z_{i_0 i,\infty}, \epsilon)\right) \bigcup \left[B(0,M) \setminus \left(\cup_{i \in \mathcal{D}(i_0)} B(z_{i_0 i,\infty}, \epsilon)\right) \right] \bigcup B(0,M)^c \\ &=: \Omega_1 \cup \Omega_2 \cup \Omega_3. \end{aligned}$$

We set

$$\begin{aligned} I_{1,m}(y) &:= \int_{\mathbb{R}^n} \frac{1}{|y-\omega|^{n-2s}} \left(U[0,1]^{p-1}\hat{f}_m\right)(\omega)\mathrm{d}\omega \\ &= \int_{\Omega_1} \cdots + \int_{\Omega_2} \cdots + \int_{\Omega_3} \cdots =: I_{11,m}(y) + I_{12,m}(y) + I_{13,m}(y), \\ I_{2,m}(y) &:= \int_{\mathbb{R}^n} \frac{1}{|y-\omega|^{n-2s}} \left[\left\{\left(\lambda_{i_0,m}^{-\frac{n-2s}{2}}\sigma_m\big(\lambda_{i_0,m}^{-1}\cdot+z_{i_0,m}\big)\right)^{p-1} - U[0,1]^{p-1}\right\}\hat{f}_m\right](\omega)\mathrm{d}\omega \\ &= \int_{\Omega_1} \cdots + \int_{\Omega_2} \cdots + \int_{\Omega_3} \cdots =: I_{21,m}(y) + I_{22,m}(y) + I_{23,m}(y). \end{aligned} \tag{B.8}$$

It is sufficient to analyze integrals $I_{11,m}, \ldots, I_{23,m}$ separately.

STEP 1. We take $l > \max\{M, \epsilon^{-1}\}$. Following Claim 1 in the proof of [24, Lemma 5.1], one can find

$$\lambda_{i_0,m}^{-\frac{n-2s}{2}}\sigma_m\big(\lambda_{i_0,m}^{-1}\cdot+z_{i_0,m}\big) \to U[0,1] \quad \text{in } L^\infty(\Omega_2) \quad \text{as } m \to \infty$$

using the fact that $\Omega_2 \subset \mathcal{K}_l$. Moreover, owing to (4.39), $\hat{f}_m \to \hat{f}_\infty$ uniformly in Ω_2 as $m \to \infty$, so

$$\begin{cases} I_{12,m}(y) \to \displaystyle\int_{\Omega_2} \frac{1}{|y-\omega|^{n-2s}} \left(U[0,1]^{p-1}\hat{f}_\infty\right)(\omega)\mathrm{d}\omega, \\ I_{22,m}(y) \to 0 \end{cases} \qquad \text{as } m \to \infty \tag{B.9}$$

for each fixed $y \in B'$.

Let us estimate $I_{11,m}(y)$. By using (4.33) and (4.34), we see that

$$\left|\hat{f}_m(\omega)\right| \lesssim L_0^{2s} + \frac{1}{\mathcal{W}_m(x_m)} \sum_{j\in\mathcal{D}(i_0)} \left(w_{j,m}^{\text{in}} + w_{j,m}^{\text{out}}\right)\left(\lambda_{i_0,m}^{-1}\omega + z_{i_0,m}\right)$$

for $\omega \in \Omega_1$ and $m \in \mathbb{N}$ large. Fix any $i \in \mathcal{D}(i_0)$. If $|y - z_{i_0 i,\infty}| \le 2\epsilon$, then

$$\int_{B(z_{i_0 i,\infty},\epsilon)} \frac{1}{|y-\omega|^{n-2s}} U[0,1]^{p-1}(\omega)\mathrm{d}\omega \lesssim \int_{B(y,3\epsilon)} \frac{1}{|y-\omega|^{n-2s}}\mathrm{d}\omega \simeq \epsilon^{2s}.$$

If $|y - z_{i_0 i,\infty}| \ge 2\epsilon$, then

$$\int_{B(z_{i_0 i,\infty},\epsilon)} \frac{1}{|y-\omega|^{n-2s}} U[0,1]^{p-1}(\omega)\mathrm{d}\omega \lesssim \epsilon^{2s-n} \int_{B(0,\epsilon)} \mathrm{d}\omega \simeq \epsilon^{2s}.$$

