SOME GLOBAL SOLUTIONS TO THE ENERGY-CRITICAL SEMILINEAR HEAT EQUATION

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In memory of Professor Marek Fila

ABSTRACT. We study the large-time asymptotics of global solutions to the semilinear heat equation in $\mathbb{R}^n$ with critical Sobolev exponent

$$\begin{cases} u_t = \Delta u + |u|^{\frac{4}{n-2}} u & \text{in } \mathbb{R}^n \times (0,\infty), \\ u(\cdot,0) = u_0 & \text{in } \mathbb{R}^n. \end{cases}$$

We show the existence of positive solutions for a class of initial value $u_0(x) \sim |x|^{-\gamma}$ as $|x| \to \infty$ with $\gamma > \frac{n-2}{2}$ such that the time decay rate of $\|u\|_{L^\infty(\mathbb{R}^n)}$ depends on γ in a precise manner, motivating by a program proposed by Fila and King [10]. We construct several global solutions in dimensions 4 and 6.

1. Introduction

The semilinear heat equation

$$\begin{cases} u_t = \Delta u + |u|^{p-1} u & \text{in } \mathbb{R}^n \times (0,\infty) \\ u(\cdot,0) = u_0 & \text{in } \mathbb{R}^n \end{cases} \tag{1.1}$$

with $p > 1$ has been widely studied since Fujita's seminal work [13]. The power nonlinearity plays a crucial role, producing rich phonomena concerning singularity formation, long-time dynamics and regularity properties. It is well known that (1.1) admits a global nontrivial solution $u \geq 0$ if and only if $p > p_F := \frac{n+2}{n}$. The stationary version of (1.1) does not have positive classical solutions if $p < p_S$, where

$$p_S = \begin{cases} \frac{n+2}{n-2} & \text{for } n \geq 3, \\ \infty & \text{for } n = 1, 2. \end{cases}$$

See [16] and [2]. For $p = p_S$, up to translations and dilations, the positive solution to the Yamabe problem

$$\Delta u + u^{\frac{n+2}{n-2}} = 0 \ \text{ in } \mathbb{R}^n, \ n \geq 3$$

is the Aubin-Talenti bubble

$$U(x) = \alpha_n (1 + |x|^2)^{-\frac{n-2}{2}}, \quad \alpha_n = [n(n-2)]^{\frac{n-2}{4}},$$

and it serves as a natural candidate for the profile of singularity formation for (1.1).

For (1.1) in the subcritical case $p < p_S$, Poláčik and Quittner [24] proved the nonexistence of positive, radially symmetric, bounded entire solution, and they showed that the global nonnegative radial solution of (1.1) decays to 0 uniformly as $t \to \infty$. Optimal Liouville-type results have been achieved by Quittner [29] for the case $1 < p < p_S$ in the sense that no further decay or decay assumptions are made. See also Poláčik, Quittner and Souplet [25] for a general scheme that connects parabolic Liouville type theorems with universal estimates of solutions.

In this paper, we are interested in the global positive solutions to

$$\begin{cases} u_t = \Delta u + u^{\frac{n+2}{n-2}} & \text{in } \mathbb{R}^n \times (t_0, \infty), \\ u(\cdot, t_0) = u_0 & \text{in } \mathbb{R}^n, \end{cases} \tag{1.2}$$

and aim to understand their possible dynamics at large time. (Problem (1.2) is simply a shift of (1.1) in time.) A solution is called global if its maximal existence time $T_{\max} = \infty$. The underlying motivation of this is due to extensive investigations of threshold solutions.

