NON-RADIAL SINGULAR SOLUTIONS OF LANE-EMDEN EQUATION IN $\mathbb{R}^N$

E.N. DANCER, ZONGMING GUO, AND JUNCHENG WEI

Abstract. We obtain infinitely many non-radial singular solutions of Lane-Emden equation

$$\Delta u + u^p = 0 \text{ in } \mathbb{R}^N \backslash \{0\}, \quad N \geq 4$$

with

$$\frac{N+1}{N-3} < p < p_c(N-1)$$

by constructing infinitely many radially symmetric regular solutions of equation on S^{N-1}

$$\Delta_{S^{N-1}} w - \frac{2}{p-1}\Big[N-2-\frac{2}{p-1}\Big] w + w^p = 0.$$

1. Introduction

We consider positive solutions of Lane-Emden equation:

$$(P) \qquad \Delta u + u^p = 0 \quad \text{in } \mathbb{R}^N, \quad N \geq 4$$

where

$$p > \frac{N+2}{N-2}.$$

Problem (P) arises both in physics and in geometry, and is a model semilinear elliptic equation. It has attracted extensive studies in the past three decades. In the subcritical case $1 < p < \frac{N+2}{N-2}$, a well-known result of Gidas and Spruck ([25]) says that (P) admits no nontrivial nonnegative solution. In the Sobolev critical case $p = \frac{N+2}{N-2}$, any positive solution of (P) can be written in the form (see [12]):

$$u_{\epsilon,\xi}(x) = C_N \Big(\frac{\epsilon}{\epsilon^2 + |x-\xi|^2}\Big)^{\frac{N-2}{2}}.$$

Therefore the structure of positive solutions in the critical or subcritical cases are completely classified. A fundamental question is to classify positive solutions in the supercritical case. This question remains largely open.

When $p > \frac{N+2}{N-2}$, the structure of positive radial solutions of (P) has been studied by Gui, Ni and Wang [26] and Wang [36]. They showed that for any $a > 0$,

---

1991 *Mathematics Subject Classification.* 35J25, 35J20, 35B33, 35B40.

Key words and phrases. Nonradial Singular Solutions, Supercritical Lane-Emden Equations, Asymptotic Analysis.

The first author is supported by Australia Research Council. The research of the second author is supported by NSFC (10871060) and NSF of Henan Province. The research of the third author is supported by an Earmarked Grant from RGC of Hong Kong, Oversea Joint Grant of NSFC, and a direct grant from CUHK.

equation (P) admits a unique positive radial solution $u = u(r)$ such that $u(0) = a$ and $u(r) \to 0$ as $r \to +\infty$. The solution u satisfies $u'(r) < 0$ for all $r > 0$ and

$$\lim_{r\to\infty} r^{\frac{2}{p-1}} u(r) = \Big[\frac{2}{p-1}\Big(N-2-\frac{2}{p-1}\Big)\Big]^{\frac{1}{p-1}} \ (:= \beta^{\frac{1}{p-1}}).$$

Moreover, if $N \le 10$ or $N \ge 11$ and

$$\frac{N+2}{N-2} < p < p_c(N),$$

then $u(r) - \beta^{\frac{1}{p-1}} r^{-\frac{2}{p-1}}$ changes sign infinitely many times. If $N \ge 11$ and $p \ge p_c(N)$, then $u(r) < \beta^{\frac{1}{p-1}} r^{-\frac{2}{p-1}}$ for all $r > 0$ and the solutions are strictly ordered with respect to the initial value $a = u(0)$. Here $p_c(M)$ (M is an integer) is the Joseph-Lundgren exponent [29]:

$$p_c(M) := \begin{cases} \infty, & \text{if } 2 \le M \le 10, \\ P(M), & \text{if } M \ge 11, \end{cases}$$

where

$$P(M) = \frac{(M-2)^2 - 4M + 8\sqrt{M-1}}{(M-2)(M-10)}.$$

When p is supercritical, it is still open if all positive solutions are radially symmetric around some point. The first result was due to Zou [38], who showed that when $p \in (\frac{N+2}{N-2}, \frac{N+1}{N-3})$ and u has the right decay $u = O(|x|^{-\frac{2}{p-1}})$, then all solutions are radially symmetric. Guo [27] extended Zou's result to $p \ge \frac{N+1}{N-3}$ by assuming $\lim_{|x|\to+\infty} |x|^{\frac{2}{p-1}} u(x) = \beta^{\frac{1}{p-1}}$.

Recently, solutions of (P) up to $p_c(N)$ are classified by using the Morse index theory. Farina [24] showed that if $\frac{N+2}{N-2} < p < p_c(N)$ and $u \in C^2(\mathbb{R}^N)$ is a positive solution of (P) that has finite Morse index, then $u \equiv 0$ in $\mathbb{R}^N$. (The condition that $u \in C^2(\mathbb{R}^N)$ can be weakened to be $H^1_{loc} \cap L^p_{loc}$. See Davila, Dupaigne and Farina [17].)

On the other hand, supercritical problems in a bounded domain

$$(D) \qquad \Delta u + u^p = 0, u > 0 \ \text{ in } \Omega, \ p > \frac{N+2}{N-2}, \ \ u = 0 \text{ on } \partial\Omega$$

have been studied by variational and perturbation methods. In case of pure non-linearity $u^{\frac{N+2}{N-2}}$, Coron [4] used a variational approach to prove that (D) is solvable if Ω exhibits *a small hole*. Bahri and Coron [1] established that solvability holds for $p = \frac{N+2}{N-2}$ whenever Ω has a non-trivial homology. On the other hand, examples in [13, 23, 33] shows that when $p \ge \frac{N+2}{N-2}$ (D) can still have a solution on some domains whose topology is trivial. If p is supercritical but close to critical, bubbling solutions are found, see [19, 20, 31, 32].

In the case of p being purely supercritical, there are very few existence results on (D). Variational machinery no longer applies, due to a lack of Sobolev inequality. In [22], del Pino and Wei extended Coron's result to supercritical problems (modulo some sequence of critical exponents) using perturbation methods. The role of the *second critical exponent* $p = \frac{N+1}{N-3}$, the Sobolev exponent in one dimension less, is investigated in the paper by del Pino, Musso and Pacard [21] in which they constructed solutions concentrating on a boundary geodesics for $p = \frac{N+1}{N-3} - \epsilon$ with $\epsilon \to 0+$. Under some symmetry assumptions, Wei and Yan [37] proved the existence of infinitely many positive solutions for some domains when $p = \frac{N+m-2}{N-m-2}, m \geq 1$. We should also mention that Davila, Del Pino and Musso [15] showed that in the case of the exterior domains $\Omega = \mathbb{R}^N \backslash \mathcal{D}$, and $p > \frac{N+2}{N-2}$, problem (D) admits infinitely many positive solutions. (See also [16].) We refer to the survey article [18] for more references.

Now we turn to singular solutions to (P)

$$\Delta u + u^p = 0, u > 0, \text{ in } \mathbb{R}^N \backslash \{0\}. \tag{1.1}$$

The singular solution in the subcritical or critical case has been completely classified. See Bidaut-Veron and Veron [6], Gidas and Spruck [25] and Korevaar-Mazzeo-Pacard-Schoen [30]. When $p > \frac{N+2}{N-2}$ and $p \neq \frac{N+1}{N-3}$, the only singular solution to (1.1) known so far is the radial singular solution

$$U(x) := U(r) = \Big[\frac{2}{p-1}\Big(N-2-\frac{2}{p-1}\Big)\Big]^{\frac{1}{p-1}} |x|^{-\frac{2}{p-1}}. \tag{1.2}$$

In [14], the authors showed that if Ω_0 is a bounded domain containing 0; u is a positive solution of (P) in $\Omega_0 \backslash \{0\}$; u has finite Morse index and $\frac{N+2}{N-2} < p < p_c(N)$, then $x = 0$ must be a removable singularity of u. They also showed that if Ω_0 is a bounded domain containing 0; u is a positive solution of (P) in $\mathbb{R}^N \backslash \Omega_0$ that has finite Morse index and $\frac{N+2}{N-2} < p < p_c(N)$, then u must be a fast decay solution. We still do not know more about the structure of positive solutions of (P) when $p \geq p_c(N)$.

Our motivation of studying (1.1) is to classify all possible singular solutions. This is important for Liouville type theorems (Polacik, Quittner, Souplet [35]). The first question is whether or not all singular solutions to (1.1) are radially symmetric. The purpose of this paper is to construct infinitely many positive nonradial singular solutions of (1.1) provided

$$\frac{N+1}{N-3} < p < p_c(N-1).$$

This gives an negative answer to the above question. Note that $p_c(M)$ is a decreasing function of M. Then $p_c(N) < p_c(N-1)$. This provides new information on the case $p \geq p_c(N)$. Note also that

$$\Big(M-2-\frac{4}{p-1}\Big)^2 - 8\Big(M-2-\frac{2}{p-1}\Big) < 0, \quad \text{for } \tfrac{M+2}{M-2} < p < p_c(M). \tag{1.3}$$

Our main result can be stated as follows.

Theorem 1.1. *Assume that*

$$\frac{N+1}{N-3} < p < p_c(N-1) \tag{1.4}$$

Then there exist infinitely many nonradial singular solutions to (1.1).