From (4.20), (4.22), (4.16), and (B.4), we know

$$\begin{aligned} &\frac{1}{\mathcal{W}_m(x_m)} \sum_{j\in\mathcal{D}(i_0)} \int_{B(z_{i_0 j,\infty},\epsilon)} \frac{1}{|y-\omega|^{n-2s}} U[0,1]^{p-1}(\omega)\left(w_{j,m}^{\text{in}} + w_{j,m}^{\text{out}}\right)\left(\lambda_{i_0,m}^{-1}\omega + z_{i_0,m}\right)\mathrm{d}\omega \\ &\lesssim \frac{1}{\lambda_{i_0}^{2s}\mathcal{W}_m(x_m)} \sum_{j\in\mathcal{D}(i_0)} \int_{B(z_{i_0 j,m},2\epsilon)} \frac{1}{|y-\omega|^{n-2s}} \Bigg[\mathscr{R}_m^{-2s} v_{j,m}^{\text{in}}\big(\lambda_{i_0,m}^{-1}\omega + z_{i_0,m}\big) \\ &\qquad\qquad + \left\{\left(\frac{\lambda_{i_0,m}}{\lambda_{j,m}}\right)^{2s} + \epsilon^{2s}\right\} v_{j,m}^{\text{out}}\big(\lambda_{i_0,m}^{-1}\omega + z_{i_0,m}\big)\Bigg]\mathrm{d}\omega \\ &\lesssim \frac{1}{\mathcal{W}_m(x_m)} \sum_{j\in\mathcal{D}(i_0)} \left[\mathscr{R}_m^{-2s} + \left(\frac{\lambda_{i_0,m}}{\lambda_{j,m}}\right)^{2s} + \epsilon^{2s}\right]\left(w_{j,m}^{\text{in}} + w_{j,m}^{\text{out}}\right)\left(\lambda_{i_0,m}^{-1}y + z_{i_0,m}\right) \end{aligned}$$

$$\lesssim L_0^{n-4s} \sum_{j\in\mathcal{D}(i_0)} \left[\mathscr{R}_m^{-2s} + \left(\frac{\lambda_{i_0,m}}{\lambda_{j,m}}\right)^{2s} + \epsilon^{2s}\right] \simeq L_0^{n-4s}\epsilon^{2s} + o(1)$$

for $m \in \mathbb{N}$ large. In addition,

$$\begin{aligned}
&\frac{1}{\mathcal{W}_m(x_m)} \sum_{\substack{i,\,j\in\mathcal{D}(i_0),\\ z_{i_0i,\infty}\neq z_{i_0j,\infty}}} \int_{B(z_{i_0i,\infty},\epsilon)} \frac{1}{|y-\omega|^{n-2s}} U[0,1]^{p-1}(\omega)\left(w_{j,m}^{\text{in}} + w_{j,m}^{\text{out}}\right)(\lambda_{i_0,m}^{-1}\omega + z_{i_0,m})\mathrm{d}\omega \\
&\lesssim \sum_{\substack{i,\,j\in\mathcal{D}(i_0),\\ z_{i_0i,\infty}\neq z_{i_0j,\infty}}} \int_{B(z_{i_0i,\infty},\epsilon)} \frac{1}{|y-\omega|^{n-2s}} U[0,1]^{p-1}(\omega)\left[\frac{L_0^{2s}}{|\omega - z_{i_0j,m}|^{2s}} + \frac{L_0^{n-4s}}{|\omega - z_{i_0j,m}|^{n-4s}}\right]\mathrm{d}\omega \\
&\lesssim L_0^{n-4s} \sum_{i\in\mathcal{D}(i_0)} \int_{B(z_{i_0i,\infty},\epsilon)} \frac{1}{|y-\omega|^{n-2s}} U[0,1]^{p-1}(\omega)\mathrm{d}\omega \lesssim L_0^{n-4s}\epsilon^{2s}
\end{aligned}$$

where we choose ϵ so small that $\epsilon < \frac{1}{2}\min\{|z_{i_0i,\infty} - z_{i_0j,\infty}| : i,j \in \mathcal{D}(i_0) \text{ and } z_{i_0i,\infty} \neq z_{i_0j,\infty}\}$ for the second inequality. Therefore,

$$|I_{11,m}(y)| \lesssim L_0^{n-4s}\epsilon^{2s} + o(1) \quad \text{for } y \in B' \text{ and } m \in \mathbb{N} \text{ large.} \tag{B.10}$$

