ON DE GIORGI'S CONJECTURE: RECENT PROGRESS AND OPEN PROBLEMS

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Abstract. In 1979, De Giorgi conjectured that the only bounded monotone solutions to Allen-Cahn equation

$$\Delta u + u - u^3 = 0 \quad \text{in } \mathbb{R}^N,$$

are one-dimensional. This conjecture and its connection with minimal surfaces and Toda systems is the subject of this survey article.

1. Motivations: Classifications of solutions

This article is a survey on recent progress on classification of solutions to Allen-Cahn equation. Classification of solutions to differential equations is fundamental and crucial in many pure and applied fields. To motivate the study, let us consider the simplest family of differential equations–the linear ordinary differential equations (ODEs). In an elementary calculus course we have learnt that if $u'(t) = 0$ then u is a constant function. More generally, any solution of

$$u^{(n)}(t) + a_1 u^{(n-1)}(t) + \cdots + a_n u = g(t)$$

must be of the form

$$u(t) = u_s(t) + \sum_{j=1}^{n} c_j u_j(t)$$

where u_s is a particular solution, c_j are arbitrary constants and u_j are the elements in a basis of the solution space of the homogeneous equation. We know exactly the structure of the solutions. In other words, all the solutions to linear ODEs are *classified*.

Let us consider a nonlinear ODE, say $u'' + u^2 = 0$. The existence-uniqueness theorem asserts that u is completely determined by the initial values $(u(0), u'(0))$. This is another type of *classification*.

A fundamental problem in (linear or nonlinear) partial differential equations (PDEs) is the *classification* of solutions of PDEs

$$F(x_1, \cdots, x_N, u, Du, D^2u, \cdots) = 0.$$

An easy example may be that of the Laplace equation

$$\Delta u = \sum_{j=1}^{N} \frac{\partial^2 u}{\partial x_j^2} = 0 \quad \text{in } \mathbb{R}^N,$$

whose solutions we call (entire) harmonic functions.

There are many, many harmonic functions like $x^2 - y^2$, $\quad xy$, $\quad e^x \sin(y)$, $\quad \ldots$, leading to the

Question 1.1. *How do we* classify *the harmonic functions?*

A significant advance was made in the nineteenth century by Liouville.

Theorem 1.1 (Liouville's theorem)**.** All bounded harmonic functions are constants.

A more general version reads

Theorem 1.2. All harmonic functions of algebraic growth are polynomials.

These are called *Liouville type theorems*. In this survey article, we will discuss some Liouville type theorems of nonlinear PDEs.

A nonlinear equation may look like

$$\Delta u + u^3 = 0 \quad \text{in } \mathbb{R}^N,$$

or more generally,

$$\Delta u + f(u) = 0 \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

which is a semilinear elliptic equation. One may ask the following

Questions. *Are all solutions to (1.1) necessarily constant, i.e. $u = 0$? Are they necessarily radially symmetric around some point $x_0 \in \mathbb{R}^N$, i.e. $u(x) = U(|x-x_0|)$? Or, are they necessarily one-dimensional, i.e. depend on one variable only?*

If the answer is YES, then the nonlinear PDE (partial differential equation) problem becomes an ODE (ordinary differential equation). This is a nonlinear version of Liouville type theorem. As illustrations, let us mention for following examples of nonlinear Liouville theorem.

Example 1.1 (Yamabe problem)**.** *In conformal geometry, the Yamabe problem asks whether a given compact Riemannian manifold M is necessarily conformally equivalent to another with constant scalar curvature. The involved PDE,*

$$\Delta u + u^{\frac{N+2}{N-2}} = 0 \quad \text{in } M, \tag{1.2}$$

was studied by Schoen [95, 96], Schoen–Yau [97] and an affirmative answer was given. When M is instead $\mathbb{R}^N$, Caffarelli–Gidas–Spruck [18] proved that all solutions are radially symmetric around some point.

Example 1.2 (Klein–Gordon equation)**.** *The equation reads*

$$\Delta u - u + u^3 = 0 \quad \text{in } \mathbb{R}^N. \tag{1.3}$$

Gidas–Ni–Nirenberg [51] showed the radial symmetry and uniqueness of solutions of the resulting nonlinear ODE was known by Kwong [67].

Example 1.3 (Monge–Ampère equation)**.** *Classical convex solutions of the Monge–Ampère equation*

$$\det(D^2 u) = 1 \quad \text{in } \mathbb{R}^N$$

is necessarily a quadratic polynomial. This is a classical result of Jörgens [63], Calabi [21] and Pogorelov [80].

2. Allen–Cahn equation and De Giorgi's Conjecture

In this article, we shall report some progress on another example of nonlinear Liouville type theorem — *De Giorgi's Conjecture*. For the simplicity of presentation, we consider one simple-looking nonlinear equation, the *Allen-Cahn Equation*:

$$\Delta u + u - u^3 = 0 \quad \text{in } \mathbb{R}^N, \tag{2.1}$$

which is the Euler–Lagrange equation of the energy

$$E(u) = \int \left[\frac{1}{2}|\nabla u|^2 + \frac{1}{2}\left(1-u^2\right)^2\right] dx. \tag{2.2}$$

We aim to classify its entire solutions, which is what the De Giorgi's conjecture is all about. This equation has deep connections with minimal surfaces as we shall see.

The Allen-Cahn equation (2.1) arises from the gradient theory of phase transitions by Cahn–Hilliard [20] and Allen–Cahn [5], in connection with the energy functional in bounded domains Ω

$$E_\epsilon(u) = \frac{\epsilon^2}{2}\int_\Omega |\nabla u|^2 + \frac{1}{4\epsilon}\int_\Omega (1-u^2)^2, \quad \int_\Omega u = m, \tag{2.3}$$

whose Euler–Lagrange equation corresponds precisely to

$$\begin{cases} \varepsilon^2\Delta u + u - u^3 = \lambda & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

The energy (2.3) is called the Ginzburg–Landau [54] energy functional. In the formula, the function $F(u) = \frac{1}{4}(1-u^2)^2$ has two minima of equal depth and is thus commonly referred to as a *double-well potential*. The trivial solutions $u = -1$ and $u = +1$ represent *two phases*. The gradient term penalizes sharp changes between the two phases and results in moderated phase transitions.

The importance of this equation lies in the theories of phase transition, superconductivity, and mathematical biology modelling interfaces. One example is pattern formation in di-block co-polymers, or polymers consisting of two types of monomers. It is modeled by the system

$$\begin{cases} u_t = \epsilon^2\Delta u + u - u^3 - \gamma v \\ v_t = D\Delta v + u - \dfrac{1}{|\Omega|}\displaystyle\int_\Omega u, \end{cases} \tag{2.4}$$

as derived from the density functional theory of di-block co-polymers of Ohta–Kawasaki [77]. The core equation for the profile is exactly Allen-Cahn type.

On a bounded domain, this can be studied as a nonlocal free boundary problem [76]. Various phases of di-block co-polymer has been studied mathematically: the lamellar phase by Ren–Wei [81, 82, 83, 84, 85], Fife–Hilhorst [47], Choksi–Ren [25], Chen–Oshita [24], Choksi–Sternberg [26], the cylindrical phase by Ren–Wei [86, 87] and the gyroid phase by Teramoto–Nishiura [104].

Another example is the interface model from mathematical biology. A model for biological pattern formations is the Gierer–Meinhardt system [53] with saturation

$$\begin{cases} u_t = \varepsilon^2\Delta u - u + \dfrac{u^2}{(1+ku^2)v} \\ v_t = D\Delta v - v + u^2 \end{cases} \tag{2.5}$$

The core of the stripe profile is governed by the simple equation

$$\Delta u - u + \frac{u^2}{1+ku^2} = 0 \tag{2.6}$$

which admits a double well potential at special value of k. This is again Allen-Cahn type eqaution. We refer to the book [107] and references therein.