For any nonnegative, smooth function $\phi(x)$ with $\phi \not\equiv 0$, we define

$$\alpha^* = \alpha^*(\phi) := \sup\{\alpha > 0 : \ T_{\max}(\alpha\phi) = \infty\},$$

and $u^* := u(x, t; \alpha^*\phi)$ is called the threshold solution associated with ϕ. The threshold solution can be viewed as the borderline between global solutions and finite-time blow-ups, as the nonlinearity dominates the Laplacian for $\alpha \gg \alpha^*$ and vice versa. At the threshold level, all the possibilities can happen for the dynamics of u^* in the L^∞-sense, and depending on p and the domain, the solutions could be global and bounded, global and unbounded, or blow up in finite time. There are innumerable studies in related literature, and we refer the readers to Ni-Sacks-Tavantzis [22], Lee-Ni [20], Galaktionov-Vázquez [15], Poláčik [23], Quittner [28], and the monograph by Quittner and Souplet [30] for the state of the art of threshold solutions. See also [11, 12, 17–20, 25–27, 32, 33] and their references.

Define

$$L^2_\rho = \{f \in L^2 \mid \int_{\mathbb{R}^n} |f|^2 e^{\frac{|x|^2}{4}} dx < \infty\}.$$

For (1.1) with $p_F < p < p_S$ and $u_0 \geq 0, u_0 \not\equiv 0$ in $L^2_\rho \cap L^\infty$, the asymptotic behavior of the nonnegative solution was studied by Kawanago [19], and in particular, he proved that

$$\|u(\cdot, t; \alpha^*\phi)\|_{L^\infty} \sim t^{-\frac{1}{p-1}}$$

for $t > 1$. For $p \geq p_S$ and radial, positive initial data u_0 with decay

$$\lim_{|x|\to\infty} u_0(x)|x|^{\frac{2}{p-1}} = 0,$$

Quittner [27] proved the nonexistence of global solution with self-similar rate $t^{-\frac{1}{p-1}}$. For $p = p_S$ and radial initial data u_0 satisfying

$$\lim_{r\to\infty} r^\gamma u_0(r) = A \ \text{ for some } \ A > 0 \ \text{ and } \ \gamma > \frac{n-2}{2} = \frac{2}{p_S - 1},$$

Fila and King [10] carried out interesting formal analysis and predicted the possible long-time dynamics of threshold solutions to (1.1). They conjectured that the threshold solution u of (1.1) with initial value u_0 must satisfy

$$\lim_{t\to\infty} \frac{\|u(\cdot, t)\|_{L^\infty(\mathbb{R}^n)}}{\varphi(t; n, \gamma)} = C$$

for some positive constant $C = C(n, u_0)$, and $\varphi(t; n, \gamma)$ depends on γ and n in a precise manner:

	$\frac{n-2}{2} < \gamma < 2$	$\gamma = 2$	$\gamma > 2$
$n = 3$	$t^{\frac{\gamma-1}{2}}$	$t^{\frac{1}{2}}(\ln t)^{-1}$	$t^{\frac{1}{2}}$
$n = 4$	$t^{-\frac{2-\gamma}{2}}\ln t$	1	$\ln t$
$n = 5$	$t^{-\frac{3(2-\gamma)}{2}}$	$(\ln t)^{-3}$	1

If $n \geq 6$ and $\gamma > (n-2)/2$, then $\varphi(t; n, \gamma) \equiv 1$.

Table 1. Fila-King [10, Conjecture 1.1]

Motivated by the program of Fila-King, there are a series of rigorous constructions via the gluing method. The case $\gamma > 1$, $n = 3$ was answered affirmatively by del Pino, Musso and the first author [8]. The cases $n = 4$ with $\gamma > 2$ and $n = 5$ with $\gamma > 3/2$ were solved respectively in [35] and [21]. The relation between (1.2) with $n = 4$ and the 1-equivariant harmonic map heat flow (HMHF) has long been known, and related trichotomy dynamics for HMHF were constructed in [34]. The current known examples can be summarized as

	$\frac{n-2}{2} < \gamma < 2$	$\gamma = 2$	$\gamma > 2$
$n = 3$	[8] if $1 < \gamma < 2$	[8]	[8]
$n = 4$	HMHF [34]	HMHF [34]	[35], HMHF [34]
$n = 5$	[21]	[21]	[21]

Table 2. Examples of global solutions

For the Cauchy-Dirichlet problem of (1.2), global unbounded solutions have been constructed by Galaktionov-King [14], Cortázar-del Pino-Musso [3], del Pino-Musso-Wei-Zheng [9], and Ageno-del Pino [1], where the Dirichlet boundary plays an important role in the blow-up dynamics.