We next turn to the integral $I_{13,m}(y)$. If $\omega \in \Omega_3$, or equivalently, $|\omega| \geq M$, then

$$\begin{aligned}
\left|\hat{f}_m(\omega)\right| &\lesssim \left(L_0^{2s}M^{-2s} + L_0^{n-4s}M^{-(n-4s)}\right) + L_0^{2s} \\
&\quad + \frac{1}{\mathcal{W}_m(x_m)} \sum_{\{j:\,|z_{i_0j,m}|\to\infty\}} \left(w_{j,m}^{\text{in}} + w_{j,m}^{\text{out}}\right)(\lambda_{i_0,m}^{-1}\omega + z_{i_0,m})
\end{aligned} \tag{B.11}$$

for $m \in \mathbb{N}$ large. On one hand, it holds that

$$\int_{\Omega_3} \frac{1}{|y-\omega|^{n-2s}} U[0,1]^{p-1}(\omega)\mathrm{d}\omega \lesssim M^{-s}\int_{\Omega_3} \frac{1}{|y-\omega|^{n-2s}}\frac{1}{1+|\omega|^{3s}}\mathrm{d}\omega \lesssim \frac{M^{-s}}{1+|y|^s} \lesssim M^{-s}.$$

On the other hand, by (4.18)–(4.21), (4.16), (4.33), and (4.34),

$$\begin{aligned}
&\frac{1}{\mathcal{W}_m(x_m)} \sum_{\{j:\,|z_{i_0j,m}|\to\infty\}} \int_{\Omega_3} \frac{1}{|y-\omega|^{n-2s}} U[0,1]^{p-1}(\omega)\left(w_{j,m}^{\text{in}} + w_{j,m}^{\text{out}}\right)(\lambda_{i_0,m}^{-1}\omega + z_{i_0,m})\mathrm{d}\omega \\
&\lesssim \frac{1}{\lambda_{i_0}^{2s}\mathcal{W}_m(x_m)} \sum_{\{j:\,|z_{i_0j,m}|\to\infty\}} \int_{\Omega_3} \frac{\mathscr{R}_m^{-2s} + \langle z_{i_0j,m}\rangle^{-2s}}{|y-\omega|^{n-2s}} \left(v_{i_0,m}^{\text{in}} + v_{i_0,m}^{\text{out}} + v_{j,m}^{\text{in}} + v_{j,m}^{\text{out}}\right)(\lambda_{i_0,m}^{-1}\omega + z_{i_0,m})\mathrm{d}\omega \\
&\lesssim \left(\mathscr{R}_m^{-2s} + \sum_{\{j:\,|z_{i_0j,m}|\to\infty\}} \langle z_{i_0j,m}\rangle^{-2s}\right) \\
&\qquad \times \frac{1}{\mathcal{W}_m(x_m)}\left[w_{i_0,m}^{\text{in}} + w_{i_0,m}^{\text{out}} + \sum_{\{j:\,|z_{i_0j,m}|\to\infty\}} \left(w_{j,m}^{\text{in}} + w_{j,m}^{\text{out}}\right)\right](\lambda_{i_0,m}^{-1}y + z_{i_0,m}) \\
&\lesssim L_0^{2s}\left(\mathscr{R}_m^{-2s} + \sum_{\{j:\,|z_{i_0j,m}|\to\infty\}} \langle z_{i_0j,m}\rangle^{-2s}\right) = o(1).
\end{aligned}$$

Summing up, we discover

$$|I_{13,m}(y)| \lesssim L_0^{2s}M^{-s} + L_0^{n-4s}M^{-(n-3s)} + o(1) \quad \text{for } y \in B' \text{ and } m \in \mathbb{N} \text{ large.} \tag{B.12}$$