To explain our idea of construction, we perform a separation of variable: it is easy to see that that any solution $u(x) := u(r, \omega)$ of (P) satisfies the equation

$$u_{rr} + \frac{N-1}{r} u_r + \frac{1}{r^2} \Delta_{S^{N-1}} u + u^p = 0 \tag{1.5}$$

where $r = |x|$. If

$$u(x) = r^{-\frac{2}{p-1}} w(\omega) \tag{1.6}$$

where w is a solution of the equation

$$\Delta_{S^{N-1}} w - \beta w + w^p = 0, \tag{1.7}$$

with

$$\beta = \frac{2}{p-1}(N-2-\frac{2}{p-1}),$$

then u is a singular solution of (P). It is clear that

$$w(\omega) \equiv \beta^{\frac{1}{p-1}}$$

is the constant solution of (1.7) and it provides a radial singular solution of (P) as given in (1.2). For $p < \frac{N+1}{N-3}$, Bidaut-Veron and Veron ([6]) proved that the only solutions to (1.7) are constants. (See also Zou [38].) On the other hand, when $p = \frac{N+1}{N-3}$, problem (1.7) becomes Yamabe problem on S^{N-1} whose solutions are all classified.

To construct positive non-radial singular solutions of (P), we need to find positive non-constant solutions $w(\omega)$ of (1.7). In this paper, we will construct infinitely many positive nonconstant radially symmetric solutions of (1.7), i.e., solutions that only depend on the geodesic distance $\theta \in [0, \pi)$. In this case, (1.7) can be written in a more convenient form (with $x = \cos\theta$), namely

$$\begin{cases} (1-x^2)^{-\frac{N-3}{2}}\Big((1-x^2)^{\frac{N-1}{2}} w_x\Big)_x - \beta w + w^p = 0, \quad w(x) > 0, \;\; -1 < x < 1 \\ w'(1) \;\; \text{exists.} \end{cases} \tag{1.8}$$

If we only consider the simple case $w(-x) = w(x)$ for $x \in (0,1)$, we see that $w'(0) = 0$. Then $w(x) := w(\theta)$ with $w(\theta) = w(\pi - \theta)$ for $0 < \theta \le \pi/2$ satisfies the problem

$$\begin{cases} \frac{1}{\sin^{N-2}\theta}\frac{d}{d\theta}\Big(\sin^{N-2}\theta\,\frac{dw}{d\theta}(\theta)\Big) - \beta w(\theta) + w^p(\theta) = 0, \quad w(\theta) > 0, \;\; 0 < \theta < \frac{\pi}{2}, \\ w'_\theta(0) \;\; \text{exists}, \quad w'_\theta(\frac{\pi}{2}) = 0. \end{cases} \tag{1.9}$$

Note that even though (1.9) is an ODE, it is still supercritical. Neither variational methods nor sub-super solution method apply. Note also that the β here is *fixed* so bifurcation argument does not work, either. A key observation is that besides the obvious constant solution $w = \beta^{\frac{1}{p-1}}$, there is another solution

$$w_*(\theta) = A_p[\sin\theta]^{-\frac{2}{p-1}}, \;\; \theta \in (0, \frac{\pi}{2}], \quad A_p^{p-1} = \frac{2}{p-1}\Big[N-3-\frac{2}{p-1}\Big] \tag{1.10}$$

which is a singular solution of (1.9) with two singularities at $\theta = 0$ and $\theta = \pi$. A crucial fact is that because of the condition $p < p_c(N-1)$, the singular solution to (1.9) has Morse index ∞. We will construct the inner and outer solutions of (1.9) and then glue them to be solutions of (1.9). Such arguments have been used in [11] for the supercritical problem $\Delta u + \lambda u + u^p = 0$ in a unit ball in $\mathbb{R}^3$ with $p > 5$.

We should mention that recently Bidaut-Veron, Ponce and Veron [8] studied solutions of (P) with boundary singularities. In particular, they obtained the existence of a singular solutions of the separated form (1.6), where w vanishes on the equator, for $\frac{N+1}{N-1} < p < \frac{N+1}{N-3}$ and nonexistence beyond. They also showed that these solutions only depend on the incidence angle $\theta \in (0, \pi)$, satisfying the ODE (1.8) and vanishing at $\frac{\pi}{2}$, and are unique.

Equation (1.9) has also been studied recently by many authors. Regarding β as a parameter, it has been shown that there are more and more nonradial solutions as $\beta \to +\infty$. We refer to Brezis-Peletier [9], Bandle-Wei [5] and the references therein. Here in this paper, β is fixed and equals $\frac{2}{p-1}(N-2-\frac{2}{p-1})$.

The nonradial singular solutions to (P) may serve as good asymptotics for nonradial entire solutions to (P). Thus we conjecture

Conjecture: *For each of the nonradial singular solutions $r^{-\frac{2}{p-1}}w(\theta)$ constructed in Theorem 1.1 there exists an entire positive solution u to (P) such that*

$$u-|x|^{-\frac{2}{p-1}}w(\theta)=o(|x|^{-\frac{2}{p-1}}),\ \textit{for } |x|\gg 1. \tag{1.11}$$

This paper is organized as follows: in Section 2, we study an intial value problem and study the asymptotic behavior of the inner solution when the initial value tends to infinity. In Section 3, we study the outer problem. Namely we solve the problem (1.9) from $\theta=\frac{\pi}{2}$. The asymptotic behavior of the outer problem will be analyzed near the origin. Finally in Section 4, we use asymptotics to match the inner and outer solutions, thereby proving Theorem 1.1.

2. Inner solutions

In this section we study solutions $w(\theta)$ of (1.9) with $w(0)=Q\gg 1$ and analyze their behaviors near $\theta=0$. This is the inner solution. Since $Q\gg 1$, we set $Q=\epsilon^{-\frac{2}{p-1}}$ $(:=\epsilon^{-\alpha})$ with ϵ sufficiently small.

Rescaling as $w(\theta)=\epsilon^{-\alpha}v(\frac{\theta}{\epsilon})$, we see that $v(0)=1$ and $v(r)$ (for $r=\frac{\theta}{\epsilon}$) satisfies the following equation

$$v_{rr}+(N-2)\epsilon\cot(\epsilon r)v_r-\beta\epsilon^2 v+v^p=0,\quad v(0)=1. \tag{2.1}$$

Observe that for $\epsilon>0$ sufficiently small,

$$\cot(\epsilon r)=\frac{\cos(\epsilon r)}{\sin(\epsilon r)}=\frac{1}{\epsilon r}-\frac{1}{3}(\epsilon r)+\sum_{k=1}^{\infty}\ell_k(\epsilon r)^{2k+1}.$$

Thus,

$$v_{rr}+\frac{N-2}{r}v_r-\frac{(N-2)}{3}(\epsilon^2 r)v_r+\Big(\sum_{k=1}^{\infty}(N-2)\ell_k\epsilon^{2(k+1)}r^{2k+1}\Big)v_r-\beta\epsilon^2 v+v^p=0,\quad v(0)=1. \tag{2.2}$$

The first approximation to the solution of (2.2) is the radial solution $v_0(r)$ of the problem

$$\Delta v+v^p=0\quad\text{in } \mathbb{R}^{N-1},\quad v(0)=1. \tag{2.3}$$

For $p\geq p_c(N-1)$, $v_0(r)$ is stable and the asymptotic expansion can be found in [26]. For $p<p_c(N-1)$, we can not find a reference for the asymptotic behavior of v. We state the following result.

Lemma 2.1. *For $\frac{N+1}{N-3}<p<p_c(N-1)$, there exist constants a_0, b_0 and $R_0\gg 1$ such that for $r\geq R_0$ the unique positive solution $v_0(r)$ of (2.3) satisfies*

$$v_0(r)=A_p r^{-\alpha}+\frac{a_0\cos(\omega\ln r)+b_0\sin(\omega\ln r)}{r^{\frac{N-3}{2}}}+O(r^{-(N-3-\frac{2}{p-1})}), \tag{2.4}$$

where

$$A_p^{p-1} = \frac{2}{p-1}\Big[N-3-\frac{2}{p-1}\Big],$$

$$\omega = \frac{1}{2}\sqrt{8\Big(N-3-\frac{2}{p-1}\Big) - \Big(N-3-\frac{4}{p-1}\Big)^2} \quad . \tag{2.5}$$

Proof. Note that

$$8\Big(N-3-\frac{2}{p-1}\Big) - \Big(N-3-\frac{4}{p-1}\Big)^2 > 0, \quad \text{for } \tfrac{N+1}{N-3} < p < p_c(N-1),$$

$$N-3-\frac{2}{p-1} > \frac{N-3}{2} \quad \text{for } p > \tfrac{N+1}{N-3}.$$

The existence and uniqueness of $v_0(r)$ can be found [26] and [28]. It is also known ([26], [28]) that

$$\lim_{r\to+\infty} r^\alpha v_0(r) = A_p.$$

To find the next order term, we use the Emden-Fowler transformation:

$$V(t) = r^\alpha v_0(r) - A_p, \quad t = \ln r \ .$$

It is easy to see that $V(t)$ satisfies the equation

$$V_{tt} + (N-3-2\alpha)V_t + 2(N-3-\alpha) + g(V) = 0, \quad \text{for } t \geq T = \ln R, \ \ R > 10, \tag{2.6}$$

where

$$g(s) = (s+A_p)^p - A_p^p - pA_p^{p-1}s.$$

By the standard argument of variation of constants we obtain the following integral equation

$$V(t) = e^{\sigma t}[a\cos\omega t + b\sin\omega t] + \frac{1}{\omega}\int_T^t e^{\sigma(t-t')}\sin\omega(t-t')g(V(t'))dt',$$

where $\sigma = -\frac{1}{2}(N-3-2\alpha)$, ω is given in (2.5). Note that $g(s) = O(s^2)$ for s sufficiently small.