The theory of Γ-convergence, developed in the 70s and 80s notably by Modica–Mortola [75], Modica [74] and Kohn–Sternberg [64], showed a deep connection between equation (2.1) and the theory of minimal surfaces. In fact, the functional E_ε converges in suitable sense as $\varepsilon \to 0$ to the perimeter functional of the limiting interface between the stable *phases* $u = -1$ and $u = +1$, so that, roughly speaking, interfaces of local minimizers of E_ε are expected to approach minimal surfaces:

$$E_\varepsilon[u] \sim c_0 \text{Perimeter}(\{u = 0\}).$$

Rigorous results may be found in Cafferalli–Cordoba [16, 17] and Röger–Tonegawa [91]. It also indicates that the critical points of $E_\varepsilon[u]$ are converging to those of the perimeter functional, which has zero mean curvature.

Definition 2.1. *Minimal surfaces* are surfaces with zero mean curvature, i.e. $H = 0$.

Typical examples of minimal surfaces include hyperplanes $\{x_1 = 0\}$, catenoids $\left\{\cosh^2(x_3) = x_1^2 + x_2^2\right\}$, helicoids $\{(r\cos t, r\sin t, t) : r, t \in \mathbb{R}\}$ and the Costa surface.

This connection led E. De Giorgi [39] to formulate in 1978 the following celebrated conjecture:

Conjecture 2.1 (De Giorgi)**.** *Let u be a bounded solution of equation of*

$$\Delta u + u - u^3 = 0 \quad \text{in } \mathbb{R}^N,$$

which is monotone in one direction, say $\frac{\partial u}{\partial x_N} > 0$. Then, ***at least*** *when $N \leq 8$, u is one-dimensional. Equivalently, u depends on just one Euclidean variable so that it must have the form*

$$u(x) = tanh\left(\frac{x \cdot a - b}{\sqrt{2}}\right),$$

for some $b \in \mathbb{R}$ and some a with $|a| = 1$ and $a_N > 0$.

We observe that the function $w(t) = \tanh\left(\frac{t}{\sqrt{2}}\right)$ is the unique solution of the one-dimensional problem

$$w'' + w - w^3 = 0, \quad w(0) = 0, \quad \text{and} \quad w(\pm\infty) = \pm 1.$$

It is an increasing function which connects -1 to $+1$.

De Giorgi's Conjecture was proved true in dimension $N = 2$ by Ghoussoub–Gui [50] and for $N = 3$ by Ambrosio–Cabré [7]. Savin [92] showed that for $4 \leq N \leq 8$, it is true under the additional limit condition

$$u(x_1, \ldots, x_N) \to \pm 1, \quad \text{as } x_N \to \pm\infty.$$

For $N \geq 9$, counterexamples were constructed by del Pino–Kowalczyk–Wei [43].

Note that the threshold dimension is 8. This has a deep connection with minimal surfaces. Let us restate De Giorgi's Conjecture in an equivalent form.

Conjecture 2.2 (De Giorgi)**.** *For $N \leq 8$, all level sets $\{u = \lambda\}$ of u must be hyperplanes.*

Then a comparison to the parallel *Bernstein Conjecture* for minimal graphs can be made. By minimal graphs we mean graphs

$$\left\{(x_1, \ldots, x_N) \in \mathbb{R}^N : x_N = F(x_1, \ldots, x_{N-1})\right\}$$

with vanishing mean curvature,

$$H = \nabla \cdot \left(\frac{\nabla F}{\sqrt{1 + |\nabla F|^2}} \right) = 0 \quad \text{ in } \mathbb{R}^{N-1}.$$

It is trivial to see that any hyperplane defined by

$$F(x_1, ..., x_{N-1}) = \sum_{j=1}^{N-1} a_j x_j + b$$

has zero mean curvature. Bernstein conjectured that these define all minimal graphs in $\mathbb{R}^N$.

Conjecture 2.3 (Bernstein)**.** *All entire minimal graphs are hyperplanes. Namely, any entire solution of $H = 0$ must be an affine function.*

It turned out to be true exactly when $N \leq 8$. It was proved by Bernstein [9] and Fleming [49] for $N = 3$, De Giorgi [38] for $N = 4$, Almgren [6] for $N = 5$ and finally Simons [99] for $N = 6, 7, 8$. For $N \geq 9$, Bombieri–De Giorgi–Giusti [12] found a counterexample.

In the construction of counterexamples of De Giorgi's Conjecture in $\mathbb{R}^9$ in [43], the authors developed an *infinite dimensional Liapunov–Schmidt reduction method* and to established an explicit and intricate connection between

Allen–Cahn $\Delta u + u - u^3 = 0$ in $\mathbb{R}^N$	$\longleftrightarrow$	Minimal Surfaces $\nabla \cdot \left(\frac{\nabla F}{\sqrt{1+\|\nabla F\|^2}} \right) = 0$ in $\mathbb{R}^{N-1}$

In this way De Giorgi's Conjecture was almost completely settled, modulo the mild limit assumption that Savin imposed in [92].

3. Beyond De Giorgi's Conjecture I – Savin's theorem on global minimizers

After the resolution of De Giorgi conjecture, one natural question is if we can replace the monotonicity condition

$$\frac{\partial u}{\partial x_N} > 0$$

by other more physical conditions. This leads us to study the so-called *global minimizers.*

We start with the definition of global minimizers.

Definition 3.1. We say that u is a *global minimizer* of the energy function E if

$$E\left(u\right) \leq E\left(u + \phi\right), \quad \text{ for all } \phi \in C_0^1\left(\mathbb{R}^N\right). \tag{3.1}$$

Monotonicity ($\partial_{x_N} u > 0$) together with the limit condition ($u \to \pm 1$ as $x_N \to \pm\infty$) imply that u is a global minimizer of the energy functional (2.2). In this setting Savin [92] proved a rigidity result.

Theorem 3.1 (Savin)**.** Let $N \leq 7$. If u is a global minimizer, then u is one-dimensional.

Since the counterexample of del Pino–Kowalczyk–Wei [43] provides a global minimizer in dimension $N = 9$, it arises the natural

Question 3.1. *Are there nontrivial global minimizers for $N = 8$?*

On the other hand, the construction in [43] is very involved. It consists of two main ingredients, namely

(1) a detailed analysis of properties of the Bombieri–De Giorgi–Giusti minimal graph $F : \mathbb{R}^8 \to \mathbb{R}$ which satisfies

$$\nabla \cdot \left(\frac{\nabla F}{\sqrt{1 + |\nabla F|^2}} \right) = 0 \quad \text{in } \mathbb{R}^8 \tag{3.2}$$

and

$$F = r^3 g(\theta) + O(r^{-\sigma});$$

as well as

(2) the infinite-dimensional Liapunov-Schmidt reduction method.

Hence we may ask the following

Question 3.2. *Are there easier proofs?*

Question 3.3. *Are there more examples other than the Bombieri–De Giorgi–Giusti minimal graph?*

All these questions can be answered, provided that the answer to the first question is a *Yes*. This is based on the Jerison–Monneau program [62].

Theorem 3.2 (Jerison–Monneau)**.** Suppose v is a global minimizer of (2.1) in $\mathbb{R}^8$ which is even in each variable. Then for each $\delta \in \left(0, 1 - v^2(0)\right)$, there is a solution u_δ in $\mathbb{R}^9$ which is monotone in x_9, even in $x_1, \ldots, x_8$, with

$$u_\delta(0) = v(0), \quad \frac{\partial u_\delta}{\partial x_9}(0) = \delta.$$

An answer to Question 3.1 was recently given by Liu–Wang–Wei [71].