Furthermore, by employing the pointwise convergence

$$\hat{f}_m(\omega) \to \hat{f}_\infty(\omega) \quad \text{for } \omega \in \mathbb{R}^n \setminus \widetilde{\mathcal{Z}}_\infty \quad \text{as } m \to \infty,$$

and Fatou's Lemma, we deduce

$$\int_{\Omega_1} \frac{1}{|y-\omega|^{n-2s}} \left(U[0,1]^{p-1}|\hat{f}_\infty|\right)(\omega)\mathrm{d}\omega \le \liminf_{m\to\infty} \int_{\Omega_1} \frac{1}{|y-\omega|^{n-2s}} \left(U[0,1]^{p-1}|\hat{f}_m|\right)(\omega)\mathrm{d}\omega \lesssim L_0^{n-4s}\epsilon^{2s} \tag{B.13}$$

and

$$\int_{\Omega_3} \frac{1}{|y-\omega|^{n-2s}} \left(U[0,1]^{p-1}|\hat{f}_\infty|\right)(\omega)\mathrm{d}\omega \le \liminf_{m\to\infty} \int_{\Omega_3} \frac{1}{|y-\omega|^{n-2s}} \left(U[0,1]^{p-1}|\hat{f}_m|\right)(\omega)\mathrm{d}\omega = L_0^{2s}M^{-s} + L_0^{n-4s}M^{-(n-3s)}. \tag{B.14}$$

By collecting (B.9), (B.10), (B.12), and (B.13)–(B.14), we conclude

$$I_{1,m}(y) = \int_{\mathbb{R}^n} \frac{1}{|y-\omega|^{n-2s}} \left(U[0,1]^{p-1}\hat{f}_\infty\right)(\omega)\mathrm{d}\omega + O\left(L_0^{n-4s}\epsilon^{2s} + L_0^{2s}M^{-s} + L_0^{n-4s}M^{-(n-3s)}\right) + o(1) \tag{B.15}$$

for each $y \in B'$ and $m \in \mathbb{N}$ large.

STEP 2. We evaluate $I_{2,m}(y)$ for $y \in B'$. Direct computations similar to the previous step yield

$$\frac{1}{\mathcal{W}_m(x_m)} \int_{\Omega_1} \frac{1}{|y-\omega|^{n-2s}} \left[\left(\sum_{i\prec i_0} \lambda_{i_0,m}^{-\frac{n-2s}{2}} U_{i,m}\right)^{p-1} \left(w_{i_0,m}^{\text{in}} + w_{i_0,m}^{\text{out}}\right)\right](\lambda_{i_0,m}^{-1}\omega + z_{i_0,m})\mathrm{d}\omega$$
$$\lesssim L_0^{2s} \sum_{i\prec i_0} \left(\frac{\lambda_{i,m}}{\lambda_{i_0,m}}\right)^{2s} \int_{\Omega_1} \frac{1}{|y-\omega|^{n-2s}}\mathrm{d}\omega \lesssim L_0^{2s} \sum_{i\prec i_0} \left(\frac{\lambda_{i,m}}{\lambda_{i_0,m}}\right)^{2s} \epsilon^{2s} \lesssim o(1) L_0^{2s}\epsilon^{2s}$$
$$\left(\text{since } \left(w_{i_0,m}^{\text{in}} + w_{i_0,m}^{\text{out}}\right)(\lambda_{i_0,m}^{-1}\omega + z_{i_0,m}) \lesssim L_0^{2s}\mathcal{W}_m(x_m) \text{ for } \omega \in \Omega_1\right),$$

$$\frac{1}{\mathcal{W}_m(x_m)} \int_{\Omega_1} \frac{1}{|y-\omega|^{n-2s}} \left[\left(\sum_{i\succ i_0} \lambda_{i_0,m}^{-\frac{n-2s}{2}} U_{i,m}\right)^{p-1} \left(w_{i_0,m}^{\text{in}} + w_{i_0,m}^{\text{out}}\right)\right](\lambda_{i_0,m}^{-1}\omega + z_{i_0,m})\mathrm{d}\omega$$
$$\lesssim L_0^{n-4s}\mathscr{R}_m^{-2s} = o(1) \quad (\text{by } (4.18), (4.19), (4.16), \text{ and } (B.4)),$$