Set $\tilde V(t) = e^{-\sigma t}V(t)$. Then $\tilde V(t)$ satisfies the integral equation

$$\tilde V(t) := \mathcal{N}\tilde V(t) = C\sin(\omega t + D) + \frac{1}{\omega}\int_{T_0}^t e^{-\sigma t'}\sin\omega(t-t')g(e^{\sigma t'}\tilde V(t'))dt', \tag{2.7}$$

where $C = \sqrt{a^2+b^2}$, $\sin D = \frac{a}{C}$, $\cos D = \frac{b}{C}$. We take t in the range $T_0 \leq t < \infty$, where $T_0 = \ln R_0$ is suitably large, and consider $\mathcal{N}\tilde V$ as a map from $C[T_0,\infty)$ into itself. We claim that, for each $C>0$ and suitable T_0, the operator $\mathcal{N}\tilde V$ maps the set

$$\mathcal{B} = \{\tilde V \in C[T_0,\infty): \ \ \|z\|_0 = \sup_{T_0<t<\infty}|\tilde V(t)| \leq 2C, \ \ C>0\}$$

into itself, and is a contraction mapping on $\mathcal{B}$. Indeed, if $\|\tilde V\|_0 < 2C$, then

$$|g(e^{\sigma t}\tilde V(t))| = e^{2\sigma t}O(1)$$

and

$$\|\mathcal{N}\tilde V - C\sin(\omega t + D)\|_0 \leq C' e^{\sigma T_0}$$

where $C' > 0$ only depends on C, N, p. Note that $\sigma < 0$ and $\|e^{\sigma t}\tilde V(t)\|_0$ is sufficiently small for $\tilde V \in \mathcal{B}$ for T_0 suitably large. Thus, if we choose $T_0 > 1$

suitably large, we see that $\|\mathcal{N}\tilde{V} - C\sin(\omega t + D)\|_0 < C$. A similar calculation shows that

$$\|\mathcal{N}\tilde{V}_1 - \mathcal{N}\tilde{V}_2\|_0 \le e^{\sigma T_0}\|\tilde{V}_1 - \tilde{V}_2\|_0.$$

Hence it is possible for each value of C to choose T_0 so that $\mathcal{N}$ is a contraction mapping of $\mathcal{B}$ to itself. Thus, we define $\tilde{V}_0 = C\sin(\omega t + D)$ and the iteration $\tilde{V}_{n+1} = \mathcal{N}\tilde{V}_n$ for $n \ge 0$. The contraction mapping theorem then ensures that this iteration converges to the unique solution $\tilde{V}_*(t)$ of (2.7) in $\mathcal{B}$. Note that

$$\Big|\frac{1}{\omega}\int_{T_0}^{t} e^{-\sigma t'}\sin\omega(t-t')g(e^{\sigma t'}\tilde{V}_*(t'))dt'\Big| = O(e^{\sigma t}).$$

Then

$$V_*(t) = e^{\sigma t}\tilde{V}_*(t) = C_0 e^{\sigma t}\sin(\omega t + D_0) + O(e^{2\sigma t}) \ \text{ for } t \in (T_0, \infty).$$

This implies that for $r \in [R_0, \infty)$,

$$v_0(r) = A_p r^{-\alpha} + r^{-\frac{N-3}{2}}[a_0\cos\omega\ln r + b_0\sin\omega\ln r] + O(r^{-(N-3-\frac{2}{p-1})})$$

and completes the proof of this lemma. □

Lemma 2.2. *Let p satisfy the conditions of Lemma 2.1 and $v_1(r)$ be the unique solution of the problem*

$$\begin{cases} v_1''(r) + \frac{N-2}{r}v_1'(r) + pv_0^{p-1}(r)v_1(r) - \frac{(N-2)}{3}rv_0'(r) - \beta v_0(r) = 0, & r \in (0,\infty), \\ v_1(0) = 0, \ \ v_1'(0) = 0. \end{cases} \tag{2.8}$$

Then for $r \in [R_0, \infty)$,

$$v_1(r) = C_p r^{2-\alpha} + r^{2-\frac{N-3}{2}}(a_1\cos(\omega\ln r) + b_1\sin(\omega\ln r)) + o(r^{2-\frac{N-3}{2}}), \tag{2.9}$$

where C_p satisfies

$$[(2-\alpha)(N-1-\alpha) + pA_p^{p-1}]C_p = A_p\Big[\beta - \frac{2(N-2)}{3(p-1)}\Big], \tag{2.10}$$

(a_1, b_1) is the solution of

$$\begin{cases} D_1a_1 + 4\omega b_1 = \beta a_0 + \frac{(N-2)}{3}b_0\omega - \frac{(N-2)(N-3)}{6}a_0 - p(p-1)A_p^{p-2}C_pa_0 \\ -4\omega a_1 + D_1b_1 = \beta b_0 - \frac{(N-2)}{3}a_0\omega - \frac{(N-2)(N-3)}{6}b_0 - p(p-1)A_p^{p-2}C_pb_0, \end{cases}$$

where $D_1 = \frac{(N+1)(7-N)}{4} - \omega^2 + pA_p^{p-1}$; a_0, b_0 and ω are given in Lemma 2.1.

Proof. Let

$$v_1(r) = C_p r^{2-\alpha} + h(r)r^{2-\frac{N-3}{2}} + o(r^{2-\frac{N-3}{2}})$$

where

$$h(r) = c_1\cos(\omega\ln r) + c_2\sin(\omega\ln r).$$

Using the expression of $v_0(r)$ in (2.4), (2.9) follows by direct calculations. Note that

$$O(r^{-(N-3-\frac{2}{p-1})}) = o(r^{-\frac{(N-3)}{2}})$$

provided $p > \frac{N+1}{N-3}$. □

Now we obtain the following proposition.

Proposition 2.3. *Let $\frac{N+1}{N-3} < p < p_c(N-1)$ and $v(r)$ be a solution of (2.1). Then for $\epsilon > 0$ sufficiently small,*

$$v(r) = v_0(r) + \sum_{k=1}^{\infty} \epsilon^{2k} v_k(r).$$

Moreover, for $r \in [R_0, \infty)$,

$$v_k(r) = \sum_{j=1}^{k} d_j^k r^{2j-\alpha} + \sum_{j=1}^{k} e_j^k r^{2j-\frac{N-3}{2}} \sin(\omega \ln r + E_j^k) + o(r^{2k-\frac{N-3}{2}}), \tag{2.11}$$

where d_j^k, e_j^k, E_j^k $(j = 1, 2, \ldots, k)$ are constants. Moreover,

$$d_1^1 = C_p, \quad e_1^1 = \sqrt{a_1^2 + b_1^2}, \quad \sin E_1^1 = \frac{a_1}{e_1^1}, \quad \cos E_1^1 = \frac{b_1}{e_1^1}$$

where C_p, a_1, b_1 are given in Lemma 2.2.

Proof. Using the Taylor's expansion of v^p and the expressions of $v_0(r), v_1(r), \ldots, v_{k-1}(r)$, we can obtain this proposition by the induction argument and direct calculations. Note that

$$O(r^{2-\frac{N-3}{2}}) = o(r^{2-\alpha}).$$

□

Now we obtain the following theorem.

Theorem 2.4. *Let $\frac{N+1}{N-3} < p < p_c(N-1)$ and $w_\epsilon^{inn}(\theta)$ be an inner solution of (1.9) with $w_\epsilon(0) = \epsilon^{-\alpha}$. Then for any sufficiently small $\epsilon > 0$ and $\theta > R_0\epsilon$ but θ is also sufficiently small,*

$$\begin{aligned} w_\epsilon^{inn}(\theta) = \quad & \frac{A_p}{\theta^\alpha} + \frac{C_p}{\theta^{\alpha-2}} + \sum_{k=2}^{\infty} \sum_{j=1}^{k} d_j^k \epsilon^{2(k-j)} \theta^{2j-\alpha} \\ & + \epsilon^{\frac{N-3}{2}-\alpha} \Big[\frac{a_0 \cos(\omega \ln \frac{\theta}{\epsilon}) + b_0 \sin(\omega \ln \frac{\theta}{\epsilon})}{\theta^{\frac{N-3}{2}}} + \frac{a_1 \cos(\omega \ln \frac{\theta}{\epsilon}) + b_1 \sin(\omega \ln \frac{\theta}{\epsilon})}{\theta^{\frac{N-3}{2}-2}} \\ & + \sum_{k=2}^{\infty} \Big(\sum_{j=1}^{k} e_j^k \epsilon^{2(k-j)} \theta^{2j-\frac{N-3}{2}} \sin(\omega \ln \frac{\theta}{\epsilon} + E_j^k) + o(\theta^{2k-\frac{N-3}{2}}) \Big) \Big]. \end{aligned}$$

Proof. This is a direcct consequence of Proposition 2.3 by setting $r = \theta/\epsilon$. □

We now obtain the following lemmas similar to Lemma 2.4 and Lemma 3.3 of [11] respectively which will be useful in the following proofs.

