COMPACTNESS OF THE Q-CURVATURE PROBLEM

LIUWEI GONG, SEUNGHYEOK KIM, AND JUNCHENG WEI

ABSTRACT. We provide a complete resolution to the question of the C^4-compactness for the solution set of the constant Q-curvature problem on a smooth closed Riemannian manifold of dimension $5 \le n \le 24$. For dimensions $n \ge 25$, an example of an L^∞-unbounded sequence of solutions for $n \ge 25$ has been known for over a decade (Wei-Zhao [43]). On the other hand, Li-Xiong [28] established compactness in dimensions $5 \le n \le 9$ (under some conditions). Our principal observation is that the linearized equation associated with the Q-curvature problem can be transformed into an overdetermined linear system, which admits a nontrivial solution due to an unexpected algebraic structure of the Paneitz operator. This key insight plays a crucial role in extending the compactness result to higher dimensions.

1. Introduction

Let (M, g) be a smooth closed Riemannian manifold of dimension $n \ge 3$, Ric_g the Ricci curvature tensor on (M, g), R_g the scalar curvature, and $A_g = \frac{1}{n-2}(\text{Ric}_g - \frac{1}{2(n-1)} R_g g)$ the Schouten tensor. Then the Branson's Q-curvature Q_g and the Paneitz operator P_g are defined as

$$Q_g = -\Delta_g \sigma_1(A_g) + 4\sigma_2(A_g) + \frac{n-4}{2}\sigma_1(A_g)^2$$

and

$$P_g u = \Delta_g^2 u + \text{div}_g \{(4A_g - (n-2)\sigma_1(A_g)g)(\nabla u, \cdot)\} + \frac{n-4}{2} Q_g u \quad \text{for } u \in C^4(M),$$

where $\sigma_k(A_g)$ denotes the k-th symmetric function of the eigenvalues of A_g and $\Delta_g = \text{div}_g \nabla_g$ is the Laplace-Beltrami operator.

The Q-curvature problem refers to the fourth-order elliptic equation

$$P_g u + 2Q_g = 2\lambda e^{4u}, \ u > 0 \quad \text{on } M \tag{1.1}$$

for $n = 4$, and

$$P_g u = \lambda u^{\frac{n+4}{n-4}}, \ u > 0 \quad \text{on } M \tag{1.2}$$

for $n = 3$ or $n \ge 5$, where $\lambda \in \mathbb{R}$ is a constant whose sign is determined by the conformal structure of (M, g).

If $n = 4$, the existence theory of (1.1) was developed by Chang and Yang [9], Djadli and Malchiodi [10], and Li, Li, and Liu [27], among others. For the existence result for (1.2) in $n = 3$, refer to Hang and Yang [17] and references therein.

Since our main focus in this paper is on the case $n \ge 5$, we provide a more detailed description on the development of the existence theory under this setting: For $n \ge 5$, the first existence result on (1.2) with $\lambda > 0$ was achieved by Qing and Raske [36]. By appealing to the work of Schoen and Yau [42] to lift the metric to a domain in the sphere via the developing map, they proved that (1.2) admits a solution if (M, g) is locally conformally flat, its Yamabe invariant Y_g is positive, and its Poincaré exponent is less than $\frac{n-4}{2}$. Later, Gursky and Malchiodi [15] established the

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existence result under the conditions that $R_g \geq 0$ and $Q_g \geq 0$ on M, and $Q_g > 0$ somewhere, which in particular imply $\lambda > 0$. They achieved this through the study of the maximum principle for P_g and a sequential convergence of a non-local flow as $t \to \infty$. The condition $R_g \geq 0$ was relaxed by Hang and Yang [18], who substituted it with the requirement that $Y_g > 0$. They deduced this by solving a maximization problem associated with a nonlinear integral equation equivalent to (1.2). Refer to also [14], where Gursky, Hang, and Lin proved that the existence of a conformal metric g with R_g, $Q_g > 0$ on M is equivalent to the positivity of both Y_g and P_g provided $n \geq 6$. Recently, several results concerning the multiplicity of solutions to equation (1.2) have appeared under specific settings: See [5, 1, 21]. This leads us to investigate the properties of the solution set itself.

In this work, we examine the $C^4(M)$-compactness of the entire solution set of (1.2) with $\lambda > 0$. Establishing this compactness property will enable us to calculate the Leray-Schauder degree for $C^4(M)$-bounded subsets of the solution set or to develop Morse theory, as already accomplished in the Yamabe case [22]. Aforementioned work of Qing and Raske [36] obtained the compactness result under their setting. Moreover, assuming $5 \leq n \leq 9$, $\mathrm{Ker}P_g = \{0\}$, the positivity of the Green's function G_g of P_g, and the validity of the positive mass theorem for P_g, Y.Y. Li and Xiong [28] proved that the solution set is $C^4(M)$-compact; we refer also to the work of G. Li [26] that studied when $5 \leq n \leq 7$. On the other hand, Wei and Zhao [43] constructed a smooth Riemannian metric g on the unit n-sphere $\mathbb{S}^n$ with $n \geq 25$, which is not conformally diffeomorphic to the standard metric on $\mathbb{S}^n$, such that the solution set of (1.2) is $L^\infty(\mathbb{S}^n)$-unbounded (and so $C^4(\mathbb{S}^n)$-noncompact). The case $10 \leq n \leq 24$ remains unsolved. In this paper, we establish the compactness result for these dimensions, thereby providing the conclusive answer to the question of the $C^4(M)$-compactness of the full solution set of the Q-curvature problem (1.2) on (M, g) with $\lambda > 0$ and $Y_g > 0$.

A simple proof of the C^4-compactness of the solution set of (1.2) with $\lambda < 0$ was given by Y.Y. Li and Xiong [28, Theorem 1.3].

In what follows, we write $\lambda = \mathfrak{c}(n) := (n-4)(n-2)n(n+2) > 0$ so that (1.2) is rewritten as

$$P_g u = \mathfrak{c}(n) u^{\frac{n+4}{n-4}}, \ u > 0 \quad \text{on } M. \tag{1.3}$$

Our main theorem is stated as follows.

Theorem 1.1. *Let (M, g) be a smooth closed Riemannian manifold of dimension $5 \leq n \leq 24$ and not conformally diffeomorphic to the standard unit n-sphere. Assume either*

(i) $\mathrm{Ker}P_g = \{0\}$*, the positivity of the Green's function G_g of the Paneitz operator P_g, and the validity of the positive mass theorem for P_g; or*

(ii) *$Q_g \geq 0$ on M, $Q_g > 0$ somewhere, and $Y_g > 0$.*

Then there exists a constant $C = C(M, g) > 0$ such that any solution $u \in C^4(M)$ of equation (1.3) satisfies

$$\|u\|_{C^4(M)} + \|u^{-1}\|_{C^4(M)} \leq C.$$

Remark 1.2. Four remarks concerning Theorem 1.1 are in order.

1. The geometric assumptions in the aforementioned existence results [36, 15, 18] imply $\mathrm{Ker}P_g = \{0\}$ and the positivity of G_g. By [16], if $Y_g > 0$, then the existence of positive Q-curvature in the conformal class $[g]$ is equivalent to $\mathrm{Ker}P_g = \{0\}$ together with the positivity of G_g.

2. We say that the positive mass theorem for P_g holds if the constant-order term A in the expansion (8.2) of G_g is positive. In Proposition 8.3, we will establish this positivity under condition (ii) (first introduced by Hang and Yang [18]) by applying the positive mass theorem of Avalos, Laurain, and Lira [4]. Subsection 2.3 contains the precise statement of their theorem and the definition of the higher-order mass. The condition $Y_g > 0$ is required solely to make sure

the validity of the positive mass theorem. An interesting question is whether this condition can be removed or replaced with alternative conditions, such as spin conditions.
3. Let $\epsilon > 0$ be an arbitrarily chosen, sufficiently small number. A slight modification of the proof of Theorem 1.1 shows that the set $\{u \in C^4(M) \mid P_g u = \mathfrak{c}(n)u^p,\ u > 0 \text{ on } M,\ 1+\epsilon \le p \le \frac{n+4}{n-4}\}$ remains $C^4(M)$-compact under the assumptions of Theorem 1.1. See [28, Theorem 1.1].
4. Remarkably, the threshold dimension 24 for the compactness of the Q-curvature problem coincides with that of the Yamabe problem. We remind that the $C^2(M)$-compactness of the Yamabe problem for $n \le 24$ was established through the successive works of Li and Zhu [31], Druet [11], Li and Zhang [29, 30], Marques [33], and Khuri, Marques, and Schoen [22]. Furthermore, the existence of a metric g on $\mathbb{S}^n$ and an $L^\infty(\mathbb{S}^n)$-unbounded sequence of solutions to the Yamabe problem on $(\mathbb{S}^n, g)$ for $n \ge 25$ was demonstrated by Brendle [7] and Brendle and Marques [8]. Recently, similar techniques have been used to study sign-changing solutions of the Yamabe equation by Premoselli and Vétois [35, Section 3].

Consequently, we may pose the following questions:

- Why does the critical dimension 24 arise in the compactness results for both the Yamabe problem and the Q-curvature problem? What is the relationship between these phenomena?
- What is the threshold dimensions of the higher-order Q-curvature problems involving the GJMS operators, recently studied by Mazumdar and Vétois [34] and Robert [38]? ⋄

Remark 1.3. Escobar [12] introduced a version of the boundary Yamabe problem on a smooth compact manifold $(\overline{X}^n, g)$ with boundary M^{n-1}, generalizing of the Riemann mapping theorem. In [2], Almaraz constructed an $L^\infty(\overline{X})$-unbounded example for $n \ge 25$. This finding suggests that the threshold dimension for the $C^2(\overline{X})$-compactness of the boundary Yamabe problem is likely 24, reaffirming its critical nature. Compactness results have so far been established only up to dimension 5 [3, 25], primarily due to serious challenges in the linear theory. A similar issue occurs in the Q-curvature problem, which we resolve in this paper. Therefore, it is natural to ask whether our approach can also be applied to the boundary Yamabe problem.

The boundary Yamabe problem on $(\overline{X}^{n+1}, g)$ can be viewed as a fractional Yamabe problem involving the half-Laplacian $(-\Delta)^{1/2}$ on M^n. Since the fractional Yamabe problem is defined on M, we redefine n to represent the boundary dimension. In particular, Almaraz's non-compactness example in [2] now holds for $n \ge 24$. In view of the non-compactness example of Brendle and Marques [8] for the Yamabe problem existing for $n \ge 25$, the critical dimension for the fractional Yamabe problems involving $(-\Delta)^\gamma$ must shift at some $\gamma^* \in (\frac{1}{2}, 1)$. In [23], Kim, Musso, and Wei identified $\gamma^* \simeq 0.940197$. The investigation of $C^2(M)$-compactness for the fractional Yamabe problem is still in its infancy, with only the works [37, 24] currently addressing this topic. ⋄

An essential step to proving Theorem 1.1 is the establishment of the following Weyl vanishing theorem for the Q-curvature problem (1.3).

Theorem 1.4. *Let g be a smooth Riemannian metric in the unit n-ball B_1^n with $8 \le n \le 24$, and $\{u_a\}_{a\in\mathbb{N}} \subset C^4(B_1^n)$ a sequence of solutions of equation (1.3) with $M = B_1^n$. Suppose that for each $\epsilon > 0$, there is a constant $C = C(\epsilon) > 0$ such that $\sup_{B_1 \setminus B_\epsilon} u_a \le C(\epsilon)$ and $\lim_{a\to\infty} \sup_{B_1} u_a = \infty$. Then the Weyl tensor W_g satisfies*

$$|W_g|(x) \le C|x|^l, \quad x \in B_1^n$$

for some integer $l > \frac{n-8}{2}$.

Remark 1.5. Thanks to Theorem 1.4, our compactness theorem also holds for $8 \le n \le 24$ without requiring the positive mass theorem for the Paneitz operator to be valid provided

$$\sum_{i=0}^{\lfloor \frac{n-8}{2} \rfloor} \left|\nabla_g^i W_g\right| > 0 \quad \text{on } M$$

where $\lfloor \frac{n-8}{2} \rfloor$ is the greatest integer that does not exceed $\frac{n-8}{2}$. ◇

In the proof of Theorems 1.1 and 1.4, we assume $n \ge 8$ unless otherwise stated. The proof of Theorem 1.1 for $5 \le n \le 7$ does not require the Weyl vanishing theorem as can be seen in [26, 28].

Organizations of the paper and novelties. We will adapt Schoen's strategy for the compactness problem of the Yamabe problem described in [39, 41] and further developed in [31, 11, 29, 30, 33, 22].

In Section 2, we present some relevant background, including the bubbles, the local Pohozaev identity, and the positive mass theorem for the Paneitz operator obtained in [4].

In Section 3, we employ Brendle's approach in [6] to expand the Ricci curvature tensors, the scalar curvature, the Q-curvature, and the Paneitz operator.

In Section 4, we introduce inhomogeneous linear problems and explicitly solve them in rational form, which are indispensable for constructing sharp approximations of blowing-up sequences of solutions in a neighborhood of each blowup point. We emphasize that this step constitutes one of our main contributions, as such problems generally do not admit explicit rational solutions unless the inhomogeneous term exhibits special properties (see Remark 4.3). Notably, the algebraic structure of the Paneitz operator ensures that the inhomogeneous term is indeed special, facilitating the existence of these explicit rational solutions. A geometric interpretation of the explicit solvability phenomenon is provided in Appendix C.

In Section 5, we build sharp approximations of blowing-up sequences and estimate their pointwise error.

In Section 6, we establish the Weyl vanishing theorem (Theorem 6.1) by defining and analyzing the Pohozaev quadratic form. The analysis requires a full understanding of the eigenspaces of the operator $\mathcal{L}_k$ defined in (6.11), which highlights an additional distinction between the Yamabe and Q-curvature problems. Refer to the paragraphs following Corollary 6.7.

In Sections 7 and 8, we prove the non-negativity of a local Pohozaev term and the positive mass theorem for P_g under Condition (ii) of Theorem 1.1, and then complete the proof of Theorems 1.1 and 1.4.

In Appendix A, we provide elementary yet useful tools for the proof of Theorems 1.1 and 1.4.

In Appendix B, we conduct technical computations needed for the proof of Theorem 6.1. We note that verifying the positivity of the matrices $(m_{qq'}^{D,s})$, $(m_{qq'}^{W,s})$, and $(m_{qq'}^{H,s})$ defined in Lemmas B.11–B.13 is too tedious to perform manually. Therefore, we carried out these calculations using Mathematica. We included the Mathematica code as the ancillary files with our arXiv submission.

In proving Theorem 1.1, we will omit several proofs concerning the blowup analysis or quantitative estimates, because the necessary arguments, with some modifications, can be derived from [22, 28]. We remark that the authors of [28] have laid the analytic foundations, resolving the technical issues stemming from the fourth-order structure of equation (1.3).

Notations. We list the notations used in the introduction and the rest of the paper.

- For a function $u \in C^4(M)$, let $\mathcal{E}_g(u) = P_g u - \Delta_g^2 u$.
- $\mathbb{S}^{n-1}$ is the $(n-1)$-dimensional unit sphere in the n-dimensional Euclidean space $\mathbb{R}^n$, g_{std} is its standard metric, and $|\mathbb{S}^{n-1}|$ is the canonical surface area of $(\mathbb{S}^{n-1}, g_{\text{std}})$.
- $\mathcal{P}_k$ denotes the space of homogeneous polynomial on $\mathbb{R}^n$ of degree k.

- $\mathcal{H}_k$ denotes the space of homogeneous harmonic polynomial on $\mathbb{R}^n$ of degree k, that is,

$$\mathcal{H}_k := \{P \in \mathcal{P}_k \mid \Delta P = 0\}.$$

- Constants: $\mathfrak{c}(n) := (n-4)(n-2)n(n+2)$ and $\tilde{\mathfrak{c}}(n) := (n-2)n(n+2)(n+4)$.
- Constant: $c_n := 2(n-2)(n-4)|\mathbb{S}^{n-1}|$.
- Constants: $\alpha_n := \frac{4+(n-2)^2}{2(n-1)(n-2)}$, $\beta_n := \frac{n-6}{2(n-1)}$, and $\gamma_n := \frac{n-4}{4(n-1)}$.
- Constants: $K := n-6$ and $d := \lfloor \frac{n-4}{2} \rfloor$.
- For i, j integers such that $i - 2j < -1$, we set

$$\mathcal{I}_j^i := \int_0^\infty \frac{r^i}{(1+r^2)^j} dr. \tag{1.4}$$

- Given $\sigma, \sigma_1, \sigma_2 \in (M, g)$ and $R > 0$, $B_R^g(\sigma)$ is the open geodesic ball of radius R centered at σ and $d_g(\sigma_1, \sigma_2)$ is the geodesic distance between σ_1 and σ_2. If (M, g) is the Euclidean space $\mathbb{R}^n$, we write $B_R^n(\sigma) = B_R^g(\sigma)$ and $B_R = B_R^n(0) = B^n(0, R)$.
- dv_g and dS_g denote the volume form and the surface measure, respectively. If (M, g) is the standard unit sphere $(\mathbb{S}^n, g_{\text{std}})$, we write $dS = dS_g$.
- δ_{ij} is the Kronecker delta, that is, $\delta_{ij} = 1$ if $i = j$ and 0 if $i \neq j$.
- Repeated indices adhere to the summation convention, and a comma denotes partial differentiation. Also, the empty sum is zero.
- Let $m \in \mathbb{N}$. For an m-tensor A with components A_α, where α is a multi-index satisfying $|\alpha| = m$, we write $A_\alpha A_\alpha = (A_\alpha)^2$.
- For a function $f \in C^4(B_1)$ and $m \in \mathbb{R}$, $f(x) = O^{(4)}(|x|^m)$ denotes that

$$|\nabla^i f|(x) = O(|x|^{m-i}), \quad i = 0, \dots, 4.$$

2. Preliminaries

Throughout this section, we assume that $n \geq 5$.

2.1. Bubbles. Given parameters $\lambda > 0$ and $\xi \in \mathbb{R}^n$, we define a bubble by

$$w_{\lambda,\xi}(x) = \left(\frac{\lambda}{\lambda^2 + |x-\xi|^2}\right)^{\frac{n-4}{2}} \quad \text{for } x \in \mathbb{R}^n.$$

Then the set $\{w_{\lambda,\xi} : \lambda > 0, \xi \in \mathbb{R}^n\}$ constitutes the set of all solutions to

$$\Delta^2 u = \mathfrak{c}(n) u^{\frac{n+4}{n-4}},\ u > 0 \quad \text{in } \mathbb{R}^n.$$

For $\xi = (\xi^1, \dots, \xi^n) \in \mathbb{R}^n$, we define

$$Z_{\lambda,\xi}^0 = -\frac{\partial w_{\lambda,\xi}}{\partial \lambda} \quad \text{and} \quad Z_{\lambda,\xi}^i = \frac{\partial w_{\lambda,\xi}}{\partial \xi^i} \quad \text{for } i = 1, \dots, n.$$

In [32, Theorem 2.2], Lu and Wei proved that the solution space of the linear problem

$$L(\Psi) := \Delta^2 \Psi - \tilde{\mathfrak{c}}(n) w_{1,0}^{\frac{8}{n-4}} \Psi = 0 \quad \text{in } \mathbb{R}^n, \quad \Psi \in \dot{H}^2(\mathbb{R}^n)$$

is spanned by $Z_{1,0}^0$, $Z_{1,0}^1, \dots,\ Z_{1,0}^n$. Here, $\tilde{\mathfrak{c}}(n) = (n-2)n(n+2)(n+4)$.

For simplicity, we will write $w = w_{1,0}$ and $Z^i = Z_{1,0}^i$ for $i = 0, \dots, n$.

2.2. **Pohozaev identity.** Let $u : B_R \to \mathbb{R}$ be a positive C^4 function satisfying $P_g u = \mathfrak{c}(n) u^{\frac{n+4}{n-4}}$ in B_R. For $0 < r < R$, we define

$$\mathbf{P}(r,u) := \int_{\partial B_r} \left[\frac{n-4}{2}(\Delta u \partial_r u - u \partial_r(\Delta u)) - \frac{r}{2}|\Delta u|^2 - r\partial_r u \partial_r(\Delta u) + \Delta u \partial_r(r\partial_r u) \right] dS, \quad (2.1)$$

where $\partial_r := \frac{x_i}{r}\partial_{x_i}$. Then a standard argument yields

$$\mathbf{P}(r,u) = \int_{B_r} \left(r\partial_r u + \frac{n-4}{2} u \right) (\mathcal{E}_g + \Delta_g^2 - \Delta^2)(u)\, dx - \frac{\mathfrak{c}(n)(n-4)r}{2n} \int_{\partial B_r} u^{\frac{2n}{n-4}} dS. \quad (2.2)$$

2.3. **Positive mass theorem.** Recall that an asymptotically flat manifold $(M^n, \hat{g})$ (with one end and decay rate $\tau > 0$) is defined by $M = M_0 \cup M_\infty$, where M_0 is compact and M_∞ is diffeomorphic to $\mathbb{R}^n \setminus \bar{B}_1^n$ such that $g_{ij}(y) - \delta_{ij} = O^{(4)}(|y|^{-\tau})$ as $|y| \to \infty$. Moreover, if $\tau > \frac{n-4}{2}$ and $Q_g \in L^1(M, \hat{g})$, we can define a higher-order mass

$$m(\hat{g}) := \lim_{r\to\infty} \int_{\{|y|=r\}} (\hat{g}_{ii,jjk} - \hat{g}_{ij,ijk}) \frac{y_k}{r} dS. \quad (2.3)$$

In [4, Theorem A], the following result was proved.

Theorem 2.1. *Assume that $(M^n, \hat{g})$ is an asymptotically flat manifold with decay rate $\tau > \frac{n-4}{2}$ satisfying*

(i) $n \geq 5$;
(ii) $Y_{\hat{g}} > 0$, $Q_{\hat{g}} \in L^1(M, \hat{g})$, *and* $Q_{\hat{g}} \geq 0$.

Then the higher-order mass $m(\hat{g})$ is nonnegative. Moreover, $m(\hat{g}) = 0$ if and only if $(M, \hat{g})$ is isometric to the Euclidean space $\mathbb{R}^n$.

Prior to [4], the positive mass theorem for manifolds of dimensions $5 \leq n \leq 7$ and for locally conformally flat manifolds was proved in [20, 15, 16].

2.4. **Properties of blowup sequences.** We assume that $\{g_a\}_{a\in\mathbb{N}} \subset [g]$ is a sequence of metrics on M converging to a metric $g_\infty \in [g]$ in $C^l(M)$ as $a \to \infty$ for any $l \in \mathbb{N}$, and $\{u_a\}_{a\in\mathbb{N}} \subset C^4(M)$ is a sequence of solutions of

$$P_{g_a} u_a = \mathfrak{c}(n) u_a^{\frac{n+4}{n-4}}, \ u_a > 0 \quad \text{on } M. \quad (2.4)$$

Definition 2.2.
1. A point $\bar{\sigma} \in M$ is called a blowup point for $\{u_a\}$ if $u_a(\sigma_a) \to \infty$ as $a \to \infty$ for some sequence $\{\sigma_a\} \subset M$ such that $\sigma_a \to \bar{\sigma}$.
2. A blowup point $\bar{\sigma} \in M$ is called isolated if there exist constants R, $C > 0$ such that

$$u_a(\sigma) \leq C d_{g_a}(\sigma, \sigma_a)^{-\frac{n-4}{2}} \quad \text{for all } y \in B_R^{g_a}(\sigma_a).$$

3. An isolated blowup point $\bar{\sigma} \in M$ is called simple if the map $r \mapsto r^{\frac{n-4}{2}} \bar{u}_a(r)$ has only one critical point in the interval $(0, R)$, where

$$\bar{u}_a(r) := |\partial B_r^{g_a}(\sigma_a)|_{g_a}^{-1} \int_{\partial B_r^{g_a}(\sigma_a)} u_a dS_{g_a}.$$

To specify the points $\{\sigma_a\}$, we will use the expression that $\sigma_a \to \bar{\sigma} \in M$ is a blowup point for $\{u_a\}$.

Fix any $\sigma_0 \in M$ and $\delta > 0$ small. By applying the representation formula for u_a, one can rewrite (2.4) as

$$u_a(\sigma) = \mathfrak{c}(n) \int_{B_\delta^{g_a}(\sigma_0)} G_{g_a}(\sigma, \tau) u_a^{\frac{n+4}{n-4}}(\tau) dv_{g_a} + \mathfrak{h}_a(\sigma) \quad \text{for } \sigma \in B_\delta^{g_a}(\sigma_0).$$

Here, $\{\mathfrak{h}_a\}_{a\in\mathbb{N}}$ is a sequence of smooth positive functions on $B^{g_a}_{\delta}(\sigma_0)$ such that for any $r \in (0, \frac{\delta}{4})$,

$$\sup_{B^{g_a}_r(\sigma)} \mathfrak{h}_a \le C \inf_{B^{g_a}_r(\sigma)} \mathfrak{h}_a \quad \text{and} \quad \sum_{m=1}^{5} r^m |\nabla^m \mathfrak{h}_a(\sigma)| \le C\|\mathfrak{h}_a\|_{L^\infty(B^{g_a}_r(\sigma))}, \quad \sigma \in B^{g_a}_{\delta/2}(\sigma_0), \tag{2.5}$$

where the constant $C > 0$ depends only on n and (M, g_∞). Building upon this observation, the proofs of the following propositions were provided in [28, Section 4].