$$\frac{1}{\mathcal{W}_m(x_m)} \int_{\Omega_1} \frac{1}{|y-\omega|^{n-2s}} \left[\left(\sum_{\{i:\, |z_{i_0 i,m}|\to\infty\}} \lambda_{i_0,m}^{-\frac{n-2s}{2}} U_{i,m}\right)^{p-1} \left(w_{i_0,m}^{\text{in}} + w_{i_0,m}^{\text{out}}\right)\right](\lambda_{i_0,m}^{-1}\omega + z_{i_0,m})\mathrm{d}\omega$$
$$\lesssim L_0^{2s}\left(\mathscr{R}_m^{-2s} + \sum_{\{i:\, |z_{i_0 i,m}|\to\infty\}} \langle z_{i_0 i,m}\rangle^{-2s}\right) = o(1) \quad (\text{by } (4.18)\text{–}(4.21), (4.16), (4.33), \text{ and } (4.34)),$$

and

$$\frac{1}{\mathcal{W}_m(x_m)} \int_{\Omega_3} \frac{1}{|y-\omega|^{n-2s}} \left[\left(\sum_{j\ne i_0} \lambda_{i_0,m}^{-\frac{n-2s}{2}} U_{j,m}\right)^{p-1} \left(w_{i_0,m}^{\text{in}} + w_{i_0,m}^{\text{out}}\right)\right](\lambda_{i_0,m}^{-1}\omega + z_{i_0,m})\mathrm{d}\omega$$
$$\lesssim \left(L_0^{2s}M^{-2s} + L_0^{n-4s}M^{-(n-4s)}\right) \int_{\mathbb{R}^n} \frac{1}{|y-\omega|^{n-2s}} \left(\sum_{j\ne i_0} \lambda_{i_0,m}^{-\frac{n-2s}{2}} U_{j,m}\right)^{p-1} (\lambda_{i_0,m}^{-1}\omega + z_{i_0,m})\mathrm{d}\omega$$
$$\lesssim \left(L_0^{2s}M^{-2s} + L_0^{n-4s}M^{-(n-4s)}\right) \sum_{j\ne i_0} \frac{1}{\left\langle \frac{\lambda_{j,m}}{\lambda_{i_0,m}}(y - z_{i_0 j,m})\right\rangle^{2s}} \lesssim L_0^{2s}M^{-2s} + L_0^{n-4s}M^{-(n-4s)}$$

for $m \in \mathbb{N}$ large. Also, considering (4.18)–(4.23), (B.4), (4.33), and (4.34), we can derive, as in (4.17), that