Lemma 2.5. *Let $\frac{N+1}{N-3} < p < p_c(N-1)$ and*

$$v(Q, \theta) = Q v_0(Q^{\frac{p-1}{2}} \theta).$$

Then for $Q^{\frac{p-1}{2}} \theta \geq R_0$, and for $n = 0, 1, 2$, $v(Q, \theta)$ satisfies

$$\begin{aligned} (i) \quad \frac{\partial^n}{\partial Q^n}(v(Q,\theta)) \quad = \quad & \frac{\partial^n}{\partial Q^n}\Big(\frac{A_p}{\theta^\alpha}\Big) \\ & + \frac{\partial^n}{\partial Q^n}\Big\{ C\theta^{-\frac{N-3}{2}} Q^{-(\frac{(p-1)(N-3)}{4}-1)} \sin\Big(\omega \ln(Q^{\frac{p-1}{2}}\theta) + D\Big) \Big\} \\ & + Q^{-n-[\frac{p-1}{2}(N-3-\alpha)-1]} O(\theta^{-(N-3-\alpha)}), \end{aligned}$$

$$\begin{aligned}(ii)\ \frac{\partial^n}{\partial Q^n}(v'_\theta(Q,\theta)) \quad = \quad & \frac{\partial^n}{\partial Q^n}\Big(\frac{-\alpha A_p}{\theta^{\alpha+1}}\Big) \\ & +\frac{\partial^{n+1}}{\partial Q^n\partial\theta}\Big\{C\theta^{-\frac{N-3}{2}}Q^{-(\frac{(p-1)(N-3)}{4}-1)}\ \sin\Big(\omega\ln(Q^{\frac{p-1}{2}}\theta)+D\Big)\Big\} \\ & +Q^{-n-[\frac{p-1}{2}(N-3-\alpha)-1]}O(\theta^{-(N-2-\alpha)}),\end{aligned}$$

where

$$D=\tan^{-1}(\frac{b_0}{a_0}),\quad C=\sqrt{a_0^2+b_0^2}.$$

Proof. These estimates are obtained by the expansion of $v_0(r)$ given above and some calculations. □

Lemma 2.6. *In the region $\theta=|O(Q^{\frac{\sigma}{(2-\sigma)\alpha}})|$, the solution $w(Q,\theta)$ of* (1.9) *with $w(Q,0)=Q$, $w'_\theta(Q,0)=0$ satisfies*

(i) $\left|\frac{\partial w}{\partial Q}(Q,\theta)-\frac{\partial v}{\partial Q}(Q,\theta)\right|=Q^{-\frac{(p-1)(N-3)}{4}}|o(\theta^{-\frac{N-3}{2}})|$;

(ii) $\left|\frac{\partial w'_\theta}{\partial Q}(Q,\theta)-\frac{\partial v'_\theta}{\partial Q}(Q,\theta)\right|=Q^{-\frac{(p-1)(N-3)}{4}}|o(\theta^{-\frac{N-1}{2}})|$;

(iii) $\left|\frac{\partial^2 w}{\partial Q^2}(Q,\theta)-\frac{\partial^2 v}{\partial Q^2}(Q,\theta)\right|=Q^{-(\frac{(p-1)(N-3)}{4}+1)}|o(\theta^{-\frac{N-3}{2}})|$;

(iv) $\left|\frac{\partial^2 w'_\theta}{\partial Q^2}(Q,\theta)-\frac{\partial^2 v'_\theta}{\partial Q^2}(Q,\theta)\right|=Q^{-(\frac{(p-1)(N-3)}{4}+1)}|o(\theta^{-\frac{N-1}{2}})|$.

Proof. This lemma can be obtained from Lemma 2.5 and Theorem 2.4. Note that

$$\epsilon=Q^{-\frac{1}{\alpha}},\quad \frac{-\sigma}{\alpha}=\frac{(p-1)(N-3)}{4}-1.$$

Moreover,

$$Q^{\frac{p-1}{2}}\theta=|O(Q^{\frac{p-1}{2-\sigma}})|>R_0$$

provided Q suitably large. □

Now we can summarize the inner solution obtained in Theorem 2.4 in the form of parameter Q:

Theorem 2.7. *Let $\frac{N+1}{N-3}<p<p_c(N-1)$ and $w_Q^{inn}(\theta)$ be an inner solution of* (1.9) *with $w_Q(0)=Q$. Then for any sufficiently large $Q>0$ and $\theta=|O(Q^{\frac{\sigma}{(2-\sigma)\alpha}})|$,*

$$\begin{aligned}& w_Q^{inn}(\theta) \\ & =\frac{A_p}{\theta^\alpha}+\frac{C_p}{\theta^{\alpha-2}}+\sum_{k=2}^\infty\sum_{j=1}^k d_j^kQ^{-(p-1)(k-j)}\theta^{2j-\alpha} \\ & +Q^{\frac{\sigma}{\alpha}}\Big[\frac{a_0\cos\Big(\omega\ln(Q^{\frac{p-1}{2}}\theta)\Big)+b_0\sin\Big(\omega\ln(Q^{\frac{p-1}{2}}\theta)\Big)}{\theta^{\frac{N-3}{2}}} \\ & \quad+\frac{a_1\cos\Big(\omega\ln(Q^{\frac{p-1}{2}}\theta)\Big)+b_1\sin\Big(\omega\ln(Q^{\frac{p-1}{2}}\theta)\Big)}{\theta^{\frac{N-3}{2}-2}} \\ & \quad+\sum_{k=2}^\infty\Big(\sum_{j=1}^k e_j^kQ^{-(p-1)(k-j)}\theta^{2j-\frac{N-3}{2}}\sin\Big(\omega\ln(Q^{\frac{p-1}{2}}\theta)+E_j^k\Big)+o(\theta^{2k-\frac{N-3}{2}})\Big)\Big].\end{aligned}$$

3. OUTER SOLUTIONS

In this section we study the asymptotic behaviors of solutions $w(\theta)$ of (1.9) far from $\theta = 0$.

Let $w_*(\theta)$ be the singular solution given in (1.10). We first obtain the following lemma.

Lemma 3.1. *Equation*

$$\frac{1}{\sin^{N-2}\theta}\frac{d}{d\theta}\Big(\sin^{N-2}\theta\,\frac{d\phi}{d\theta}(\theta)\Big) - \beta\phi(\theta) + pw_*^{p-1}(\theta)\phi(\theta) = 0, \qquad 0 < \theta < \frac{\pi}{2}, \quad (3.1)$$

admits two fundamental solutions $\phi_1(\theta)$ and $\phi_2(\theta)$. Moreover, any solution $\phi(\theta)$ of (3.1) *can be written in the form*

$$\phi(\theta) = c_1\phi_1(\theta) + c_2\phi_2(\theta), \quad \textit{where } c_1 \textit{ and } c_2 \textit{ are constants,}$$

which satisfies that as $\theta \to 0$,

$$\phi(\theta) = \theta^{-\frac{N-3}{2}}\Big[c_1\cos(\omega\ln\frac{\theta}{2}) + c_2\sin(\omega\ln\frac{\theta}{2})\Big] + O(\theta^{2-\frac{N-3}{2}}). \qquad (3.2)$$

Proof. Let $\tilde\phi(\theta) = [\sin\theta]^\alpha\phi(\theta)$. We see that $\tilde\phi(\theta)$ satisfies the equation

$$\sin^2\theta\tilde\phi''(\theta) + (N-2-2\alpha)\sin\theta\cos\theta\tilde\phi'(\theta) + (p-1)A_p^{p-1}\tilde\phi(\theta) = 0. \qquad (3.3)$$

Under the Emden-Fowler transformations:

$$\psi(t) = \tilde\phi(\theta), \quad t = \ln\tan\frac{\theta}{2},$$

we obtain that for $t \in (-\infty, 0)$,

$$\psi''(t) + \Big(N-3-2\alpha\Big)\Big(1 - \frac{2e^{2t}}{1+e^{2t}}\Big)\psi'(t) + 2\Big(N-3-\alpha\Big)\psi(t) = 0. \qquad (3.4)$$

Note that

$$\sin\theta = \frac{2e^t}{1+e^{2t}}, \quad \cos\theta = \frac{1-e^{2t}}{1+e^{2t}} = 1 - \frac{2e^{2t}}{1+e^{2t}}.$$

We can obtain solutions of (3.4) by shooting backwards under the conditions $\psi(0) = a$, $\psi'(0) = 0$. The standard ODE arguments imply that (3.4) admits two fundamental solutions $\psi_1, \psi_2 \in C^2(-\infty, 0)$ such that any solution $\psi(t)$ of (3.4) satisfies

$$\psi(t) = \ell_1\psi_1(t) + \ell_2\psi_2(t)$$

where ℓ_1 and ℓ_2 are two constants. Now we show that as $t \to -\infty$,

$$\psi(t) = e^{\sigma t}[\ell_3\cos\omega t + \ell_4\sin\omega t] + O(e^{(\sigma+2)t})$$

where $\sigma = -\frac{N-3}{2} + \alpha$.