Theorem 3.3 (Liu–Wang–Wei)**.** There exists a global minimizer to the Allen–Cahn equation in $\mathbb{R}^8$ satisfying

$$\Delta u + u - u^3 = 0 \quad \text{in } \mathbb{R}^8. \tag{3.3}$$

More precisely, u is invariant under $O(3) \times O(3)$ and its zero level set is asymptotic to the Simons' cone, i.e.

$$u = u(r, s)$$

where

$$r = \sqrt{x_1^2 + \cdots + x_4^2}, \quad s = \sqrt{x_5^2 + \cdots + x_8^2},$$

with

$$\{u(r, s) = 0\} \sim \{r = s\}.$$

This completes the classification of global minimizers of the Allen–Cahn equation. To summarize, we have:

- If $N \leq 7$, then any global minimizer of (2.1) is one-dimensional [92].
- If $N \geq 8$, then there exist global minimizers of (2.1) that is not one-dimensional [71].

In [71] a more general cone is considered. Consider the minimal Lawson's cone $C_{p,q}$ over $S^p \times S^q$ in $\mathbb{R}^{p+q+2} = \mathbb{R}^{n+1}$,

$$C_{p,q} = \left\{ (x_1, ..., x_{p+q+2}) : x_1^2 + ... + x_{p+1}^2 = \frac{p}{q} \left(x_{p+2}^2 + ... + x_{p+q+2}^2 \right) \right\}.$$

It is known that $C_{p,q}$ is a strictly area minimizer if $p+q \geq 7$ or $p+q = 6$, $|p-q| < 4$. The readers may refer to, among others, Alencar–Barros–Palmas–Reyes–Santos [3], Davini [37], De Philippis–Paolini [40], Lawlor [68], Lawson [69], Miranda [73] and the references therein.

More global minimizers were obtained in [71] in a similar way.

Theorem 3.4 (Liu–Wang–Wei)**.** Suppose either $p + q \geq 7$ or $p + q = 6$ with $|p - q| < 4$. Then there is a global minimizer $u(r, s)$ with $\{u = 0\} \sim C_{p,q}$.

According to the program of Jerison–Monneau [62], these results on the existence of global minimizers lead to more counterexamples of De Giorgi's Conjecture.

Corollary 3.1. *For each $N \geq 9$, there are families of monotone solutions in $\mathbb{R}^N$ which are different from those constructed by del Pino–Kowalczyk–Wei.*

This is in parallel to a result of Simon [98] in minimal surfaces theory.

Theorem 3.5 (Simon)**.** Suppose C is a strict minimizing isoparametric cone in $\mathbb{R}^N$ with $\operatorname{sing} C = \{0\}$. Then there is an entire solution u of the minimal surface equation in $\mathbb{R}^N$ having $C \times \mathbb{R}$ as tangent cylinder at ∞.

4. Beyond De Giorgi Conjecture II – stable solutions

After global minimizers, another important question is the classification of *local minimizers*, i.e. stable solutions.

Definition 4.1. Given a bounded solution u to the nonlinear elliptic equation

$$\Delta u + f(u) = 0 \quad \text{in } \mathbb{R}^N,$$

we say that u is *stable* if

$$\int_{\mathbb{R}^N} (|\nabla \varphi|^2 - f'(u)\varphi^2) \geq 0, \qquad \text{for all } \varphi \in C_0^\infty(\mathbb{R}^N). \tag{4.1}$$

Related to De Giorgi's Conjecture is the

Conjecture 4.1 (Stability Conjecture)**.** *Let u be a bounded stable solution of equation*

$$\Delta u + u - u^3 = 0 \quad \text{in } \mathbb{R}^N.$$

Then the level sets $\{u = \lambda\}$ are all hyperplanes.

The conjecture is true when $N = 2$, as proved by Ambrosio–Cabré [7]. At the other extreme, the papers by Pacard–Wei [79] Liu–Wang–Wei [71] disproved it when $N = 8$. This leaves the Stability Conjecture open for $3 \leq N \leq 7$.

5. Beyond De Giorgi's Conjecture III – finite Morse index solutions

So far we have an almost complete classification for monotone solutions and a complete one for global minimizers. The problem concerning stable solutions, or local minimizers, is still largely open. But then one may ask the

Question 5.1. *How does one classify* unstable solutions*?*

After global minimizers and local minimizers (stable solutions), the next important question is the classification of *finite Morse index solutions*, i.e. solutions that are not too unstable.

Definition 5.1. A solution u is said to have a *finite Morse index* if there exists a compact set K such that

$$\int \left(|\nabla\phi|^2 - f'(u)\phi^2\right) \geq 0, \qquad \text{for all } C_0^\infty(\mathbb{R}^N \setminus K).$$

This is equivalent to saying that the number of negative eigenvalues is finite.

Classifying stable or finite Morse index solutions is a difficult problem in higher dimensions. We concentrate on the lowest dimension $N = 2$, where (2.1) reads

$$\Delta u + u - u^3 = 0 \quad \text{ in } \mathbb{R}^2. \tag{5.1}$$

De Giorgi's Conjecture, asserting that monotone solutions are one-dimensional, is true according to [50]. Stable solutions also possess one-dimensional symmetry [7]. This means that all solutions that are not one-dimensional are necessarily unstable. In order to classify them, it is tempting to use the so-called *Morse index*. Unfortunately, such notion seems to be inconvenient to utilize. A handy condition turns out to be the number of *ends*. In what follows we will consider *finite-end solutions*.

Intuitively, around a large circle, the number of ends at infinity is equal to the number of sign-changes. Let us denote the set of k-end solutions by $\mathcal{M}_k$. By definition, the number of ends must be even. Observe that finite-end solutions have finite Morse index. A rigorous definition is given below.

Definition 5.2. We say that u is a *k-end solution* of (5.1) if there exist k oriented half lines $\{\mathbf{a}_j \cdot \mathbf{x} + b_j = 0\}$, $j = 1, \ldots, k$ (for some choice of $\mathbf{a}_j \in \mathbb{R}^2$, $|\mathbf{a}_j| = 1$ and $b_j \in \mathbb{R}$) such that along these half lines and away from a compact set K containing the origin, the solution is asymptotic to $w(\mathbf{a}_j \cdot \mathbf{x} + b_j)$, that is, there exist positive constants C, c such that

$$\|u(\mathbf{x}) - \sum_{j=1}^{k} (-1)^j w(\mathbf{a}_j \cdot \mathbf{x} + b_j)\|_{L^\infty(\mathbb{R}^2 \setminus K)} \leq C e^{-c|\mathbf{x}|}.$$

The balancing formula

$$\sum_{j=1}^{k} \mathbf{a}_j = 0 \tag{5.2}$$

follows from the translation invariance. A Hamiltonian identity was derived by Gui [55]. If $k = 2$, then by the balancing formula and the method of moving plane, u must be the one-dimensional profile. This forces $k \geq 3$. But since k is even, $k = 2m \geq 4$.

Let us mention some examples of elements in $\mathcal{M}_k$, starting with symmetric $2m$-end solutions. For $m = 2$ Dang–Fife–Peletier [33] constructed a cross saddle solution whose nodal set consists of the two axes (giving four ends), hence an element in $\mathcal{M}_4$.

Theorem 5.1 (Dang–Fife–Peletier)**.** There exists a solution u (5.1) such that $u(x_1, x_2)$ is positive for x_1, $x_2 > 0$ and has the symmetry

$$u(x_1, x_2) = -u(-x_1, x_2) = -u(x_1, -x_2).$$

The cross solution was constructed using sub- and super-solutions in the first quadrant. An extension to this result was given by Alessio–Calamai–Montecchiari [4] who constructed saddle solutions with dihedral symmetry. The nodal set has m lines (hence $2m$ ends), where $m \geq 2$. This is an element in $\mathcal{M}_{2m}$.

For non-symmetric $2m$-end solutions, del Pino–Kowalczyk–Wei [42] found a relation between solutions with $2m$ parallel ends and *Toda system*.

Theorem 5.2 (del Pino–Kowalczyk–Pacard–Wei)**.** Let $f_1 < f_2 < \cdots < f_m$ be a solution to the (integrable) Toda system

$$f_j'' = e^{f_{j-1}-f_j} - e^{f_j - f_{j+1}}, \quad j = 1, \ldots, m \quad f_0 = -\infty, \quad f_{m+1} = +\infty. \tag{5.3}$$

Then there exists a solution to (5.1) with the asymptotic behavior

$$u(x, y) \sim \sum_{j=1}^{m} (-1)^j w(x - f_j(\alpha y)) + (-1)^{m-1}, \tag{5.4}$$

where $\alpha > 0$ is small.