Proposition 2.3. *Let $\sigma_a \to \bar{\sigma} \in M$ be an isolated blowup point for a sequence $\{u_a\}$ of solutions to* (2.4)*, and $\epsilon_a = u_a(\sigma_a)^{-\frac{2}{n-4}} > 0$. Suppose that $\{R_\ell\}_{\ell\in\mathbb{N}}$ and $\{\tau_\ell\}_{\ell\in\mathbb{N}}$ are arbitrary sequences of positive numbers such that $R_\ell \to \infty$ and $\tau_\ell \to 0$ as $\ell \to \infty$. Then $\{u_a\}$ has a subsequence $\{u_{a_\ell}\}_{\ell\in\mathbb{N}}$ such that*

$$\left\| \epsilon_{a_\ell}^{\frac{n-4}{2}} u_{a_\ell}(\epsilon_{a_\ell} \cdot) - w \right\|_{C^3(B_{R_{a_\ell}})} \le \tau_\ell \tag{2.6}$$

in g_{a_ℓ}-normal coordinates centered at σ_{a_ℓ}, and $R_\ell \epsilon_{a_\ell} \to 0$ as $\ell \to \infty$. Here, $w = w_{1,0}$.

Thus, we can select $\{R_\ell\}_{\ell\in\mathbb{N}}$ and $\{u_{a_\ell}\}_{\ell\in\mathbb{N}}$ satisfying (2.6) and $R_\ell\epsilon_{a_\ell} \to 0$. To simplify notations, we will write $\{u_a\}_{a\in\mathbb{N}}$ instead of $\{u_{a_\ell}\}_{\ell\in\mathbb{N}}$, and so on.

Proposition 2.4. *If $\sigma_a \to \bar{\sigma} \in M$ is an isolated simple blowup point for a sequence $\{u_a\}$ of solutions to* (2.4)*, then there exist constants δ_0, $C > 0$ such that*

$$u_a(\sigma_a)u_a(\sigma) \le C d_{g_a}(\sigma, \sigma_a)^{4-n} \quad \textit{for all } 0 < d_{g_a}(\sigma, \sigma_a) < \delta_0. \tag{2.7}$$

Moreover, $u_a(\sigma_a)u_a \to \mathfrak{g} := c_n G_{g_\infty}(\cdot, \bar{\sigma}) + \mathfrak{h}$ in $C^3_{\mathrm{loc}}(B^{g_\infty}_{\delta_0}(\bar{\sigma}) \setminus \{\bar{\sigma}\})$, where $c_n = 2(n-2)(n-4)|\mathbb{S}^{n-1}|$, G_{g_∞} is the Green's function of the Paneitz operator P_{g_∞}, and $\mathfrak{h} \in C^5(B^{g_\infty}_{\delta_0}(\bar{\sigma}))$ is a nonnegative function satisfying (2.5)*.*

3. Expansions of Curvatures

Let σ_0 be an arbitrarily fixed point on M. By a conformal change, one can assume that

$$\det g(x) = 1 \quad \text{for } x \in B^n(0, \delta),$$

where $x = (x_1, \ldots, x_n)$ are normal coordinates centered at σ_0 and $\delta > 0$ is a small number determined by (M, g).

Adopting the idea of Brendle [6] (see also Khuri, Marques, and Schoen [22]), we consider the metric $g = \exp(h)$, where h is a symmetric 2-tensor whose norm $|h|$ is small. Then we have the following expansion:

$$g^{ij} = \delta_{ij} - h_{ij} + \frac{1}{2} h_{ik}h_{kj} + O(|h|^3).$$

Because of $\det g(x) = 1$ and Gauss's lemma, h has two properties:

$$h_{ii}(x) = 0 \quad \text{and} \quad x_i h_{ij}(x) = 0. \tag{3.1}$$

A direct and useful corollary is

$$x_i h_{ij,k}(x) = \partial_k(x_i h_{ij}(x)) - \delta_{ki}h_{ij} = -h_{jk}, \tag{3.2}$$

where $\partial_k := \partial_{x_k}$.

The proofs of Lemma 3.1–Lemma 3.4 are straightforward, relying only on $\mathrm{tr}(h) = 0$ and the definition of the curvatures. We skip them.

Lemma 3.1. *Let $g = \exp(h)$, $\mathrm{tr}(h) = 0$, and Ric_g be the Ricci curvature tensor on (M, g). Then*

$$\mathrm{Ric}_g = \dot{\mathrm{Ric}}[h] + \ddot{\mathrm{Ric}}[h,h] + O(|h|^2|\partial^2 h| + |h||\partial h|^2),$$

where

$$\begin{aligned} 2(\dot{\mathrm{Ric}}[h])_{ij} &:= h_{jk,ik} + h_{ik,jk} - h_{ij,kk}; \\ 4(\ddot{\mathrm{Ric}}[h,h])_{ij} &:= h_{lk,l}(2h_{ij,k} - h_{kj,i} - h_{ki,j}) \\ &\quad + h_{lk,i}h_{kj,l} + h_{lk,j}h_{ki,l} - h_{kl,i}h_{kl,j} - 2h_{ik,l}h_{jl,k} \\ &\quad + h_{kl}(2h_{ij,kl} - h_{ki,lj} - h_{kj,il}) \\ &\quad + h_{ki}h_{kl,lj} + h_{kj}h_{kl,li} - h_{ki}h_{kj,ll} - h_{kj}h_{ki,ll}. \end{aligned} \tag{3.3}$$

Note that each term in $\ddot{\mathrm{Ric}}[h,h]$ involves two factors of h. For instance, the term $h_{lk,l}(2h_{ij,k} - h_{kj,i} - h_{ki,j})$ includes one factor of h in $h_{lk,l}$ and another in $2h_{ij,k} - h_{kj,i} - h_{ki,j}$. Accordingly, the bracket $[h,h]$ in $\ddot{\mathrm{Ric}}[h,h]$ indicates these two distinct appearance of h. One can define

$$\ddot{\mathrm{Ric}}[h,h'] = \frac{1}{2}\left(\ddot{\mathrm{Ric}}[h+h', h+h'] - \ddot{\mathrm{Ric}}[h,h] - \ddot{\mathrm{Ric}}[h',h']\right) \tag{3.4}$$

for two symmetric 2-tensors h, h' so that $\ddot{\mathrm{Ric}}[h,h']$ is symmetric with respect to h and h'. If no ambiguity arises, we will simply write $\dot{\mathrm{Ric}} = \dot{\mathrm{Ric}}[h]$ and $\ddot{\mathrm{Ric}} = \ddot{\mathrm{Ric}}[h,h]$. The same rule will apply below.

Lemma 3.2. *Let $g = \exp(h)$, $\mathrm{tr}(h) = 0$, and R_g be the scalar curvature on (M, g). Then*

$$R_g = \dot{R}[h] + \ddot{R}[h,h] + O(|h|^2|\partial^2 h| + |h||\partial h|^2),$$

where

$$\dot{R}[h] := h_{ij,ij}; \tag{3.5}$$

$$\ddot{R}[h,h] := -\partial_i(h_{ij}h_{kj,k}) + \frac{1}{2}h_{ij,i}h_{kj,k} - \frac{1}{4}(h_{jk,i})^2. \tag{3.6}$$

Corollary 3.3. *Under the same assumption, we have*

$$\begin{aligned} \dot{R} &= \dot{\mathrm{Ric}}_{ii}; \\ \ddot{R} &= \ddot{\mathrm{Ric}}_{ii} - h_{ij}\dot{\mathrm{Ric}}_{ij}. \end{aligned}$$

Lemma 3.4. *Let $g = \exp(h)$, $\mathrm{tr}(h) = 0$, and Q_g be the Q-curvature on (M, g). Then*

$$Q_g = \dot{Q}[h] + \ddot{Q}[h,h] + O(|h|^2|\partial^4 h| + |h||\partial h||\partial^3 h| + |h||\partial^2 h|^2 + |\partial h|^2|\partial^2 h|),$$

where

$$\begin{aligned} \dot{Q}[h] &:= -\frac{1}{2(n-1)}\Delta\dot{R}[h]; \\ \ddot{Q}[h,h] &:= -\frac{1}{2(n-1)}(\Delta\ddot{R}[h,h] - \partial_i(h_{ij}\partial_j\dot{R}[h])) \\ &\quad + \frac{n^3 - 4n^2 + 16n - 16}{8(n-1)^2(n-2)^2}(\dot{R}[h])^2 - \frac{2}{(n-2)^2}(\dot{\mathrm{Ric}}_{ij}[h])^2. \end{aligned}$$

Lemma 3.5. *Let $g = \exp(h)$, $\mathrm{tr}(h) = 0$, and P_g be the Paneitz operator on (M, g). Then*

$$\begin{aligned} \mathcal{E}_g := P_g - \Delta_g^2 = L_1[h] + L_2[h,h] &+ O([|h|^2|\partial^2 h| + |h||\partial h|^2]\partial^2) \\ &+ O([|h|^2|\partial^3 h| + |h||\partial h||\partial^2 h| + |\partial h|^3]\partial) \\ &+ O(|h|^2|\partial^4 h| + |h||\partial h||\partial^3 h| + |h||\partial^2 h|^2 + |\partial h|^2|\partial^2 h|), \end{aligned}$$

where $L_1[h]$ and $L_2[h,h]$ are differential operators defined by

$$L_1[h] := \frac{4}{n-2}(\dot{\mathrm{Ric}}[h])_{ij}\partial_{ij} - \alpha_n \dot{R}[h]\Delta - \beta_n \partial_k \dot{R}[h]\partial_k + \frac{n-4}{2}\dot{Q}[h]; \tag{3.7}$$

$$\begin{aligned} L_2[h,h] := &\left[\frac{4}{n-2}(\ddot{\mathrm{Ric}}[h,h])_{ij} - \frac{8}{n-2}(\dot{\mathrm{Ric}}[h])_{ik}h_{kj} + \alpha_n \dot{R}[h]h_{ij}\right]\partial_{ij} - \alpha_n \ddot{R}[h,h]\Delta \\ &+ \Bigg[-\frac{2}{n-2}(\dot{\mathrm{Ric}}[h])_{ij}(h_{kj,i}+h_{ki,j}-h_{ij,k}) + \alpha_n \dot{R}[h]h_{ik,i} \\ &\qquad - \beta_n(\partial_k \ddot{R}[h,h] - \partial_i \dot{R}[h]h_{ik})\Bigg]\partial_k + \frac{n-4}{2}\ddot{Q}[h,h]. \end{aligned}$$

Here, $\alpha_n = \frac{4+(n-2)^2}{2(n-1)(n-2)}$ *and* $\beta_n = \frac{n-6}{2(n-1)}$.

Proof. By [15, Lemma 2.8], we know that

$$\mathcal{E}_g = \frac{4}{n-2}g^{ik}g^{jl}(\mathrm{Ric}_g)_{ij}\nabla_k\nabla_l - \alpha_n R_g \Delta_g - \beta_n g^{ij}\partial_i R_g \partial_j + \frac{n-4}{2}Q_g.$$

Then, owing to the expansion of g, $\mathrm{tr}(h) = 0$, and the definitions of the Hessian tensor and Δ_g, we have

$$\begin{aligned} \nabla_k\nabla_l &= \partial_{kl} - \frac{1}{2}(h_{ki,l}+h_{li,k}-h_{kl,i})\partial_i + O(|h|^2\partial^2 + |h||\partial h|\partial + |\partial^2 h| + |\partial h|^2), \\ \Delta_g &= \Delta - (h_{ij}\partial_{ij} + h_{ij,j}\partial_i) + O(|h|^2\partial^2 + |h||\partial h|\partial + |\partial^2 h| + |\partial h|^2). \end{aligned}$$

Combining these relations with Lemmas 3.1–3.4, we obtain the conclusion. □

Lemma 3.6. *Let u be radial, namely, $u(x) = u(r)$ where $r = |x|$. Then,*

$$L_1 u = -\alpha_n \dot{R}u'' + \left[\left(\frac{2}{n-2} - \frac{n-2}{2}\right)\dot{R} - \beta_n x_i \partial_i \dot{R}\right]\frac{u'}{r} - \gamma_n \Delta\dot{R}u; \tag{3.8}$$

$$\begin{aligned} L_2 u = &-\alpha_n \ddot{R}u'' - \frac{(x_i h_{kl,i})^2}{n-2}\left(\frac{u''}{r^2} - \frac{u'}{r^3}\right) \\ &+ \left[\left(\frac{2}{n-2} - \frac{n-2}{2}\right)\ddot{R} - \beta_n x_i\partial_i \ddot{R} + \frac{2x_i h_{kl,i}}{n-2}\dot{\mathrm{Ric}}_{kl}\right]\frac{u'}{r} \\ &- \gamma_n\left[\Delta\ddot{R} - \partial_i(h_{ij}\partial_j \dot{R}) - \frac{n^3-4n^2+16n-16}{4(n-1)(n-2)^2}\dot{R}^2 + \frac{4(n-1)}{(n-2)^2}(\dot{\mathrm{Ric}}_{ij})^2\right]u, \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} \mathcal{E}_g u = L_1 u + L_2 u &+ O([|h|^2|\partial^2 h| + |h||\partial h|^2]|u''|) \\ &+ O([|h|^2|\partial^3 h| + |h||\partial h||\partial^2 h| + |\partial h|^3 + r^{-1}(|h|^2|\partial^2 h| + |h||\partial h|^2)]|u'|) \\ &+ O([|h|^2|\partial^4 h| + |h||\partial h||\partial^3 h| + |h||\partial^2 h|^2 + |\partial h|^2|\partial^2 h|]|u|). \end{aligned} \tag{3.10}$$

Here, $\alpha_n = \frac{4+(n-2)^2}{2(n-1)(n-2)}$, $\beta_n = \frac{n-6}{2(n-1)}$, $\gamma_n = \frac{n-4}{4(n-1)}$, *and* $u' = \partial_r u$.

Proof. We shall first prove two identities:

$$h_{ij,kl}x_i x_j = 2h_{kl}; \tag{3.11}$$

$$h_{ik,jl}x_i x_j = h_{kl} - x_i h_{kl,i}. \tag{3.12}$$

Indeed, by equations (3.1) and (3.2), it holds that

$$\begin{aligned} h_{ij,kl}x_i x_j &= \partial_l(h_{ij,k}x_i x_j) - h_{lj,k}x_j - h_{il,k}x_i \\ &= \partial_{lk}(h_{ij}x_i x_j) - \partial_l(h_{kj}x_j) - \partial_l(h_{ik}x_i) + 2h_{kl} \\ &= 2h_{kl}. \end{aligned}$$

and

$$\begin{aligned} h_{ik,jl}x_ix_j &= \partial_l(h_{ik,j}x_ix_j) - h_{lk,j}x_j - h_{ik,l}x_i \\ &= \partial_{lj}(h_{ik}x_ix_j) - \partial_l(h_{jk}x_j) - \partial_l(nh_{ik}x_i) + h_{kl} - x_ih_{kl,i} \\ &= h_{kl} - x_ih_{kl,i}. \end{aligned}$$

To evaluate $L_1[h]$, we only have to compute $\dot{\mathrm{Ric}}_{ij}x_ix_j$, since $\dot{\mathrm{Ric}}_{ii} = \dot{R}$. As a matter of fact, by (3.1), (3.11), and (3.12), we have

$$2\dot{\mathrm{Ric}}_{ij}x_ix_j = (h_{jk,ik} + h_{ik,jk} - h_{ij,kk})x_ix_j = 0.$$

To evaluate $L_2[h,h]$, we need to compute $\ddot{\mathrm{Ric}}_{ij}x_ix_j$, since $\ddot{\mathrm{Ric}}_{ii} = \ddot{R} + h_{ij}\dot{\mathrm{Ric}}_{ij}$. As the following claim shows, it has a simple expression even though $\ddot{\mathrm{Ric}}$ itself is complicated.

Claim 3.7.

$$4\ddot{\mathrm{Ric}}_{ij}x_ix_j = -(x_k\partial_kh_{ij})^2.$$

Proof of Claim 3.7. By (3.1) and (3.2), we know that

$$4\ddot{\mathrm{Ric}}_{ij}x_ix_j = -2x_ih_{lk,i}h_{kl} - 2(h_{kl})^2 - (x_ih_{lk,i})^2 + h_{kl}(2h_{ij,kl}x_ix_j - 2h_{ki,lj}x_ix_j).$$

Plugging this in (3.11) and (3.12), we can prove Claim 3.7.

Since u is radial, we know that $\partial_ku = x_k\frac{u'}{r}$, $\partial_{ij}u = \frac{x_ix_j}{r^2}u'' + \left(\delta_{ij} - \frac{x_ix_j}{r^2}\right)\frac{u'}{r}$ and $\Delta u = u'' + (n-1)\frac{u'}{r}$. Combining all the computations made here with Lemma 3.5, we finish the proof. □

4. The Correction Term

Let h be a symmetric 2-tensor on $B^n(0,\delta)$ whose entry is a smooth function and that satisfies (3.1). Given $K = n-6$ and a multi-index α, we define

$$H_{ij}^{(k)}(x) = \sum_{|\alpha|=k}\frac{h_{ij,\alpha}(0)}{\alpha!}x_\alpha, \quad H_{ij}(x) = \sum_{2\le|\alpha|\le K}\frac{h_{ij,\alpha}(0)}{\alpha!}x_\alpha = \sum_{k=2}^{K}H_{ij}^{(k)}(x), \tag{4.1}$$

and

$$|H^{(k)}| = \left(\sum_{|\alpha|=k}|h_{ij,\alpha}(0)|^2\right)^{\frac{1}{2}}, \tag{4.2}$$

where $0 \in \mathbb{R}^n$ is identified with $\sigma_0 \in M$. Then, $h_{ij}(x) = H_{ij}(x) + O(|x|^{K+1})$, $H_{ij}(x) = H_{ji}(x)$, $H_{ij}(x)x_j = 0$, and $\mathrm{tr}(H(x)) = 0$. Although the following result is well-known (see [22, Section 4]), we include a proof for the reader's convenience.

Lemma 4.1. *For $l = 1,\ldots,n$,*

$$\int_{\mathbb{S}^{n-1}} H_{ij,ij} = 0 \quad \text{and} \quad \int_{\mathbb{S}^{n-1}} x_lH_{ij,ij} = 0.$$

Proof. Because of equations (3.1) and (3.2), we have

$$x_iH_{ij,j}^{(k)} = 0 \quad \text{for any } k = 2,\ldots,K.$$

Then, according to Lemma A.3, we have

$$\int_{\mathbb{S}^{n-1}} H_{ij,ij}^{(k)} = \int_{\mathbb{S}^{n-1}} \partial_jH_{ij,i}^{(k)} = (n+k-2)\int_{\mathbb{S}^{n-1}} x_iH_{ij,j}^{(k)} = 0,$$

and for any $l = 1,\ldots,n$,

$$\int_{\mathbb{S}^{n-1}} x_lH_{ij,ij}^{(k)} = \int_{\mathbb{S}^{n-1}} \partial_j\left(x_lH_{ij,i}^{(k)} - H_{jl}^{(k)}\right) = (n+k-1)\int_{\mathbb{S}^{n-1}} x_j\left(x_lH_{ij,i}^{(k)} - H_{jl}^{(k)}\right) = 0. \quad □$$

Since $H^{(k+2)}_{ij,ij}$ is a homogeneous polynomial on $\mathbb{R}^n$ of degree k, i.e., $H^{(k+2)}_{ij,ij} \in \mathcal{P}_k$, we can write it by using the spherical harmonic decomposition:

$$H^{(k+2)}_{ij,ij}(x) = \sum_{s=0}^{\lfloor \frac{k-2}{2} \rfloor} r^{2s} p^{(k-2s)}(x), \tag{4.3}$$

where $r = |x|$ and $p^{(k-2s)} \in \mathcal{H}_{k-2s}$. By Lemma 4.1 and Lemma A.3, $p^{(0)}$ and $p^{(1)}$ vanish.

Remark 4.2. Let $\gamma_1^* := \frac{2(n-1)}{n-2} - \frac{(n-1)(n-2)}{2} + 6 - n$, $\gamma_2^* := -\frac{n-2}{2} - \frac{2}{n-2}$, and $\gamma_n = \frac{n-4}{4(n-1)}$. For smooth functions u, v on $\mathbb{R}^n$, where u is radial, we define

$$\mathfrak{L}_1[v]u := \frac{v}{n-1}\left[\gamma_2^* u'' + \gamma_1^* \frac{u'}{r} - \frac{(n-6)(k-2)}{2}\frac{u'}{r}\right] - \gamma_n \Delta v u.$$

Then, $\mathfrak{L}_1[v]u$ is linear in both u and v. By Lemma 3.6, (3.5) and Lemma A.1, we have

$$L_1[H^{(k+2)}]u = \mathfrak{L}_1[H^{(k+2)}_{ij,ij}]u. \tag{4.4}$$

When $k = 2$, it holds that $H^{(4)}_{ij,ij} = p^{(2)} \in \mathcal{H}_2$ because $p^{(0)} = 0$. In this case, our computation agrees with [28, Lemma 2.1]. $\diamond$

Given the normalized bubble $w = w_{1,0}$, we decompose $L_1[H^{(k+2)}]w$ into

$$L_1[H^{(k+2)}]w = \mathfrak{L}_1\left[\sum_{s=0}^{\lfloor \frac{k-2}{2} \rfloor} r^{2s} p^{(k-2s)}\right] w = \sum_{s=0}^{\lfloor \frac{k-2}{2} \rfloor} \mathfrak{L}_1[r^{2s}p^{(k-2s)}]w.$$

Then,

$$\begin{aligned} &\mathfrak{L}_1[r^{2s}p^{(k-2s)}]w \\ &= \frac{n-4}{2(n-1)}\left[\frac{(k+2)(n-6)r^{2s}}{(1+r^2)^{\frac{n-2}{2}}} + \frac{(n^2-4n+8)r^{2s}}{(1+r^2)^{\frac{n}{2}}} - \frac{s(2k-2s+n-2)r^{2s-2}}{(1+r^2)^{\frac{n-4}{2}}}\right] p^{(k-2s)}. \end{aligned} \tag{4.5}$$

In this section, we will find an explicit solution of the linearized equation

$$\Delta^2 \Psi^{k,s} - \tilde{\mathfrak{c}}(n) w^{\frac{8}{n-4}} \Psi^{k,s} = -\mathfrak{L}_1[r^{2s}p^{(k-2s)}]w \tag{4.6}$$

for each $k = 2, \dots, K-2$ and $s = 0, \dots, \lfloor \frac{k-2}{2} \rfloor$.

Remark 4.3. It is a surprising fact that (4.6) possesses a solution of the form

$$\Psi(x) = F(x)(1+r^2)^{-\frac{n-2}{2}}, \tag{4.7}$$

where $F(x)$ is a polynomial. To explain why, we assume the form of Ψ in (4.7) and define

$$T_Q(F) := (1+r^2)^{\frac{n+4}{2}} \left(\Delta^2 - \tilde{\mathfrak{c}}(n) w^{\frac{8}{n-4}}\right) \Psi.$$

Then,

$$\begin{aligned} T_Q(F) &= (1+r^2)^3 \Delta^2 F - 4(n-2)(1+r^2)^2(\Delta F + x_i \partial_i \Delta F) \\ &\quad - 2n(n-2)(1+r^2)(\Delta F - 2x_i x_j \partial_i \partial_j F) + 4n(n-2)(n+2)(x_i \partial_i F - F). \end{aligned}$$

We can see that $\deg T_Q(F) = \deg F + 2$.

For the case of Yamabe equation, the map corresponding to T_Q is

$$\begin{aligned} T_Y(F) &:= (1+r^2)^{\frac{n+2}{2}} \left(\Delta + n(n+2)(1+r^2)^{-2}\right)\left(F(x)(1+r^2)^{-\frac{n}{2}}\right) \\ &= (1+r^2)\Delta F - 2n x \cdot \nabla F + 2nF, \end{aligned}$$

which appeared in the proof of [22, Proposition 4.1].

Unlike the Yamabe case where $\deg T_Y(F) = \deg F$, if $T_Q(F)$ is equal to a general polynomial, there may be no polynomial solution F with degree 2 less. This is where we need the algebraic structure of the Paneitz operator.