We see that the characteristic equation of (3.4) admits a pair roots $\lambda_1 = \sigma + i\omega$, $\lambda_2 = \sigma - i\omega$ as $t \to -\infty$ since

$$(N-3-2\alpha)^2 - 8(N-3-\alpha) < 0 \quad \text{for } \tfrac{N+1}{N-3} < p < p_c(N-1).$$

By the standard argument of variation of constants we obtain the following integral equation

$$\psi(t) = e^{\sigma t}[\ell_3\cos\omega t + \ell_4\sin\omega t] + \frac{1}{\omega}\int_T^t e^{\sigma(t-t')}\sin\omega(t-t')j(\psi)(t')dt',$$

where $T \in (-\infty, 0)$ with sufficiently large $|T|$, $j(\psi)(t') = -(N-3-2\alpha)\frac{2e^{2t'}}{1+e^{2t'}}\psi'(t')$. Setting $\hat{\psi}(t) = e^{-\sigma t}\psi(t)$, we see

$$\hat{\psi}(t) = [\ell_3 \cos \omega t + \ell_4 \sin \omega t] + \frac{1}{\omega}\int_T^t \sin \omega(t-t') j(\hat{\psi})(t')dt', \tag{3.5}$$

where

$$j(\hat{\psi})(t') = -(N-3-2\alpha)\frac{2e^{2t'}}{1+e^{2t'}}(\sigma\hat{\psi}(t') + \hat{\psi}'(t')).$$

It follows from (3.5) that

$$\|\hat{\psi} - [\ell_3 \cos \omega t + \ell_4 \sin \omega t]\|_0 \le \tau(|\sigma|\|\hat{\psi}\|_0 + \|\hat{\psi}'\|_0), \tag{3.6}$$

where $0 < \tau := \tau(T) \to 0$ as $T \to -\infty$ and $\|\rho\|_0 = \sup_{-\infty<t<T} |\rho(t)|$.

On the other hand, we see that $z(t) := \psi'(t)$ satisfies the equation

$$z''(t) + (N-3-2\alpha)z'(t) + 2(N-3-\alpha)z(t) + h(t, \psi(t), \psi'(t)) = 0$$

where

$$\begin{aligned} h(t, \psi(t), \psi'(t)) &= (N-3-2\alpha)^2\frac{2e^{2t}}{(1+e^{2t})}\Big(1 - \frac{2e^{2t}}{(1+e^{2t})}\Big)\psi'(t) \\ &-2(N-3-2\alpha)\frac{2e^{2t}}{(1+e^{2t})^2}\psi'(t) + 2(N-3-\alpha)(N-3-2\alpha)\frac{2e^{2t}}{(1+e^{2t})}\psi(t). \end{aligned}$$

Thus,

$$e^{-\sigma t}\psi'(t) = [\ell_5 \cos \omega t + \ell_6 \sin \omega t] + \frac{1}{\omega}\int_T^t \sin \omega(t-t') h(t', \hat{\psi}(t'), \hat{\psi}'(t'))dt', \tag{3.7}$$

where

$$\ell_5 = \ell_3\sigma + \ell_4\omega, \quad \ell_6 = \ell_4\sigma - \omega\ell_3$$

and

$$\begin{aligned} h(t, \hat{\psi}(t), \hat{\psi}'(t)) &= (N-3-2\alpha)^2\frac{2e^{2t}}{(1+e^{2t})}\Big(1 - \frac{2e^{2t}}{(1+e^{2t})}\Big)(\sigma\hat{\psi}(t) + \hat{\psi}'(t)) \\ &-2(N-3-2\alpha)\frac{2e^{2t}}{(1+e^{2t})^2}(\sigma\hat{\psi}(t) + \hat{\psi}'(t)) \\ &+2(N-3-\alpha)(N-3-2\alpha)\frac{2e^{2t}}{(1+e^{2t})}\hat{\psi}(t). \end{aligned}$$

It follows from (3.7) that

$$\|e^{-\sigma t}\psi'(t) - [\ell_5 \cos \omega t + \ell_6 \sin \omega t]\|_0 \le \tau(|\sigma|\|\hat{\psi}\|_0 + \|\hat{\psi}'\|_0), \tag{3.8}$$

where τ is as in (3.6). Since $\hat{\psi}'(t) = e^{-\sigma t}\psi'(t) - \sigma\hat{\psi}(t)$, it follows from (3.6) and (3.8) that by choosing $|T|$ suitably large,

$$\|\hat{\psi}\|_0 \le C, \quad \|\hat{\psi}'\|_0 \le C \tag{3.9}$$

where $C = C(p, N, T, \ell_3, \ell_4)$. Both (3.9) and (3.5) imply that as $t \to -\infty$,

$$\hat{\psi}(t) = [\ell_3 \cos \omega t + \ell_4 \sin \omega t] + O(e^{2t}). \tag{3.10}$$

Therefore, as $t \to -\infty$,

$$\psi(t) = e^{\sigma t}[\ell_3 \cos \omega t + \ell_4 \sin \omega t] + O(e^{(\sigma+2)t}). \tag{3.11}$$

This implies that as $\theta \to 0^+$,

$$\phi(\theta) = [\sin\theta]^{-\alpha}\Big(\tan\frac{\theta}{2}\Big)^{\sigma}\Big[\ell_3\cos(\omega\ln\frac{\theta}{2})+\ell_4\sin(\omega\ln\frac{\theta}{2})\Big]+O\Big([\sin\theta]^{-\alpha}\Big(\tan\frac{\theta}{2}\Big)^{\sigma+2}\Big). \tag{3.12}$$

Then the Taylor's expansions of $\sin\theta$ and $\tan\frac{\theta}{2}$ imply that (3.2) holds. This completes the proof. □

Remark 3.2. For any $\delta > 0$ sufficiently small, if c_1 and c_2 in (3.2) satisfy that $c_1 = \tilde{c}_1\delta$, $c_2 = \tilde{c}_2\delta$, where $\tilde{c}_1$ and $\tilde{c}_2$ are constants, then as $\theta \to 0^+$,

$$\phi(\theta) := \phi_\delta(\theta) = \delta\theta^{-\frac{N-3}{2}}\Big[\tilde{c}_1\cos(\omega\ln\frac{\theta}{2}) + \tilde{c}_2\sin(\omega\ln\frac{\theta}{2})\Big] + O(\delta)\theta^{2-\frac{N-3}{2}}. \tag{3.13}$$

Indeed, if $\ell_3 = \tilde{\ell}_3\delta$, $\ell_4 = \tilde{\ell}_4\delta$, where $\tilde{\ell}_3$ and $\tilde{\ell}_4$ are constants, we see from (3.8) that

$$|\sigma|\|\hat{\psi}\|_0 + \|\hat{\psi}'\|_0 \le C\delta$$

where $C := C(p, N, T, \tilde{\ell}_3, \tilde{\ell}_4) > 0$ is independent of δ. Hence

$$\psi(t) := \psi_\delta(t) = e^{\sigma t}\delta[\tilde{\ell}_3\cos\omega t + \tilde{\ell}_4\sin\omega t] + O(\delta)e^{(\sigma+2)t}.$$

For any $\delta > 0$ sufficiently small, if $w \in C^2(0, \frac{\pi}{2})$ is a solution of (1.9) and

$$w(\theta) = w_*(\theta) + \delta\phi_\delta(\theta) + \delta^2\psi_\delta(\theta),$$

where

$$\phi_\delta(\theta) = \tilde{c}_1\delta\phi_1(\theta) + \tilde{c}_2\delta\phi_2(\theta)$$

is a solution of (3.1) with

$$c_1 = \tilde{c}_1\delta, \quad c_2 = \tilde{c}_2\delta,$$

then $\psi_\delta(\theta)$ satisfies the problem

$$\begin{cases} \frac{1}{\sin^{N-2}\theta}\frac{d}{d\theta}\Big(\sin^{N-2}\theta\,\frac{d\psi}{d\theta}(\theta)\Big) - \beta\psi(\theta) + pw_*^{p-1}\psi(\theta) \\ \quad +\delta^{-2}\Big[(w_* + \delta\phi_\delta + \delta^2\psi)^p - w_*^p - pw_*^{p-1}\delta\phi_\delta - \delta^2 pw_*^{p-1}\psi\Big] = 0, \quad 0 < \theta < \pi/2, \\ \psi'(\frac{\pi}{2}) = -(\tilde{c}_1\phi_1'(\frac{\pi}{2}) + \tilde{c}_2\phi_2'(\frac{\pi}{2})). \end{cases} \tag{3.14}$$

Lemma 3.3. *For any $\delta > 0$ sufficiently small and each fixed pair $(\tilde{c}_1, \tilde{c}_2)$, (3.14) admits solutions $\psi_\delta \in C^2(0, \pi/2)$.*

Proof. We set the initial value conditions on ψ of (3.14) at $\theta = \pi/2$: $\psi(\pi/2) = 1$ provided

$$\psi'(\frac{\pi}{2}) = -(\tilde{c}_1\phi_1'(\frac{\pi}{2}) + \tilde{c}_2\phi_2'(\frac{\pi}{2})) = 0;$$

$\psi(\pi/2) = 0$ provided

$$\psi'(\frac{\pi}{2}) = -(\tilde{c}_1\phi_1'(\frac{\pi}{2}) + \tilde{c}_2\phi_2'(\frac{\pi}{2})) \ne 0.$$

Then, the standard shooting argument in ODE implies that (3.14) admits a unique nontrivial solution ψ_δ in $C^2(0, \pi/2)$. Note that there is no singularity of (3.14) for $\theta \in (0, \pi/2)$. Note also that ψ_δ depends on $\tilde{c}_1$ and $\tilde{c}_2$. □

Now we obtain the following proposition.