For $m \geq 3$, general $2m$-end solutions were constructed by Kowalczyk–Liu–Pacard–Wei [66] using moduli space theory.

Theorem 5.3 (Kowalczyk–Liu–Pacard–Wei)**.** Given any m (generic) lines in $\mathbb{R}^2$ where $m \geq 3$, there exists a solution to (5.1) whose zero level set approaches these m lines.

Let us turn to the question of classification of $\mathcal{M}_k$.

Question 5.2. *Regarding $\mathcal{M}_k$, what can we say about the dimension, connectedness, Morse index, or topology?*

All these questions are completely open except when $k = 4$. In this case, we have

Theorem 5.4 (Gui [55])**.** Any four-end solution is even symmetric with respect to two orthogonal lines.

Theorem 5.5 (Kowalczyk–Liu–Pacard [65])**.** Modulo rigid motions, the solutions constitute a one parameter family which is diffeomorphic to $\mathbb{R}$. The solutions can be parameterized by the angle A, i.e. u_A.

This proves that the cross and the Toda solution do connect. Recently Gui–Liu–Wei [56] gave a new proof of by a variational mountain-pass lemma.

Let us introduce the

Definition 5.3. A solution u of (5.1) is said to have *finite energy* if

$$\int_{B_R} \left(\frac{1}{2}|\nabla u|^2 + \frac{1}{4}(1 - u^2)^2 \right) \leq CR. \tag{5.5}$$

A long standing open question in classifying solutions to Allen–Cahn in $\mathbb{R}^2$ is the following:

Question 5.3. *Must a solution with finite Morse index be finite-ended or have finite energy?*

A recent result by Wang–Wei [106] answers it affirmatively.

Theorem 5.6 (Wang–Wei)**.** Finite Morse index solutions to (5.1) have finitely many ends and finite energy. In particular, in $\mathbb{R}^2$, finite Morse index, finite ends and finite energy are all equivalent.

In the same paper it was also proved that

Theorem 5.7 (Wang–Wei)**.** Any solution to Allen-Cahn in $\mathbb{R}^2$ with Morse index 1 has four ends.

let us make comments on the proofs in [**?**]. First of all it is not easy to use the stability condition

$$\int |\nabla\varphi|^2 + (3u^2-1)\varphi^2 \geq 0,$$

from which we can only derive the trivial energy bound

$$\int_{B_R} \left(\frac{1}{2}|\nabla u|^2 + \frac{1}{4}(1-u^2)^2\right) \leq CR^N.$$

Instead the authors used the equivalent Sternberg–Zumbrun's Poincaré inequality [103]

$$\text{Stability} \quad \Leftrightarrow \quad \int |\nabla\varphi|^2|\nabla u|^2 - \varphi^2\left[|\nabla^2 u|^2 - |\nabla|\nabla u||^2\right] \geq 0.$$

The main idea is to such inequality to show that the following basic curvature decay at ∞:

$$B(x)^2 = \frac{|\nabla^2 u|^2 - |\nabla^2 u\cdot\nu|^2}{|\nabla u|^2} \leq \frac{C}{|x|^2} \tag{5.6}$$

This curvature decay is similar to Schoen's curvature estimate for stable minimal surfaces [94], however the proof is quite different. This is mainly due to the lack of a suitable Simons type inequality [99] for semilinear elliptic equations. Hence an indirect method is employed, by introducing a blow-up procedure due to Colding–Minnicozzi [27, 28, 29, 30, 31, 32] and reducing the curvature decay estimate to a second order estimate on interfaces of solutions. Using doubling lemma, the problem is reduced to the stability of the Toda System

$$f_j'' = e^{-(f_{j+1}-f_j)} - e^{-(f_j-f_{j-1})}.$$

The classification of multiple-ended solutions (number of ends ≥ 6) seems to be extremely hard and out of reach for the Allen–Cahn equation. However for the related elliptic sine–Gordon equation

$$\Delta u + \sin(\pi u) = 0 \quad \text{in } \mathbb{R}^2 \tag{SG}$$

with another double-well potential, one can explicitly write down all the multiple-ended solutions using integrable system. This gives rise to a complete classification of all $2k$-ended solutions, including the Morse index and the dimension.

Theorem 5.8 (Liu–Wei [72])**.** The set $\mathcal{M}_k$ of solutions of (SG) is a connected, analytic manifold of dimension $\frac{k}{2}-1$. It is non-degenerate with Morse index $\frac{k(k-2)}{8}$.

6. Beyond De Giorgi's Conjecture IV – two-ended solutions in $\mathbb{R}^3$

As we already see the classification of unstable solutions is challenging in $\mathbb{R}^2$. Now let us increase the dimension and consider

$$\Delta u + u - u^3 = 0 \quad \text{in } \mathbb{R}^3. \tag{6.1}$$

As mentioned before, the classification of stable solutions in $\mathbb{R}^3$ is still an open question.

The existence of finite Morse index solutions was established by del Pino–Kowalczyk–Wei [44].

Theorem 6.1 (del Pino–Kowalczyk–Wei)**.** Let M be an embedded non-degenerate complete minimal surfaces with finite total curvature.Then there exists a family of solutions to the Allen–Cahn equation whose nodal set is close to a scaling of M. Moreover, the Morse index of the solution equals the index of M.

A particular example is the catenoid

$$z = \log\left(r + \sqrt{r^2 - 1}\right)$$

which has two ends. The solutions constructed by del Pino–Kowalczyk–Wei arise from large catenoids and the level set $\{u = 0\}$ is asymptotically

$$z \sim \frac{1}{\varepsilon}\log r.$$

Another example is given by del Pino–Agudelo–Wei [1].

Theorem 6.2 (Agudelo–del Pino–Wei)**.** There exists a two-ended solution with asymptotic nodal set

$$z \sim \sqrt{2}\log r.$$

This was constructed using two-dimensional Toda system

$$\begin{cases} \Delta q_1 = -\beta e^{\sqrt{2}(q_1 - q_2)}, \\ \Delta q_2 = \beta e^{\sqrt{2}(q_1 - q_2)}, \end{cases} \tag{6.2}$$

and the zero level set $\{u = 0\}$ consists of two disjoint leaves with a slope different slope:

$$z \sim \sqrt{2}\log r.$$

Concerning the classification, one may ask the

Question 6.1. *Are these two branches of solutions* connected*?*

A positive answer was given by Gui–Liu–Wei [57].

Theorem 6.3. For each slope $k \in (\sqrt{2}, +\infty)$ there exists a solution to (6.1) with

$$\|u_k(r,\cdot) - w(\cdot - k\log r - c_k)\|_{L^\infty(0,+\infty)} \to 0,$$

as $r \to +\infty$, where c_k is a constant depending on k.

The proof is quite different from the usual reduction method. The existence is proved by moduli space theory and bifurcation of real analytical varieties.

7. Beyond De Giorgi's Conjecture V: two-ended solutions in $\mathbb{R}^N$, $N \geq 4$

With the established connection, it is tempting to align the Allen–Cahn equation with minimal surface theory in every respect. However, this is *not true*, according a theorem of Agudelo–del Pino–Wei [2].

Let $N \geq 4$ and consider the N-dimensional catenoid defined by

$$F = F(r), \quad \nabla \cdot \left(\frac{\nabla F}{\sqrt{1 + |\nabla F|^2}} \right) = 0, \tag{7.1}$$

so that

$$F(r) \sim 1 + O(r^{2-N}).$$

It is known that the N-dimensional catenoid has Morse index one. In this setting we have

Theorem 7.1 (Agudelo–del Pino–Wei)**.** For every $N \geq 4$ and any sufficiently small $\varepsilon > 0$, there exist a solution u_ε to (2.1) with $\{u_\varepsilon = 0\}$ asymptotic to the largely dilated N-dimensional catenoid. For $\varepsilon > 0$ small and for dimensions $4 \leq N \leq 10$, the Morse index of u_ε is $+\infty$. For $N \geq 11$, the Morse index of u_ε is one.