$$\begin{aligned}
&\frac{1}{\mathcal{W}_m(x_m)}\int_{\Omega_1\cup\Omega_3}\frac{1}{|y-\omega|^{n-2s}}\left[\left(\sum_{j\neq i_0}\lambda_{i_0,m}^{-\frac{n-2s}{2}}U_{j,m}\right)^{p-1}\sum_{i\neq i_0}\left(w_{i,m}^{\text{in}}+w_{i,m}^{\text{out}}\right)\right](\lambda_{i_0,m}^{-1}\omega+z_{i_0,m})\mathrm{d}\omega \\
&\lesssim \frac{1}{\mathcal{W}_m(x_m)}\sum_{i\neq i_0}\left(w_{i,m}^{\text{in}}+w_{i,m}^{\text{out}}\right)(\lambda_{i_0,m}^{-1}y+z_{i_0,m})\cdot\Big[M_1^{4s}\mathscr{R}_m^{-2s}+M_1^{-2s} \\
&+M_1^{3n}\left(\frac{1}{\left\langle\frac{\lambda_{i,m}}{\lambda_{i_0,m}}(y-z_{i_0i,m})\right\rangle^{2s}}\mathbf{1}_{\left\{\left|\frac{\lambda_{i,m}}{\lambda_{i_0,m}}(y-z_{i_0i,m})\right|<\mathscr{R}_m\right\}}+\frac{\log\left|\frac{\lambda_{i,m}}{\lambda_{i_0,m}}(y-z_{i_0i,m})\right|}{\left|\frac{\lambda_{i,m}}{\lambda_{i_0,m}}(y-z_{i_0i,m})\right|^{2s}}\mathbf{1}_{\left\{\left|\frac{\lambda_{i,m}}{\lambda_{i_0,m}}(y-z_{i_0i,m})\right|\geq\mathscr{R}_m\right\}}\right)\Bigg] \\
&\lesssim L_0^{n-4s}\left(M_1^{4s}\mathscr{R}_m^{-2s}+M_1^{-2s}\right)+\frac{o(1)M_1^{3n}}{\mathcal{W}_m(x_m)}\left(w_{i_0,m}^{\text{in}}+w_{i_0,m}^{\text{out}}\right)(\lambda_{i_0,m}^{-1}y+z_{i_0,m}) \\
&+M_1^{3n}\sum_{i\in\mathcal{D}(i_0)}\left[\frac{L_0^{2s}}{\left\langle\frac{\lambda_{i,m}}{\lambda_{i_0,m}}(y-z_{i_0i,m})\right\rangle^{2s}}\mathbf{1}_{\left\{\left|\frac{\lambda_{i,m}}{\lambda_{i_0,m}}(y-z_{i_0i,m})\right|<\mathscr{R}_m\right\}}\right. \\
&\qquad\qquad\left.+\frac{L_0^{n-4s}\log\left|\frac{\lambda_{i,m}}{\lambda_{i_0,m}}(y-z_{i_0i,m})\right|}{\left|\frac{\lambda_{i,m}}{\lambda_{i_0,m}}(y-z_{i_0i,m})\right|^{2s}}\mathbf{1}_{\left\{\left|\frac{\lambda_{i,m}}{\lambda_{i_0,m}}(y-z_{i_0i,m})\right|\geq\mathscr{R}_m\right\}}\right] \\
&\lesssim L_0^{n-4s}\left(M_1^{4s}\mathscr{R}_m^{-2s}+M_1^{-2s}\right)+o(1)L_0^{2s}M_1^{3n}+M_1^{3n}\sum_{i\in\mathcal{D}(i_0)}\left[L_0^{2s}\left(\frac{\lambda_{i,m}}{\lambda_{i_0,m}}\right)^{-2s}+L_0^{n-4s}\left(\frac{\lambda_{i,m}}{\lambda_{i_0,m}}\right)^{-2s}\left|\log\frac{\lambda_{i,m}}{\lambda_{i_0,m}}\right|\right] \\
&\lesssim L_0^{n-4s}M_1^{-2s}+o(1)L_0^{n-4s}M_1^{3n}
\end{aligned}$$

for any $M_1 > 1$ and $m \in \mathbb{N}$ large. Hence, with (B.9), we have proven that

$$|I_{2,m}(y)| \lesssim o(1)L_0^{2s}\epsilon^{2s}+o(1)+L_0^{2s}M^{-2s}+L_0^{n-4s}M^{-(n-4s)}+L_0^{n-4s}M_1^{-2s}+o(1)L_0^{n-4s}M_1^{3n} \tag{B.16}$$

for $y \in B'$ and $m \in \mathbb{N}$ large.

By gathering (B.8), (B.15), and (B.16), selecting M, $M_1 > 0$ sufficiently large and $\epsilon > 0$ small, and then taking $m \to \infty$, we establish (B.7).

B.3. **Derivation of (4.41).** We will adopt the strategy in Appendix B.2.

First, because $\hat{f}_m \to \hat{f}_\infty$ uniformly in Ω_2 as $m \to \infty$, we have

$$\int_{\Omega_2} U[0,1]^{p-1}Z^a[0,1]\hat{f}_m\mathrm{d}y \to \int_{\Omega_2} U[0,1]^{p-1}Z^a[0,1]\hat{f}_\infty\mathrm{d}y \quad \text{as } m\to\infty.$$

Also, in light of (4.35), we find that

$$\begin{aligned}
&\int_{\Omega_1}\left|U[0,1]^{p-1}Z^a[0,1]\hat{f}_m\right|\mathrm{d}y \\
&\lesssim \int_{\Omega_1}U[0,1]^p\left[L_0^{2s}+\sum_{j\in\mathcal{D}(i_0)}\left(\frac{L_0^{2s}}{|y-z_{i_0j,m}|^{2s}}+\frac{L_0^{n-4s}}{|y-z_{i_0j,m}|^{n-4s}}\right)\right]\mathrm{d}y \\
&\lesssim L_0^{n-4s}\left[\int_{\Omega_1}dy+\sum_{j\in\mathcal{D}(i_0)}\int_{B(z_{i_0j,\infty},\epsilon)}\left(\frac{1}{|y-z_{i_0j,m}|^{2s}}+\frac{1}{|y-z_{i_0j,m}|^{n-4s}}\right)\mathrm{d}y\right. \\
&\qquad\left.+\sum_{\substack{i,j\in\mathcal{D}(i_0),\\ z_{i_0i,\infty}\neq z_{i_0j,\infty}}}\int_{B(z_{i_0i,\infty},\epsilon)}\left(\frac{1}{|y-z_{i_0j,m}|^{2s}}+\frac{1}{|y-z_{i_0j,m}|^{n-4s}}\right)\mathrm{d}y\right]\lesssim L_0^{n-4s}\epsilon^{4s}
\end{aligned}$$