Proposition 3.4. *For any $\delta > 0$ sufficiently small and ψ_δ being given in Lemma 3.3, then for $\theta = |O(\delta^{\frac{2}{(2-\sigma)}})|$,*

$$\psi_\delta(\theta) = \theta^{-\frac{N-3}{2}}\Big[\tilde{d}_1\cos(\omega\ln\frac{\theta}{2}) + \tilde{d}_2\sin(\omega\ln\frac{\theta}{2})\Big] + O(\theta^{2-\frac{N-3}{2}}), \tag{3.15}$$

where $\tilde{d}_1$ and $\tilde{d}_2$ are constants depending on $\tilde{c}_1$ and $\tilde{c}_2$ but independent of δ.

Proof. Setting $\psi_\delta(\theta) = [\sin\theta]^{-\alpha}\tilde{\psi}_\delta(\theta)$, we see that $\tilde{\psi}_\delta(\theta)$ satisfies the problem

$$\begin{cases} \sin^2\theta\tilde{\psi}''(\theta) + (N-3-2\alpha)\cos\theta\sin\theta\tilde{\psi}'(\theta) + 2(N-3-\alpha)\tilde{\psi}(\theta) + G(\tilde{\psi}(\theta)) = 0, \\ \tilde{\psi}'(\frac{\pi}{2}) = \psi'_\delta(\frac{\pi}{2}), \end{cases} \tag{3.16}$$

where

$$\begin{aligned} G(\tilde{\psi}(\theta)) = {} & [\sin\theta]^{2+\alpha}\delta^{-2}\Big[w_*(\theta) + \delta\phi_\delta(\theta) + \delta^2[\sin\theta]^{-\alpha}\tilde{\psi}(\theta)\Big]^p \\ & -w_*^p - pw_*^{p-1}\delta\phi_\delta(\theta) - \delta^2 pw_*^{p-1}[\sin\theta]^{-\alpha}\tilde{\psi}(\theta). \end{aligned}$$

Under the Emden-Fowler transformations:

$$z(t) = \tilde{\psi}(\theta), \quad t = \ln\tan\frac{\theta}{2}$$

we obtain

$$z''(t) + (N-3-2\alpha)\Big(1 - \frac{2e^{2t}}{1+e^{2t}}\Big)z'(t) + 2(N-3-\alpha)z(t) + G(z(t)) = 0. \tag{3.17}$$

By the standard argument of variation of constants and Lemma 3.1, if

$$\tilde{\phi}_1(t) = [\sin\theta]^\alpha\phi_1(\theta), \quad \tilde{\phi}_2(t) = [\sin\theta]^\alpha\phi_2(\theta),$$

then we obtain the following integral equation for $T \in (-\infty, 0)$ and $|T|$ suitably large,

$$\begin{aligned} z(t) &= \vartheta_1\tilde{\phi}_1(t) + \vartheta_2\tilde{\phi}_2(t) + \int_T^t \frac{-\tilde{\phi}_1(t)\tilde{\phi}_2(t') + \tilde{\phi}_2(t)\tilde{\phi}_1(t')}{\tilde{\phi}_1(t')\tilde{\phi}'_2(t') - \tilde{\phi}'_1(t')\tilde{\phi}_2(t')}dt' \\ &= e^{\sigma t}[\vartheta_1\cos\omega t + \vartheta_2\sin\omega t] + O(e^{(\sigma+2)t}) \\ &\quad + \frac{1}{\omega}\int_T^t e^{\sigma(t-t')}\frac{\sin\omega(t-t') + O(e^{2t'})}{1+O(e^{2t'})}G(z(t'))dt' \\ &= e^{\sigma t}[\vartheta_1\cos\omega t + \vartheta_2\sin\omega t] + O(e^{(\sigma+2)t}) \\ &\quad + \frac{p(p-1)}{2\omega}\int_T^t e^{\sigma t}\sin\omega(t-t')\Big[e^{\sigma t'}\delta^2\Big][\rho(t')]^2dt' \\ &\quad + \frac{1}{\omega}\int_T^t e^{\sigma t}\sin\omega(t-t')O\Big(\Big[e^{\sigma t'}\delta^2\Big]^2[\rho(t')]^3\Big)dt' \\ &\quad + \frac{1}{\omega}\int_T^t e^{\sigma t}\sin\omega(t-t')O(e^{2t'})\Big[e^{\sigma t'}\delta^2\Big][\rho(t')]^2dt' \\ &\quad + \frac{1}{\omega}\int_T^t e^{\sigma t}\sin\omega(t-t')O(e^{2t'})O\Big(\Big[e^{\sigma t'}\delta^2\Big]^2[\rho(t')]^3\Big)dt' \end{aligned}$$

where

$$\rho(t') = \tilde{c}_1\cos\omega t' + \tilde{c}_2\sin\omega t' + e^{-\sigma t'}z(t').$$

Setting $\hat{z}(t) = e^{-\sigma t} z(t)$, arguments similar to those in the proof of Lemma 3.1 imply that there exists $C := C(N, p, T) > 0$ but independent of δ such that

$$\|\hat{z} - [\vartheta_1 \cos \omega t + \vartheta_2 \sin \omega t]\|_0 \le C \tag{3.18}$$

provided that for $t \in [2T, 10T]$,

$$\delta^2 = |O(e^{(2-\sigma)t})|. \tag{3.19}$$

Therefore,

$$z(t) = e^{\sigma t}[\vartheta_1 \cos \omega t + \vartheta_2 \sin \omega t] + O(e^{(\sigma+2)t}) \tag{3.20}$$

provided that (3.19) holds. Therefore,

$$\psi_\delta(\theta) = [\sin\theta]^{-\alpha}\Big(\tan\frac{\theta}{2}\Big)^{\sigma}\Big[\vartheta_1 \cos\omega \ln\frac{\theta}{2} + \vartheta_2 \sin \omega \ln \frac{\theta}{2}\Big] + O\Big([\sin\theta]^{-\alpha}\Big(\tan\frac{\theta}{2}\Big)^{\sigma+2}\Big) \tag{3.21}$$

provided

$$\theta = |O(\delta^{\frac{2}{2-\sigma}})|. \tag{3.22}$$

Taylor's expansions of $\sin\theta$ and $\tan\frac{\theta}{2}$ imply that (3.15) holds provided that (3.22) holds. This completes the proof of this proposition. □

Now we are in the position to obtain the following theorem.

Theorem 3.5. *For any $\delta > 0$ sufficiently small, problem* (1.9) *admits outer solutions $w_\delta^{out} \in C^2(0, \pi/2)$ satisfying*

$$w_\delta^{out}(\theta) = w_*(\theta) + \delta\phi_\delta(\theta) + \delta^2\psi_\delta(\theta), \quad \theta \in (0, \frac{\pi}{2}), \quad w_\delta'(\frac{\pi}{2}) = 0. \tag{3.23}$$

Moreover,

$$w_\delta^{out}(\theta) = \frac{A_p}{\theta^\alpha} + \frac{A_p}{3(p-1)}\frac{1}{\theta^{\alpha-2}} + \delta^2\Big[\frac{\vartheta_3\cos(\omega\ln\frac{\theta}{2}) + \vartheta_4 \sin(\omega \ln\frac{\theta}{2})}{\theta^{\frac{N-3}{2}}} + O\Big(\frac{1}{\theta^{\frac{N-3}{2}-2}}\Big)\Big] \tag{3.24}$$

provided

$$\theta = |O(\delta^{\frac{2}{2-\sigma}})|, \tag{3.25}$$

where ϑ_3 and ϑ_4 are constants which are independent of δ.

Proof. This theorem can be obtained from the expression of $w(\theta)$, (3.21) and the Taylor's expansions of $\sin\theta$ and $\tan\frac{\theta}{2}$. Note that for $\delta > 0$ sufficiently small and $\theta = |O(\delta^{\frac{2}{2-\sigma}})|$,

$$O(\theta^{4-\alpha}) = o(\delta^2\theta^{2-\frac{N-3}{2}}).$$

□

Remark 3.6. From (3.19) we see that $\delta^2 = |O(\theta^{2-\sigma})|$. Thus $w_\delta^{\rm out}$ can also be expressed by

$$w_\delta^{\rm out}(\theta) = \frac{A_p}{\theta^\alpha} + \frac{A_p}{3(p-1)}\frac{1}{\theta^{\alpha-2}} + \delta^2\Big[\frac{\vartheta_3\cos(\omega\ln\frac{\theta}{2}) + \vartheta_4 \sin(\omega \ln\frac{\theta}{2})}{\theta^{\frac{N-3}{2}}} + \delta^2 O\Big(\theta^{\sigma-\frac{N-3}{2}}\Big)\Big]. \tag{3.26}$$

4. Infinitely many solutions of (1.9) and Proof of Theorem 1.1

In this section we will construct infinitely many regular solutions for (1.9) by combining the inner and outer solutions.