Based on previous works, this paper aims to extend the study for the cases $n = 4$ with $1 < \gamma \leq 2$ and $n = 6$ with $\gamma > 2$, and we construct global solutions with rates predicted in Table 1.

Theorem 1.1. *For $n = 6$ in (1.2), if $\gamma > 2$ and t_0 sufficiently large, then there exists a positive solution of the form*

$$u(x,t) = 24\left(\frac{\mu(t)}{\mu^2(t) + |x-\xi(t)|^2}\right)^2 \eta\left(\frac{x-\xi(t)}{\sqrt{t}}\right) + \text{h.o.t.},$$

where $\mu(t) \sim 1$ and $\xi(t) \to 0$ as $t \to \infty$. Here, $\eta(x)$ is a smooth cut-off function with $\eta(x) = 1$ for $|x| \leq 1$, $\eta(x) = 0$ for $|x| \geq 2$.

Theorem 1.2. *For $n = 4$ in (1.2), if $1 < \gamma \leq 2$ and for t_0 sufficiently large, then there exists a positive solution of the form*

$$u(x,t) = 2\sqrt{2}\frac{\mu(t)}{\mu^2(t) + |x|^2}\eta\left(\frac{x}{\sqrt{t}}\right) + \text{h.o.t.},$$

where

$$\mu(t) \sim \begin{cases} (\ln t)^{-1} t^{\frac{2-\gamma}{2}}, & 1 < \gamma < 2 \\ 1, & \gamma = 2 \end{cases}$$

as $t \to \infty$.

Theorem 1.1 and Theorem 1.2 thus provide examples of global solution in the remaining cases for $n = 4,\ 6$ in Table 1. The approach of deriving the desired asymptotics is to balance the bubble and the contribution from the initial data with decay like $u_0(x) \sim \langle x \rangle^{-\gamma}$. Here $\langle x \rangle = \sqrt{|x|^2 + 1}$. We combine Fila-King's formal analysis with gluing technique to complete the rigorous construction. The gluing method, developed in [3,7], turns out to be versatile and can be systematically used to detect singularity formation for a lot of evolution PDEs with precise blow-up dynamics captured. We refer to [3–8, 31] and the references therein. We finally remark that the constructions for $n \geq 7$ with $\gamma > \frac{n-2}{2}$ and for $n = 3$ with $\frac{1}{2} < \gamma \leq 1$ in Table 1 will be very similar to those carried out already. See also Remark 4.1.

The rest of this paper is organized as follows. In Section 2, we present the basic ansatz and necessary ingredients for general dimension $n \geq 3$. In Section 3, we focus on the case $n = 6$ and prove Theorem 1.1. For the four-dimensional case, substantial change will be made in the ansatz of approximate solution, and the non-local nature is due to the lack of decay in lower dimensions. We shall only point out the key parts and omit other details in the proof of Theorem 1.2, as this case is similar to those done in [35] and [34].