Here the level set of the solution resembles the catenoid, but its Morse index does not agree with that of the catenoid.

Let us summarize the results on the classical Allen–Cahn Equation (2.1). The classification of monotone solutions is almost complete and that of global minimizers is complete, while that of stable solutions (local minimizers) is widely open. On the plane $\mathbb{R}^2$, having a finite Morse index is equivalent to having finitely many ends. The classifications of 2-ended and 4-ended solutions in $\mathbb{R}^2$ are complete, so is that of 2-ended solutions in $\mathbb{R}^3$. The questions of existence and classification of other unstable solutions are still open to a very large extent.

8. Beyond De Giorgi's Conjecture VI: Fractional De Giorgi's Conjecture

Now we turn to recent exciting developments on fractional De Giorgi conjecture. Let $0 < s < 1$. The fractional Allen–Cahn equation

$$(-\Delta)^s u = u - u^3 \quad \text{in } \mathbb{R}^N, \tag{8.1}$$

is the Euler–Lagrange equation of the energy functional

$$J(u) = \frac{c_{N,s}}{2} \int \int \left[\frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \right] dxdy + \frac{1}{4} \int \left(1 - u^2\right)^2 dx. \tag{8.2}$$

Nonlocal interactions and nonlocal equations arise in many pure and applied fields, including differential geometry (like fractional Yamabe problem and fractional minimal surfaces), probability (notably Lévy process), and applied physics.

A fractional version of De Giorgi's Conjecture may be stated as follows.

Conjecture 8.1 (Fractional De Giorgi's Conjecture)**.** *Let u be a bounded solution to (8.1) which is monotone in one direction, say $\frac{\partial u}{\partial x_N} > 0$. Then u is one-dimensional.*

When $N = 1$, the existence and uniqueness of heteroclinic solutions

$$(-\partial_{zz}^2)^s w = w - w^3, \quad w(\pm\infty) = \pm 1, \quad w(0) = 0, \quad w' > 0 \tag{8.3}$$

was shown by Cabré–Sire [14, 15].

Since then much progress has been made on the Fractional De Giorgi's Conjecture. It was proved by Sire–Valdinoci [100] for $N = 2$, $s \in (0,1)$ and Cabré–Cinti for $N = 3$, $s \in [\frac{1}{2}, 1)$. Under the limit assumption

$$u(x_1, \ldots, x_N) \to \pm 1, \quad \text{as } x_N \to \pm\infty,$$

the conjecture was showed true for $N = 3$, $s \in (0,1)$ by Dipierro–Serra–Valdinoci [45] and $4 \leq N \leq 8$, $s \in (\frac{1}{2}, 1)$ by Savin [93].

One would next classify global minimizers, which are defined in the following way.

Definition 8.1. A solution u is called a *global minimizer* on a compact set Ω in $\mathbb{R}^N$ if

$$J_\Omega(u) \leq J_\Omega(u + \phi), \quad \text{for all } \phi \in C_0^1(\Omega),$$

where

$$J_\Omega(u) = \frac{c_{N,s}}{2} \int\int_{\mathbb{R}^N \times \mathbb{R}^N \setminus (\Omega^c \times \Omega^c)} \left[\frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \right] dxdy + \frac{1}{4} \int_\Omega \left(1 - u^2\right)^2 dx. \tag{8.4}$$

Note that the part $\Omega^c \times \Omega^c$ of the domain is taken away, in order that the energy is well-defined. In fact, the value of u remains unchanged by the perturbation in the region Ω.

The known classification results for global minimizers are due to Dipierro–Serra–Valdinoci [45] in the case $N \leq 3$, $s \in (0, \frac{1}{2})$ and Savin [93] for $N \leq 7$, $s \in (\frac{1}{2}, 1)$.

Definition 8.2. A solution u is called *stable* in Ω if for all $\phi \in C_0^1(\Omega)$,

$$c_{N,s} \int\int_{\mathbb{R}^N \times \mathbb{R}^N \setminus (\Omega^c \times \Omega^c)} \left[\frac{|\phi(x) - \phi(y)|^2}{|x - y|^{N+2s}} \right] dxdy + \int_\Omega (3u^2 - 1)\phi^2 dx \geq 0$$

The one-dimensional symmetry of stable solutions was proved for $N = 2$, $s \in (0,1)$ in [100, 45]. A similar result was obtained by Hamel–Ros-Oton–Sire–Valdinoci [61] for more general nonlocal equations. A breakthrough was made by Figalli–Serra [48] when $N = 3$ and $s = \frac{1}{2}$.

Very recently, counterexamples to Fractional De Giorgi Conjecture are constructed by Chan–Liu–Wei [22, 23].

Theorem 8.1 (Chan–Liu–Wei)**.** Consider fractional Allen–Cahn equation

$$(-\Delta)^s u = u - u^3 \quad \text{in } \mathbb{R}^N.$$

For $N = 8$, $s \in (\frac{1}{2}, 1)$, there exists a global minimizer which is NOT one-dimensional. For $N = 9$, $s \in (\frac{1}{2}, 1)$ there exists a monotone solution with $\frac{\partial u}{\partial x_N} > 0$, which is NOT one-dimensional.

This implies that Savin's classification results [93] concerning global minimizers and monotone solutions are both optimal.

The key idea of the proof is to developing an *fractional infinite dimensional gluing method* for higher dimensional concentration phenomena for the fractional elliptic equations.

Let us briefly sketch the main ideas. In finite dimensional reduction methods, one varies the points, which are zero dimensional objects; in infinite dimensional reduction methods, one moves instead curves or surfaces.

For fractional elliptic equations, the finite dimensional Liapunov–Schmidt reduction method has been applied to obtain solutions with spikes [34] of the fractional nonlinear Schrödinger equation

$$\varepsilon^{2s}(-\Delta)^s u + V(x)u - u^P = 0, \quad \text{in } \mathbb{R}^N, \tag{8.5}$$

layered solutions [46] of the inhomogeneous fractional Allen–Cahn equation

$$\varepsilon^{2s}(-\partial_{xx}^2)^s u + V(x)(u - u^3) = 0 \quad \text{in } \mathbb{R}^1, \tag{8.6}$$

as well as bubbles [8] of the fractional Yamabe problem

$$(-\Delta)^s u = u^{\frac{N+2s}{N-2s}} \quad \text{in } \mathbb{R}^N \setminus \{p_1, \ldots, p_m\}. \tag{8.7}$$

However, there are no constructions of higher dimensional concentrations for fractional elliptic equations. This is the content of next section.

9. The infinite dimensional gluing method: classical and fractional

9.1. The classical case. Let us describe the infinite dimensional Liapunov–Schmidt reduction procedure, first reviewing the classical case $s = 1$.

To this end, we need some geometric backgrounds. Let $\Gamma \subset \mathbb{R}^N$ be a hypersurface. Under the Fermi coordinates associated with Γ we can write

$$\Delta = \partial_{zz}^2 + \Delta_{\Gamma_z} - \sum_j \frac{k_j}{1 - zk_j}\partial_z,$$

where z is the signed distance to Γ and Γ_z is the hypersurface $\{d(x,\Gamma) = z\}$, and k_j is the principal curvature. Note that this is only defined in a neighborhood of Γ.