In Appendix C, we will offer a geometric intuition for the explicit solvability of (4.6). $\diamond$

Considering the form of the right-hand side of equation (4.6), we expect that the solution $\Psi^{k,s}$ can be written as a linear combination of

$$\left\{ r^{2j}(1+r^2)^{-\frac{n-2}{2}} p^{(k-2s)} \mid j = 0, \ldots, s+2,\ \Delta p^{(k-2s)} = 0 \right\}.$$

However, when we apply the linearized operator $\Delta^2 - \tilde{\mathfrak{c}}(n)(1+r^2)^{-4}$ to $r^{2j}(1+r^2)^{-\frac{n-2}{2}} p^{(k-2s)}$, the computations become lengthy. So we decide to use another equivalent linear combinations of

$$\left\{ (1+r^2)^{-\frac{n-2j}{2}} p^{(k-2s)} \mid j = 1, \ldots, s+3,\ \Delta p^{(k-2s)} = 0 \right\}. \tag{4.8}$$

Let us express the term $\mathfrak{L}_1[r^{2s}p^{(k-2s)}]w$ from the right-hand side of (4.6) and one more term $\mathfrak{L}_1[r^{2s}p^{(k-2s)}]Z^0$ in terms of the new basis given in (4.8). The latter term will arise when we analyze the Pohozaev quadratic form as shown in the proof of Lemma B.13.

Definition 4.4. We define two $(s+4) \times 1$ column vectors $\vec{b} = (b_i)$ and $\vec{b}' = (b'_i)$, where the elements $b_i = b_i(n,k,s) \in \mathbb{R}$ and $b'_i = b'_i(n,k,s) \in \mathbb{R}$ are defined by the relation

$$\mathfrak{L}_1[r^{2s}p^{(k-2s)}](1+r^2)^{-\frac{n-4}{2}} := \frac{n-4}{2(n-1)} \sum_{i=1}^{s+4} b_i (1+r^2)^{-\frac{n+6-2i}{2}} p^{(k-2s)};$$

$$\mathfrak{L}_1[r^{2s}p^{(k-2s)}](1+r^2)^{-\frac{n-2}{2}} := \frac{1}{2(n-1)} \sum_{i=1}^{s+4} b'_i (1+r^2)^{-\frac{n+8-2i}{2}} p^{(k-2s)}.$$

In view of equation (4.5), we have $b_1 = 0$, $b_2 = 0$,

$$\begin{aligned} b_3 &= (-1)^s (n^2 - 4n + 8), \\ b_4 &= (-1)^{s+1} (s(n^2 - 4n + 8) - (k+2)(n-6)), \\ b_{s+4} &= (k+2)(n-6) - s(2k - 2s + n - 2), \end{aligned}$$

and when $s \geq 2$, for $i = 5, \ldots, s+3$,

$$\begin{aligned} b_i = (-1)^{s+3-i} \Bigg[(n^2 - 4n + 8)\binom{s}{i-3} &+ s(2k - 2s + n - 2)\binom{s-1}{i-5} \\ &- (k+2)(n-6)\binom{s}{i-4} \Bigg]. \end{aligned}$$

Similarly to (4.5), we can calculate

$$\begin{aligned} &2(n-1)\mathfrak{L}_1[r^{2s}p^{(k-2s)}](1+r^2)^{-\frac{n-2}{2}} \\ &\quad = \left[\frac{(k(n-2)(n-6) - 8(n-1))r^{2s}}{(1+r^2)^{\frac{n}{2}}} + \frac{n(n^2-4n+8)r^{2s}}{(1+r^2)^{\frac{n+2}{2}}} - \frac{(n-4)s(2k-2s+n-2)r^{2s-2}}{(1+r^2)^{\frac{n-2}{2}}} \right] p^{(k-2s)}. \end{aligned}$$

Then, we have $b'_1 = 0$, $b'_2 = 0$,

$$\begin{aligned} b'_3 &= (-1)^s n (n^2 - 4n + 8), \\ b'_4 &= (-1)^{s+1} [sn(n^2 - 4n + 8) - (k(n-2)(n-6) - 8(n-1))], \\ b'_{s+4} &= (k(n-2)(n-6) - 8(n-1)) - (n-4)s(2k - 2s + n - 2), \end{aligned}$$

and when $s \geq 2$, for $i = 5, \ldots, s+3$,

$$b_i' = (-1)^{s+3-i}\left[n(n^2-4n+8)\binom{s}{i-3} + (n-4)s(2k-2s+n-2)\binom{s-1}{i-5}\right.$$
$$\left. -(k(n-2)(n-6)-8(n-1))\binom{s}{i-4}\right].$$

Applying the linearized operator $\Delta^2 - \tilde{\mathfrak{c}}(n)(1+r^2)^{-4}$ to the new basis given in (4.8), we obtain

Lemma 4.5. *Let $a > 0$, $b \geq 2$ and $\Delta p^{(b)} = 0$. Then*

$$\begin{aligned}
&(\Delta^2 - \tilde{\mathfrak{c}}(n)(1+r^2)^{-4})((1+r^2)^{-a}p^{(b)})\\
&= (2a(2a+2)(2a+4)(2a+6) - \tilde{\mathfrak{c}}(n))(1+r^2)^{-a-4}p^{(b)}\\
&\ + 2a(2a+2)(2a+4)2(n-2a+2b-4)(1+r^2)^{-a-3}p^{(b)}\\
&\ + 2a(2a+2)(n-2a+2b-2)(n-2a+2b-4)(1+r^2)^{-a-2}p^{(b)}.
\end{aligned}$$

We are now ready to solve the linearized equation (4.6).

Proposition 4.6. *Given $n \geq 10$, $k = 2, \ldots, K-2$, and $s = 0, \ldots, \lfloor \frac{k-2}{2} \rfloor$, we assume $\Delta p^{(k-2s)} = 0$. Then equation (4.6) has a solution of the form*

$$\Psi^{k,s} = -\frac{n-4}{2(n-1)}\sum_{j=1}^{s+3}\Gamma_j(1+r^2)^{-\frac{n-2j}{2}}p^{(k-2s)},$$

where the precise value of $\Gamma_j = \Gamma_j(n,k,s) \in \mathbb{R}$ is determined by (4.13)–(4.16).

Proof. For $j = 1, \ldots, s+3$, let $a = \frac{n-2j}{2}$ and $b = k-2s$. By Lemma 4.5, we find

$$\begin{aligned}
&(\Delta^2 - \tilde{\mathfrak{c}}(n)(1+r^2)^{-4})((1+r^2)^{-\frac{n-2j}{2}}p^{(k-2s)})\\
&= -8(j-1)(n+2-j)(n^2-2(j-2)n+2j(j-3))(1+r^2)^{-\frac{n+8-2j}{2}}p^{(k-2s)}\\
&\ + 4(n-2j)(n+2-2j)(n+4-2j)(k-2s+j-2)(1+r^2)^{-\frac{n+6-2j}{2}}p^{(k-2s)}\\
&\ + 4(n-2j)(n+2-2j)(k-2s+j-1)(k-2s+j-2)(1+r^2)^{-\frac{n+4-2j}{2}}p^{(k-2s)}.
\end{aligned}$$

Then we define an $(s+4) \times (s+3)$ matrix $A = (a_{i,j})$:

$$(\Delta^2 - \tilde{\mathfrak{c}}(n)(1+r^2)^{-4})((1+r^2)^{-\frac{n-2j}{2}}p^{(k-2s)}) = \sum_{i=1}^{s+4} a_{i,j}(1+r^2)^{-\frac{n+6-2i}{2}}p^{(k-2s)}.$$

Note that the matrix A has non-zero entries only on the main diagonal, the superdiagonal (i.e., upper secondary diagonal), and the subdiagonal (i.e., lower secondary diagonal). We have

$$\begin{aligned}
a_{1,1} &= 4(n-2)n(n+2)(k-2s-1),\\
a_{2,1} &= 4(n-2)n(k-2s)(k-2s-1),
\end{aligned}$$

and for $j = 2, \ldots, s+3$,

$$\begin{aligned}
a_{j-1,j} &= -8(j-1)(n+2-j)(n^2-2(j-2)n+2j(j-3)),\\
a_{j,j} &= 4(n-2j)(n+2-2j)(n+4-2j)(k-2s+j-2),\\
a_{j+1,j} &= 4(n-2j)(n+2-2j)(k-2s+j-1)(k-2s+j-2).
\end{aligned}$$

To prove this proposition, we just need to solve the following overdetermined linear system:

$$A\Gamma = \vec{b}, \tag{4.9}$$

where $\Gamma = (\Gamma_j)$ is an $(s+3)\times 1$ column vector. By direct computation, we obtain

$$\begin{aligned}\Gamma_{s+3} &= a^{-1}_{s+4,s+3} b_{s+4};\\ \Gamma_{s+2} &= a^{-1}_{s+3,s+2}(b_{s+3} - a_{s+3,s+3}\Gamma_{s+3}),\end{aligned} \tag{4.10}$$

and for $j = 1, \ldots, s+1$, the following recurrence relations

$$\Gamma_j = a^{-1}_{j+1,j}(b_{j+1} - a_{j+1,j+1}\Gamma_{j+1} - a_{j+1,j+2}\Gamma_{j+2}). \tag{4.11}$$

System (4.9) has a solution if the condition

$$a_{1,1}\Gamma_1 + a_{1,2}\Gamma_2 = 0 \tag{4.12}$$

holds. For the rest of the proof, we will verify (4.12).

By plugging the values of A and $\vec{b}$ into (4.10) and (4.11), we obtain

$$\begin{aligned}\Gamma_{s+3} &= \frac{(k+2)(n-6) - s(2k-2s+n-2)}{4(n-2s-6)(n-2s-4)(k-s+2)(k-s+1)};\\ \Gamma_{s+2} &= \frac{n^2-4n+8+s(s-1)(2k-2s+n-2)-s(k+2)(n-6)}{4(n-2s-4)(n-2s-2)(k-s+1)(k-s)} - \frac{n-2s-6}{k-s}\Gamma_{s+3};\\ \Gamma_1 &= -\frac{n-4}{k-2s-1}\Gamma_2 + \frac{4(n-1)}{(k-2s)(k-2s-1)}\Gamma_3.\end{aligned} \tag{4.13}$$

When $s \geq 1$, we obtain

$$\Gamma_2 = \frac{(-1)^s(n^2-4n+8)}{4(n-4)(n-2)(k-2s+1)(k-2s)} - \frac{n-6}{k-2s}\Gamma_3 + \frac{6(n^2-4n+8)}{(n-4)(k-2s+1)(k-2s)}\Gamma_4. \tag{4.14}$$

When $s \geq 2$, we obtain

$$\begin{aligned}\Gamma_3 = &\frac{(-1)^{s+1}(s(n^2-4n+8)-(k+2)(n-6))}{4(n-6)(n-4)(k-2s+2)(k-2s+1)}\\ &- \frac{n-8}{k-2s+1}\Gamma_4 + \frac{8(n-3)(n^2-6n+20)}{(n-6)(n-4)(k-2s+2)(k-2s+1)}\Gamma_5.\end{aligned} \tag{4.15}$$

When $s \geq 3$, for $j = 4, \ldots, s+1$, we obtain the following recurrence relations

$$\begin{aligned}\Gamma_j = &(-1)^{s+2-j}\frac{(s-j+3)(n^2-4n+8)+(j-2)(j-3)(2k-2s+n-2)-(j-2)(k+2)(n-6)}{4(j-2)(n-2j)(n+2-2j)(k-2s+j-1)(k-2s+j-2)}\binom{s}{j-3}\\ &- \frac{n-2j-2}{k-2s+j-2}\Gamma_{j+1} + \frac{2(j+1)(n-j)(n^2-2jn+2(j+2)(j-1))}{(n-2j)(n+2-2j)(k-2s+j-1)(k-2s+j-2)}\Gamma_{j+2}.\end{aligned} \tag{4.16}$$

In the end, using (4.13)–(4.16) and a computer software such as Mathematica, we see that the following cancellation appears:

$$(k-2s-1)\Gamma_1 - 2\Gamma_2 = 0,$$

which is exactly (4.12). □

Remark 4.7. In the Yamabe case, the analogous matrix to A is a **square** matrix with non-vanishing entries only on the diagonal and superdiagonal. This is the reason that the proof of [22, Proposition 4.1] is relatively simple. ◇

Given (4.3) and (4.6), we define

$$\Psi[H^{(k+2)}] := \sum_{s=0}^{\lfloor \frac{k-2}{2} \rfloor} \Psi^{k,s}. \tag{4.17}$$

From Proposition 4.6, we immediately deduce

$$\Delta^2\Psi[H^{(k+2)}] - \tilde{\mathfrak{c}}(n)w^{\frac{8}{n-4}}\Psi[H^{(k+2)}] = -L_1[H^{(k+2)}]w = -\mathfrak{L}_1[H^{(k+2)}_{ij,ij}]w \quad \text{in } \mathbb{R}^n$$

and the following estimate.

Corollary 4.8. *For $n \geq 10$ and $k = 2, \ldots, K-2$, there exists a constant $C = C(n,k) > 0$ such that*

$$\left|\partial_\beta \Psi[H^{(k+2)}]\right|(x) \leq C|H^{(k+2)}|(1+|x|)^{k+6-n-|\beta|}$$

for a multi-index β with $|\beta| = 0, \ldots, 4$. Here, $|H^{(k+2)}|$ is the quantity defined in (4.2).

5. Refined Blowup Analysis

Throughout Sections 5 and 6, we assume that $\sigma_a \to \bar{\sigma} \in M$ is an isolated simple blowup point for a sequence $\{u_a\}_{a\in\mathbb{N}}$ of solutions to (2.4). We work in g_a-normal coordinates x centered at σ_a, chosen such that $\det g_a = 1$.

We write $u_a(x) = u_a(\exp^{g_a}_{\sigma_a}(x))$ by slight abuse of notation and $r = |x| = d_{g_a}(\exp^{g_a}_{\sigma_a}(x), \sigma_a)$, where $\exp^{g_a}$ is the exponential map on (M, g_a). We set a rescaling factor $\epsilon_a = u_a(0)^{-\frac{2}{n-4}} > 0$ so that $\epsilon_a \to 0$ as $a \to \infty$, $\tilde{g}_a(y) = g_a(\epsilon_a y)$, $P_a = P_{\tilde{g}_a}$, $\mathcal{E}_a = \mathcal{E}_{\tilde{g}_a}$, and the natural normalization of u_a by

$$U_a(y) = \epsilon_a^{\frac{n-4}{2}} u_a(\epsilon_a y).$$

We note that $K = n-6 \geq d = \lfloor \frac{n-4}{2} \rfloor$ for all $n \geq 8$. Let $g_a = \exp(h_a)$ and $\tilde{g}_a = \exp(\tilde{h}_a)$ so that $\tilde{h}_a(y) = h_a(\epsilon_a y)$. We define the 2-tensor H_a by (4.1), where h is replaced by h_a, and write

$$\widetilde{H}_a^{(k)}(y) := H_a^{(k)}(\epsilon_a y) = \epsilon_a^k H_a^{(k)}(y), \quad k = 2, \ldots, K$$

and $\widetilde{H}_a := \sum_{k=2}^K \widetilde{H}_a^{(k)}$. For $n = 8, 9$, we set $\widetilde{\Psi}_a := 0$. For $n \geq 10$, we set

$$\widetilde{\Psi}_a := \sum_{k=4}^K \Psi[\widetilde{H}_a^{(k)}],$$

where $\Psi[\widetilde{H}_a^{(k)}]$ is the function in (4.17) with $H^{(k+2)}$ replaced by $\widetilde{H}_a^{(k)}$. Then,

$$\Delta^2 \widetilde{\Psi}_a - \tilde{\mathfrak{c}}(n) w^{\frac{8}{n-4}} \widetilde{\Psi}_a = -\sum_{k=4}^K L_1[\widetilde{H}_a^{(k)}] w \quad \text{in } \mathbb{R}^n$$

and $\widetilde{\Psi}_a$ is an explicit rational function on $\mathbb{R}^n$ satisfying

$$\left|\partial_\beta \widetilde{\Psi}_a\right|(y) \leq C \sum_{k=4}^K \epsilon_a^k |H_a^{(k)}|(1+|y|)^{k+4-n-|\beta|} \tag{5.1}$$

for a multi-index β with $|\beta| = 0, \ldots, 4$ (see Corollary 4.8).

The following result improves estimate (2.7) in Proposition 2.4.

Proposition 5.1. *Let $n \geq 8$ and $\sigma_a \to \bar{\sigma} \in M$ be an isolated simple blowup point for a sequence $\{u_a\}$ of solutions to* (2.4). *Then there exist $\delta_0 \in (0,1)$ and $C > 0$ such that*

$$\left|\partial_\beta (U_a - w - \widetilde{\Psi}_a)\right|(y) \leq C \sum_{k=2}^{d-1} \epsilon_a^{2k} |H_a^{(k)}|^2 (1+|y|)^{2k+4-n-|\beta|} + C\epsilon_a^{n-5}(1+|y|)^{-1-|\beta|}$$

for all $a \in \mathbb{N}$, multi-indices β with $|\beta| = 0, 1, \ldots, 4$, and $|y| \leq \delta_0 \epsilon_a^{-1}$.

In order to deduce this proposition, one may suitably modify the argument in [28, Section 5] for general manifolds of dimension $5 \leq n \leq 9$, so we omit the details. Interested readers may consult [22] to see how the arguments in [33], originally formulated for the low-dimensional Yamabe problem ($3 \leq n \leq 7$), can be extended to the higher-dimensional range $3 \leq n \leq 24$.

Corollary 5.2. *Scaling back to the x-coordinates, we define* $w_a(x) := w_{\epsilon_a,0}(x) = \epsilon_a^{-\frac{n-4}{2}} w(\epsilon_a^{-1}x)$ *and* $\Psi_a(x) = \epsilon_a^{-\frac{n-4}{2}} \widetilde{\Psi}(\epsilon_a^{-1}x)$ *so that*

$$\Delta^2 \Psi_a - \tilde{\mathfrak{c}}(n) w_a^{\frac{8}{n-4}} \Psi_a = -\sum_{k=4}^{K} L_1[H_a^{(k)}] w_a \quad \text{in } \mathbb{R}^n.$$

Under the conditions of Proposition 5.1, there exist $\delta_0 \in (0,1)$ *and* $C > 0$ *such that*

$$|\partial_\beta (u_a - w_a - \Psi_a)|(x) \le C \epsilon_a^{\frac{n-4}{2}+|\beta|} \left(\sum_{k=2}^{d-1} |H_a^{(k)}|^2 (\epsilon_a + |x|)^{2k+4-n-|\beta|} + (\epsilon_a + |x|)^{-1-|\beta|} \right)$$

for all $a \in \mathbb{N}$*, multi-indices* β *with* $|\beta| = 0, 1, \dots, 4$*, and* $|x| \le \delta_0$*.*

6. Weyl Vanishing Theorem

In this section, we prove the Weyl vanishing theorem, which is necessary to apply the positive mass theorem for the proof of Theorem 1.1. We set $\theta_k = 1$ if $k = \frac{n-4}{2}$ and $\theta_k = 0$ otherwise.

Theorem 6.1. *Let* $8 \le n \le 24$ *and* $\sigma_a \to \bar{\sigma} \in M$ *be an isolated simple blowup point for a sequence* $\{u_a\}$ *of solutions to* (2.4)*. Then for all* $a \in \mathbb{N}$ *and* $k = 0, \dots, \lfloor \frac{n-8}{2} \rfloor$*,*

$$|\nabla_{g_a}^k W_{g_a}|^2(\sigma_a) \le C \epsilon_a^{n-8-2k} |\log \epsilon_a|^{-\theta_{k+2}}.$$

Consequently,

$$\nabla_g^k W_g(\bar{\sigma}) = 0. \tag{6.1}$$

The corresponding result for $n = 8, 9$ was deduced in [28, Proposition 6.1]. Also, as shown in [19], (6.1) holds for $k = 0, \dots, d-2 = \lfloor \frac{n-8}{2} \rfloor$ if and only if $\nabla_g^k H(\bar{\sigma}) = 0$ for $k = 2, \dots, d$, provided $\det g = 1$ near $\bar{\sigma}$. The proof of Theorem 6.1 is long and will be conducted in Subsections 6.1–6.3.

6.1. Pohozaev quadratic form.

We remind that $w(y) = (1+|y|^2)^{-\frac{n-4}{2}}$ and $Z^0(y) = \frac{n-4}{2}(1-|y|^2)(1+|y|^2)^{-\frac{n-2}{2}}$ for $y \in \mathbb{R}^n$. Because they are radial and $\det \tilde{g}_a = 1$, it holds that

$$\Delta_{\tilde{g}_a} w = \Delta w \quad \text{and} \quad \Delta_{\tilde{g}_a} Z^0 = \Delta Z^0. \tag{6.2}$$

Let $\mathcal{E}_a = \mathcal{E}_{\tilde{g}_a} = P_{\tilde{g}_a} - \Delta_{\tilde{g}_a}^2$ be the differential operator analyzed in Lemma 3.5, $\mathcal{E}_a' := \mathcal{E}_a + \Delta_{\tilde{g}_a}^2 - \Delta^2$, and $\mathcal{R}_a := U_a - w - \widetilde{\Psi}_a$. By employing the local Pohozaev identity (2.1)–(2.2) with the selection $(r, u) = (\delta_0 \epsilon_a^{-1}, U_a)$, (5.1), Proposition 5.1, and (6.2), we discover

$$\begin{aligned} O(\epsilon_a^{n-4}) &\ge \int_{B^n(0,\delta_0\epsilon_a^{-1})} \left(y \cdot \nabla U_a + \frac{n-4}{2} U_a \right) \mathcal{E}_a'(U_a) dy \\ &= \mathcal{I}_{1,\epsilon_a,a} + \mathcal{I}_{2,\epsilon_a,a} + \mathcal{I}_{3,\epsilon_a,a} + O(\epsilon_a^{n-4}) + \int_{B^n(0,\delta_0\epsilon_a^{-1})} \left(y \cdot \nabla + \frac{n-4}{2} \right) \widetilde{\Psi}_a \mathcal{E}_a'(\widetilde{\Psi}_a) dy \\ &\quad + \left[\int_{B^n(0,\delta_0\epsilon_a^{-1})} Z^0 \mathcal{E}_a(\mathcal{R}_a) dy + \int_{B^n(0,\delta_0\epsilon_a^{-1})} \left(y \cdot \nabla + \frac{n-4}{2} \right) \widetilde{\Psi}_a \mathcal{E}_a'(\mathcal{R}_a) dy \right. \\ &\quad \left. + \int_{B^n(0,\delta_0\epsilon_a^{-1})} \left(y \cdot \nabla + \frac{n-4}{2} \right) \mathcal{R}_a \mathcal{E}_a'(w + \widetilde{\Psi}_a + \mathcal{R}_a) dy \right]. \end{aligned} \tag{6.3}$$

Here, $\delta_0 \in (0,1)$ is the small number appearing in Proposition 5.1, and for $\epsilon > 0$ small,

$$\mathcal{I}_{1,\epsilon,a} := \int_{B^n(0,\delta_0\epsilon^{-1})} Z^0 \mathcal{E}_a(w) dy,$$

$$\mathcal{I}_{2,\epsilon,a} := \int_{B^n(0,\delta_0\epsilon^{-1})} \left(y \cdot \nabla \widetilde{\Psi}_a + \frac{n-4}{2} \widetilde{\Psi}_a \right) \mathcal{E}_a(w) dy,$$
$$\mathcal{I}_{3,\epsilon,a} := \int_{B^n(0,\delta_0\epsilon^{-1})} Z^0 \mathcal{E}_a(\widetilde{\Psi}_a) dy.$$

Using the estimate

$$\begin{aligned}
&(|\mathcal{E}_a| + |\Delta_{\tilde{g}_a}^2 - \Delta^2|)(u) \\
&= \sum_{m=0}^4 O\left(\left[\sum_{k=2}^{n-4} \epsilon_a^k |H_a^{(k)}||y|^{k-m} + O(\epsilon_a^{n-3}|y|^{n-3-m}) \right] |\nabla_{g_a}^{4-m} u| \right) \\
&= O\left(\epsilon_a^2 |y|^2 |\nabla_{g_a}^4 u| + \epsilon_a^2 |y||\nabla_{g_a}^3 u| + \sum_{m=0}^2 \epsilon_a^{2+m} |\nabla_{g_a}^{2-m} u| \right) \quad \text{for } u \in C^4(B^n(0,\delta_0\epsilon_a^{-1})),
\end{aligned}$$

we further reduce (6.3) to

$$\begin{aligned}
O(\epsilon_a^{n-4}) &\geq \mathcal{I}_{1,\epsilon_a,a} + \mathcal{I}_{2,\epsilon_a,a} + \mathcal{I}_{3,\epsilon_a,a} + O(\epsilon_a^{n-4}) \\
&\quad + O\left(\sum_{k=2}^{\lfloor \frac{n-4}{3} \rfloor} \epsilon_a^{3k} |\log \epsilon_a| |H_a^{(k)}|^3 \right) + O\left(\epsilon_a^2 |\log \epsilon_a| \sum_{k=2}^{d-1} \epsilon_a^{2k} |H_a^{(k)}|^2 \right).
\end{aligned} \tag{6.4}$$

To control the terms $\mathcal{I}_{1,\epsilon_a,a}$, $\mathcal{I}_{2,\epsilon_a,a}$ and $\mathcal{I}_{3,\epsilon_a,a}$, we first observe from (3.5) and Lemmas 3.6, 4.1 and A.3 that

$$\int_{B^n(0,\delta_0\epsilon_a^{-1})} Z^0 L_1[\widetilde{H}_a] w = 0 \quad \text{for all } a \in \mathbb{N}.$$

We then introduce several symmetric bilinear forms defined on symmetric 2-tensors whose entries are smooth functions.