2. Basic set-up

The global solutions for the critical semilinear heat equation

$$u_t = \Delta u + |u|^{\frac{4}{n-2}} u \quad \text{in} \quad \mathbb{R}^n \times (t_0, \infty) \tag{2.1}$$

with $n \geq 3$ are built on its ground state. Recall that the positive solution, called Aubin-Talenti bubble, to the stationary equation

$$\Delta u + u^{\frac{n+2}{n-2}} = 0$$

is unique up to translations and dilations:

$$U(x) = \alpha_n (1 + |x|^2)^{-\frac{n-2}{2}}, \quad \alpha_n = [n(n-2)]^{\frac{n-2}{4}},$$

and it is non-degenerate in the sense that the corresponding linearized operator $\Delta + \frac{n+2}{n-2}U^{\frac{4}{n-2}}$ has only $n+1$ bounded kernel functions

$$Z_i(x) = \partial_{x_i} U(x), \quad i = 1, \cdots, n, \quad Z_{n+1}(x) = \frac{n-2}{2}U(x) + x \cdot \nabla U(x).$$

Moreover, the linearized operator $\Delta + \frac{n+2}{n-2}U^{\frac{4}{n-2}}$ has only one positive eigenvalue $\gamma_0 > 0$:

$$\Delta Z_0 + \frac{n+2}{n-2}U^{\frac{4}{n-2}} Z_0 = \gamma_0 Z_0,$$

and the corresponding eigenfunction $Z_0 \in L^\infty(\mathbb{R}^n)$ is radially symmetric and has exponential decay at spatial infinity.

The first approximate solution of (2.1) is chosen as

$$u_1(x,t) = \mu^{-\frac{n-2}{2}} U(y)\, \eta(\tilde{y}) + \Psi_0(x,t), \quad \text{where} \quad y := \frac{x - \xi}{\mu}, \quad \tilde{y} := \frac{x - \xi}{\sqrt{t}},$$

$\mu=\mu(t)>0,\ \xi=\xi(t)\in C^1[t_0,\infty)$ will be determined later, and

$$\partial_t\Psi_0=\Delta\Psi_0,\quad \Psi_0(\cdot,0)=\psi_0\sim\langle x\rangle^{-\gamma},$$

namely

$$\Psi_0(x,t)=(4\pi t)^{-\frac{n}{2}}\int_{\mathbb{R}^n}e^{-\frac{|x-z|^2}{4t}}\psi_0(z)dz.$$

For later purposes, we need the following estimate for $\Psi_0(0,t)$.

Lemma 2.1. *Assume $\gamma\in\mathbb{R}$, $t\geq 1$. Then*

$$(4\pi t)^{-\frac{n}{2}}\int_{\mathbb{R}^n}e^{-\frac{|y|^2}{4t}}\langle y\rangle^{-\gamma}dy=v_{n,\gamma}(t)(C_{n,\gamma}+\text{h.o.t.}),$$

where

$$v_{n,\gamma}(t)=\begin{cases}t^{-\frac{\gamma}{2}}, & \gamma<n\\ t^{-\frac{n}{2}}\ln(1+t), & \gamma=n\ ,\\ t^{-\frac{n}{2}}, & \gamma>n\end{cases}\qquad C_{n,\gamma}=\begin{cases}(4\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n}e^{-\frac{|z|^2}{4}}|z|^{-\gamma}dz, & \gamma<n\\ (4\pi)^{-\frac{n}{2}}\frac{1}{2}\left|S^{n-1}\right|, & \gamma=n\\ (4\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n}\langle y\rangle^{-\gamma}dy, & \gamma>n\end{cases}\tag{2.2}$$

Proof. See [21, Lemma 2.1]. □

Hereafter, we always assume $t_0\geq 1$ is sufficiently large and $t\geq t_0$. By Lemma 2.1, we have

$$\Psi_0(0,t)\sim v_{n,\gamma}(t)\left(C_{n,\gamma}+g_{n,\gamma}(t)\right).\tag{2.3}$$

For $\gamma\geq 0$, by similar calculations as in [35, Lemma A.3], we have

$$\|\nabla\Psi_0(\cdot,t)\|_{L^\infty(\mathbb{R}^n)}\lesssim t^{-\frac{1}{2}}v_{n,\gamma}(t).\tag{2.4}$$

Here the notation $a\lesssim b$ means that there exists a constant $C>0$ independent of t_0 such that $a\leq Cb$.