The basic profile $w(z) = \tanh(\frac{z}{\sqrt{2}})$ satisfies

$$w'' + w - w^3 = 0$$
$$w' \sim e^{-\sqrt{2}|z|}$$

An inner-outer gluing scheme is to be employed as we look for a solution to

$$\Delta u + u - u^3 = 0 \quad \text{in } \mathbb{R}^N$$

concentrating on a given hypersurface Γ. We introduce a small parameter $\varepsilon > 0$ and consider the rescaled equation

$$\varepsilon^2 \Delta u + u - u^3 = 0,$$

making the change of variables

$$z \to \varepsilon z \quad \text{(signed distance to } \Gamma)$$
$$y \to \varepsilon y \quad \text{(local coordinate on } \Gamma).$$

The hypersurface is shifted in the normal direction by a function h, as in

$$\bar{z} = z - \varepsilon h(\varepsilon y).$$

We use the approximate solution

$$w(z - h(\varepsilon y))$$

and seek a solution of the following form

$$u = w(z - h(\varepsilon y)) + \phi(z)\eta(z) + \psi(x) \tag{9.1}$$

where ϕ solve an inner problem (near the hypersurface) and ψ solves an outer problem; η is a cut-off function near Γ.

The inner problem is

$$\partial^2_{zz}\phi + \Delta_{\epsilon^{-1}\Gamma}\phi + f'(w)\phi + E + (f'(w) + 2)\psi\eta = 0, \tag{9.2}$$

with the error $E = \Delta w(z) + w - w^3$ and the outer-to-inner coupling term $(f'(w) + 2)\psi\eta$ as the differential operator in the outer problem is made coercive.

The outer problem reads

$$\Delta\psi + f'(w)(1 - \eta)\psi + 2\nabla\phi \cdot \nabla\eta + \psi\Delta\eta = 0, \tag{9.3}$$

where the inner-to-outer coupling term $2\nabla\phi\nabla\eta + \psi\Delta\eta$ is induced from the cut-off function η.

The outer problem is solved first since the operator is basically

$$\Delta\psi - (2 - \delta)\psi = \text{Error}$$

for which maximum principles and elliptic regularities apply easily.

The inner problem is solved in two steps:

Step 1: We solve the inner problem up to a function $c(y)$:

$$\partial^2_{zz}\phi + \Delta_{\epsilon^{-1}\Gamma}\phi + f'(w)\phi + E + (f'(w) + 2)\psi\eta = c(y)w_z;$$

Step 2: We perturb the hypersurface Γ to $\Gamma_h = \{y + zh(y) | y \in \Gamma\}$ so that

$$c(y) = 0,$$

which is equivalent to

$$\Delta_\Gamma h + |A|^2 h = G.$$

The main ingredients in Step 1 are the *non-degeneracy* of $w(z)$, namely the only bounded solution to

$$\phi_{zz} + \Delta_y\phi + f'(w)\phi = 0$$

is be $cw_z(z)$, and the *solvability* of the linear equation

$$\phi_{zz} + \Delta_y\phi + f'(w)\phi = g$$

for

$$\int g(z, y)w_z(z)dz = 0 \quad \forall y$$

in the space of functions

$$\int \phi(z, y)w_z(z)dz = 0 \quad \forall y.$$

In Step 2, a linear theory of *Jacobi operator* $\Delta_\Gamma + |A|^2$ is required.

A good introduction to this method can be found in [44].

9.2. **The fractional gluing.** Now let us turn to the fractional case. Immediately one sees the difficulties from the lack of Fermi coordinates and linear theory, algebraically slow decay of the profile, and nonlocal interactions. We will briefly describe how these difficulties are tackled.

We first derive the following *fractional Fermi coordinates*. Let Γ be a hypersurface and z be the signed distance to $\varepsilon^{-1}\Gamma$. Then for $s > \frac{1}{2}$,

$$(-\Delta)^s w|_{(y_0,z_0)} = (-\partial_{zz}^2)^s w(z_0) + \varepsilon H(\varepsilon y_0) c(z_0) + O(R_1^{-2s})$$

where R_1 is the maximal radius of a spherical region in which the Fermi coordinates can be defined and

$$c(z_0) = c_{1,s} \int_{\mathbb{R}} \frac{w(z_0) - w(z)}{|z_0 - z|^{1+2s}} (z_0 - z)\, dz.$$

In partciular,

$$c(z_0) \sim |z_0|^{1-2s}$$

so $s > \frac{1}{2}$ is crucially used in order to obtain a decay in the normal direction.

Next, the basic profile, satisfying

$$(-\partial_{zz}^2)^s w = w - w^3, \quad w' > 0, \quad \lim_{z\to\pm\infty} w(z) = \pm\infty, \quad w(0) = 0,$$

was shown in [14, 15] to be unique and has the asymptotic behavior

$$1 - |w(z)| \sim \frac{1}{|z|^{2s}}, \quad \text{as } |z| \to \infty.$$

The non-degeneracy in one dimension was proved by Du–Gui–Sire–Wei [46]. If $\phi(z)$ is a bounded solution of

$$(-\partial_{zz}^2)^s \phi + f'(w)\phi = 0 \quad \text{in } \mathbb{R}^1,$$

then $\phi(z) = cw_z$, provided $s > \frac{1}{2}$. In higher dimensions, we use the extension of Caffarelli–Silvestre [19] and an idea in del Pino–Kowalczyk–Pacard–Wei [41] to show that the Fourier transform in y must be a sum of derivatives of dirac masses

$$\hat{\phi}(z,\xi,t) = \sum_{j=0}^{N_0} a_j(z,t)\delta_0^{(j)}(\xi),$$

so that $\phi(z,y,t)$ is a polynomial of some order N_0 in y. Again, the fact that $s > \frac{1}{2}$ is used.

In the fractional linear theory one is concerned with the

Question. *For what (g,ϕ) does the linear problem*

$$(-\partial_{zz}^2 - \Delta_y)^s \phi + f'(w)\phi = g$$

has a unique solution?

We recall that when $s = 1$, $N \geq 2$, the condition [44] is

$$\int g(z,y) w_z(z)\, dz = 0 \quad \text{and} \quad \int \phi(z,y) w_z(z)\, dz = 0, \quad \forall y;$$

and when $s \in (0,1)$, $N = 1$, one needs

$$\int g(z) w_z(z)\, dz = 0 \quad \text{and} \quad \int \phi(z) w_z(z)\, dz = 0, \quad \forall y.$$

However, when $s \in (0,1)$ and $N \geq 2$, we can only have

Lemma 9.1. *For each g satisfying*

$$\int g(z,y)w_z(z)\,dz = 0 \tag{9.4}$$

there exists a unique solution such that

$$\iint_{\mathbb{R}^2_+} t^a\bar{\phi}(t,z,y)\bar{w_z}(t,z)\,dtdz = 0, \tag{9.5}$$

where $a = 1-2s$ and the bars denote s-harmonic extensions, i.e. convolution with the Poisson kernel $P(z,t) = \frac{t^{1-a}}{(|z|^2+t^2)^{\frac{2-a}{2}}}$.

Testing

$$\begin{cases} \nabla_{t,z}(t^a\nabla_{t,z}\bar{\phi}) + \Delta_y(t^a\bar{\phi}) = 0, & (t,z,y) \in \mathbb{R}^{N+1}_+, \\ t^a\dfrac{\partial\phi}{\partial t} + f'(w)\phi = g, & (z,y) \in \partial\mathbb{R}^N_+ \end{cases}$$

with $\bar{w}_z(t,z)$ we obtain

$$\Delta_y\left(\iint_{\mathbb{R}^2_+} t^a\bar{\phi}\bar{w}_z\,dtdz\right) = 0, \quad \forall y,$$

hence the necessary condition

$$\iint_{\mathbb{R}^2_+} t^a\bar{\phi}\bar{w}_z\,dtdz = 0, \quad \forall y.$$

This condition turns out to be also sufficient. Consider the Fourier-transformed problem

$$\begin{cases} \nabla\cdot(t^a\nabla\bar{\phi}) - |\xi|^2t^a\bar{\phi} = 0 & (t,z,y) \in \mathbb{R}^2_+ \\ t^a\dfrac{\partial\bar{\phi}}{\partial t} + f'(w)\bar{\phi} = g & (z,y) \in \partial\mathbb{R}^2_+. \end{cases}$$

in the space

$$X = \left\{\bar{\phi} \in H^1(\mathbb{R}^2_+, t^a) : \iint_{\mathbb{R}^2_+} t^a\bar{\phi}\bar{w}_z\,dtdz = 0\right\}. \tag{9.6}$$

We minimize the energy

$$E(\bar{\phi}) = \iint_{\mathbb{R}^2_+}\left(t^a|\nabla\bar{\phi}|^2 + |\xi|^2t^a\bar{\phi}^2\right)\,dzdt + \int_{\partial\mathbb{R}^2_+}\left(f'(w)\phi^2 - 2g\phi\right)\,dz \tag{9.7}$$

for $\bar{\phi} \in X$. The minimizer exists and the Euler–Lagrange multiplier equals zero.