Definition 6.2. Given any symmetric 2-tensors h, h', we define $L_2[h,h']$ by using (3.9) and the polarization identity as in (3.4). For $\epsilon > 0$ small, let

$$I_{1,\epsilon}[h,h'] := \int_{B^n(0,\delta_0\epsilon^{-1})} Z^0 L_2[h,h'] w. \tag{6.5}$$

Whenever $\Psi[h]$ and $\Psi[h']$ are well-defined via (4.17), we also set

$$I_{2,\epsilon}[h,h'] := \frac{1}{2} \int_{B^n(0,\delta_0\epsilon^{-1})} \left[\left(y_i\partial_i + \frac{n-4}{2} \right) \Psi[h] L_1[h'] w + \left(y_i\partial_i + \frac{n-4}{2} \right) \Psi[h'] L_1[h] w \right], \tag{6.6}$$

$$I_{3,\epsilon}[h,h'] := \frac{1}{2} \int_{B^n(0,\delta_0\epsilon^{-1})} \left(\Psi[h] L_1[h'] Z^0 + \Psi[h'] L_1[h] Z^0 \right), \tag{6.7}$$

and

$$I_\epsilon[h,h'] := I_{1,\epsilon}[h,h'] + I_{2,\epsilon}[h,h'] + I_{3,\epsilon}[h,h']. \tag{6.8}$$

Similarly to [22, Lemmas A.2–A.3], we can invoke (3.10) to derive the following estimate. The proof is omitted.

Lemma 6.3. *Let $d = \lfloor \frac{n-4}{2} \rfloor$. Given any small $\eta \in (0,1)$, there exists $C > 0$ depending only on n and (M,g) such that*

$$\left| \mathcal{I}_{i,\epsilon_a,a} - \sum_{k,m=2}^d I_{i,\epsilon_a}\left[\widetilde{H}_a^{(k)}, \widetilde{H}_a^{(m)}\right] \right| \leq C\eta \sum_{k=2}^d \epsilon_a^{2k} |\log \epsilon_a|^{\theta_k} |H_a^{(k)}|^2 + C\delta_0 \eta^{-1} \epsilon_a^{n-4}$$

for all $a \in \mathbb{N}$ and $i = 1, 2, 3$.

We introduce the Pohozaev quadratic form for the Q-curvature problem:

$$I_{\epsilon_a}\big[\widetilde{H}_a,\widetilde{H}_a\big] = \sum_{k,m=2}^{d} I_{\epsilon_a}\big[\widetilde{H}_a^{(k)},\widetilde{H}_a^{(m)}\big]. \tag{6.9}$$

In the next subsection, we will establish the following key estimate.

Proposition 6.4. *For any $8 \le n \le 24$, there exists a constant $C = C(n) > 0$ such that*

$$I_{\epsilon_a}\big[\widetilde{H}_a,\widetilde{H}_a\big] \ge C^{-1}\sum_{k=2}^{d} \epsilon_a^{2k}|\log\epsilon_a|^{\theta_k}|H_a^{(k)}|^2 + O(\epsilon_a^{n-4}) \quad \textit{for all } a \in \mathbb{N}.$$

For brevity, we will often drop the subscript a, writing e.g. $I_{1,\epsilon}[\widetilde{H}^{(k)},\widetilde{H}^{(m)}] = I_{1,\epsilon_a}[\widetilde{H}_a^{(k)},\widetilde{H}_a^{(m)}]$, and assume that both k and m range from 2 to d unless stated otherwise.

6.2. **Proof of Proposition 6.4.** Let $\bar{H}$ and $\bar{H}'$ be matrices of polynomials on $\mathbb{R}^n$. We set the inner product of $\bar{H}$ and $\bar{H}'$ and the norm of $\bar{H}$ as

$$\langle \bar{H},\bar{H}'\rangle := \int_{\mathbb{S}^{n-1}} \bar{H}_{ij}\bar{H}'_{ij} \quad \text{and} \quad \|\bar{H}\| = \sqrt{\langle \bar{H},\bar{H}\rangle},$$

respectively. Also, for $k \in \mathbb{N}\cup\{0\}$, we define $\mathcal{V}_k$ to be a subspace of symmetric matrices whose elements are homogeneous polynomials on $\mathbb{R}^n$ of degree k such that

$$\bar{H}_{ii} = 0 \quad \text{and} \quad x_i\bar{H}_{ij} = 0$$

for each $\bar{H} \in \mathcal{V}_k$. We set the divergence $\delta\bar{H}$ and the double divergence $\delta^2\bar{H}$ of the matrix $\bar{H}$ as

$$\delta_j\bar{H} := \bar{H}_{ij,i} \quad \text{and} \quad \delta^2\bar{H} := \bar{H}_{ij,ij},$$

respectively.

By (3.1) and (4.1), we have that $H^{(k)}, \widetilde{H}^{(k)} \in \mathcal{V}_k$ for $k = 2,\dots,K$. As a preliminary step, we evaluate $I_{1,\epsilon}[\widetilde{H}^{(k)},\widetilde{H}^{(m)}]$ in polar coordinates.

Proposition 6.5. *For $k,m = 2,\dots,d$, let $\ddot{R}^{(k,m)} := \ddot{R}[H^{(k)},H^{(m)}]$, $\dot{\mathrm{Ric}}^{(k)} := \dot{\mathrm{Ric}}[H^{(k)}]$, and*

$$\begin{aligned} J_1'[H^{(k)},H^{(m)}] := \frac{(n-4)^2}{4(n-1)(n-2)}\Bigg[&-c_1(n,k,m)\int_{\mathbb{S}^{n-1}}\ddot{R}^{(k,m)} - c_2(n,k,m)\int_{\mathbb{S}^{n-1}} H_{ij}^{(k)}H_{ij}^{(m)} \\ &+c_3(n,k,m)\int_{\mathbb{S}^{n-1}}\dot{\mathrm{Ric}}_{ij}^{(k)}\dot{\mathrm{Ric}}_{ij}^{(m)} - c_4(n,k,m)\int_{\mathbb{S}^{n-1}}\delta^2H^{(k)}\delta^2H^{(m)}\Bigg], \end{aligned}$$

where

$$\begin{aligned} c_1(n,k,m) &:= (k+m)(n^3-(k+m+2)n^2+(6(k+m)-4)n-4(k+m)+8), \\ c_2(n,k,m) &:= 2(n-1)(k+m)(n+k+m-2)km, \\ c_3(n,k,m) &:= \frac{8(n-1)(n-3)(k+m)}{(n-2)(n+k+m-4)}, \\ c_4(n,k,m) &:= \frac{(n-3)(n^3-4n^2+16n-16)(k+m)}{2(n-1)(n-2)(n+k+m-4)}. \end{aligned}$$

Let also

$$\begin{aligned} &J_1[H^{(k)},H^{(m)}] \\ &:= \begin{cases} \dfrac{1}{n-k-m-4}\mathcal{I}_{n-2}^{n+k+m-3}J_1'[H^{(k)},H^{(m)}] & \text{if } k+m < n-4, \\ \dfrac{1}{2(n-3)}J_1'[H^{(d)},H^{(d)}] & \text{if } n \ge 8 \text{ is even and } k = m = d = \frac{n-4}{2}. \end{cases} \end{aligned}$$

Then, for $k, m = 2, \dots, d$, it holds that

$$I_{1,\epsilon}\big[\widetilde{H}^{(k)}, \widetilde{H}^{(m)}\big] = \epsilon^{k+m}|\log\epsilon|^{\theta_{\frac{k+m}{2}}} J_1[H^{(k)}, H^{(m)}] + O(\epsilon^{n-4}), \tag{6.10}$$

where $\mathcal{I}_{n-2}^{n+k+m-3}$ is the quantity defined by (1.4).

Remark 6.6. In the Yamabe case, it holds that

$$I_{1,\epsilon}\big[\widetilde{H}^{(k)}, \widetilde{H}^{(m)}\big] = -\frac{n-2}{8(n-1)}(k+m)\,\mathcal{I}_{n-2}^{n+k+m-3}\epsilon^{k+m}\int_{\mathbb{S}^{n-1}} \ddot{R}^{(k,m)} + O(\epsilon^{n-2})$$

for $k, m = 2, \dots, \lfloor \frac{n-2}{2} \rfloor$ such that $k + m < n - 2$. $\diamond$

We observe from the definition of $\dot{\mathrm{Ric}}[\bar{H}]$ in (3.3) that

$$\begin{aligned} (\dot{\mathrm{Ric}}[\bar{H}])_{ii} &= \delta^2 \bar{H}; \\ x_i(\dot{\mathrm{Ric}}[\bar{H}])_{ij} &= \frac{k}{2}\delta_j \bar{H}; \\ x_i x_j(\dot{\mathrm{Ric}}[\bar{H}])_{ij} &= 0 \end{aligned}$$

for $\bar{H} \in \mathcal{V}_k$. Thus, we can apply Lemma B.4 to define the linear operator $\mathcal{L}_k : \mathcal{V}_k \to \mathcal{V}_k$:

$$\mathcal{L}_k \bar{H} := \mathrm{Proj}\left[|x|^2 \dot{\mathrm{Ric}}[\bar{H}]\right] \quad \text{for } \bar{H} \in \mathcal{V}_k. \tag{6.11}$$

We call $\mathcal{L}_k$ as the "stability" operator because it essentially comes from the second variation formula for the total scalar curvature functional (see [40]).

By (B.4) and (3.3),

$$\begin{aligned} (\mathcal{L}_k \bar{H})_{ij} &= \frac{|x|^2}{2}(\bar{H}_{jl,il} + \bar{H}_{il,jl} - \bar{H}_{ij,ll}) - \frac{k}{2}(x_j \bar{H}_{il,l} + x_i \bar{H}_{jl,l}) \\ &\quad + \frac{1}{n-1}\delta^2 \bar{H}(x_i x_j - |x|^2 \delta_{ij}). \end{aligned}$$

From this, one can check that $\mathcal{L}_k : \mathcal{V}_k \to \mathcal{V}_k$ is symmetric with respect to the inner product $\langle \cdot, \cdot \rangle$ (the proof is given in [22, Page 182]), so the spectral theorem is applicable. Using $\mathcal{L}_k$, Proposition 6.5 is rephrased as follows.

Corollary 6.7. *For $k, m = 2, \dots, d$, it holds that*

$$\begin{aligned} J_1'[H^{(k)}, H^{(m)}] = \frac{(n-4)^2}{4(n-1)(n-2)} \Bigg[&\frac{1}{4}c_1(n,k,m) \int_{\mathbb{S}^{n-1}} \left\{ (\mathcal{L}_k H^{(k)})_{ij} H_{ij}^{(m)} + H_{ij}^{(k)} (\mathcal{L}_m H^{(m)})_{ij} \right\} \\ &+ c_3(n,k,m) \int_{\mathbb{S}^{n-1}} (\mathcal{L}_k H^{(k)})_{ij} (\mathcal{L}_m H^{(m)})_{ij} \\ &+ \left\{ \frac{1}{8}(n+k+m-2)(k+m)c_1(n,k,m) - c_2(n,k,m) \right\} \int_{\mathbb{S}^{n-1}} H_{ij}^{(k)} H_{ij}^{(m)} \\ &+ \frac{km}{2} c_3(n,k,m) \int_{\mathbb{S}^{n-1}} \delta_i H^{(k)} \delta_i H^{(m)} \\ &+ \left\{ \frac{1}{n-1} c_3(n,k,m) - c_4(n,k,m) \right\} \int_{\mathbb{S}^{n-1}} \delta^2 H^{(k)} \delta^2 H^{(m)} \Bigg]. \end{aligned}$$

We defer the proofs of Proposition 6.5 and Corollary 6.7 to Appendix B.1.

Before proving Proposition 6.4, we consider a decomposition of the space $\mathcal{V}_k$ arising from the eigenspace decomposition of $\mathcal{L}_k$ for each $k \geq 2$:

$$\mathcal{V}_k = \mathcal{V}_k / \mathcal{W}_k \oplus \mathcal{W}_k / \mathcal{D}_k \oplus \mathcal{D}_k, \tag{6.12}$$

where

$$\begin{aligned}\mathcal{W}_k &:= \{W \in \mathcal{V}_k \mid \delta^2 W = 0\},\\ \mathcal{D}_k &:= \{D \in \mathcal{V}_k \mid \delta D = 0\},\\ \mathcal{V}_k/\mathcal{W}_k &:= \{\hat{H} \in \mathcal{V}_k \mid \langle \hat{H}, W\rangle = 0 \text{ for all } W \in \mathcal{W}_k\},\\ \mathcal{W}_k/\mathcal{D}_k &:= \{\hat{W} \in \mathcal{W}_k \mid \langle \hat{W}, D\rangle = 0 \text{ for all } D \in \mathcal{D}_k\}.\end{aligned}$$

We remark that this decomposition, introduced in [22], shares many similarities with the decomposition mentioned in [13].

From Corollary 6.7, we see that a key distinction between the Pohozaev quadratic form for the Q-curvature problem and the Yamabe problem is the presence of the following three terms:

$$\int_{\mathbb{S}^{n-1}} (\mathcal{L}_k H^{(k)})_{ij}(\mathcal{L}_m H^{(m)})_{ij}, \quad \int_{\mathbb{S}^{n-1}} \delta_i H^{(k)} \delta_i H^{(m)} \quad \text{and} \quad \int_{\mathbb{S}^{n-1}} \delta^2 H^{(k)} \delta^2 H^{(m)}.$$

In [22], the eigenvectors of $\mathcal{L}_k$ in $\mathcal{V}_k/\mathcal{W}_k$ and $\mathcal{D}_k$ were found by using the spherical harmonics. Readers can refer to Lemmas B.5–B.6 for their explicit expressions. In contrast, the eigenvectors of $\mathcal{L}_k$ in $\mathcal{W}_k/\mathcal{D}_k$ were not written out in [22]. As we will see, they all share the same eigenvalue $-\frac{(n+k-2)k}{2}$, and only that eigenvalue information was necessary there. For the Q-curvature problem, the appearance of the extra terms

$$\int_{\mathbb{S}^{n-1}} \delta_i H^{(k)} \delta_i H^{(m)} \quad \text{and} \quad \int_{\mathbb{S}^{n-1}} \delta^2 H^{(k)} \delta^2 H^{(m)}$$

in the expansion of $I_{1,\epsilon}[\widetilde{H}^{(k)}, \widetilde{H}^{(m)}]$ forces us to find the explicit expressions of the eigenvectors in $\mathcal{W}_k/\mathcal{D}_k$. This is the content of Lemma 6.8, which will be reinforced in Lemma B.7 below.

Let $V = (V_1, \ldots, V_n)$ be a smooth vector field on $\mathbb{R}^n$ and $\delta V := V_{i,i}$ be its divergence. We define the conformal Killing operator by

$$(\mathscr{D}V)_{ij} := \partial_i V_j + \partial_j V_i - \frac{2}{n}\delta V \delta_{ij}. \tag{6.13}$$

Lemma 6.8. *For $k \geq 2$ and $q = 1, \ldots, \lfloor \frac{k-1}{2} \rfloor$, we write $s = k - 2q$. We choose a smooth vector field on $\mathbb{R}^n$,*

$$V^{(s+1)} \in \{V = (V_1, \ldots, V_n) \mid V_i \in \mathcal{H}_{s+1} \text{ for } i = 1, \ldots, n,\ \delta V = 0,\ x_i V_i = 0\},$$

and set $\hat{W} := \mathrm{Proj}[|x|^{2q} \mathscr{D} V^{(s+1)}] \in \mathcal{V}_k$. Then $\hat{W} \in \mathcal{W}_k$. In addition, it holds that

$$\mathcal{L}_k \hat{W} = -\frac{(n+k-2)k}{2} \hat{W}. \tag{6.14}$$

Proof. We infer from Lemmas A.1 and B.4 that

$$\hat{W}_{ij} = |x|^{2q} (\mathscr{D}V^{(s+1)})_{ij} - s|x|^{2q-2}(x_j V_i^{(s+1)} + x_i V_j^{(s+1)}).$$

A straightforward computation yield

$$\begin{aligned}\delta_j \hat{W} &= -(n+s)s|x|^{2q-2} V_j^{(s+1)},\\ \Delta \hat{W}_{ij} &= (2q(2q+2s+n-2) - 2s)\, |x|^{2q-2} (\mathscr{D}V^{(s+1)})_{ij}\\ &\quad - s(2q-2)(2q+2s+n)|x|^{2q-4}(V_i x_j + V_j x_i).\end{aligned}$$

It follows that $\delta^2 \hat{W} = 0$ so $\hat{W} \in \mathcal{W}_k$.

Also, by combining these identities with the definition of $\mathcal{L}_k$ in (6.11), we derive

$$(\mathcal{L}_k \hat{W})_{ij} = \frac{|x|^2}{2}(\partial_i \delta_j \hat{W} + \partial_j \delta_i \hat{W} - \Delta \hat{W}_{ij}) - \frac{2q+s}{2}(x_j \delta_i \hat{W} + x_i \delta_j \hat{W})$$

$$
\begin{aligned}
&= \frac{|x|^{2q}}{2}\left[-(n+s)s(\partial_i V_j + \partial_j V_i) - (2q(2q+2s+n-2) - 2s)(\mathscr{D}V)_{ij}\right] \\
&\quad + \frac{s|x|^{2q-2}}{2}[-(n+s)(2q-2) + (2q-2)(2q+2s+n) + (2q+s)(n+s)](V_i x_j + V_j x_i) \\
&= -\frac{(n+2q+s-2)(2q+s)}{2}\hat{W}_{ij} = -\frac{(n+k-2)k}{2}\hat{W}_{ij},
\end{aligned}
$$

which reads (6.14). □

Remark 6.9.
1. Note that

$$-\frac{(n+k-2)k}{2} \neq -q(n-2q+2k-2) \quad \text{for } q = 0, \dots, \lfloor \tfrac{k-2}{2} \rfloor.$$

In light of (6.14) and (B.7), it follows that $\hat{W} = \mathrm{Proj}[|x|^{2q}\mathscr{D}V^{(s+1)}] \in \mathcal{W}_k/\mathcal{D}_k \subset \mathcal{W}_k$.
2. In Lemma B.7, we verify that every element of $\mathcal{W}_k/\mathcal{D}_k$ can be expressed as a linear combination of matrices $\hat{W}$'s. ⋄

The following corollary comes from Lemmas B.5–B.7.

Corollary 6.10. *Given any $\bar{H} \in \mathcal{V}_k$ for $k \geq 2$, there exist $\hat{H} \in \mathcal{V}_k/\mathcal{W}_k$, $\hat{W} \in \mathcal{W}_k/\mathcal{D}_k$, and $\hat{D} \in \mathcal{D}_k$ such that*

$$\bar{H} = \hat{H} + \hat{W} + \hat{D}.$$

In fact, one can pick $\hat{H}_1, \dots, \hat{H}_{\lfloor \frac{k-2}{2} \rfloor} \in \mathcal{V}_k/\mathcal{W}_k$ satisfying (B.5)–(B.6)*, $\hat{W}_1, \dots, \hat{W}_{\lfloor \frac{k-1}{2} \rfloor} \in \mathcal{W}_k/\mathcal{D}_k$ satisfying* (B.10)–(B.11)*, and $\hat{D}_0, \dots, \hat{D}_{\lfloor \frac{k-2}{2} \rfloor} \in \mathcal{D}_k$ satisfying* (B.7)–(B.8) *such that*

$$\bar{H} = \sum_{q=1}^{\lfloor \frac{k-2}{2} \rfloor} \hat{H}_q + \sum_{q=1}^{\lfloor \frac{k-1}{2} \rfloor} \hat{W}_q + \sum_{q=0}^{\lfloor \frac{k-2}{2} \rfloor} \hat{D}_q.$$

In particular, $\mathcal{V}_2 = \mathcal{D}_2$ and $\mathcal{V}_3 = \mathcal{W}_3$.

We are now ready to prove Proposition 6.4. Due to its technical nature, we will outline the main structure of the proof here and postpone several detailed computations to Appendix B.

Proof of Proposition 6.4. Given $k = 2, \dots, d$, we write

$$H^{(k)} = \hat{H}^{(k)} + \hat{W}^{(k)} + \hat{D}^{(k)},$$

where $\hat{H}^{(k)} \in \mathcal{V}_k/\mathcal{W}_k$, $\hat{W}^{(k)} \in \mathcal{W}_k/\mathcal{D}_k$ and $\hat{D}^{(k)} \in \mathcal{D}_k$. Then $\hat{H}^{(2)} = \hat{H}^{(3)} = 0$ and $\hat{W}^{(2)} = 0$. We also set

$$\check{H}^{(k)}(y) := \hat{H}^{(k)}(\epsilon y), \quad \check{W}^{(k)}(y) := \hat{W}^{(k)}(\epsilon y), \quad \text{and} \quad \check{D}^{(k)}(y) := \hat{D}^{(k)}(\epsilon y)$$

so that $\widetilde{H}^{(k)} = \check{H}^{(k)} + \check{W}^{(k)} + \check{D}^{(k)}$. As shown in Lemma B.9, it holds that

$$I_{1,\epsilon}\big[\widetilde{H}^{(k)}, \widetilde{H}^{(m)}\big] = I_{1,\epsilon}\big[\check{H}^{(k)}, \check{H}^{(m)}\big] + I_{1,\epsilon}\big[\check{W}^{(k)}, \check{W}^{(m)}\big] + I_{1,\epsilon}\big[\check{D}^{(k)}, \check{D}^{(m)}\big]. \tag{6.15}$$

By (3.5), we have that $\dot{R}[\check{W}^{(k)}] = \delta^2 \check{W}^{(k)} = 0$ and $\dot{R}[\check{D}^{(k)}] = \delta(\delta \check{D}^{(k)}) = 0$. This together with (3.8) leads to

$$L_1[\check{W}^{(k)}]u = L_1[\check{D}^{(k)}]u = 0 \quad \text{for } u \text{ radial} \quad \text{and} \quad \Psi[\check{W}^{(k)}] = \Psi[\check{D}^{(k)}] = 0. \tag{6.16}$$

From (6.9), (6.8), (6.15), (6.6)–(6.7), and (6.16), we readily observe

$$I_\epsilon\big[\widetilde{H}, \widetilde{H}\big] = \sum_{k,m=2}^{d} I_{1,\epsilon}\big[\check{D}^{(k)}, \check{D}^{(m)}\big] + \sum_{k,m=3}^{d} I_{1,\epsilon}\big[\check{W}^{(k)}, \check{W}^{(m)}\big] + \sum_{k,m=4}^{d} I_\epsilon\big[\check{H}^{(k)}, \check{H}^{(m)}\big]. \tag{6.17}$$

For the remainder of the proof, we analyze each term on the right-hand side of (6.17).