By [35, Lemma A.3], we get

$$\Psi_0(x,t)\lesssim t^{-\frac{\tilde\gamma}{2}}\mathbf{1}_{|x|\leq t^{\frac{1}{2}}}+|x|^{-\tilde\gamma}\mathbf{1}_{|x|>t^{\frac{1}{2}}},$$

where $\tilde\gamma$ is defined as

$$\tilde\gamma:=\min\{\gamma,3-\}.$$

Here, for any $c\in\mathbb{R}$, the notation $c-$ means a constant less than c and can be chosen arbitrarily close to c.

Define the error of f as

$$E[f]:=-\partial_t f+\Delta f+|f|^{\frac{4}{n-2}}f.$$

Straightforward computation implies

$$E[u_1]=\mu^{-\frac{n}{2}}\dot\mu Z_{n+1}(y)\eta(\tilde y)+\mu^{-\frac{n}{2}}\dot\xi\cdot(\nabla U)(y)\,\eta(\tilde y)+\mathcal{E}_\eta+|u_1|^{\frac{4}{n-2}}u_1-\mu^{-\frac{n+2}{2}}U(y)^{\frac{n+2}{n-2}}\eta(\tilde y),\tag{2.5}$$

where

$$\mathcal{E}_\eta:=\mu^{-\frac{n-2}{2}}U(y)\left(2^{-1}t^{-1}\tilde y+t^{-\frac{1}{2}}\dot\xi\right)\cdot(\nabla\eta)(\tilde y)+2\mu^{-\frac{n}{2}}t^{-\frac{1}{2}}(\nabla U)(y)\cdot(\nabla\eta)(\tilde y)+\mu^{-\frac{n-2}{2}}t^{-1}U(y)\,(\Delta\eta)(\tilde y).\tag{2.6}$$

3. The case $n = 6$

We look for an exact solution u of (2.1) in the case $n = 6$ in the form

$$u = u_1 + \psi(x,t) + \mu^{-2}\phi\left(\frac{x-\xi}{\mu}, t\right)\eta_R, \quad \eta_R := \eta\left(\frac{x-\xi}{\mu R}\right), \quad R = R(t) = t^{1/1000}.$$

We assume that

$$2\mu R \le \sqrt{t}/9. \tag{3.1}$$

We compute

$$\begin{aligned} E[u] = & \left(\mu^{-3}\dot{\mu}Z_{n+1}(y) + \mu^{-3}\dot{\xi}\cdot(\nabla U)(y)\right)\eta(\tilde{y}) + \mathcal{E}_\eta + \mu^{-4}U^2(y)\left(\eta^2(\tilde{y}) - \eta(\tilde{y})\right) \\ & + \mathcal{N}[\psi,\phi,\mu,\xi] + 2\mu^{-2}U(y)\eta(\tilde{y})\left(\Psi_0 + \psi + \mu^{-2}\phi(y,t)\eta_R\right) \\ & - \partial_t\psi + \Delta\psi - \mu^{-2}\partial_t\phi(y,t)\eta_R + \mu^{-4}\Delta_y\phi(y,t)\eta_R + \Lambda_1[\phi,\mu,\xi] + \Lambda_2[\phi,\mu,\xi], \end{aligned}$$

where

$$\begin{aligned} \Lambda_1[\phi,\mu,\xi] := & \mu^{-4}R^{-2}\phi(y,t)(\Delta\eta)(\frac{y}{R}) + 2\mu^{-4}R^{-1}\nabla_y\phi(y,t)\cdot(\nabla\eta)(\frac{y}{R}) \\ & + \mu^{-2}\phi(y,t)(\nabla\eta)(\frac{y}{R})\cdot\left(\frac{\dot{\xi}}{\mu R} + \frac{y}{R}\frac{\partial_t(\mu R)}{\mu R}\right), \end{aligned} \tag{3.2}$$

$$\Lambda_2[\phi,\mu,\xi] := \mu^{-3}\dot{\mu}\left(2\phi(y,t) + y\cdot\nabla_y\phi(y,t)\right)\eta_R + \mu^{-3}\dot{\xi}\cdot\nabla_y\phi(y,t)\eta_R, \tag{3.3}$$

and

$$\mathcal{N}[\psi,\phi,\mu,\xi] := |u|u - \mu^{-4}U^2(y)\eta^2(\tilde{y}) - 2\mu^{-2}U(y)\eta(\tilde{y})\left(\Psi_0 + \psi + \mu^{-2}\phi(y,t)\eta_R\right). \tag{3.4}$$