for $\epsilon > 0$ small and $m \in \mathbb{N}$ large.

Next, by carrying out computations with the help of (B.11), we obtain

$$\begin{aligned}
&\int_{\Omega_3} \left| U[0,1]^{p-1} Z^a[0,1] \hat{f}_m \right| \mathrm{d}y \\
&\lesssim \int_{\Omega_3} U[0,1]^p(y) \Bigg[\left(L_0^{2s} M^{-2s} + L_0^{n-4s} M^{-(n-4s)} \right) + L_0^{2s} \\
&\qquad\qquad + \sum_{\{j :\, |z_{i_0 j,m}| \to \infty\}} \left(\frac{|z_{i_0 j,m}|^{2s}}{|y - z_{i_0 j,m}|^{2s}} + \frac{|z_{i_0 j,m}|^{n-4s}}{|y - z_{i_0 j,m}|^{n-4s}} \right) \Bigg] \mathrm{d}y \\
&\lesssim \left(L_0^{2s} + L_0^{n-4s} M^{-(n-4s)} \right) M^{-2s} \\
&\quad + \sum_{\{j :\, |z_{i_0 j,m}| \to \infty\}} \int_{B\left(z_{i_0 j,m}, \frac{|z_{i_0 j,m}|}{2}\right)} \left[\frac{1}{|z_{i_0 j,m}|^n} \frac{1}{|y - z_{i_0 j,m}|^{2s}} + \frac{1}{|z_{i_0 j,m}|^{6s}} \frac{1}{|y - z_{i_0 j,m}|^{n-4s}} \right] \mathrm{d}y \\
&\quad + \sum_{\{j :\, |z_{i_0 j,m}| \to \infty\}} \int_{B\left(z_{i_0 j,m}, \frac{|z_{i_0 j,m}|}{2}\right)^c \cap \Omega_3} U[0,1]^p(y) \left[\frac{|z_{i_0 j,m}|^{2s}}{|y - z_{i_0 j,m}|^{2s}} + \frac{|z_{i_0 j,m}|^{n-4s}}{|y - z_{i_0 j,m}|^{n-4s}} \right] \mathrm{d}y \\
&\lesssim \left(L_0^{2s} + L_0^{n-4s} M^{-(n-4s)} \right) M^{-2s} + \sum_{\{j :\, |z_{i_0 j,m}| \to \infty\}} |z_{i_0 j,m}|^{-2s} + M^{-2s}
\end{aligned}$$

for $m \in \mathbb{N}$ large.

By making use of Fatou's Lemma, we easily get

$$\int_{\Omega_1} \left| U[0,1]^{p-1} Z^a[0,1] \hat{f}_\infty \right| \mathrm{d}y \lesssim L_0^{n-4s} \epsilon^{4s}$$

and

$$\int_{\Omega_3} \left| U[0,1]^{p-1} Z^a[0,1] \hat{f}_\infty \right| \mathrm{d}y \lesssim L_0^{2s} M^{-2s} + L_0^{n-4s} M^{-(n-2s)}.$$

All the information above and the second equality of (4.32) present

$$0 = \lim_{m \to \infty} \int_{\mathbb{R}^n} U[0,1]^{p-1} Z^a[0,1] \hat{f}_m \mathrm{d}y = \int_{\mathbb{R}^n} U[0,1]^{p-1} Z^a[0,1] \hat{f}_\infty \mathrm{d}y.$$

The proof of (4.41) is completed.

Remark B.2. This proof essentially gives

$$\left\| \langle \cdot \rangle^{-(n+2s)} \hat{f}_m \right\|_{L^1(\mathbb{R}^n)} \lesssim L_0^{n-4s},$$

which is necessary to deduce (4.38).

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