Case 1. Let us study the term $\sum_{k,m=2}^{d} I_{1,\epsilon}[\check{D}^{(k)}, \check{D}^{(m)}]$. Thanks to Lemma B.6, we can write $\hat{D}^{(k)} = \sum_{q=0}^{\lfloor \frac{k-2}{2} \rfloor} |x|^{2q} M^{(k-2q)}$. Setting $s = k - 2q$ for $q = 0, \ldots, \lfloor \frac{k-2}{2} \rfloor$ and

$$\begin{cases} M_q^{(s)} := M^{(k-2q)}, & E_s^D := \displaystyle\sum_{q=0}^{\lfloor \frac{d-s}{2} \rfloor} |x|^{2q} M_q^{(s)}, \\ \widetilde{M}_q^{(s)}(y) := M_q^{(s)}(x), & \widetilde{E}_s^D(y) := E_s^D(x) \quad \text{for } x = \epsilon y \in \mathbb{R}^n, \end{cases} \tag{6.18}$$

we obtain

$$\sum_{k=2}^{d} \check{D}^{(k)} = \sum_{k=2}^{d} \sum_{q=0}^{\lfloor \frac{k-2}{2} \rfloor} |x|^{2q} \widetilde{M}^{(k-2q)} = \sum_{s=2}^{d} \sum_{q=0}^{\lfloor \frac{d-s}{2} \rfloor} |x|^{2q} \widetilde{M}_q^{(s)} = \sum_{s=2}^{d} \widetilde{E}_s^D.$$

As pointed out in Lemma B.10, we have that $I_{1,\epsilon}[\widetilde{E}_s^D, \widetilde{E}_{s'}^D] = 0$ for $s \neq s'$, which combined with (6.10) implies that

$$\begin{aligned} \sum_{k,m=2}^{d} I_{1,\epsilon}\big[\check{D}^{(k)}, \check{D}^{(m)}\big] &= \sum_{s=2}^{d} I_{1,\epsilon}\big[\widetilde{E}_s^D, \widetilde{E}_s^D\big] = \sum_{s=2}^{d} \sum_{q,q'=0}^{\lfloor \frac{d-s}{2} \rfloor} I_{1,\epsilon}\left[|x|^{2q} \widetilde{M}_q^{(s)}, |x|^{2q'} \widetilde{M}_{q'}^{(s)}\right] \\ &= \sum_{s=2}^{d} \sum_{q,q'=0}^{\lfloor \frac{d-s}{2} \rfloor} |\log \epsilon|^{\theta_{q+q'+s}} J_1\left[\epsilon^{2q+s}|x|^{2q} M_q^{(s)}, \epsilon^{2q'+s}|x|^{2q'} M_{q'}^{(s)}\right] + O(\epsilon^{n-4}). \end{aligned}$$

Lemma B.11 tells us that if $8 \le n \le 24$, then there exists a constant $C = C(n) > 0$ such that

$$\begin{aligned} \sum_{k,m=2}^{d} I_{1,\epsilon}\big[\check{D}^{(k)}, \check{D}^{(m)}\big] &\ge C^{-1} \sum_{s=2}^{d} \sum_{q=0}^{\lfloor \frac{d-s}{2} \rfloor} |\log \epsilon|^{\theta_{2q+s}} \left\|\epsilon^{2q+s}|x|^{2q} M_q^{(s)}\right\|^2 + O(\epsilon^{n-4}) \\ &= C^{-1} \sum_{k=2}^{d} \epsilon^{2k} |\log \epsilon|^{\theta_k} \big\|\hat{D}^{(k)}\big\|^2 + O(\epsilon^{n-4}). \end{aligned} \tag{6.19}$$

Case 2. Let us study the term $\sum_{k,m=3}^{d} I_{1,\epsilon}[\check{W}^{(k)}, \check{W}^{(m)}]$. Thanks to Lemma B.7, we can write $\hat{W}^{(k)} = \sum_{q=1}^{\lfloor \frac{k-1}{2} \rfloor} \mathrm{Proj}[|x|^{2q} \mathscr{D} V^{(k-2q+1)}]$. Setting $s = k - 2s$ for $q = 1, \ldots, \lfloor \frac{k-1}{2} \rfloor$ and

$$\begin{cases} V_q^{(s+1)} := V^{(k-2q+1)}, & E_s^W := \displaystyle\sum_{q=1}^{\lfloor \frac{d-s}{2} \rfloor} \mathrm{Proj}\left[|x|^{2q} \mathscr{D} V_q^{(s+1)}\right], \\ \widetilde{V}_q^{(s+1)}(y) := V_q^{(s+1)}(x), & \widetilde{E}_s^W(y) := E_s^W(x) \quad \text{for } x = \epsilon y \in \mathbb{R}^n, \end{cases} \tag{6.20}$$

we obtain

$$\sum_{k=3}^{d} \check{W}^{(k)} = \sum_{k=3}^{d} \sum_{q=1}^{\lfloor \frac{k-1}{2} \rfloor} \mathrm{Proj}\left[|x|^{2q} \mathscr{D} \widetilde{V}^{(k-2q+1)}\right] = \sum_{s=1}^{d-2} \sum_{q=1}^{\lfloor \frac{d-s}{2} \rfloor} \mathrm{Proj}\left[|x|^{2q} \mathscr{D} \widetilde{V}_q^{(s+1)}\right] = \sum_{s=1}^{d-2} \widetilde{E}_s^W.$$

By Lemma B.10 and (6.10), we deduce

$$\begin{aligned} \sum_{k,m=3}^{d} I_{1,\epsilon}\big[\check{W}^{(k)}, \check{W}^{(m)}\big] = &\sum_{s=1}^{d-2} \sum_{q,q'=1}^{\lfloor \frac{d-s}{2} \rfloor} |\log \epsilon|^{\theta_{q+q'+s}} J_1\left[\mathrm{Proj}\left[|x|^{2q} \mathscr{D} V_q^{(s+1)}\right], \mathrm{Proj}\left[|x|^{2q'} \mathscr{D} V_{q'}^{(s+1)}\right]\right] \\ &+ O(\epsilon^{n-4}). \end{aligned}$$

If $n=8$ or 9, then $d=2$ so that $\sum_{k,m=3}^d I_{1,\epsilon}[\check{W}^{(k)},\check{W}^{(m)}]=0$. If $10\le n\le 28$, then Lemma B.12 tells us that there exists a constant $C=C(n)>0$ such that

$$\sum_{k,m=3}^{d} I_{1,\epsilon}\big[\check{W}^{(k)},\check{W}^{(m)}\big]\ge C^{-1}\sum_{k=3}^{d}\epsilon^{2k}|\log\epsilon|^{\theta_k}\big\|\hat{W}^{(k)}\big\|^2+O(\epsilon^{n-4}). \tag{6.21}$$

Case 3. Let us study the term $\sum_{k,m=4}^d I_\epsilon[\check{H}^{(k)},\check{H}^{(m)}]$. Thanks to Lemma B.5, we can write $\hat{H}^{(k)}=\sum_{q=1}^{\lfloor\frac{k-2}{2}\rfloor}\mathrm{Proj}[|x|^{2q+2}\nabla^2P^{(k-2q)}]$. Setting $s=k-2s$ for $q=1,\dots,\lfloor\frac{k-2}{2}\rfloor$ and

$$\begin{cases} P_q^{(s)}:=P^{(k-2q)}, \quad \hat{H}_q^{(2q+s)}:=\mathrm{Proj}\left[|x|^{2q+2}\nabla^2P_q^{(s)}\right], \quad E_s^H:=\displaystyle\sum_{q=1}^{\lfloor\frac{d-s}{2}\rfloor}\hat{H}_q^{(2q+s)}, \\ \widetilde{P}_q^{(s)}:=\widetilde{P}^{(k-2q)}, \quad \check{H}_q^{(2q+s)}:=\mathrm{Proj}\left[|x|^{2q+2}\nabla^2\widetilde{P}_q^{(s)}\right], \quad \widetilde{E}_s^H:=\displaystyle\sum_{q=1}^{\lfloor\frac{d-s}{2}\rfloor}\check{H}_q^{(2q+s)}, \end{cases} \tag{6.22}$$

we obtain

$$\sum_{k=4}^d\check{H}^{(k)}=\sum_{k=4}^d\sum_{q=1}^{\lfloor\frac{k-2}{2}\rfloor}\mathrm{Proj}\left[|x|^{2q+2}\nabla^2\widetilde{P}^{(k-2q)}\right]=\sum_{s=2}^{d-2}\sum_{q=1}^{\lfloor\frac{d-s}{2}\rfloor}\check{H}_q^{(2q+s)}=\sum_{s=2}^{d-2}\widetilde{E}_s^H.$$

We note that $\Psi[\widetilde{E}_s^H]=\sum_{q=1}^{\lfloor\frac{d-s}{2}\rfloor}\Psi[\check{H}_q^{(2q+s)}]$ is well-defined, so are $I_{2,\epsilon}[\widetilde{E}_s^H,\widetilde{E}_{s'}^H]$ and $I_{3,\epsilon}[\widetilde{E}_s^H,\widetilde{E}_{s'}^H]$. By Lemma B.10, we deduce

$$\sum_{k,m=4}^d I_{i,\epsilon}\big[\check{H}^{(k)},\check{H}^{(m)}\big]=\sum_{s=2}^{d-2}\sum_{q,q'=1}^{\lfloor\frac{d-s}{2}\rfloor}I_{i,\epsilon}\left[\check{H}_q^{(2q+s)},\check{H}_{q'}^{(2q'+s)}\right]\quad\text{for } i=1,2,3.$$

If $8\le n\le 11$, then $d=2$ or 3 so that $\sum_{k,m=4}^d I_{i,\epsilon}[\check{H}^{(k)},\check{H}^{(m)}]=0$ for $i=1,2,3$. If $12\le n\le 32$, then Lemma B.13 tells us that there exists a constant $C=C(n)>0$ such that

$$\sum_{k,m=4}^{d} I_{\epsilon}\big[\check{H}^{(k)},\check{H}^{(m)}\big]\ge C^{-1}\sum_{k=4}^{d}\epsilon^{2k}|\log\epsilon|^{\theta_k}\big\|\hat{H}^{(k)}\big\|^2+O(\epsilon^{n-4}). \tag{6.23}$$

As a consequence, by putting (6.19), (6.21) and (6.23) into (6.17), and employing the identity $\|H^{(k)}\|^2=\|\hat{H}^{(k)}\|^2+\|\hat{W}^{(k)}\|^2+\|\hat{D}^{(k)}\|^2$ for $k=2,\dots,d$ as well as the equivalence between two norms $|H^{(k)}|$ and $\|H^{(k)}\|$, we conclude the proof of Proposition 6.4. □

6.3. **Proof of Theorem 6.1.** By virtue of (6.4), Lemma 6.3 with $\eta\in(0,1)$ small, and Proposition 6.4, there exists a constant $C=C(n)>0$ such that

$$O(\epsilon_a^{n-4})\ge C^{-1}\sum_{k=2}^d\epsilon_a^{2k}|\log\epsilon_a|^{\theta_k}|H_a^{(k)}|^2.$$

Therefore,

$$|H_a^{(k)}|^2\le C\epsilon_a^{n-4-2k}|\log\epsilon_a|^{-\theta_k}\quad\text{for } k=2,\dots,d, \tag{6.24}$$

which implies

$$|\nabla_{g_a}^kW_{g_a}|^2(\sigma_a)\le C\epsilon_a^{n-8-2k}|\log\epsilon_a|^{-\theta_{k+2}}\quad\text{for } k=0,\dots,d-2.$$

This concludes the proof of Theorem 6.1.

6.4. **A corollary.** By Theorem 6.1 (or (6.24)) and Corollary 5.2, a direct corollary follows.

Corollary 6.11. *Under the conditions of Theorem 6.1, there exist $\delta_0 \in (0,1)$ and $C > 0$ such that*

$$|\partial_\beta(u_a - w_a - \Psi_a)|(x) \le C\epsilon_a^{\frac{n-4}{2}+|\beta|}(\epsilon_a + |x|)^{-1-|\beta|}$$

for all $a \in \mathbb{N}$, multi-indices β with $|\beta| = 0, 1, \ldots, 4$, and $|x| \le \delta_0$.

7. Local Sign Restriction

A slight modification of the proof of Theorem 6.1 yields the following local sign restriction of the function $\mathbf{P}$ in (2.1). We skip the proof.

Proposition 7.1. *Let $8 \le n \le 24$ and $\sigma_a \to \bar{\sigma} \in M$ be an isolated simple blowup point for a sequence $\{u_a\}$ of solutions to* (2.4)*, where* $\det g_a = 1$ *near $\sigma_a \in M$. Assume that $u_a(\sigma_a)u_a \to \mathfrak{g}$ in $C^3_{\text{loc}}(B^{g_\infty}_{\delta_0}(\bar{\sigma}) \setminus \{\bar{\sigma}\})$, where $\delta_0 \in (0,1)$ is the number in Proposition 5.1 and $\mathfrak{g}$ is the function in Proposition 2.4. Then*

$$\liminf_{r\to 0} \mathbf{P}(r, \mathfrak{g}) \ge 0.$$

We also recall a well-known result, which can be found in e.g. [28, Proposition 7.1].

Proposition 7.2. *For any given $\epsilon, R > 0$, there exists a constant $C = C(M, g, \epsilon, R) > 0$ such that if a solution u of equation (1.3) satisfies* $\max_{\sigma\in M} u(\sigma) \ge C$*, then there exist local maximum points $\sigma^1, \ldots, \sigma^N \in M$, where $N \in \mathbb{N}$ depends on u, such that*

1. *$\{B_{r_i}(\sigma^i)\}_{1\le i\le N}$ are disjoint, where $r_i = Ru(\sigma^i)^{-\frac{2}{n-4}}$;*
2. *For each $i = 1, \ldots, N$, it holds that $\|u(\sigma^i)^{-1}u(u(\sigma^i)^{-\frac{2}{n-4}}x) - w(x)\|_{C^4(B_R(0))} < \epsilon$;*
3. *$u(\sigma) \le Cd_g(\sigma, \{\sigma^1, \ldots, \sigma^N\})^{-\frac{n-4}{2}}$ for any $\sigma \in M \setminus \{\sigma^1, \ldots, \sigma^N\}$.*

Finally, the following corollaries can be achieved as in [28, Proposition 6.2, Proposition 7.2].

Corollary 7.3. *Let $8 \le n \le 24$ and $\sigma_a \to \bar{\sigma} \in M$ be an isolated blowup point for a sequence $\{u_a\}$ of solutions to* (2.4)*, where* $\det g_a = 1$ *near $\sigma_a \in M$. Then $\sigma_a \to \bar{\sigma} \in M$ is isolated simple.*

Corollary 7.4. *Let $8 \le n \le 24$. Let ϵ, R, $C = C(M, g, \epsilon, R)$, u and $\{\sigma^1, \ldots, \sigma^N\}$ be as in Proposition 7.2. Suppose that $\epsilon > 0$ is sufficiently small and $R > 0$ is sufficiently large. Then, there exists $\bar{C} = \bar{C}(M, g, \epsilon, R)$ such that if* $\max_{\sigma\in M} u(\sigma) \ge C$*, then $d_g(\sigma^i, \sigma^j) \ge \bar{C}$ for any $i \ne j \in \{1, \ldots, N\}$.*

8. Proof of The Compactness Theorem

Because $\text{Ker}P_g = \{0\}$, the Green's function G_g of the Paneitz operator P_g has the expansion

$$G_g(x, 0) = c_n^{-1}|x|^{4-n}\left(1 + \sum_{k=4}^{n-4} \psi^{(k)}(x)\right) + A + B\log|x| + O^{(4)}(|x|) \tag{8.1}$$

in normal coordinates x centered at a point $\bar{\sigma} \in M$, provided $\det g = 1$ near $\bar{\sigma}$. Here, $c_n = 2(n-2)(n-4)|\mathbb{S}^{n-1}|$, $A, B \in \mathbb{R}$, $\psi^{(k)} \in \mathcal{P}_k$, and $\int_{\mathbb{S}^{n-1}} \psi^{(n-4)} = 0$. If n is odd, we also have $B = 0$. For its proof, refer to [18, Subsection 2.4].

Lemma 8.1. *Let $8 \le n \le 24$ and $\sigma_a \to \bar{\sigma} \in M$ be an isolated simple blowup point for a sequence $\{u_a\}$ of solutions to* (2.4)*, where* $\det g_a = 1$ *near $\sigma_a \in M$. Assume that $u_a(\sigma_a)u_a \to c_nG_{g_\infty}(\cdot, \bar{\sigma}) + \mathfrak{h}$ in $C^3_{\text{loc}}(B^{g_\infty}_{\delta_0}(\bar{\sigma}) \setminus \{\bar{\sigma}\})$, where $\delta_0 \in (0,1)$ is the number in Proposition 5.1, G_{g_∞} is* *the Green's function of the Paneitz operator P_{g_∞}, and $\mathfrak{h} \in C^5(B^{g_\infty}_{\delta_0}(\bar\sigma))$ is a nonnegative function from Proposition 2.4. Then it holds that*

$$G_{g_\infty}(x,0) = c_n^{-1}|x|^{4-n}\left(1 + \sum_{k=d+1}^{n-4} \psi^{(k)}(x)\right) + A + O^{(4)}(|x|),$$
$$\text{where} \quad \int_{\mathbb{S}^{n-1}} \psi^{(k)} = 0 \quad \text{and} \quad \int_{\mathbb{S}^{n-1}} x_i\psi^{(k)} = 0. \tag{8.2}$$

Proof. First, we prove that for $k = 4, \dots, d$, $\psi^{(k)} = 0$. Let $\Psi_a^{(\geq d+1)}$ be the solution to equation

$$\Delta^2 \Psi_a^{(\geq d+1)} - \tilde{\mathfrak{c}}(n) w_a^{\frac{8}{n-4}} \Psi_a^{(\geq d+1)} = -\sum_{k=d+1}^{K} L_1[H_a^{(k)}]w_a \quad \text{in } \mathbb{R}^n.$$

From (5.1), Theorem 6.1 and Corollary 6.11, we obtain that

$$\lim_{a\to 0} \epsilon_a^{-\frac{n-4}{2}} \left| u_a - w_a - \Psi_a^{(\geq d+1)} \right|(x) \leq C|x|^{-1}$$

for $0 < |x| \leq \delta_0$. Since

$$\epsilon_a^{-\frac{n-4}{2}} u_a \to c_n G_{g_\infty}(\cdot, 0) + \mathfrak{h} \quad \text{and} \quad \epsilon_a^{-\frac{n-4}{2}} w_a \to |\cdot|^{4-n} \quad \text{in } C^3_{\text{loc}}(B_{\delta_0} \setminus \{0\})$$

and

$$\epsilon_a^{-\frac{n-4}{2}} \left| \widetilde{\Psi}_a^{(\geq d+1)} \right|(x) \leq C \sum_{k=d+1}^{K} (\epsilon_a + |x|)^{4-n+k} \to C \sum_{k=d+1}^{K} |x|^{4-n+k},$$

the function $\psi^{(k)}$ in (8.1) vanishes for $k = 4, \dots, d$.

Next, for $k = d+1, \dots, n-3$, we prove that $\int_{\mathbb{S}^{n-1}} \psi^{(k)} = B = 0$. On one hand, due to Theorem 6.1 and Lemmas 3.6 and 4.1,

$$\int_{\{|x|=r\}} (P_{g_\infty} - \Delta^2)(|x|^{4-n}) = O(r^{2d+1}) \leq Cr^{n-4},$$
$$\int_{\{|x|=r\}} (P_{g_\infty} - \Delta^2)(|x|^{4-n}\psi^{(k)}) = O(r^{d+k}) \leq Cr^{n-4}.$$

On the other hand, it holds that $\int_{\{|x|=r\}} \Delta^2(|x|^{4-n}) = 0$, $\int_{\{|x|=r\}} \Delta^2(|x|^{4-n}\psi^{(n-4)}) = 0$, and

$$\int_{\{|x|=r\}} \Delta^2(|x|^{4-n}\psi^{(k)}) = 4(n-4)(n-2)k(k+1)r^{k-1} \int_{\mathbb{S}^{n-1}} \psi^{(k)},$$
$$\int_{\{|x|=r\}} \Delta^2(B\log|x|) = -2(n-2)(n-4)|\mathbb{S}^{n-1}|Br^{n-5}.$$

Because G_{g_∞} is the Green's function of P_{g_∞}, we also have

$$\int_{\{|x|=r\}} (P_{g_\infty} - \Delta^2)G_{g_\infty}(\cdot, 0)dS = -\int_{\{|x|=r\}} \Delta^2 G_{g_\infty}(\cdot, 0)dS$$

for any $r \in (0, \delta_0)$. It follows that $\int_{\mathbb{S}^{n-1}} \psi^{(k)} = O(r)$ and $B = O(r)$. Taking $r \to 0$, we conclude that $\int_{\mathbb{S}^{n-1}} \psi^{(k)} = B = 0$.

Finally, in a similar manner, we can also derive that $\int_{\mathbb{S}^{n-1}} x_i\psi^{(k)} = 0$ for $k = d+1, \dots, n-4$. □

Remark 8.2. Note that in the proof, we have

$$\epsilon_a^{-\frac{n-4}{2}} \Psi_a^{(\geq d+1)} \to \sum_{k=d+1}^{K} \psi^{(k)} \text{ in } C^3_{\text{loc}}(B_{\delta_0} \setminus \{0\}).$$

By explicitly solving of the linearized equations according to Proposition 4.6, we could provide an alternative method for computing the expansion of the Green's function. $\diamond$

The following lemma generalizes [4, Proposition 6]. The analogous result for the Yamabe problem is well-known; see e.g. [6, Proposition 19].

Proposition 8.3. *Let $n \geq 5$ and (M^n, g) be a closed manifold such that $\ker P_g = 0$. Suppose that the Weyl tensor vanishes up to an order greater than or equal to $d-2$ at a point $\bar{\sigma} \in M$, and that the Green's function G_g of P_g satisfies the expansion* (8.2) *in a small neighborhood of $\bar{\sigma}$. Then, $(M \setminus \{\bar{\sigma}\}, \hat{g} := G_g^{\frac{4}{n-4}} g)$ is an asymptotically flat manifold with decay rate $\tau > \frac{n-4}{2}$ such that $Q_{\hat{g}} = 0$. Moreover, the higher-order mass of $\hat{g}$ defined in* (2.3) *is a positive multiple of the constant A in* (8.2)*:*

$$m(\hat{g}) = \frac{4(n-1)}{n-4} A.$$

Proof. In conformal normal coordinates centered at $\bar{\sigma}$, we have $\mathrm{tr} g = n$ and $x_i g_{ij}(x) = x_j$.

We introduce an inversion variable given by $y = c_n^{-\frac{2}{n-4}} \frac{x}{|x|^2}$, whose transition relation is $\partial_{y_i} = c_n^{\frac{2}{n-4}} (|x|^2 \delta_{ij} - 2x_i x_j) \partial_{x_j}$. Then we have

$$\begin{aligned} \hat{g}_{ij}(y) &= c_n^{\frac{4}{n-4}} |x|^4 G(x)^{\frac{4}{n-4}} g_{ij}(x) \\ &= \left(1 + \sum_{k=d+1}^{n-4} \psi^{(k)}(x) + c_n A |x|^{n-4} + O^{(4)}(|x|^{n-3}) \right)^{\frac{4}{n-4}} g_{ij}(x) \\ &:= (F(x))^{\frac{4}{n-4}} g_{ij}(x), \end{aligned}$$

where $g_{ij}(x) = \delta_{ij} + O(|x|^{d+1})$ by the Weyl vanishing condition. Hence $\hat{g}_{ij}(y) = \delta_{ij} + O(|y|^{-(d+1)})$. Since $d+1 = \lfloor \frac{n-4}{2} \rfloor + 1 > \frac{n-4}{2}$, we deduce that $(M \setminus \{\bar{\sigma}\}, \hat{g})$ is asymptotic flat.

Next, we evaluate the higher-order mass:

$$\begin{aligned} m(\hat{g}) &= \lim_{\varepsilon \to 0} \int_{\{|y| = c_n^{-\frac{2}{n-4}} \varepsilon^{-1}\}} \partial_{y_k} (\hat{g}_{ii,jj} - \hat{g}_{ij,ij}) \frac{y_k}{|y|} dS \\ &= \lim_{\varepsilon \to 0} c_n^{-\frac{2(n-1)}{n-4}} \int_{\{|x| = \varepsilon\}} \partial_{y_k} (\hat{g}_{ii,jj} - \hat{g}_{ij,ij}) \frac{x_k}{|x|} |x|^{2-2n} dS \\ &= -\lim_{\varepsilon \to 0} c_n^{-\frac{2(n-2)}{n-4}} \int_{\{|x| = \varepsilon\}} x_k \partial_{x_k} (\hat{g}_{ii,jj} - \hat{g}_{ij,ij}) |x|^{3-2n} dS. \end{aligned} \tag{8.3}$$

It holds that

$$\begin{aligned} c_n^{-\frac{2}{n-4}} \hat{g}_{ii,j} &= n(|x|^2 \delta_{ij} - 2x_i x_j) \partial_{x_i} (F^{\frac{4}{n-4}}), \\ c_n^{-\frac{4}{n-4}} \hat{g}_{ii,jj} &= n|x|^4 \Delta(F^{\frac{4}{n-4}}) - 2n(n-2)|x|^2 x_i \partial_{x_i} (F^{\frac{4}{n-4}}) \end{aligned}$$

and

$$\begin{aligned} c_n^{-\frac{2}{n-4}} \hat{g}_{ij,i} &= (|x|^2 g_{ij} - 2x_i x_j) \partial_{x_i} (F^{\frac{4}{n-4}}) + |x|^2 g_{ij,i} F^{\frac{4}{n-4}}, \\ c_n^{-\frac{4}{n-4}} \hat{g}_{ii,jj} &= |x|^4 g_{ij} \partial_{x_i x_j} (F^{\frac{4}{n-4}}) - 2(n-2)|x|^2 x_i \partial_{x_i} (F^{\frac{4}{n-4}}) \\ &\quad + 2|x|^4 g_{ij,i} \partial_{x_j} (F^{\frac{4}{n-4}}) + |x|^4 g_{ij,ij} F^{\frac{4}{n-4}}. \end{aligned}$$

Plugging the above computation of $\hat{g}_{ii,jj} - \hat{g}_{ij,ij}$ into (8.3), we obtain

$$m(\hat{g}) = 2(n-1)(n-2)c_n^{-2} \lim_{\varepsilon\to 0} \int_{\{|x|=\varepsilon\}} (x_k\partial_{x_k}+2)\circ(x_l\partial_{x_l})(F^{\frac{4}{n-4}})|x|^{5-2n}dS$$
$$- c_n^{-2} \lim_{\varepsilon\to 0} \int_{\{|x|=\varepsilon\}} (x_k\partial_{x_k}+4)\left(n\Delta(F^{\frac{4}{n-4}}) - \partial_{x_ix_j}(g_{ij}F^{\frac{4}{n-4}})\right)|x|^{7-2n}dS.$$

We now make use of the following assertion, whose derivation is deferred to the end of the proof.