Now we construct a solution of problem

$$\begin{cases} \frac{1}{\sin^{N-2}\theta}\frac{d}{d\theta}\Big(\sin^{N-2}\theta\frac{dw}{d\theta}(\theta)\Big)-\beta w(\theta)+w^p(\theta)=0, \;\; w(\theta)>0, \;\; 0<\theta<\frac{\pi}{2}, \\ w(0)=Q\ (:=\epsilon^{-\alpha}), \quad w'_\theta(\frac{\pi}{2})=0 \end{cases} \tag{4.1}$$

by using the expressions in Theorems 2.7 and 3.5. The variables Q and δ are then chosen to ensure that, at a fixed $\theta=\Theta$ chosen to satisfy

$$\Theta=O(Q^{\frac{\sigma}{(2-\sigma)\alpha}})$$

$$w_Q^{\mathrm{inn}}(\Theta)=w_\delta^{\mathrm{out}}(\Theta) \tag{4.2}$$

$$[w_Q^{\mathrm{inn}}(\theta)-w_\delta^{\mathrm{out}}(\theta)]'_\theta\Big|_{\theta=\Theta}=0. \tag{4.3}$$

These will be done by arguments similar to those in the proof of Lemma 6.1 of [11]. From the choice of Q and δ we deduce the existence of a C^2 function $w(\theta)$ defined by $w(\theta)=w_Q^{\mathrm{inn}}(\theta)$ for $\theta\le\Theta$ and by $w(\theta)=w_\delta^{\mathrm{out}}(\theta)$ for $\theta\ge\Theta$. Thus $w(\theta)$ satisfies (4.1).

We first observe that

$$\frac{A_p}{3(p-1)}=C_p \tag{4.4}$$

where C_p is given in Theorem 2.4. Note that

$$\begin{aligned}&(2-\alpha)(N-1-\alpha)+pA_p^{p-1}\\ &\quad=\frac{2(p-2)}{p-1}\Big(\frac{2(p-2)}{p-1}+N-3\Big)+\frac{2p}{p-1}\Big(N-3-\frac{2}{p-1}\Big)\\ &\quad=\frac{4(p^2-5p+4)}{(p-1)^2}+4(N-3)\\ &\quad=4\Big(N-2-\frac{3}{p-1}\Big).\end{aligned}$$

$$\beta-\frac{2(N-2)}{3(p-1)}=\frac{4}{3(p-1)}\Big(N-2-\frac{3}{p-1}\Big).$$

It follows from (2.10) that (4.4) holds.

Define Q_* and δ_*^2 by

$$\omega\ln Q_*^{\frac{p-1}{2}}+D=\omega\ln 2^{-1}+\phi+2m\pi, \tag{4.5}$$

$$\delta_*^2=\sqrt{\frac{a_0^2+b_0^2}{\vartheta_3^2+\vartheta_4^2}}Q_*^{\frac{\sigma}{\alpha}}, \tag{4.6}$$

where ϕ given by

$$\phi=\tan^{-1}(\frac{\vartheta_4}{\vartheta_3})$$

and $m\gg1$ is a large positive integer. The integer m is chosen such that the results in Section 2 and Section 3 hold.

Note that

$$O(\delta_*^{\frac{2}{2-\sigma}})=O(Q_*^{\frac{\sigma}{\alpha(2-\sigma)}}),$$

$$a_0 \cos\Big(\omega \ln(Q^{\frac{p-1}{2}}\theta)\Big)+b_0 \sin\Big(\omega \ln(Q^{\frac{p-1}{2}}\theta\Big) = \sqrt{a_0^2+b_0^2}\ \sin\Big(\omega\ln\theta+\omega\ln Q^{\frac{p-1}{2}}+D\Big), \tag{4.7}$$

$$\vartheta_3 \cos(\omega \ln\frac{\theta}{2})+\vartheta_4\sin(\omega\ln\frac{\theta}{2}) = \sqrt{\vartheta_3^2+\vartheta_4^2}\ \sin\Big(\omega\ln\theta+\omega\ln 2^{-1}+\phi\Big). \tag{4.8}$$

Then we claim that the values of Q and δ^2 required to satisfy the matching conditions (4.2)-(4.3) may be obtained as small perturbations of the values of Q_* and δ_*^2 given in (4.5) and (4.6), i.e.,

$$Q = Q_*\Big(1+O\Big(Q_*^{\frac{2\sigma}{(2-\sigma)\alpha}}\Big)\Big), \tag{4.9}$$

$$\delta^2 = \delta_*^2\Big(1+O\Big(Q_*^{\frac{2\sigma}{(2-\sigma)\alpha}}\Big)\Big). \tag{4.10}$$

To show this we define the function $\mathbf{F}(Q,\delta)$ by

$$\mathbf{F}^T(Q,\delta^2) = \Big(\Theta^{\frac{N-3}{2}}(w_Q^{\rm inn}(\Theta)-w_\delta^{\rm out}(\Theta)),\ \ [\theta^{\frac{N-3}{2}}(w_Q^{\rm inn}(\theta)-w_\delta^{\rm out}(\theta))]'_\theta\Big|_{\theta=\Theta}\Big).$$

(We treat δ^2 as a new variable.) Taking $Q=Q_*$ and $\delta^2=\delta_*^2$ we find a bound for $\mathbf{F}(Q_*,\delta_*^2)$ by making use of the behavior of $w_Q^{\rm inn}(\theta)$ determined by Theorem 2.7, and the behavior of $w_\delta^{\rm out}(\theta)$ given in Theorem 3.5. Accordingly we find for some $M>1$ suitably large,

$$\Big|\Theta^{-\frac{N-3}{2}}\mathbf{F}(Q_*,\delta_*^2)\Big| \le M\delta_*^4\Theta^{\sigma-\frac{N-3}{2}} + \text{small terms}. \tag{4.11}$$

We now seek values of Q and δ^2 which are small perturbations of Q_* and δ_*^2 and for which $\mathbf{F}(Q,\delta^2)=0$. As in [11], we need to evaluate the Jacobian of $\mathbf{F}$ at (Q_*,δ_*^2). We can obtain the following estimates from Lemmas 2.5, 2.6 and Theorems 2.7, 3.5:

$$\frac{\partial\mathbf{F}(Q,\delta^2)}{\partial(Q,\delta^2)} = \begin{bmatrix} C\Big(\frac{\sigma}{\alpha}\sin\tau+\frac{\omega(p-1)}{2}\cos\tau\Big)Q_*^{\frac{\sigma}{\alpha}-1}, & -E\sin\tau \\ C\Big(\frac{\sigma}{\alpha}\cos\tau-\frac{\omega(p-1)}{2}\sin\tau\Big)Q_*^{\frac{\sigma}{\alpha}-1}, & -E\cos\tau \end{bmatrix}$$
$$+\text{small order terms},$$

where

$$C=\sqrt{a_0^2+b_0^2},\quad E=\sqrt{\vartheta_3^2+\vartheta_4^2},$$

$$\tau = \omega\ln\Theta+\omega\ln Q_*^{\frac{p-1}{2}}+D = \omega\ln\Theta+\omega\ln 2^{-1}+\phi+2m\pi.$$

Note that

$$\frac{\sigma}{\alpha}-1 = -\frac{(N-3)(p-1)}{4}.$$

To simplify this expression we define the function $\mathbf{G}(x,y)$ by

$$\mathbf{G}(x,y) = \mathbf{F}(Q_*+xQ_*^{1-\frac{\sigma}{\alpha}},\delta_*^2+y).$$

Using the bounds for $\mathbf{F}$ given in (4.11) and (3.26) and the results in Lemmas 2.5, 2.6, we express $\mathbf{G}(x,y)$ in the form

$$\begin{aligned}&\mathbf{G}(x,y)\\ &= \mathbf{C} + \left[\begin{matrix} C\Big(\frac{\sigma}{\alpha}\sin\tau+\frac{\omega(p-1)}{2}\cos\tau\Big), & -E\sin\tau \\ C\Big(\frac{\sigma}{\alpha}\cos\tau-\frac{\omega(p-1)}{2}\sin\tau\Big), & -E\cos\tau\end{matrix} \quad +\text{small terms}\right]\begin{pmatrix} x\\ y\end{pmatrix}\\ &+\mathbf{E}(x^2(\delta_*^2)^{-1}+y^2\Theta^\sigma),\end{aligned}$$

where $\mathbf{C}$ is a constant vector independent of (x, y) which is bounded above by $M\delta_*^4\Theta^\sigma$. Also $|\mathbf{E}|$ is bounded independently of x, y, Q and δ. Thus,

$$\mathbf{G}(x,y) = \mathbf{C} + L\begin{pmatrix} x \\ y \end{pmatrix} + \mathbf{T}(x,y),$$

where L is a linear operator which, from a direct calculation, is seen to be invertible. If we define the operator $\mathbf{J}$ mapping $\mathbb{R}^2$ into itself by

$$\mathbf{J}(x,y) = -(L^{-1}\mathbf{C} + L^{-1}\mathbf{T}(x,y)),$$

then, provided that Q_* is suitably large, a direct calculation shows that $\mathbf{J}$ maps the set B into itself, where B is the ball

$$B = \Big\{(x,y):\ (x^2+y^2)^{1/2} \le \frac{4\delta_*^4\Theta^\sigma M}{(p-1)\omega E\sqrt{a_0^2+b_0^2}}\Big\}.$$

We may therefore apply the Brouwer Fixed Point Theorem to conclude that $\mathbf{J}$ has a fixed point in B. This point (x, y) satisfies both $\mathbf{G}(x, y) = 0$ and

$$(x^2+y^2)^{1/2} \le A\delta_*^4\Theta^\sigma,$$

where A is a constant independent of δ_*, Q_* and Θ. By substituting for Q and δ, and then taking Θ to have the upper limiting value of $Q_*^{\frac{\sigma}{(2-\sigma)\alpha}}$, we obtain (4.9) and (4.10).