The assumption (3.1) gives $\eta(\tilde{y}) = \eta(\frac{x-\xi}{\sqrt{t}}) = 1$ if $|x-\xi| \le 2\mu R$, i.e., $\eta_R\eta(\tilde{y}) = \eta_R$, In order for $E[u] = 0$, it suffices to solve the coupled system for ϕ and ψ, as well as the parameter functions μ and ξ:

$$\partial_t\psi = \Delta\psi + \mathcal{G}[\psi,\phi,\mu,\xi] \quad \text{in} \quad \mathbb{R}^6\times(t_0,\infty), \quad \psi(\cdot,t_0) = 0 \quad \text{in} \quad \mathbb{R}^6, \tag{3.5}$$

and

$$\mu^2\partial_t\phi = \Delta_y\phi + 2U(y)\phi + \mathcal{H}[\psi,\mu,\xi] \qquad \text{for} \quad t > t_0, \quad y\in B_{4R(t)}, \tag{3.6}$$

and the former is called the outer problem and the latter is called the inner problem. Here

$$\begin{aligned} \mathcal{G}[\psi,\phi,\mu,\xi] := & \Lambda_1[\phi,\mu,\xi] + \Lambda_2[\phi,\mu,\xi] + \left(\mu^{-3}\dot{\mu}Z_7(y) + \mu^{-3}\dot{\xi}\cdot(\nabla U)(y)\right)\eta(\tilde{y})(1-\eta_R) + \mathcal{E}_\eta \\ & + \mu^{-4}U^2(y)\left(\eta^2(\tilde{y}) - \eta(\tilde{y})\right) + \mathcal{N}[\psi,\phi,\mu,\xi] + 2\mu^{-2}U(y)\eta(\tilde{y})(\Psi_0 + \psi)(1-\eta_R), \end{aligned} \tag{3.7}$$

and

$$\mathcal{H}[\psi,\mu,\xi] := \mu\dot{\mu}Z_{n+1}(y) + \mu\dot{\xi}\cdot(\nabla U)(y) + 2\mu^2U(y)\left(\Psi_0(\mu y+\xi,t) + \psi(\mu y+\xi,t)\right). \tag{3.8}$$

Problem (3.6) can be expressed in a new time variable

$$\tau = \tau(t) := \int_{t_0}^t \mu^{-2}(s)ds + C_\tau t_0\mu^{-2}(t_0), \quad \tau_0 := \tau(t_0), \tag{3.9}$$

with a sufficiently large constant C_τ independent of t_0, such that

$$\partial_\tau\phi = \Delta_y\phi(y,t(\tau)) + 2U(y)\phi(y,t(\tau)) + \mathcal{H}[\psi,\mu,\xi](y,t(\tau)) \qquad \text{for} \quad \tau > \tau_0, \quad y\in B_{4R(t(\tau))}. \tag{3.10}$$