Claim 8.4.

$$\int_{\{|x|=\varepsilon\}} (x_k\partial_{x_k}+4)\circ\partial_{x_ix_j}((g_{ij}-\delta_{ij})F^{\frac{4}{n-4}})dS = 0$$

With Claim 8.4, we obtain that

$$\begin{aligned}
&(n-1)^{-1}c_n^2 m(\hat{g})\\
&= \lim_{\varepsilon\to 0}\int_{\{|x|=\varepsilon\}} \left[2(n-2)(x_k\partial_{x_k}+2)\circ(x_l\partial_{x_l})(F^{\frac{4}{n-4}})\varepsilon^{5-2n} - (x_k\partial_{x_k}+4)\circ\Delta(F^{\frac{4}{n-4}})\varepsilon^{7-2n}\right]dS\\
&= \lim_{\varepsilon\to 0}\int_{\{|x|=\varepsilon\}} \Bigg[\frac{8(n-2)}{n-4}(x_k\partial_{x_k}+2)(x_l\partial_{x_l}F)\varepsilon^{5-2n} - \frac{4}{n-4}(x_k\partial_{x_k}+4)(\Delta F)\varepsilon^{7-2n}\\
&\qquad + \frac{4(n-8)}{(n-4)^2}(x_k\partial_{x_k}+4)((F_{,i})^2)\varepsilon^{7-2n}\Bigg]dS.
\end{aligned}$$

Because $(F_{,i})^2 = O(|x|^{2d})$ and $2d \geq n-5 > n-6$, the limit of the last term vanishes. So,

$$\begin{aligned}
&(n-1)^{-1}c_n^2 m(\hat{g})\\
&= \frac{8(n-2)}{n-4}\lim_{\varepsilon\to 0}\varepsilon^{5-2n}\int_{\{|x|=\varepsilon\}}(x_k\partial_{x_k}+2)\circ(x_l\partial_{x_l})\left(\sum_{k=d+1}^{n-4}\psi^{(k)}(x) + c_nA|x|^{n-4}\right)dS\\
&- \frac{4}{n-4}\lim_{\varepsilon\to 0}\varepsilon^{7-2n}\int_{\{|x|=\varepsilon\}}(x_k\partial_{x_k}+4)\circ\Delta\left(\sum_{k=d+1}^{n-4}\psi^{(k)}(x) + c_nA|x|^{n-4}\right)dS.
\end{aligned}$$

Using Lemma A.3 and $\int_{\mathbb{S}^{n-1}}\psi^{(k)} = 0$ for $k = d+1,\dots,n-4$, we arrive at

$$m(\hat{g}) = 8(n-1)(n-2)|\mathbb{S}^{n-1}|c_n^{-1}A.$$

Proof of Claim 8.4. Define

$$I(\varepsilon) := \int_{B^n(0,\varepsilon)} (x_k\partial_{x_k}+4)\circ\partial_{x_ix_j}((g_{ij}-\delta_{ij})F^{\frac{4}{n-4}})dx.$$

Since

$$\begin{aligned}
&(x_k\partial_{x_k}+4)\circ\partial_{x_ix_j}((g_{ij}-\delta_{ij})F^{\frac{4}{n-4}})\\
&= \partial_{x_ix_j}\circ(x_k\partial_{x_k}+2)((g_{ij}-\delta_{ij})F^{\frac{4}{n-4}})\\
&= \partial_{x_ix_j}\left((g_{ij}-\delta_{ij})(x_k\partial_{x_k}+2)(F^{\frac{4}{n-4}})\right) + \partial_{x_ix_j}\left(x_kg_{ij,k}F^{\frac{4}{n-4}}\right),
\end{aligned}$$

we can employ the divergence theorem to find

$$I(\varepsilon) = \frac{1}{\varepsilon}\int_{\{|x|=\varepsilon\}} x_i\partial_{x_j}\left((g_{ij}-\delta_{ij})(x_k\partial_{x_k}+2)(F^{\frac{4}{n-4}})\right) + x_i\partial_{x_j}\left(x_kg_{ij,k}F^{\frac{4}{n-4}}\right)dS.$$

Using $x_i(g_{ij}-\delta_{ij}) = 0$, $\mathrm{tr}(g-\delta) = 0$, and $x_ix_kg_{ij,k} = x_k(\delta_{kj}-g_{kj}) = 0$, we obtain that $I(\varepsilon) \equiv 0$. Then we can apply $I'(\varepsilon) \equiv 0$ and the coarea formula to deduce Claim 8.4. □

We now complete the proof of our main result.

Proof of Theorem 1.4. It is a simple consequence of Theorem 6.1 and Corollaries 7.3–7.4. □

Proof of Theorem 1.1. By elliptic regularity theory, it suffices to prove the uniform $L^\infty(M)$-boundedness of solutions of (1.3). Suppose by contradiction that there exist a sequence $\{g_a\}_{a\in\mathbb{N}} \subset [g]$ of metrics on M, a sequence $\{u_a\}_{a\in\mathbb{N}} \subset C^4(M)$, and a sequence $\{\sigma_a\}_{a\in\mathbb{N}} \subset M$ such that (2.4) holds, $\sigma_a \to \bar\sigma \in M$ is a blowup point for $\{u_a\}$, $\det g_a = 1$ near $\sigma_a \in M$, and g_a converges to a metric $g_\infty \in [g]$ in $C^l(M)$ as $a \to \infty$ for any $l \in \mathbb{N}$. Corollaries 7.3–7.4 imply that $\sigma_a \to \bar\sigma \in M$ is an isolated simple blowup point for $\{u_a\}$.

Owing to Proposition 2.4, $u_a(\sigma_a)u_a \to \mathfrak{g} := c_n G_{g_\infty}(\cdot,\bar\sigma) + \mathfrak{h}$ in $C^3_{\mathrm{loc}}(B^{g_\infty}_{\delta_0}(\bar\sigma) \setminus \{\bar\sigma\})$, where G_{g_∞} is the Green's function of the Paneitz operator P_{g_∞} and $\mathfrak{h} \in C^5(B^{g_\infty}_{\delta_0}(\bar\sigma))$ is a nonnegative function satisfying (2.5). Combining this fact with Proposition 7.1, we reach

$$\liminf_{r\to0} \mathbf{P}(r, G_{g_\infty}) = c_n^{-2} \liminf_{r\to0} \mathbf{P}(r, \mathfrak{g}) \geq 0. \tag{8.4}$$

By virtue of Lemma 8.1, G_{g_∞} has the expansion (8.2). If Condition (i) holds, then the positivity of the constant A of G_{g_∞} in (8.2) follows immediately. Under Condition (ii), we can deduce $A > 0$ from Proposition 8.3, Theorem 2.1 and the condition that M is not conformally diffeomorphic to the standard sphere $\mathbb{S}^n$. However, a direct computation using (8.2) tells us

$$\lim_{r\to0} \mathbf{P}(r, G_{g_\infty}) = -\frac{n-4}{2}A < 0,$$

which contradicts (8.4). This finishes the proof of Theorem 1.1. □

Appendix A. Useful Tools

In this appendix, we collect several useful results needed for the proof of Theorems 1.1 and 1.4. The proofs of Lemma A.1–Corollary A.5 follow directly from elementary computations, so we omit them.

Lemma A.1 (Euler's homogeneous function theorem)**.** *If $p^{(k)} \in \mathcal{P}_k$, then*

$$x_i\partial_i p^{(k)} = kp^{(k)}.$$

Lemma A.2. *If $|x|^{2s}p^{(k-2s)} \in \mathcal{P}_k$ for some $p^{(k-2s)} \in \mathcal{H}_{k-2s}$, then*

$$\Delta\left[|x|^{2s}p^{(k-2s)}\right] = 2s(2k-2s+n-2)|x|^{2s-2}p^{(k-2s)}.$$

Lemma A.3. *Assume that $n \geq 2$. For $F_i^{(k)} \in \mathcal{P}_k$,*

$$\int_{\mathbb{S}^{n-1}} x_iF_i^{(k)} = \int_{B_1} \partial_iF_i^{(k)} = \frac{1}{n+k-1}\int_{\mathbb{S}^{n-1}} \partial_iF_i^{(k)}.$$

Therefore, for $p^{(k)} \in \mathcal{P}_k$ with $k \in \mathbb{N}$,

$$\int_{\mathbb{S}^{n-1}} p^{(k)} = \frac{1}{k(n+k-2)}\int_{\mathbb{S}^{n-1}} \Delta p^{(k)}.$$

Lemma A.4. *Let $\mathcal{I}_n^l$ be the integral defined in (1.4). It holds that*

$$\mathcal{I}_n^l = \frac{2n-l-3}{2n-2}\mathcal{I}_{n-1}^l,$$
$$\mathcal{I}_n^l = \frac{2n-l-3}{l+1}\mathcal{I}_n^{l+2}.$$

Corollary A.5. *It holds that*

$$\mathcal{I}_n^l = \frac{l-1}{2n-l-1}\mathcal{I}_n^{l-2} = \frac{l-1}{2n-2}\mathcal{I}_{n-1}^{l-2},$$
$$\int_0^\infty \frac{(1-r^2)r^l}{(1+r^2)^n}dr = \frac{2n-2l-4}{2n-l-3}\mathcal{I}_n^l.$$

Lemma A.6. *For $l \in \mathbb{N}$, let A be an $l \times l$ positive-definite symmetric matrix and B an $l \times l$ positive semi-definite matrix. Then*

$$\mathrm{tr}(AB) \geq \lambda_1(A)\,\mathrm{tr}(B),$$

where $\lambda_1(A) > 0$ is the least eigenvalue of A.

Proof. We express A as $A = Q\Lambda Q^T$, where $\Lambda := \mathrm{diag}(\lambda_1, \ldots, \lambda_l)$ is a diagonal matrix and Q is an orthogonal matrix. We may assume that $0 < \lambda_1 \leq \cdots \leq \lambda_l$. Set $B' = Q^T BQ$. Then, B' is positive semi-definite, so

$$\mathrm{tr}(AB) = \mathrm{tr}(Q\Lambda Q^T B) = \mathrm{tr}(\Lambda Q^T BQ) = \mathrm{tr}(\Lambda B') = \sum_{i=1}^l \lambda_i B'_{ii} \geq \lambda_1 \mathrm{tr}(B') = \lambda_1 \mathrm{tr}(B)$$

as claimed. □

Appendix B. Technical Calculations for the Proof of Theorem 6.1

B.1. **Representation of $I_1[H^{(k)}, H^{(m)}]$ in polar coordinates.** We prove Proposition 6.5 and Corollary 6.7. We begin by establishing two preliminary results: Lemmas B.1–B.2.

Lemma B.1. *It holds that*

$$\int_{\mathbb{S}^{n-1}} \ddot{R}^{(k,m)} = \int_{\mathbb{S}^{n-1}} \left[\frac{1}{2}\delta_i H^{(k)}\delta_i H^{(m)} - \frac{1}{4}H_{ij,l}^{(k)}H_{ij,l}^{(m)}\right], \tag{B.1}$$

$$\int_{\mathbb{S}^{n-1}} \dot{\mathrm{Ric}}_{ij}^{(k)} H_{ij}^{(m)} = -\int_{\mathbb{S}^{n-1}} \left[2\ddot{R}^{(k,m)} + \frac{k(n+k+m-2)}{2}H_{ij}^{(k)}H_{ij}^{(m)}\right]. \tag{B.2}$$

Proof. Identity (B.1) is a direct corollary of (3.1), the definition of $\ddot{R}$ in (3.6), and Lemma A.3.

We next take (B.2) into account. Due to the definition of $\dot{\mathrm{Ric}}$ in (3.3), we have

$$\begin{aligned}\dot{\mathrm{Ric}}_{ij}^{(k)} H_{ij}^{(m)} &= H_{il,jl}^{(k)}H_{ij}^{(m)} - \frac{1}{2}H_{ij,ll}^{(k)}H_{ij}^{(m)} \\ &= \partial_j(H_{il,l}^{(k)}H_{ij}^{(m)}) - \frac{1}{2}\partial_l(H_{ij,l}^{(k)}H_{ij}^{(m)}) - H_{il,l}^{(k)}H_{ij,j}^{(m)} + \frac{1}{2}H_{ij,l}^{(k)}H_{ij,l}^{(m)}.\end{aligned}$$

Integrate it over $\mathbb{S}^{n-1}$, we discover

$$\int_{\mathbb{S}^{n-1}} \dot{\mathrm{Ric}}_{ij}^{(k)} H_{ij}^{(m)} = -\int_{\mathbb{S}^{n-1}} \left[2\ddot{R}^{(k,m)} + \frac{1}{2}\partial_l(H_{ij,l}^{(k)}H_{ij}^{(m)})\right].$$

Then, by invoking Lemma A.3 and Lemma A.1, we deduce (B.2). □

Lemma B.2. *It holds that*

$$\begin{aligned}\int_{\mathbb{S}^{n-1}} \ddot{R}^{(k,m)} = -\frac{1}{8}\int_{\mathbb{S}^{n-1}} \Big[&2(\mathcal{L}_k H^{(k)})_{ij}H_{ij}^{(m)} + 2H_{ij}^{(k)}(\mathcal{L}_m H^{(m)})_{ij} \\ &+(k+m)(n+k+m-2)H_{ij}^{(k)}H_{ij}^{(m)}\Big],\end{aligned}$$

$$\int_{\mathbb{S}^{n-1}} \dot{\mathrm{Ric}}_{ij}^{(k)}\dot{\mathrm{Ric}}_{ij}^{(m)} = \int_{\mathbb{S}^{n-1}} \left[(\mathcal{L}_k H^{(k)})_{ij}(\mathcal{L}_m H^{(m)})_{ij} + \frac{km}{2}\delta_i H^{(k)}\delta_i H^{(m)} + \frac{1}{n-1}\delta^2 H^{(k)}\delta^2 H^{(m)}\right].$$

Proof. Let us derive the first identity. According to the previous lemma, we have

$$\int_{\mathbb{S}^{n-1}} \ddot{R}^{(k,m)} = \frac{1}{4}\int_{\mathbb{S}^{n-1}} \left[\partial_l(H^{(k)}_{ij,j}H^{(m)}_{il}) + \partial_i(H^{(k)}_{lj,j}H^{(m)}_{il}) - \partial_j(H^{(k)}_{il,j}H^{(m)}_{il}) - 2\dot{\mathrm{Ric}}^{(k)}_{il}H^{(m)}_{il}\right].$$

By employing the definition of $\mathcal{L}_k$ in (6.11), (3.1), Lemma A.3, and Lemma A.1, we derive

$$\int_{\mathbb{S}^{n-1}} \ddot{R}^{(k,m)} = -\frac{1}{4}\int_{\mathbb{S}^{n-1}} \left[2(\mathcal{L}_kH^{(k)})_{ij}H^{(m)}_{ij} + k(n+k+m-2)H^{(k)}_{ij}H^{(m)}_{ij}\right].$$

Then, using the symmetry of $\ddot{R}^{(k,m)}$ with respect to k and m, we can obtain the first identity.

We next turn to the second identity. We use the definition of $\mathcal{L}_k$ and $\mathcal{L}_m$ to obtain

$$\begin{aligned}
&\int_{\mathbb{S}^{n-1}} (\mathcal{L}_kH^{(k)})_{ij}(\mathcal{L}_mH^{(m)})_{ij} \\
&= \int_{\mathbb{S}^{n-1}} \left[\dot{\mathrm{Ric}}^{(k)}_{ij}\dot{\mathrm{Ric}}^{(m)}_{ij} + \frac{km}{2}\delta_iH^{(k)}\delta_iH^{(m)} + \frac{1}{n-1}\delta^2H^{(k)}\delta^2H^{(m)}\right] \\
&\quad - \int_{\mathbb{S}^{n-1}} \dot{\mathrm{Ric}}^{(k)}_{ij}\left[mx_j\delta_iH^{(m)} - \frac{1}{n-1}\delta^2H^{(m)}(x_ix_j - |x|^2\delta_{ij})\right] \\
&\quad - \int_{\mathbb{S}^{n-1}} \dot{\mathrm{Ric}}^{(m)}_{ij}\left[kx_j\delta_iH^{(k)} - \frac{1}{n-1}\delta^2H^{(k)}(x_ix_j - |x|^2\delta_{ij})\right] \\
&\quad + \int_{\mathbb{S}^{n-1}} \left[\frac{k}{n-1}x_j\delta_iH^{(k)}\delta^2H^{(m)} + \frac{m}{n-1}x_j\delta_iH^{(m)}\delta^2H^{(k)}\right](x_ix_j - |x|^2\delta_{ij}).
\end{aligned}$$

Here, the fourth term vanishes because of (3.1) and (3.2). For the second and third terms, we need the following claim:

Claim B.3.

$$2\dot{\mathrm{Ric}}^{(k)}_{ij}x_j = k\delta_iH^{(k)}.$$

PROOF OF CLAIM B.3. By the definition of $\dot{\mathrm{Ric}}$ in (3.3), we know that

$$2\dot{\mathrm{Ric}}^{(k)}_{ij}x_j = x_j\partial_jH^{(k)}_{il,l} + \left[\partial_{li}(x_jH^{(k)}_{jl}) - H^{(k)}_{ll,i} - \delta_iH^{(k)}\right] - \left[\Delta(x_jH^{(k)}_{ij}) - 2\delta_iH^{(k)}\right].$$

Then, using (3.1) and Lemma A.1, we finish the proof of the claim.

Combining Claim B.3 with (3.1), (3.2) and $\dot{\mathrm{Ric}}^{(k)}_{ii} = \delta^2H^{(k)}$, we can compute the second term as

$$\begin{aligned}
&\int_{\mathbb{S}^{n-1}} \dot{\mathrm{Ric}}^{(k)}_{ij}\left[mx_j\delta_iH^{(m)} - \frac{1}{n-1}\delta^2H^{(m)}(x_ix_j - |x|^2\delta_{ij})\right] \\
&= \int_{\mathbb{S}^{n-1}} \left[\frac{km}{2}\delta_iH^{(k)}\delta_iH^{(m)} + \frac{1}{n-1}\delta^2H^{(k)}\delta^2H^{(m)}\right].
\end{aligned}$$

By symmetry, the third term can be handled similarly to the second term.

Finally, by putting all four terms together, we complete the proof of the second identity. □

Proofs of Proposition 6.5 and Corollary 6.7. Applying (3.9) and (B.2), we observe

$$\begin{aligned}
&\epsilon^{-(k+m)}I_{1,\epsilon}\left[\widetilde{H}^{(k)}, \widetilde{H}^{(m)}\right] \\
&= \int_{\mathbb{S}^{n-1}} \left[-\alpha_n\ddot{R}^{(k,m)} - \frac{km}{n-2}H^{(k)}_{ij}H^{(m)}_{ij}\right]dS\int_0^{\delta_0\epsilon^{-1}} r^{k+m+n-1}Z^0\left(\frac{w''}{r^2} - \frac{w'}{r^3}\right)dr \\
&\quad + \int_{\mathbb{S}^{n-1}} \left[\frac{2}{n-2} - \frac{n-2}{2} - \alpha_n - (k+m-2)\beta_n - \frac{2(k+m)}{n-2}\right]\ddot{R}^{(k,m)}dS\int_0^{\delta_0\epsilon^{-1}} r^{k+m+n-3}Z^0\frac{w'}{r}dr
\end{aligned}$$

$$
\begin{aligned}
&-\int_{\mathbb{S}^{n-1}} \frac{(n+k+m-2)km}{n-2} H_{ij}^{(k)} H_{ij}^{(m)} dS \int_0^{\delta_0\epsilon^{-1}} r^{k+m+n-3} Z^0 \frac{w'}{r} dr \\
&-\gamma_n \int_{\mathbb{S}^{n-1}} \left[\Delta \ddot{R}^{(k,m)} - \frac{n^3-4n^2+16n-16}{4(n-1)(n-2)^2} \delta^2 H^{(k)} \delta^2 H^{(m)}\right] dS \int_0^{\delta_0\epsilon^{-1}} r^{k+m+n-5} Z^0 w dr \\
&-\frac{n-4}{(n-2)^2} \int_{\mathbb{S}^{n-1}} \dot{\mathrm{Ric}}_{ij}^{(k)} \dot{\mathrm{Ric}}_{ij}^{(m)} dS \int_0^{\delta_0\epsilon^{-1}} r^{k+m+n-5} Z^0 w dr.
\end{aligned} \tag{B.3}
$$

Then, Proposition 6.5 follows from Lemma A.3, Lemma A.4, and Corollary A.5.

Corollary 6.7 is a consequence of Proposition 6.5 and Lemma B.2. □

B.2. **Eigenspaces of the operator $\mathcal{L}_k$.** We study the eigenspaces of the operator $\mathcal{L}_k$ in (6.11), which induces the orthogonal decomposition (6.12) of the space $\mathcal{V}_k$.

Lemma B.4 (Lemma A.5 in [22])**.** *Let $\bar{H}$ be a symmetric matrix whose elements are all homogeneous polynomials on $\mathbb{R}^n$ of degree k. Suppose there are $p, t \in \mathcal{P}_{k-2}$, $q_j \in \mathcal{P}_{k-1}$ such that*

$$
\begin{aligned}
\bar{H}_{ii} &= p|x|^2; \\
x_i \bar{H}_{ij} &= q_j |x|^2; \\
x_i x_j \bar{H}_{ij} &= t|x|^4.
\end{aligned}
$$

Then

$$
\bar{H}_{ij} + \mu_i x_j + \mu_j x_i + \nu \delta_{ij} \in \mathcal{V}_k,
$$

where $\mu_j := -q_j + \frac{p+(n-2)t}{2(n-1)} x_j$ and $\nu := \frac{t-p}{n-1}|x|^2$. We define

$$
(\mathrm{Proj}\bar{H})_{ij} := \bar{H}_{ij} + \mu_i x_j + \mu_j x_i + \nu \delta_{ij}. \tag{B.4}
$$

Lemma B.5 (Lemma A.6 in [22])**.** *Let $\bar{H} \in \mathcal{V}_k$. Then there exist $\bar{W} \in \mathcal{W}_k$ and $\hat{H}_q \in \mathcal{V}_k/\mathcal{W}_k$ for $q = 1, \dots, \lfloor \frac{k-2}{2} \rfloor$ such that*

$$
\bar{H} = \bar{W} + \sum_{q=1}^{\lfloor \frac{k-2}{2} \rfloor} \hat{H}_q,
$$

where $\langle \hat{H}_{q_1}, \hat{H}_{q_2} \rangle = 0$ for $q_1 \neq q_2$. Furthermore, it holds that

$$
\mathcal{L}_k \hat{H}_q = A_{k,q} \hat{H}_q, \tag{B.5}
$$

$$
\hat{H}_q = \mathrm{Proj}\left[|x|^{2q+2} \nabla^2 P^{(k-2q)}\right] \tag{B.6}
$$

for some $P^{(k-2q)} \in \mathcal{H}_{k-2q}$, where $\nabla^2 P^{(k-2q)}$ denotes the Hessian of $P^{(k-2q)}$ and

$$
A_{k,q} := (k-2q-1)\left[2 - \frac{n-2}{n-1}(n+k-2q-1)\right] - (q+1)(n+2k-2q-4).
$$

Lemma B.6 (Lemma A.7 in [22])**.** *Let $\bar{D} \in \mathcal{D}_k$. Then there exist $\hat{D}_q \in \mathcal{D}_k$ for $q = 0, \dots, \lfloor \frac{k-2}{2} \rfloor$ such that*

$$
\bar{D} = \sum_{q=0}^{\lfloor \frac{k-2}{2} \rfloor} \hat{D}_q,
$$

where $\langle \hat{D}_{q_1}, \hat{D}_{q_2} \rangle = 0$ for $q_1 \neq q_2$. Furthermore, it holds that

$$
\mathcal{L}_k \hat{D}_q = -q(n-2q+2k-2)\hat{D}_q, \tag{B.7}
$$

$$
\hat{D}_q = |x|^{2q} M^{(k-2q)}, \tag{B.8}
$$

for some $M^{(k-2q)} \in \{M \in \mathcal{D}_{k-2q} \mid \Delta M_{ij} = 0 \text{ for each } i, j = 1, \dots, n\}$.