Here we have used $s > \frac{1}{2}$ for L^2 integrability. This is also related to the *a priori* estimates, as we shall discuss.

In the linear theory, we need *a priori* estimates. Consider the equation

$$(-\partial^2_{zz} - \Delta_y)^s\phi(z,y) + f'(w(z))\phi(z,y) = h(z,y) \quad \text{ for } (z,y) \in \mathbb{R}^N. \tag{9.8}$$

Let $\langle y\rangle = \sqrt{1+|y|^2}$ and define the norm

$$\|\phi\|_{\mu,\sigma} = \sup_{(y,z)\in\mathbb{R}^N} \langle y\rangle^{\mu}\langle z\rangle^{\sigma}\|\phi\|_{L^\infty(B_1(y,z))}$$

for $0 \le \mu \le N-1$ and $2-2s < \sigma \le 1$. Using the maximum principle and a blow-up argument, we have

$$\|\phi\|_{\mu,\sigma} \le C\|\phi\|_{L^\infty(B_1(0))}\|h\|_{\mu,\sigma}.$$

While the idea is standard, the key is the following surprisingly tedious computation of the innocent-looking super-solution,

$$(-\partial_{zz}^2 - \Delta_y)^s \left(\langle y \rangle^{-\mu} \langle z \rangle^{-\sigma} \right) = o\left(|y|^{-\mu} |z|^{-\sigma} \right), \qquad \text{as } \min(|y|, |z|) \to +\infty.$$

With the linear theory at hand, let us describe the gluing scheme. In contrast to the local case, the inner problem has to be localized into many. In fact, we consider a (finite) partition of unity η_j of $\mathbb{R}^N$ such that the support of each η_j is either like a ball of radius $R = o(\varepsilon^{-1})$ on $\varepsilon^{-1}\Gamma$, a tubular neighborhood of width R around the ends of the rescaled hypersurface, or away from the interface. By enlarging the support of η_j, we obtain a family of cut-off functions ζ_j, which does not form a partition of unity, and look for a solution of the form

$$u = w(z) + \sum_j \zeta_j \phi_j. \tag{9.9}$$

Note that the outer problem is included in the above Ansatz; for consistent notation we may call $\psi = \phi_0$.

By the smallness of the supports of η_j and ζ_j in comparison to ε^{-1}, the inner problems can be written in terms of Fermi coordinates:

$$\text{(inner)} \qquad (-\Delta_{y,z})^s \phi_j + f'(w)\phi_j = \eta_j(E + N)$$

$$\text{(outer)} \qquad (-\Delta)^s \psi + 2\psi = \eta_0(E + N)$$

where E is the error and N contains all kinds of commutator terms including

$$\int \frac{(\eta_j(x) - \eta_j(y))\phi_j(y)}{|x - y|^{N+2s}}\, dy,$$

which corresponds to the inner-to-outer coupling term when $s = 1$. Yet, they no longer have compact support and have to be analyzed carefully.

These equations are highly coupled. Nonetheless, having computed the integrals on the right hand side which is cut-off, the decay properties of ϕ_j (and ψ) follow.

We point out that the fractional inner-outer gluing scheme has also been used successfully to treat the 1/2-harmonic map flows:

$$u : R \times [0, \infty) \to S^1,$$
$$u_t = -(-\Delta)^{\frac{1}{2}}(u) + \frac{1}{2\pi} \left(\int \frac{|u(x) - u(y)|^2}{|x - y|^2} dy \right) u,$$
$$u(x, 0) = u_0(x).$$

In [101], Sire–Wei–Zheng studied infinite time blow-up for half-harmonic maps into S^1.

Finally, one may ask

Question. *What is the reduced geometric problem? Is it still a minimal surface?*

The first error seems to be

$$\begin{aligned}(-\Delta)^s w|_{(y_0,z_0)} + f(w(z_0)) &= (-\partial_{zz}^2)^s w(z_0) + f(w(z_0)) + \varepsilon H(\varepsilon y_0)c(z_0) + O(R_1^{-2s}) \\ &= \varepsilon H(\varepsilon y_0)c(z_0) + \cdots\end{aligned}$$

hence the solvability condition is

$$\int \left(\varepsilon H(\epsilon y_0)c(z_0) + \cdots \right) w_z(z_0) dz_0 = 0,$$

i.e.

$$H(\epsilon y_0) \sim 0.$$

To answer this question, we consider

$$(-\Delta)^s u = u - u^3 \quad \text{in } \mathbb{R}^3$$

where Γ is the catenoid and $s > \frac{1}{2}$. We have

Theorem 9.1 (Chan–Liu–Wei [22])**.** For $s > \frac{1}{2}$ there exists a family of solutions u_ε to the fractional Allen–Cahn equation in $\mathbb{R}^3$

$$(-\Delta)^s u = u - u^3 \quad \text{in } \mathbb{R}^3$$

such that

$$\{u_\varepsilon = 0\} \sim \{z = f_\varepsilon(r)\}$$

where f_ϵ satisfies

$$\frac{1}{r}\left(\frac{r f_\varepsilon'(r)}{\sqrt{1+(f_\varepsilon')^2}}\right)' = \frac{\varepsilon^{2s-1}}{f_\varepsilon^{2s}}$$

$$f_\varepsilon \sim \varepsilon^{\frac{2s-1}{2s+1}} r^{\frac{2}{1+2s}}$$

Remarkably, the nonlinear term

$$\frac{\epsilon^{2s-1}}{f^{2s}}$$

is precisely the algebraic interaction between the leaves of the level set. The new reduced geometric problem is

$$H_f = \frac{\varepsilon^{2s-1}}{f^{2s}} \tag{9.10}$$

$$f \sim \begin{cases} \log r, & r < \varepsilon^{-\beta} \\ \varepsilon^{\frac{2s-1}{2s+1}} r^{\frac{2}{1+2s}}, & r > \varepsilon^{-\beta} \end{cases}$$

This phenomena also appears in the construction of fractional catenoid minimal surfaces by Dávila–del Pino–Wei [36].

Concerning the Simons' cone, we will address the

Question. *In the fractional case, does Simons' cone get deformed to*

$$F \sim r^{\frac{2}{1+2s}}?$$

The Simons' cone is the hypersurface $\{r = s\}$ in $\mathbb{R}^8$ with

$$r = \sqrt{x_1^2 + \cdots + x_4^2}, \quad s = \sqrt{x_5^2 + \cdots + x_8^2}.$$

Its foliations are given by $\{s = F(r)\}$ with

$$F \sim r - \frac{c}{r^2} + o(r^{-2})$$

Let us conclude with a sketch of the proof of Theorem 8.1. Following the fractional gluing approach, we construct a family of solutions concentrating on foliations of Simons'cones:

$$\{s = F(r)\}, \quad F \sim r - \frac{c_0}{r^2}$$

Because of the symmetry of the Simons' cones, the algebraic interactions of the perturbations of the ends *cancelled out*. We obtain equations like

$$H \sim \frac{\epsilon^{2s-1}}{r^{3+2s}},$$

so the adjustment of F is at a lower order. We then show that this solution is stable and ordered.