The parameter functions μ and ξ are certainly not arbitrary, and their asymptotics are determined by orthogonality conditions that ensure a good solution to (3.6) with fast enough decay. In fact, if one considers

$$\begin{cases} \partial_\tau\phi = \Delta\phi + 2U(y)\phi + h & \text{for} \quad \tau > \tau_0, \quad y\in B_{4R(t(\tau))}, \\ \phi(y,\tau_0) = e_0Z_0(y) & \text{for} \quad y\in B_{4R(t(\tau_0))}, \end{cases} \tag{3.11}$$

and uses the norms

$$\|h\|_* := \sup_{\tau>\tau_0,\ y\in B_{2R(t(\tau))}} \tau^{\nu}\langle y\rangle^{2+a}|h(y,\tau)|, \quad \nu>0,\ 0<a<1,$$

$$\|\phi\|_{\text{in}} := \sup_{\tau>\tau_0, y\in B_{2R(t(\tau))}} \tau^{\nu}R^{a-7}(t(\tau))\langle y\rangle^{7}\big(\langle y\rangle|\nabla\phi(y,\tau)|+|\phi(y,\tau)|\big), \tag{3.12}$$

then the following estimates hold.

Proposition 3.1. *For (3.11), assume that* $\|h\|_*<\infty$ *and*

$$\int_{B_{4R(t(\tau))}} h(y,\tau)Z_j(y)dy=0, \quad \forall \tau\in(\tau_0,\infty), \quad j=1,2,\cdots,7. \tag{3.13}$$

Then for τ_0 *suffciently large, there exists a solution* $(\phi,e_0)=(\mathcal{T}_{\text{in}}[h],\mathcal{T}_{e_0}[h])$ *linear in* h*, that satisfies the estimates*

$$\|\phi\|_{\text{in}}\lesssim\|h\|_*, \quad |e_0|\lesssim \tau_0^{-\nu}R^{2-a}(t(\tau_0))\|h\|_*.$$

Proof. See [3, Proposition 7.1]. □

Then the desired long-time asymptotics is given by (3.13). The leading term of μ, denoted by μ_0, will be determined by the orthogonality condition

$$\int_{B_{4R}} \left(\mu_0\dot\mu_0 Z_7(y)+2\mu_0^2U(y)\Psi_0(0,t)\right)Z_7(y)dy=0, \tag{3.14}$$

yielding

$$\dot\mu_0=C(R)\mu_0\Psi_0(0,t), \tag{3.15}$$

where

$$C(R):=-\frac{2\int_{B_{4R}}U(y)Z_7(y)dy}{\int_{B_{4R}}Z_7^2(y)dy}=\frac{2\int_{\mathbb{R}^6}U^2(y)dy}{\int_{\mathbb{R}^6}Z_7^2(y)dy}\left(1+O\left(R^{-2}\right)\right) \tag{3.16}$$

for $t\ge M$ sufficiently large. We take a solution of (3.15) as

$$\mu_0(t)=e^{\int_M^t C(R(s))\Psi_0(0,s)ds}.$$

By (2.3) and the fact $\gamma>2$, for $t_0\ge 9M$ sufficiently large, we have

$$\mu_0(t)\sim 1, \quad \dot\mu_0(t)\sim t^{-\kappa},\ 1<\kappa\le 3, \tag{3.17}$$

and clearly the range of κ depends on γ. We now make the following ansatz

$$\mu=\mu_0+\mu_1, \quad \text{where} \quad \mu_1=\mu_1(t)\in C^1[t_0,\infty), \quad |\mu_1|\le\mu_0/9, \quad |\dot\mu_1|\le\frac{t^{-\nu}}{9}, \tag{3.18}$$

which yields $\frac{8}{9}\mu_0\le\mu\le\frac{10}{9}\mu_0$ and realizes (3.1) for t_0 sufficiently large. Here we assume that $1<\nu<2$. The relation (3.9) then gives

$$\tau(t)\sim t. \tag{3.19}$$

By (3.18), $\mathcal{H}[\psi,\mu,\xi]$ is controlled by

$$|\mu\dot\mu Z_7(y)|+\left|2\mu^2U(y)\Psi_0(0,t)\right|\lesssim t^{-\nu}\langle y\rangle^{-4}. \tag{3.20}$$

By Proposition 3.1, we will solve (3.10) in the space

$$B_{\text{in}}:=\left\{\phi(x,\tau)\mid\phi(\cdot,\tau)\in C^1\left(B_{2R(t(\tau))}\right) \text{ for } \tau>\tau_0, \quad \|\phi\|_{\text{in}}\le 1\right\}.$$

For $\phi\in B_{\text{in}}$, we solve the outer problem (3.5) first. In what follows, ϵ will be denoted as a positive small constant, whose value might change from line to line.