Lemma B.7 (cf. Lemma 6.8)**.** *Let $\bar{W} \in \mathcal{W}_k$. Then there exist $\bar{D} \in \mathcal{D}_k$ and $\hat{W}_q \in \mathcal{W}_k/\mathcal{D}_k$ for $q = 1, \dots, \lfloor \frac{k-1}{2} \rfloor$ such that*

$$\bar{W} = \bar{D} + \sum_{q=1}^{\lfloor \frac{k-1}{2} \rfloor} \hat{W}_q, \tag{B.9}$$

where $\langle \hat{W}_{q_1}, \hat{W}_{q_2} \rangle = 0$ for $q_1 \neq q_2$. Furthermore, it holds that

$$\mathcal{L}_k \hat{W}_q = -\frac{(n+k-2)k}{2} \hat{W}_q, \tag{B.10}$$

$$\hat{W}_q = \mathrm{Proj}\left[|x|^{2q} \mathscr{D} V^{(k-2q+1)} \right] \tag{B.11}$$

for some vector field $V^{(k-2q+1)}$ on $\mathbb{R}^n$ such that

$$V^{(k-2q+1)} \in \{V = (V_1, \dots, V_n) \mid V_i \in \mathcal{H}_{k-2q+1} \text{ for } i = 1, \dots, n,\ \delta V = 0,\ x_i V_i = 0\}.$$

Proof. For each $j = 1, \dots, k$, we consider the spherical harmonic decomposition of $\delta_j \bar{W}$: There exist $\bar{V}_j^{(k-2q-1)} \in \mathcal{H}_{k-2q-1}$ for $q = 0, \dots, \lfloor \frac{k-1}{2} \rfloor$ such that

$$\delta_j \bar{W} = \sum_{q=0}^{\lfloor \frac{k-1}{2} \rfloor} |x|^{2q} \bar{V}_j^{(k-2q-1)}.$$

Let us define a property for a vector field V:

$$\delta V = V_{i,i} = 0 \quad \text{and} \quad x_i V_i = 0. \tag{B.12}$$

From $\bar{W} \in \mathcal{W}_k$, we see that the vector field δW satisfies property (B.12). We assert that $\bar{V}^{(k-2q-1)}$ satisfies (B.12) for all $q = 0, \dots, \lfloor \frac{k-1}{2} \rfloor$.

First, since

$$x_j \Delta \delta_j \bar{W} = \Delta(x_j \delta_j \bar{W}) - 2\partial_j \delta_j \bar{W} = 0,$$

$\Delta \delta \bar{W}$ satisfies (B.12). An induction argument shows that $\Delta^q \delta \bar{W}$ satisfies (B.12) for all $q \in \mathbb{N} \cup \{0\}$. Picking $q = \lfloor \frac{k-1}{2} \rfloor$, we observe that $\bar{V}^{(0)}$ and $\bar{V}^{(1)}$ satisfy (B.12).

Second, by applying the identity

$$\partial_j \left(|x|^{2q'} \bar{V}_j^{(k-2q-1)} \right) = 2q' |x|^{2q'-2} x_j \bar{V}_j^{(k-2q-1)} + |x|^{2q'} \partial_j \bar{V}_j^{(k-2q-1)} = 0,$$

and an induction argument, we can deduce that $\bar{V}^{(k-2q-1)}$ satisfies (B.12) for all $q = 0, \dots, \lfloor \frac{k-1}{2} \rfloor$. This proves the assertion.

As a result, $\bar{V}^{(0)} = 0$ when k is odd. Moreover, the following claim holds, which will be proved at the end of the proof.

Claim B.8. Assume that $\bar{W}^* \in \mathcal{W}_k/\mathcal{D}_k$, $k = 2m+2 \in \mathbb{N}$ even, and $\delta_j \bar{W}^* = |x|^{2m} \bar{V}_j^{(1)}$. If $\bar{V}^{(1)}$ satisfies (B.12), then $\bar{V}^{(1)} = 0$.

For $q = 0, \dots, \lfloor \frac{k-3}{2} \rfloor$, we define $V^{(k-2q-1)} = (V_1^{(k-2q-1)}, \dots, V_n^{(k-2q-1)})$, where

$$V_i^{(k-2q-1)} := -\tfrac{1}{(n+k-2q-2)(k-2q-2)} \bar{V}_i^{(k-2q-1)} \in \mathcal{H}_{k-2q-1}.$$

We also set $\hat{W}_{q+1} = \mathrm{Proj}[|x|^{2q+2} \mathscr{D} V^{(k-2q-1)}]$. It follows from Lemma 6.8 and Remark 6.9 that $\hat{W}_{q+1} \in \mathcal{W}_k/\mathcal{D}_k$, $\delta_j \hat{W}_{q+1} = |x|^{2q} \bar{V}_j^{(k-2q-1)}$, and (B.10) holds. Let

$$\bar{W}^\sharp = \bar{W} - \sum_{q=1}^{\lfloor \frac{k-1}{2} \rfloor} \hat{W}_q \in \mathcal{W}_k.$$

Then, $\bar{W}^\sharp$ can be decomposed as $\bar{W}^\sharp = \hat{W}^\sharp + \hat{D}^\sharp$ with $\hat{W}^\sharp \in \mathcal{W}_k/\mathcal{D}_k$ and $\hat{D}^\sharp \in \mathcal{D}_k$. By applying Claim B.8 for $\hat{W}^\sharp$, we find that $\delta_j \hat{W}^\sharp = 0$, i.e., $\hat{W}^\sharp \in \mathcal{D}_k$. Thus, $\hat{W}^\sharp = 0$ and $\bar{W} = \sum_{q=1}^{\lfloor \frac{k-1}{2} \rfloor} \hat{W}_q + \hat{D}^\sharp$. Setting $\bar{D} = \hat{D}^\sharp \in \mathcal{D}_k$, we obtain (B.9) as desired.

Also, a straight calculation using Lemma A.3 shows that $\langle \hat{W}_{q_1}, \hat{W}_{q_2} \rangle = 0$ for $q_1 \neq q_2$.

Proof of Claim B.8. Let $\bar{V}_i^{(1)}(x) = a_{ij}x_j$ for $a_{ij} \in \mathbb{R}$. From property (B.12), we know that $a_{ij} + a_{ji} = 0$. Combining this with the definition of $\mathcal{L}_k$ in (6.11), we get

$$\begin{aligned}(\mathcal{L}_{2m+2}\bar{W}^*)_{ij} = \frac{|x|^2}{2}\left[2m|x|^{2m-2}\left(x_i\bar{V}_j^{(1)} + x_j\bar{V}_i^{(1)}\right) - \Delta \bar{W}_{ij}^*\right] \\ - (m+1)|x|^{2m}\left(x_i\bar{V}_j^{(1)} + x_j\bar{V}_i^{(1)}\right).\end{aligned}$$

On the other hand, every element in $\mathcal{W}_k/\mathcal{D}_k$ is an eigenvector of $\mathcal{L}_k$ with the eigenvalue $-\frac{(n+k-2)k}{2}$, as shown in [22, Page 184]. Particularly, $(\mathcal{L}_{2m+2}\bar{W}^*)_{ij} = -(n+2m)(m+1)\bar{W}_{ij}^*$. Therefore,

$$2|x|^{2m}\left(x_i\bar{V}_j^{(1)} + x_j\bar{V}_i^{(1)}\right) + |x|^2\Delta\bar{W}_{ij}^* = 2(n+2m)(m+1)\bar{W}_{ij}^*. \tag{B.13}$$

Now, by exploiting the spherical harmonic decomposition of $\bar{W}_{ij}^*$, Lemma A.2, and (B.13), we can assume

$$\bar{W}_{ij}^* = b_{ij}|x|^{2m+2} + \frac{1}{n}|x|^{2m}\left(x_i\bar{V}_j^{(1)} + x_j\bar{V}_i^{(1)}\right),$$

where $b_{ij} = b_{ji} \in \mathbb{R}$. Then, given any $j = 1, \ldots, n$,

$$x_i\bar{W}_{ij}^* = \left(b_{ij} + \frac{1}{n}a_{ij}\right)x_i|x|^{2m+2} = 0 \quad \text{for all } x \in \mathbb{R}^n,$$

so $b_{ij} = -\frac{1}{n}a_{ij}$. Consequently, we conclude that $a_{ij} = b_{ij} = 0$, thereby confirming Claim B.8. □

B.3. **Orthogonality properties of $I_{1,\epsilon}$.** We prove orthogonality properties of the bilinear form $I_{1,\epsilon}$ in (6.5). Recall $d = \lfloor \frac{n-4}{2} \rfloor$.

Lemma B.9. *Given $k = 2, \ldots, d$, we write $H^{(k)} = \hat{H}^{(k)} + \hat{W}^{(k)} + \hat{D}^{(k)}$, where $\hat{H}^{(k)} \in \mathcal{V}_k/\mathcal{W}_k$, $\hat{W}^{(k)} \in \mathcal{W}_k/\mathcal{D}_k$, and $\hat{D}^{(k)} \in \mathcal{D}_k$. Then it holds that*

$$I_{1,\epsilon}\big[H^{(k)}, H^{(m)}\big] = I_{1,\epsilon}\big[\hat{H}^{(k)}, \hat{H}^{(m)}\big] + I_{1,\epsilon}\big[\hat{W}^{(k)}, \hat{W}^{(m)}\big] + I_{1,\epsilon}\big[\hat{D}^{(k)}, \hat{D}^{(m)}\big]$$

for $k, m = 2, \ldots, d$.

Proof. We must prove that

$$I_{1,\epsilon}\big[\hat{H}^{(k)}, \hat{W}^{(m)}\big] = I_{1,\epsilon}\big[\hat{H}^{(k)}, \hat{D}^{(m)}\big] = I_{1,\epsilon}\big[\hat{W}^{(k)}, \hat{D}^{(m)}\big] = 0. \tag{B.14}$$

We first observe that, whether the indices k_1, k_2, and k_3 are the same or not, the inner products satisfy

$$\big\langle \hat{H}, \hat{W} \big\rangle = \big\langle \hat{H}, \hat{D} \big\rangle = \big\langle \hat{W}, \hat{D} \big\rangle = 0 \tag{B.15}$$

for any $\hat{H} \in \mathcal{V}_{k_1}/\mathcal{W}_{k_1}$, $\hat{W} \in \mathcal{W}_{k_2}/\mathcal{D}_{k_2}$, and $\hat{D} \in \mathcal{D}_{k_3}$. Indeed, the proof that the first two inner products are 0 is provided in [22, Page 186]. The third inner product is also 0, because of Lemma B.7 and

$$\begin{aligned}\int_{\mathbb{S}^{n-1}} \text{Proj}\left[|x|^{2q}\mathscr{D}V^{(k_2-2q+1)}\right]_{ij}\hat{D}_{ij} &= \int_{\mathbb{S}^{n-1}}\left(\partial_i V_j^{(k_2-2q+1)} + \partial_j V_i^{(k_2-2q+1)}\right)\hat{D}_{ij} \\ &= (n+k_2-2q+k_3)\int_{B_1}\left(\partial_i V_j^{(k_2-2q+1)} + \partial_j V_i^{(k_2-2q+1)}\right)\hat{D}_{ij} \\ &= -2(n+k_2-2q+k_3)\int_{B_1} V_j^{(k_2-2q+1)}\delta_j\hat{D} = 0\end{aligned}$$

for $q = 1, \dots, \lfloor \frac{k_2-1}{2} \rfloor$, where we used integration by parts and $\hat{D} \in \mathcal{D}_{k_3}$.

In the remainder of the proof, we will show that $I_{1,\epsilon}[\hat{H}^{(k)}, \hat{W}^{(m)}] = 0$ only, since one can deal with the other terms in (B.14) analogously once (B.15) is known. Owing to Lemma B.2, (B.3), (B.5), and (B.10), it suffices to check that

$$\int_{\mathbb{S}^{n-1}} \hat{H}^{(k)}_{ij} \hat{W}^{(m)}_{ij} = \int_{\mathbb{S}^{n-1}} \delta_i \hat{H}^{(k)} \delta_i \hat{W}^{(m)} = \int_{\mathbb{S}^{n-1}} \delta^2 \hat{H}^{(k)} \delta^2 \hat{W}^{(m)} = 0.$$

By (B.15), it holds that $\int_{\mathbb{S}^{n-1}} \hat{H}^{(k)}_{ij} \hat{W}^{(m)}_{ij} = \langle \hat{H}^{(k)}, \hat{W}^{(m)} \rangle = 0$. Because $\delta^2 \hat{W}^{(m)} = 0$, we also have that $\int_{\mathbb{S}^{n-1}} \delta^2 \hat{H}^{(k)} \delta^2 \hat{W}^{(m)} = 0$. On the other hand, by using (B.6) and (B.11), we can write

$$\hat{H}^{(k)} = \sum_{q=1}^{\lfloor \frac{k-2}{2} \rfloor} \mathrm{Proj}\left[|x|^{2q+2} \nabla^2 P^{(k-2q)} \right] \quad \text{and} \quad \hat{W}^{(m)} = \sum_{q'=1}^{\lfloor \frac{m-1}{2} \rfloor} \mathrm{Proj}\left[|x|^{2q'} \mathscr{D} V^{(m-2q'+1)} \right].$$

A direct computation yields

$$\begin{aligned}
\delta_i \hat{H}^{(k)} &= \sum_{q=1}^{\lfloor \frac{k-2}{2} \rfloor} \left[(c_1)_{k,q} |x|^{2q} \partial_i P^{(k-2q)} + (c_2)_{k,q} |x|^{2q-2} P^{(k-2q)} x_i \right], \\
\delta_i \hat{W}^{(m)} &= \sum_{q'=1}^{\lfloor \frac{m-1}{2} \rfloor} c_{m,q'} |x|^{2q'-2} V_i^{(m-2q'+1)}
\end{aligned}$$

for some $(c_1)_{k,q}$, $(c_2)_{k,q}$, $c_{m,q'} \in \mathbb{R}$. Employing Lemma B.2 and $x_i V_i^{(m-2q'+1)} = \delta V^{(m-2q'+1)} = 0$, we deduce that

$$\begin{aligned}
\int_{\mathbb{S}^{n-1}} \delta_i \hat{H}^{(k)} \delta_i \hat{W}^{(m)} &= \sum_{q=1}^{\lfloor \frac{k-2}{2} \rfloor} \sum_{q'=1}^{\lfloor \frac{m-1}{2} \rfloor} (c_1)_{k,q} c_{m,q'} \int_{\mathbb{S}^{n-1}} \partial_i P^{(k-2q)} V_i^{(m-2q'+1)} \\
&= - \sum_{q=1}^{\lfloor \frac{k-2}{2} \rfloor} \sum_{q'=1}^{\lfloor \frac{m-1}{2} \rfloor} (c_1)_{k,q} c_{m,q'} \int_{\mathbb{S}^{n-1}} P^{(k-2q)} \delta V^{(m-2q'+1)} = 0.
\end{aligned}$$

This completes the proof of $I_{1,\epsilon}[\hat{H}^{(k)}, \hat{W}^{(m)}] = 0$. □

Following the approach from the proof of the previous lemma and using (B.20), (B.23)–(B.24) below, along with the orthogonality of harmonic polynomials on $\mathbb{S}^{n-1}$ with different degrees such as

$$\int_{\mathbb{S}^{n-1}} (M_q^{(s_1)})_{ij} (M_{q'}^{(s_2)})_{ij} = \int_{\mathbb{S}^{n-1}} (V_q^{(s_1)})_i (V_{q'}^{(s_2)})_i = \int_{\mathbb{S}^{n-1}} P_q^{(s_1)} P_{q'}^{(s_2)} = 0 \quad \text{for } s_1 \neq s_2,$$

we obtain the following result. We omit the details.

Lemma B.10. *Let $\widetilde{E}_s^D$, $\widetilde{E}_s^W$ and $\widetilde{E}_s^H$ be the matrices in* (6.18), (6.20) *and* (6.22), *respectively. Then $I_{1,\epsilon}[\widetilde{E}_s^D, \widetilde{E}_{s'}^D] = 0$ for any $2 \le s \neq s' \le d$, $I_{1,\epsilon}[\widetilde{E}_s^W, \widetilde{E}_{s'}^W] = 0$ for any $1 \le s \neq s' \le d-2$, and $I_{i,\epsilon}[\widetilde{E}_s^H, \widetilde{E}_{s'}^H] = 0$ for any $i = 1, 2, 3$ and $2 \le s \neq s' \le d-2$.*

B.4. **Evaluation of the Pohozaev quadratic form.** We examine the positivity of the Pohozaev quadratic form in three mutually exclusive cases, which is crucial in the proof of Proposition 6.4. We set $\theta_k = 1$ if $k = \frac{n-4}{2}$ and $\theta_k = 0$ otherwise.

Lemma B.11. *Assume that* $8 \le n \le 24$ *and* $s = 2, \dots, d$. *Let* $M_q^{(s)}$ *be the matrix defined in* (6.18) *and* $\epsilon > 0$ *small. Then there exists a constant* $C = C(n,s) > 0$ *such that*

$$\sum_{q,q'=0}^{\lfloor \frac{d-s}{2} \rfloor} |\log \epsilon|^{\theta_{q+q'+s}} J_1 \left[\epsilon^{2q+s}|x|^{2q} M_q^{(s)}, \epsilon^{2q'+s}|x|^{2q'} M_{q'}^{(s)}\right] \ge C^{-1} \sum_{q=0}^{\lfloor \frac{d-s}{2} \rfloor} \epsilon^{2(2q+s)} |\log \epsilon|^{\theta_{2q+s}} \left\| |x|^{2q} M_q^{(s)} \right\|^2.$$

Proof. Fixing any $s = 2, \dots, d$, let $k = 2q+s$, $m = 2q'+s$, $\lambda_q = -q(n+2q+2s-2)$, and $\lambda_{q'} = -q'(n+2q'+2s-2)$. By applying Corollary 6.7 and (B.7)–(B.8), we compute

$$\begin{aligned} &J_1' \left[|x|^{2q} M_q^{(s)}, |x|^{2q'} M_{q'}^{(s)}\right] \\ &= \frac{(n-4)^2}{4(n-1)(n-2)} \left[\frac{8(n-1)(n-3)(k+m)}{(n-2)(n+k+m-4)} \lambda_q \lambda_{q'} \right. \\ &\qquad \left. + \frac{1}{8} c_1(n,k,m) \big(2(\lambda_q + \lambda_{q'}) + (n+k+m-2)(k+m)\big) - c_2(n,k,m)\right] \left\langle M_q^{(s)}, M_{q'}^{(s)} \right\rangle \\ &:= (m_{qq'}^{D,s})' \left\langle M_q^{(s)}, M_{q'}^{(s)} \right\rangle, \end{aligned}$$

where

$$\begin{aligned} c_1(n,k,m) &= (k+m)(n^3 - (k+m+2)n^2 + (6(k+m)-4)n - 4(k+m) + 8), \\ c_2(n,k,m) &= 2(n-1)(k+m)(n+k+m-2)km. \end{aligned} \tag{B.16}$$

Then we set

$$m_{qq'}^{D,s} := \begin{cases} \dfrac{1}{n-k-m-4} \mathcal{I}_{n-2}^{n+k+m-3} (m_{qq'}^{D,s})' & \text{if } k+m = 2(q+q'+s) < n-4, \\ \dfrac{N_0}{2(n-3)} (m_{qq'}^{D,s})' & \text{if } k+m = 2(q+q'+s) = n-4, \end{cases} \tag{B.17}$$

where $N_0 \in \mathbb{N}$ is taken to be large enough; for example, $N_0 = 10^{10}$ suffices.

Using Mathematica, we observe that matrices $(m_{qq'}^{D,s})$ are positive-definite for all $s = 2, \dots, d$ when $n \le 24$. In addition, $(m_{qq'}^{D,2})$ has a negative eigenvalue when $n \ge 25$.

Let B' be a Gram matrix defined as $B'_{qq'} = \langle \epsilon^k M_q^{(s)}, \epsilon^m M_{q'}^{(s)} \rangle$, which is always positive semi-definite. Then, we set a matrix B by $B_{qq'} = B'_{qq'}$ if $k+m < n-4$ and $B_{qq'} = |\log \epsilon| N_0^{-1} B'_{qq'}$ if $k+m = n-4$, which is also positive semi-definite for ϵ so small that $\epsilon \le e^{-N_0}$. At this stage, we apply Lemma A.6 with $l = \lfloor \frac{d-s}{2} \rfloor + 1$, $A = m^{D,s}$, and the aforementioned B. It follows that

$$\begin{aligned} \sum_{q,q'=0}^{\lfloor \frac{d-s}{2} \rfloor} |\log \epsilon|^{\theta_{\frac{k+m}{2}}} J_1 \left[\epsilon^k |x|^{2q} M_q^{(s)}, \epsilon^m |x|^{2q'} M_{q'}^{(s)}\right] &= \mathrm{tr}(AB) \ge C^{-1} \mathrm{tr}(B) \\ &= C^{-1} \sum_{q=0}^{\lfloor \frac{d-s}{2} \rfloor} |\log \epsilon|^{\theta_k} \left\| \epsilon^k |x|^{2q} M_q^{(s)} \right\|^2. \end{aligned}$$ □

Lemma B.12. *Assume that* $10 \le n \le 28$ *and* $s = 1, \dots, d-2$. *Let* $V_q^{(s+1)}$ *be the vector field defined in* (6.18) *and* $\epsilon > 0$ *small. Then there exists a constant* $C = C(n,s) > 0$ *such that*

$$\sum_{q,q'=1}^{\lfloor\frac{d-s}{2}\rfloor} |\log\epsilon|^{\theta_{q+q'+s}} J_1\left[\operatorname{Proj}\left[|x|^{2q}\mathscr{D}V_q^{(s+1)}\right], \operatorname{Proj}\left[|x|^{2q'}\mathscr{D}V_{q'}^{(s+1)}\right]\right]$$
$$\geq C^{-1}\sum_{q=1}^{\lfloor\frac{d-s}{2}\rfloor} \epsilon^{2(2q+s)}|\log\epsilon|^{\theta_{2q+s}}\left\|\operatorname{Proj}\left[|x|^{2q}\mathscr{D}V_q^{(s+1)}\right]\right\|^2.$$

Proof. Fixing any $s=1,\dots,d-2$, let $k=2q+s$, $m=2q'+s$, $\lambda_q=-\frac{1}{2}(n+s+2q-2)(s+2q)$, and $\lambda_{q'}=-\frac{1}{2}(n+s+2q'-2)(s+2q')$. By applying Corollary 6.7, (B.10)–(B.11), and

$$\delta_j\left(\operatorname{Proj}\left[|x|^{2q}\mathscr{D}V_q^{(s+1)}\right]\right) = -(n+s)s|x|^{2q-2}(V_q^{(s+1)})_j,$$
$$\left\langle\operatorname{Proj}\left[|x|^{2q}\mathscr{D}V_q^{(s+1)}\right], \operatorname{Proj}\left[|x|^{2q'}\mathscr{D}V_{q'}^{(s+1)}\right]\right\rangle = 2s(n+s+1)\left\langle V_q^{(s+1)}, V_{q'}^{(s+1)}\right\rangle,$$

we compute

$$\begin{aligned}
&J_1'\left[\operatorname{Proj}\left[|x|^{2q}\mathscr{D}V_q^{(s+1)}\right], \operatorname{Proj}\left[|x|^{2q'}\mathscr{D}V_{q'}^{(s+1)}\right]\right]\\
&= \frac{(n-4)^2}{4(n-1)(n-2)}\left[\frac{4(n-1)(n-3)(k+m)}{(n-2)(n+k+m-4)}\left(4s(n+s+1)\lambda_q\lambda_{q'}+kms^2(n+s)^2\right)\right.\\
&\left.+2s(n+s+1)\left\{\frac{1}{8}c_1(n,k,m)\left(2(\lambda_q+\lambda_{q'})+(n+k+m-2)(k+m)\right)-c_2(n,k,m)\right\}\right]\left\langle V_q^{(s+1)}, V_{q'}^{(s+1)}\right\rangle\\
&:= (m_{qq'}^{W,s})'\left\langle V_q^{(s+1)}, V_{q'}^{(s+1)}\right\rangle,
\end{aligned}$$

where $c_1(n,k,m)$ and $c_2(n,k,m)$ are the numbers in (B.16). Then we set $m_{qq'}^{W,s}$ by exploiting (B.17) in which all the superscripts D are replaced with W.

Using Mathematica, we observe that matrices $(m_{qq'}^{W,s})$ are positive-definite for all $s=1,\dots,d-2$ when $n\leq 28$. In addition, $(m_{qq'}^{W,1})$ has a negative eigenvalue when $n\geq 29$.