Using ideas from Liu–Wang–Wei [71], this solution is shown to be a global minimizer. We then extend Jerison–Monneau's program [62] to construct a counterexample of Fractional De Giorgi's Conjecture in $\mathbb{R}^9$.

10. De Giorgi's conjecture for overdetermined problem: Berestycki-Caffarelli-Nirenberg Conjecture

In the next two sections, we discuss De Giorgi type conjectures for two classical problems.

First we consider the following so-called Serrin's overdetermined problem:

$$\begin{cases} \Delta u + f(u) = 0, & \text{in } \Omega \\ u > 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \\ \dfrac{\partial u}{\partial \nu} = const., & \text{on } \partial\Omega \end{cases} \tag{10.1}$$

where f is a Lipschitz nonlinearity, ν is the outer normal at $\partial\Omega$, and $\frac{\partial u}{\partial \nu}$ is a constant which is not prescribed a priori.

A classical result of Serrin's [102] asserts that if Ω is a bounded and smooth domain for which there is a positive solution to the overdetermined equation (10.1) then Ω is a sphere and u is radially symmetric. In the analysis of blown up version of free boundary problem, it is natural also to consider Serrin's overdetermined problem in unbounded domains. (See Berestycki-Caffarelli-Nirenberg [10].) In [11, pp.1110], the following De Giorgi's type conjecture on Serrin's overdetermined problem in unbounded domains was raised.

Conjecture 10.1 (Berestycki-Caffarelli-Nirenberg)**.** *Assume that Ω is a smooth domain with Ω^c connected and that there is a bounded positive solution of (10.1) for some Lipschitz function f then Ω is either a half-space, or a cylinder $\Omega = B_k \times \mathbb{R}^{N-k}$, where B_k is a k-dimensional Euclidean ball, or the complement of a ball or a cylinder.*

The conjecture, in the case of cylindrical domains, was disproved by Sicbaldi in [90], where he provided a counterexample in the case when $N \geq 3$ and $f(t) = \lambda t$, $\lambda > 0$ by constructing a periodic perturbation of the cylinder $B^{N-1} \times R$ which supports a bounded solution to (10.2). In the two-dimensional case, Hauswirth, Hèlein and Pacard in [78] provided a counterexample in a strip-like domain for the case $f = 0$. Explicitly, Serrin's overdetermined problem is found to be solvable in the domain

$$\Omega = \{x \in \mathbb{R}^2 \ / \ |x_2| < \frac{\pi}{2} + \cosh(x_1)\},$$

where the solution found is unbounded. Necessary geometric and topological conditions on Ω for solvability in the two-dimensional case have been found by Ros and

Sicbaldi in [88]. Recently the two-dimensional case is almost completely solved in [89].

An analogue case to De Giorgi's conjecture for Allen-Cahn equation is the epigraph case, namely the following overdetermined problem

$$\begin{cases} \Delta u + f(u) = 0, \ u > 0 \ \text{ in } \Omega = \{x_N > \varphi(x^{'})\} \\ u = 0, \quad \text{on } \ \{x_N = \varphi(x^{'})\}, \\ \dfrac{\partial u}{\partial \nu} = const., \ \text{on} \quad \{x_N = \varphi(x^{'})\}. \end{cases} \tag{10.2}$$

In this case, the BCN conjecture states that if Serrin's problem (10.2) is solvable, then Ω must be an half-space. In a recent paper, del Pino, Pacard and the second author [35] constructed an epigraph, which is a perturbation of the Bombieri-De Giorgi-Giusti minimal graph, such that problem (10.2) admits a solution. This counterexample requires dimension $N \geq 9$.

In the low dimensions, we have

Theorem 10.1 (Wang-Wei [105])**.** Let $N = 2$. If Serrin's overdetermined problem (10.2) has a solution then $\Omega = \{x_N > \varphi(x^{'})\}$ must be a half space and up to isometry $u(x) \equiv g(x \cdot e)$ for some unit vector e.

Theorem 10.2 (Wang-Wei [105])**.** Assume that φ is globally Lipschitz. If Serrin's overdetermined problem (10.2) has a solution then $\Omega = \{x_N > \varphi(x^{'})\}$ must be a half space and up to isometry $u(x) \equiv g(x \cdot e)$ for some unit vector e.

Theorem 10.3 (Wang-Wei [105])**.** Assume that φ is coercive, i.e.

$$\lim_{x' \to \infty} \varphi(x') = +\infty. \tag{10.3}$$

Then there is no solution to Serrin's overdetermined problem (10.2) in $\Omega = \{x_N > \varphi(x^{'})\}$.

Theorem 10.4 (Wang-Wei [105])**.** Let u be a solution of (10.2) satisfying the following monotonicity assumption in one direction

$$\frac{\partial u}{\partial x_N} > 0, \quad \text{in } \Omega. \tag{10.4}$$

If $N \leq 8$ and $0 \in \partial\Omega$, then $u(x) \equiv g(x \cdot e)$ for some unit vector e and Ω is an half space.

The extra condition (10.4) in Theorem 10.4 is a natural one. (This is similar to Savin's extra condition in De Giorgi's conjecture.) This condition is always satisfied if the epigraph is globally Lipschitz or coercive ([11]). As for De Giorgi's conjecture, it will be an interesting question to remove or prove this condition in general setting.

11. De Giorgi's conjecture for Lane-Emden equation

The classification of positive solutions to Lane-Emden equation

$$\Delta u + u^p = 0, u > 0 \ \text{ in } \mathbb{R}^N \tag{11.1}$$

has attracted lots of attentions since 1970's and has remained as one of the most difficult and challenging problem in nonlinear partial differential equations.

It is known that the the Sobolev and Joseph-Lundgren exponents

$$p_S = \begin{cases} \infty, & \text{if } 1 \leq N \leq 2, \\ \frac{N+2}{N-2}, & \text{if } N \geq 3. \end{cases}, \quad p_{JL} = \begin{cases} \infty, & \text{if } 3 \leq N \leq 10, \\ 1 + \frac{4}{N-4-2\sqrt{N-1}}, & \text{if } N \geq 11. \end{cases}$$

play a central role in existence and non-existence of solutions.

In the subcritical case $1 < p < p_S$, a well-known result of Gidas and Spruck ([52]) says that (11.1) admits no nontrivial nonnegative solution. In the Sobolev critical case $p = p_S$ any positive solution of (11.1) can be written in the form (see [18]):

$$u_{a,\lambda}(x) = C_N \left(\frac{\lambda}{\lambda^2 + |x-a|^2}\right)^{\frac{N-2}{2}}$$

Therefore the structure of positive solutions in the critical or subcritical cases are completely classified. A fundamental question is to classify positive solutions in the supercritical case. This question remains largely open.

When $p > p_S$ the structure of positive radial solutions of (11.1) has been completely classified by Gui-Ni–Wang [58]. It is shown thatfor any $a > 0$, equation (11.1) admits a unique positive radial solution $u = u_a(r)$ with $u_a(0) = a$ and $u_a(r) \to 0$ as $r \to +\infty$.

Recently, solutions of (11.1) up to $p_{JL}(N)$ are classified by using the Morse index theory. Farina [60] showed that if $p_S < p < p_{JL}$ and u is a positive solution of (11.1) that has finite Morse index, then $u \equiv 0$. This result is optimal since the radial solution u_a is shown to be stable for $p > p_{JL}$ ([58]).

When p is supercritical, it is still open if all positive solutions are radially symmetric around some point. The first result was due to Zou [108], who showed that when $p_S(N) < p < p_S(N-1)$ and u has the right decay $u = O(|x|^{-\frac{2}{p-1}})$ then all solutions are radially symmetric. Guo [59] extended Zou's result to $p \geq p_S(N-1)$ by assuming $\lim_{|x|\to+\infty} |x|^{\frac{2}{p-1}} u(x) \equiv C$.

The following question is a natural extension of De Giorgi type conjecture

Conjecture 11.1. *Let $p_{JL}(N) \leq p < p_{JL}(N-1)$. All stable solutions to (11.1) must be radially symmetric around some point.*

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