We have shown that (4.2)-(4.3) has a solution for each fixed m large. This yields a solution of (1.9). This also gives the proof of Theorem 1.1. Hence we have

Theorem 4.1. *For $m \gg 1$ large and Q and δ given in* (4.9) *and* (4.10)*, problem* (4.1) *admits a C^2 solution $w_{Q,\delta}(\theta)$. Moreover, there is $\Theta = |O(Q^{\frac{\sigma}{(2-\sigma)\alpha}})|$ such that*

$$w_Q^{inn}(\Theta) = w_\delta^{out}(\Theta),$$
$$(w_Q^{inn})'_\theta(\Theta) = (w_\delta^{out})'_\theta(\Theta).$$

As a consequence, Problem (1.9) *admits infinitely many nonconstant positive radially symmetric solutions.*

References

[1] A. Bahri and J.M. Coron, On a nonlinear elliptic equation involving the critical Sobolev exponent: the effect of the topology of the domain, *Comm. Pure Appl. Math.* 41 (1988), 255–294.

[2] H. Brezis, Elliptic equations with limiting Sobolev exponent-The impact of Topology, Proceedings 50th Anniv. Courant Inst.,*Comm. Pure Appl. Math.* 39 (1986).

[3] H. Brezis, L. Nirenberg, Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents. *Comm. Pure Appl. Math.* **36** (1983), 437–47.

[4] J.M. Coron, Topologie et cas limite des injections de Sobolev, *C.R. Acad. Sc. Paris, 299, Series I* (1984), 209–212.

[5] C. Bandle and J. Wei, Non-radial clustered spike solutions for semilinear elliptic problems on S^n, *J. Anal. Math.* 102(2007), 181-208.

[6] M.F. Bidaut-Veron and L. Veron, Nonlinear elliptic equations on compact Riemannian manifolds and asymptotics of Emden equations, *Invent. Math.* 106(1991), 489-539.

[7] M. F. Bidaut-Veron, A. Ponce and L. Veron, Boundary singularities of positive solutions of some nonlinear elliptic equations. *C.R. Math. Acad. Sci. Paris* 334(2007), no.2, 83-88.

[8] M. F. Bidaut-Veron, A. Ponce and L. Veron, Isolated boundary singularities of semilinear elliptic equations, *Cal. Var. PDE* 40(2011), 183-233.

[9] H. Brezis and L. A. Peletier, Elliptic equations with critical exponent on spherical caps of $\mathbf{S^3}$, *J'Analyse Math.* 98(2006), 279-316.

[10] H. Brezis and Y.Y. Li, Some nonlinear elliptic equations have only constant solutions, *J. Partial Differential Equations* 19(2006), no.3, 208-217.

[11] C. Budd and J. Norbury, Semilinear elliptic equations and supercritical growth, *J. Differential Equations* 68 (1987), 169-197.

[12] L.A. Caffarelli, B. Gidas, J. Spruck, Asymptotic symmetry and local behavior of semilinear elliptic equations with critical Sobolev growth, *Comm. Pure Appl. Math.* 42 (1989), 271-297.

[13] E.N. Dancer, A note on an equation with critical exponent, *Bull london Math Soc* 20(1988),600-602.

[14] E.N. Dancer, Y.H. Du and Z.M. Guo, Finite Morse index solutions of an elliptic equation with supercritical exponent, *J. Differential Equations* 250 (2011), 3281-3310.

[15] J. Davila, M. del Pino and M. Musso, The supercritical Lane-Emden-Fowler equations in exterior domains, *Comm. Part. Diff. Eqns.* 32(2007), 1225-1243.

[16] J. Davila, M. del Pino, M. Musso and J. Wei, Fast and slow decay solutions of supercritical problems in exterior domains, *Cal. Var. PDE* 32(2008), no.4, 765-786.

[17] J. Davila, L. Dupaigne and A. Farina, Partial regularity of finite Morse index solutions to the Lane-Emden equation, *J. Funct. Anal.* 261(2011), 218-232.

[18] M. del Pino, Supercritical elliptic problems from a perturbation point of view, *Disc. Cont. Dyn. Systems-A* 21(2008), 69-89.

[19] M. del Pino, P. Felmer and M. Musso, Two-bubble solutions in the super-critical Bahri-Coron's problem. *Calc. Var. Partial Differential Equations* 16 (2003), 113–145.

[20] M. del Pino, P. Felmer and M. Musso, Multi-bubble solutions for slightly super-critical elliptic problems in domains with symmetries. *Bull. London Math. Soc.* 35(2003), 513–521.

[21] M. del Pino, M. Musso and F. Pacard, Bubbling along boundary geodesics near the second critical exponents, *J. Eur. Math. Soc.*, to appear.

[22] M. del Pino and J. Wei, Supercritical elliptic problems in domains with small holes, *Ann. Inst. H. Poincar Anal. Non Linaire* 24 (2007), 507–520.

[23] W.Y. Ding, Positive solutions of $\Delta u + u^{(n+2)/(n-2)} = 0$ on contractible domains. *J. Partial Differential Equations* 2 (1989), 83–88.

[24] A. Farina, On the classification of solutions of Lane-Emden equation on unbounded domains in $\mathbb{R}^N$, *J. Math. Pures Appl.* 87 (2007), 537-561.

[25] B. Gidas and J. Spruck, Global and local behavior of positive solutions of nonlinear elliptic equations, *Comm. Pure Appl. Math.* 24 (1981), 525-598.

[26] C. Gui, W.M. Ni and X.F. Wang, On the stability and instability of positive steady states of a semilinear heat equation in $\mathbb{R}^n$, *Comm. Pure Appl. Math.* 45 (1992), 1153-1181.

[27] Z.M. Guo, On the symmetry of positive solutions of the Lane-Emden equation with supercritical exponent, *Adv. Differential Equations* 7 (2002), 641-666.

[28] R.A. Johnson, X.B. Pan and Y.F. Yi, Positive solutions of super-critical elliptic equations and asymptotics, *Comm. Partial Differential Equations* 18 (1993), 977-1019.

[29] D.D Joseph and T.S. Lundgren, Quasilinear Dirichlet problems driven by positive sources, *Arch. Rational Mech. Anal.* 49 (1973), 241-269.

[30] N. Korevaar, R. Mazzeo, F. Pacard and R. Schoen, Refined asmptotics for constant scalar curvature metrics with isolated singularities. *Invent. Math.* 135(1999), no.2, 233-272.

[31] A. Micheletti, A. Pistoia, On the effect of the domain geometry on the existence of sign changing solutions to elliptic problems with critical and supercritical growth. *Nonlinearity* 17 (2004), no. 3, 851–866

[32] R. Molle, D. Passaseo, Positive solutions of slightly supercritical elliptic equations in symmetric domains. *Ann. Inst. H. Poincare Anal. Non Lineaire* 21 (2004), no. 5, 639–656.

[33] D. Passaseo, New nonexistence results for elliptic equations with supercritical nonlinearity, *Differential and Integral Equations* 8, no. 3 (1995), 577–586.

[34] D. Passaseo, Nontrivial solutions of elliptic equations with supercritical exponent in contractible domains, *Duke Mathematical Journal* 92, no. 2 (1998), 429–457.

[35] P. Polacik, P. Quittner and Ph. Souplet, Singularity and decay estimates in superlinear problems via Liouville-type theorems. Part I: Elliptic systems, *Duke Math. J.* 139(2007), 555-579.

[36] X.F. Wang, On the Cauchy problem for reaction-diffusion equations, *Trans. Amer. Math. Soc.* 337 (1993), 549-590.

[37] J. Wei and S. Yan, Infinitely many positive solutions for an elliptic problem with critical or super-critical growth, *J. Math. Pures Appl.*, to appear.

[38] H.H. Zou, Symmetry of positive solutions of $\Delta u + u^p = 0$ in $\mathbb{R}^n$, *J. Differential Equations* 120 (1995), 46-88.

School of Mathematics and Statistics, The University of Sydney, Sydney, 2006, Australia
E-mail address: normd@maths.usyd.edu.au

Department of Mathematics, Henan Normal University, Xinxiang, 453007, China
E-mail address: gzm@htu.cn

Department of Mathematics, The Chinese University of Hong Kong, Shatin, Hong Kong
E-mail address: wei@math.cuhk.edu.hk