Following the rest of the proof of Lemma B.11, we complete the proof. □

Lemma B.13. *Assume that $12\leq n\leq 32$ and $s=2,\dots,d-2$. Let $\check{H}_q^{(2q+s)}$ be the matrix defined in (6.22) and $\epsilon>0$ small. Then there exists a constant $C=C(n,s)>0$ such that*

$$\sum_{q,q'=1}^{\lfloor\frac{d-s}{2}\rfloor} I_\epsilon\left[\check{H}_q^{(2q+s)}, \check{H}_{q'}^{(2q'+s)}\right] \geq C^{-1}\sum_{q=1}^{\lfloor\frac{d-s}{2}\rfloor}\epsilon^{2(2q+s)}|\log\epsilon|^{\theta_{2q+s}}\left\|\hat{H}_q^{(2q+s)}\right\|^2.$$

Proof. We continue to use the notations introduced in (6.22). Fix any $s=2,\dots,d-2$. We recall that $\hat{H}_q^{(2q+s)}=\operatorname{Proj}[|x|^{2q+2}\nabla^2 P_q^{(s)}]$,

$$\begin{aligned}
\mathcal{L}_k\hat{H}_q^{(2q+s)} &= A_{2q+s,q}\hat{H}_q^{(2q+s)},\\
\delta_j\hat{H}_q^{(2q+s)} &= -\frac{n-2}{n-1}(s-1)(n+s-1)|x|^{2q}\partial_j P_q^{(s)}\\
&\quad+\frac{n-2}{n-1}s(s-1)(n+s-1)|x|^{2q-2}x_jP_q^{(s)},\\
\delta^2\hat{H}_q^{(2q+s)} &= \kappa_s|x|^{2q-2}P_q^{(s)},
\end{aligned}\tag{B.18}$$

where

$$A_{2q+s,q}=(s-1)\left[2-\frac{n-2}{n-1}(n+s-1)\right]-(q+1)(n+2q+2s-4),$$
$$\kappa_s := \frac{n-2}{n-1}s(s-1)(n+s-1)(n+s-2).$$

We also note

$$\left\langle \hat{H}_q^{(2q+s)}, \hat{H}_{q'}^{(2q'+s)} \right\rangle = \kappa_s \left\langle P_q^{(s)}, P_{q'}^{(s)} \right\rangle,$$
$$\left\langle \delta\hat{H}_q^{(2q+s)}, \delta\hat{H}_{q'}^{(2q'+s)} \right\rangle = \frac{\kappa_s^2}{s(n+s-2)} \left\langle P_q^{(s)}, P_{q'}^{(s)} \right\rangle,$$
$$\left\langle \delta^2\hat{H}_q^{(2q+s)}, \delta^2\hat{H}_{q'}^{(2q'+s)} \right\rangle = \kappa_s^2 \left\langle P_q^{(s)}, P_{q'}^{(s)} \right\rangle.$$

Let $k = 2q + s$, $m = 2q' + s$, $\lambda_q = A_{2q+s,q}$, and $\lambda_{q'} = A_{2q'+s,q'}$. By applying Corollary 6.7 and (B.5)–(B.6), we find

$$\begin{aligned}
&J_1' \left[\hat{H}_q^{(k)}, \hat{H}_{q'}^{(m)}\right] \\
&= \frac{(n-4)^2\kappa_s}{4(n-1)(n-2)} \left[\frac{8(n-1)(n-3)(k+m)\lambda_q\lambda_{q'}}{(n-2)(n+k+m-4)}\right. \\
&\quad + \frac{4(n-3)(k+m)(s-1)(n+s-1)km}{n+k+m-4} - \frac{(n-3)n^2(n-4)(k+m)\kappa_s}{2(n-1)(n-2)(n+k+m-4)} \\
&\quad \left. + \frac{1}{8}c_1(n,k,m)\left(2(\lambda_q+\lambda_{q'}) + (n+k+m-2)(k+m)\right) - c_2(n,k,m)\right] \left\langle P_q^{(s)}, P_{q'}^{(s)} \right\rangle \\
&:= (m_{qq'}^{H,s,1})' \left\langle P_q^{(s)}, P_{q'}^{(s)} \right\rangle,
\end{aligned} \tag{B.19}$$

where $c_1(n,k,m)$ and $c_2(n,k,m)$ are the numbers in (B.16). Then we set $m_{qq'}^{H,s,1}$ as in (B.17).

For the $I_{2,\epsilon}[\check{H}_q^{(k)}, \check{H}_{q'}^{(m)}]$ term, we remind from (4.4), (B.18), and $s = (k-2) - 2(q-1)$ that

$$\begin{aligned}
L_1[\hat{H}_q^{(k)}]w = \mathfrak{L}_1[\delta^2\hat{H}_q^{(k)}]w &= \kappa_s \mathfrak{L}_1[|x|^{2(q-1)}P_q^{(s)}]w \\
&= \frac{(n-4)\kappa_s}{2(n-1)} \sum_{i=1}^{q+3} b_i(n,k-2,q-1)(1+r^2)^{-\frac{n+6-2i}{2}} P_q^{(s)}, \\
\Psi[\hat{H}_q^{(k)}] &= -\frac{(n-4)\kappa_s}{2(n-1)} \sum_{j=1}^{q+2} \Gamma_j(n,k-2,q-1)(1+r^2)^{-\frac{n-2j}{2}} P_q^{(s)},
\end{aligned} \tag{B.20}$$

where $b_i(n,k-2,q-1)$ and $\Gamma_j(n,k-2,q-1)$ are the numbers appearing in Definition 4.4 and Proposition 4.6, respectively. We also observe

$$\begin{aligned}
&\left(x_i\partial_i + \frac{n-4}{2}\right)\left((1+r^2)^{-\frac{n-2j}{2}} P_q^{(s)}\right) \\
&\qquad = (n-2j)(1+r^2)^{-\frac{n+2-2j}{2}} P_q^{(s)} - \frac{1}{2}(n+4-4j-2s)(1+r^2)^{-\frac{n-2j}{2}} P_q^{(s)}.
\end{aligned}$$

Making use of polar coordinates, we determine

$$\begin{aligned}
&\int_{B^n(0,\delta_0\epsilon^{-1})} \left(x_i\partial_i + \frac{n-4}{2}\right) \Psi[\check{H}_q^{(k)}] L_1[\check{H}_{q'}^{(m)}]w \\
&= \epsilon^{k+m} \left[\frac{(n-4)\kappa_s}{2(n-1)}\right]^2 \sum_{i=1}^{q'+3}\sum_{j=1}^{q+2} b_i\Gamma_j \left[-(n-2j)\int_0^{\delta_0\epsilon^{-1}} \frac{r^{n-1+2s}}{(1+r^2)^{n+4-i-j}}dr\right. \\
&\qquad \left. + \frac{1}{2}(n+4-4j-2s)\int_0^{\delta_0\epsilon^{-1}} \frac{r^{n-1+2s}}{(1+r^2)^{n+3-i-j}}dr\right] \left\langle P_q^{(s)}, P_{q'}^{(s)} \right\rangle \\
&= \epsilon^{k+m}|\log\epsilon|^{\theta_{\frac{k+m}{2}}} (m_{qq'}^{H,s,2})' \left\langle P_q^{(s)}, P_{q'}^{(s)} \right\rangle + O(\epsilon^{n-4}),
\end{aligned}$$

where $b_i = b_i(n, m-2, q'-1)$, $\Gamma_j = \Gamma_j(n, k-2, q-1)$, and

$$(m_{qq'}^{H,s,2})' := \begin{cases} -\frac{1}{2}\left[\frac{(n-4)\kappa_s}{2(n-1)}\right]^2 \sum_{i=1}^{q'+3}\sum_{j=1}^{q+2} b_i\Gamma_j \left[\frac{(n-2s-2i-2j+6)(n-2j)}{n+3-i-j} - (n+4-4j-2s)\right]\mathcal{I}_{n+3-i-j}^{n-1+2s} \\ \qquad\qquad \text{if } k+m = 2(q+q'+s) < n-4, \\ 0 \qquad\qquad \text{if } k+m = 2(q+q'+s) = n-4. \end{cases}$$

We define

$$m_{qq'}^{H,s,2} := \begin{cases} \frac{1}{2}\left[(m_{qq'}^{H,s,2})' + (m_{q'q}^{H,s,2})'\right] & \text{if } k+m < n-4, \\ \frac{N_0}{2}\left[(m_{qq'}^{H,s,2})' + (m_{q'q}^{H,s,2})'\right] & \text{if } k+m = n-4, \end{cases} \tag{B.21}$$

where $N_0 \in \mathbb{N}$ is taken to be large enough; for example, $N_0 = 10^{10}$ suffices. Clearly, $m_{qq'}^{H,s,2} = 0$ when $k+m = n-4$. Then, by (6.6),

$$I_{2,\epsilon}\left[\check{H}_q^{(k)}, \check{H}_{q'}^{(m)}\right] = \epsilon^{k+m}\left[\frac{|\log\epsilon|}{N_0}\right]^{\theta\frac{k+m}{2}} m_{qq'}^{H,s,2}\left\langle P_q^{(s)}, P_{q'}^{(s)}\right\rangle + O(\epsilon^{n-4}). \tag{B.22}$$

For the $I_{3,\epsilon}[\check{H}_q^{(k)}, \check{H}_{q'}^{(m)}]$ term, we need to evaluate

$$L_1[\hat{H}_q^{(k)}]Z^0 = -\frac{n-4}{2}L_1[\hat{H}_q^{(k)}](1+r^2)^{-\frac{n-4}{2}} + (n-4)L_1[\hat{H}_q^{(k)}](1+r^2)^{-\frac{n-2}{2}}. \tag{B.23}$$

We have (B.20) and

$$L_1[\hat{H}_q^{(k)}](1+r^2)^{-\frac{n-2}{2}} = \frac{\kappa_s}{2(n-1)}\sum_{i=1}^{q+3} b_i'(n, k-2, q-1)(1+r^2)^{-\frac{n+8-2i}{2}} P_q^{(s)}. \tag{B.24}$$

It follows that

$$\begin{aligned} &\int_{B^n(0,\delta_0\epsilon^{-1})} \Psi[\check{H}_q^{(k)}]L_1[\check{H}_{q'}^{(m)}]Z^0 \\ &= \epsilon^{k+m}\left[\frac{(n-4)\kappa_s}{2(n-1)}\right]^2 \sum_{i=1}^{q'+3}\sum_{j=1}^{q+2}\Gamma_j\left[-b_i'\int_0^{\delta_0\epsilon^{-1}}\frac{r^{n-1+2s}}{(1+r^2)^{n+4-i-j}}dr\right. \\ &\qquad\qquad \left. + \frac{n-4}{2}b_i\int_0^{\delta_0\epsilon^{-1}}\frac{r^{n-1+2s}}{(1+r^2)^{n+3-i-j}}dr\right]\left\langle P_q^{(s)}, P_{q'}^{(s)}\right\rangle \\ &= \epsilon^{k+m}|\log\epsilon|^{\theta\frac{k+m}{2}}(m_{qq'}^{H,s,3})'\left\langle P_q^{(s)}, P_{q'}^{(s)}\right\rangle + O(\epsilon^{n-4}), \end{aligned}$$

where $b_i = b_i(n, m-2, q'-1)$, $b_i' = b_i'(n, m-2, q'-1)$, $\Gamma_j = \Gamma_j(n, k-2, q-1)$, and

$$(m_{qq'}^{H,s,3})' := \begin{cases} -\frac{1}{2}\left[\frac{(n-4)\kappa_s}{2(n-1)}\right]^2 \sum_{i=1}^{q'+3}\sum_{j=1}^{q+2}\Gamma_j\left[\frac{n-2s-2i-2j+6}{n+3-i-j}b_i' - (n-4)b_i\right]\mathcal{I}_{n+3-i-j}^{n-1+2s} \\ \qquad\qquad \text{if } k+m = 2(q+q'+s) < n-4, \\ \frac{n-4}{2}\left[\frac{(n-4)\kappa_s}{2(n-1)}\right]^2\Gamma_{q+2}b_{q'+3} \qquad \text{if } k+m = 2(q+q'+s) = n-4. \end{cases}$$

If we define $m_{qq'}^{H,s,3}$ as in (B.21), then by (6.7),

$$I_{3,\epsilon}\left[\check{H}_q^{(k)}, \check{H}_{q'}^{(m)}\right] = \epsilon^{k+m}\left[\frac{|\log\epsilon|}{N_0}\right]^{\theta\frac{k+m}{2}} m_{qq'}^{H,s,3}\left\langle P_q^{(s)}, P_{q'}^{(s)}\right\rangle + O(\epsilon^{n-4}). \tag{B.25}$$

Adding up (6.10), (B.19), (B.22), and (B.25), we arrive at

$$(I_{1,\epsilon}+I_{2,\epsilon}+I_{3,\epsilon})\left[\check{H}_q^{(k)},\check{H}_{q'}^{(m)}\right]=\epsilon^{k+m}\left[\frac{|\log\epsilon|}{N_0}\right]^{\theta\frac{k+m}{2}} m_{qq'}^{H,s}\left\langle P_q^{(s)},P_{q'}^{(s)}\right\rangle+O(\epsilon^{n-4}),$$

where $m_{qq'}^{H,s}:=m_{qq'}^{H,s,1}+m_{qq'}^{H,s,2}+m_{qq'}^{H,s,3}$. Using Mathematica, we observe that matrices $(m_{qq'}^{H,s})$ are positive-definite for all $s=2,\dots,d-2$ when $n\le 32$. In addition, $(m_{qq'}^{H,2})$ has a negative eigenvalue when $n\ge 33$.

Following the rest of the proof of Lemma B.11, we complete the proof. □

Remark B.14. In the Yamabe case, the Pohozaev quadratic form on the subspace $\oplus_{k=2}^d\mathcal{D}_k$ fails to be positive for $n\ge 25$, as shown in [22, Proposition A.8]. In the proof of Lemma B.11, we saw that the same phenomenon happens for the Q-curvature problem (1.3). Wei and Zhao [43] implicitly used this fact to construct blowup examples for (1.3) provided $n\ge 25$. ◇

Appendix C. A Geometric Explanation for the Cancellation Phenomenon

We can observe the following proposition inspired by [6, Proposition 5]. Recall that $w(y)=(1+|y|^2)^{-\frac{n-4}{2}}$.

Proposition C.1. *Let $S=\mathscr{D}V$ where $\mathscr{D}$ is the conformal Killing operator defined in* (6.13) *and*

$$\Psi=V_i\partial_i w+\frac{n-4}{2n}\delta V w.$$

Then,

$$\Delta^2\Psi-\tilde{\mathfrak{c}}(n)w^{\frac{8}{n-4}}\Psi=-L_1[S]w+\Delta(\partial_i(S_{ij}\partial_j w))+\partial_i(S_{ij}\partial_j\Delta w), \tag{C.1}$$

where $L_1[S]$ is defined by (3.7)*.*

Proof. We have

$$\delta_j S=\Delta V_j+\frac{n-2}{n}\partial_j\delta V,$$
$$\delta^2 S=\frac{2(n-1)}{n}\Delta\delta V,$$
$$(\dot{\mathrm{Ric}}[S])_{ij}=\frac{n-2}{n}\partial_{ij}\delta V+\frac{1}{n}\Delta\delta V\delta_{ij}.$$

Then, there holds

$$\begin{aligned}&-L_1[S]w+\Delta(\partial_i(S_{ij}\partial_j w))+\partial_i(S_{ij}\partial_j\Delta w)\\&=4\partial_iV_j\partial_{ij}\Delta w-\frac{4}{n}\delta V\Delta^2 w+2\Delta V_j\partial_j\Delta w+\frac{2(n-4)}{n}\partial_j\delta V\partial_j\Delta w+4\partial_{jk}V_i\partial_{ijk}w\\&\quad+\frac{2(n-4)}{n}\partial_{ij}\delta V\partial_{ij}w+\frac{n-4}{n}\Delta\delta V\Delta w+4\partial_i\Delta V_j\partial_{ij}w\\&\quad+\Delta^2V_j\partial_j w+\frac{2(n-4)}{n}\partial_j\Delta\delta V\partial_j w+\frac{n-4}{2n}\Delta^2\delta V w.\end{aligned}$$

Continuing the computation and plugging in $\Delta^2 w=\mathfrak{c}(n)w^{\frac{n+4}{n-4}}$, we deduce

$$\begin{aligned}&-L_1[S]w+\Delta(\partial_i(S_{ij}\partial_j w))+\partial_i(S_{ij}\partial_j\Delta w)\\&=\Delta^2\left(V_i\partial_i w+\frac{n-4}{2n}\delta V w\right)-V_i\partial_i\Delta^2 w-\frac{n+4}{2n}\delta V\Delta^2 w,\\&=\Delta^2\left(V_i\partial_i w+\frac{n-4}{2n}\delta V w\right)-\tilde{\mathfrak{c}}(n)w^{\frac{8}{n-4}}\left(V_i\partial_i w+\frac{n-4}{2n}\delta V w\right),\end{aligned}$$

which is (C.1). □

Remark C.2.
1. The role of Proposition 4.6 cannot be replaced by this proposition because in our proof, we not only need the estimates on Ψ, but also need the explicit expression of Ψ to compute the Pohozaev quadratic form.
2. Nonetheless, this proposition is interesting by itself because it can explain the special relationship between the linearized equation at the standard bubble and the expansion of the Paneitz operator.
3. The idea of using conformal Killing operator to solve the linearized equation and study the energy expansion is robust. It is possible that similar cancellation to Proposition 4.6 can also be seen in higher order Q-curvature problems. $\diamond$

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References

[1] S. Alarcón, J. Petean, and C. Rey, *Multiplicity results for constant Q-curvature conformal metrics*, Calc. Var. Partial Differential Equations **63** (2024), Paper No. 146, 30 pp.
[2] S. Almaraz, *Blow-up phenomena for scalar-flat metrics on manifolds with boundary*, J. Differential Equations **251** (2011), 1813–1840.
[3] S. Almaraz, O. S. de Queiroz, and S. Wang, *A compactness theorem for scalar-flat metrics on 3-manifolds with boundary*, J. Funct. Anal. **277** (2019), 2092–2116.
[4] R. Avalos, P. Laurain, and J. H. Lira, *A positive energy theorem for fourth-order gravity*, Calc. Var. Partial Differential Equations **61** (2022), Paper No. 48, 33 pp.
[5] R. G. Bettiol, P. Piccione, and Y. Sire, *Nonuniqueness of conformal metrics with constant Q-curvature*, Int. Math. Res. Not. IMRN **2021** (2021), 6967–6992.
[6] S. Brendle, *Convergence of the Yamabe flow in dimension 6 and higher*, Invent. Math. **170** (2007), 541–576.
[7] ———, *Blow-up phenomena for the Yamabe equation*, J. Amer. Math. Soc. **21** (2008), 951–979.
[8] S. Brendle and F. C. Marques, *Blow-up phenomena for the Yamabe equation. II*, J. Differential Geom. **81** (2009), 225-250.
[9] S.-Y. A. Chang and P. C. Yang, *Extremal metrics of zeta function determinants on 4-manifolds*, Ann. of Math. **142** (1995), 171–212.
[10] Z. Djadli and A. Malchiodi, *Existence of conformal metrics with constant Q-curvature*, Ann. of Math. **168** (2008), 813–858.
[11] O. Druet, *Compactness for Yamabe metrics in low dimensions*, Int. Math. Res. Not. IMRN **23** (2004), 1143–1191.
[12] J. F. Escobar, *Conformal deformation of a Riemannian metric to a scalar flat metric with constant mean curvature on the boundary*, Ann. of Math. **136** (1992), 1–50.
[13] A. E. Fischer and J. E. Marsden, *The manifold of conformally equivalent metrics*, Canad. J. Math. **29** (1977), 193–209.
[14] M. J. Gursky, F. Hang, and Y.-J. Lin, *Riemannian manifolds with positive Yamabe invariant and Paneitz operator*, Int. Math. Res. Not. IMRN **2016** (2016), 1348–1367.
[15] M. J. Gursky and A. Malchiodi, *A strong maximum principle for the Paneitz operator and a non-local flow for the Q-curvature*, J. Eur. Math. Soc. (JEMS) **17** (2015), 2137–2173.
[16] F. Hang and P. Yang, *Sign of Green's function of Paneitz operators and the Q curvature*, Int. Math. Res. Not. IMRN **2015** (2015), 9775–9791.
[17] ———, *Q curvature on a class of 3-manifolds*, Comm. Pure Appl. Math. **69** (2016) 734–744.
[18] ———, *Q-curvature on a class of manifolds of dimension at least 5*, Comm. Pure Appl. Math. **69** (2016) 1452–1491.
[19] E. Hebey and M. Vaugon, *Le problème de Yamabe équivariant*, Bull. Sci. Math. **117** (1993), 241–286.

[20] E. Humbert and S. Raulot, *Positive mass theorem for the Paneitz-Branson operator*, Calc. Var. Partial Differential Equations **36** (2009), 525–531.

[21] J. Julio-Batalla and J. Petean, *Global bifurcation for Paneitz type equations and constant Q-curvature metrics*, J. Differential Equations **410** (2024), 278–300.

[22] M. A. Khuri, F. C. Marques, and R. Schoen, *A compactness theorem for the Yamabe problem*, J. Differential Geom. **81** (2009), 143–196.

[23] S. Kim, M. Musso, and J. Wei, *A non-compactness result on the fractional Yamabe problem in large dimensions*, J. Funct. Anal. **273** (2017), 3759–3830.

[24] ———, *A compactness theorem for the fractional Yamabe problem, Part I: The nonumbilic conformal infinity*, J. Eur. Math. Soc. (JEMS) **23** (2021), 3017–3073.

[25] ———, *Compactness of scalar-flat conformal metrics on low-dimensional manifolds with constant mean curvature on boundary*, Ann. Inst. H. Poincaré C Anal. Non Linéaire **38** (2021), 1763–1793.

[26] G. Li, *A compactness theorem on Branson's Q-curvature equation*, Pacific J. Math. **302** (2019), 119–179.

[27] J. Li, Y. Li, and P. Liu, *The Q-curvature on a 4-dimensional Riemannian manifold* (M,g) *with* $\int_M Q dV_g = 8\pi^2$, Adv. Math. **231** (2012), 2194–2223.

[28] Y.Y. Li and J. Xiong, *Compactness of conformal metrics with constant Q-curvature. I*, Adv. Math. **345** (2019), 116–160.

[29] Y.Y. Li and L. Zhang, *Compactness of solutions to the Yamabe problem II*, Calc. Var. and Partial Differential Equations **25** (2005), 185–237.

[30] ———, *Compactness of solutions to the Yamabe problem III*, J. Funct. Anal. **245** (2006), 438–474.

[31] Y.Y. Li and M. Zhu, *Yamabe type equations on three dimensional Riemannian manifolds*, Comm. Contemp. Math. **1** (1999), 1–50.

[32] G. Lu and J. Wei, *On a Sobolev inequality with remainder terms*, Proc. Amer. Math. Soc. **128** (1999), 75–84.

[33] F. C. Marques, *A priori estimates for the Yamabe problem in the non-locally conformally flat case*, J. Diff. Geom. **71** (2005), 315–346.

[34] S. Mazumdar and J. Vétois, *Existence results for the higher-order Q-curvature equation*, Calc. Var. Partial Differential Equations **63** (2024), Paper No. 151, 29 pp.

[35] B. Premoselli and J. Vétois, *Non-existence of minimizers for the second conformal eigenvalue near the round sphere in low dimensions*, preprint, arXiv:2408.07823.

[36] J. Qing and D. Raske, *Compactness for conformal metrics with constant Q curvature on locally conformally flat manifolds*, Calc. Var. Partial Differential Equations **26** (2006) 343–356.

[37] ———, *On positive solutions to semilinear conformally invariant equations on locally conformally flat manifolds*, Int. Math. Res. Not. **2006** (2006), Article ID 94172, 1–20.

[38] F. Robert, *Localization of bubbling for high order nonlinear equations*, preprint, arXiv:2501.00531.

[39] R. Schoen, *Course notes on 'Topics in differential geometry' at Stanford University*, (1988), available at `https://www.math.washington.edu/~pollack/research/Schoen-1988-notes.html`

[40] ———, *Variational theory for the total scalar curvature functional for Riemannian metrics and related topics*, in 'Topics in Calculus of Variations', Lecture Notes in Mathematics, Springer-Verlag, New York, 1365, 1989.

[41] ———, *On the number of constant scalar curvature metrics in a conformal class*, Differential Geometry: A symposium in honor of Manfredo Do Carmo (H.B. Lawson and K. Tenenblat, eds.), Wiley (1991), 311–320.

[42] R. Schoen and S.-T. Yau, *Conformally flat manifolds, Kleinian groups and scalar curvature*, Invent. Math. **92** (1988) 47–71.

[43] J. Wei and C. Zhao, *Non-compactness of the prescribed Q-curvature problem in large dimensions*, Calc. Var. Partial Differential Equations **46** (2013), 123–164.

(Liuwei Gong) Department of Mathematics, Chinese University of Hong Kong, Shatin, NT, Hong Kong

Email address: `lwgong@math.cuhk.edu.hk`

(Seunghyeok Kim) Department of Mathematics and Research Institute for Natural Sciences, College of Natural Sciences, Hanyang University, 222 Wangsimni-ro Seongdong-gu, Seoul 04763, Republic of Korea, School of Mathematics, Korea Institute for Advanced Study, 85 Hoegiro Dongdaemun-gu, Seoul 02455, Republic of Korea.

Email address: `shkim0401@hanyang.ac.kr shkim0401@gmail.com`

(Juncheng Wei) Department of Mathematics, Chinese University of Hong Kong, Shatin, NT, Hong Kong

Email address: `wei@math.cuhk.edu.hk`