Proposition 3.2. *Assume that* $\phi \in B_{\text{in}}$, $\mu_1,\ \xi \in C^1[t_0,\infty)$ *satisfying*

$$|\mu_1| \le t^{-\epsilon}, \quad |\dot{\mu}_1| \le t^{-\nu-\epsilon}, \quad |\xi| \le t^{-\epsilon}, \quad |\dot{\xi}| \le t^{-\nu-\epsilon} \tag{3.21}$$

for small $\epsilon > 0$. *Then for* t_0 *sufficiently large, there exists a unique solution* $\psi = \psi[\phi,\mu_1,\xi]$ *for the outer problem* (3.5), *which satisfies the following estimates:*

$$|\psi| \lesssim t^{-\nu} R^{-a_1} \left(\mathbf{1}_{|x|\le\sqrt{t}} + t|x|^{-2}\mathbf{1}_{|x|>\sqrt{t}} \right), \tag{3.22}$$

$$\|\nabla\psi(\cdot,t)\|_{L^\infty(\mathbb{R}^6)} \lesssim t^{-\nu} R^{-1-a_1} \tag{3.23}$$

for some $0 < a_1 < a$.

Proof. We only need to find a fixed point for

$$\psi = \mathcal{T}_6^{\text{out}}\left[\mathcal{G}\left[\psi,\phi,\mu,\xi\right]\right],$$

where $\mathcal{G}\left[\psi,\phi,\mu,\xi\right]$ is given in (3.7), and

$$\mathcal{T}_6^{\text{out}}\left[f\right] := \int_{t_0}^{t}\int_{\mathbb{R}^6} \left[4\pi(t-s)\right]^{-3} e^{-\frac{|x-z|^2}{4(t-s)}} f(z,s)dzds.$$

We assume in the sequel $\int_{t_2}^{t_1}\cdots ds = 0$ if $t_1 \le t_2$. Clearly, the assumptions (3.18) and (3.1) are ensured by (3.21). Combining these with (3.17), there exists a constant $C_\mu > 9$ sufficiently large such that

$$9C_\mu^{-1} < \mu < C_\mu/9.$$

Recall $\Lambda_1\left[\phi,\mu,\xi\right]$ in (3.2). Using (3.1) and (3.21), we have

$$\left|\frac{\dot{\xi}}{\mu R}\right| + \left|\frac{\partial_t(\mu R)}{\mu R}\right| = \left|\frac{\dot{\xi}}{\mu R}\right| + \left|\frac{\dot{\mu}}{\mu} + \frac{\dot{R}}{R}\right| \lesssim t^{-\nu}.$$

Since $\phi \in B_{\text{in}}$ and $t \sim \tau$, one has

$$\langle y\rangle|\nabla_y\phi| + |\phi| \lesssim t^{-\nu}R^{7-a}\langle y\rangle^{-7},$$

and thus

$$|\Lambda_1\left[\phi,\mu,\xi\right]| \lesssim t^{-\nu}R^{-2-a}\mathbf{1}_{R\le|y|\le 2R} \le t^{-\nu}R^{-2-a}\mathbf{1}_{|x|\le C_\mu R}.$$

Then

$$\begin{aligned} &\mathcal{T}_6^{\text{out}}\left[t^{-\nu}R^{-2-a}\mathbf{1}_{|x|\le C_\mu R}\right] \\ \lesssim\ & t^{-3}e^{-\frac{|x|^2}{16t}}\int_{t_0}^{\frac{t}{2}} s^{-\nu}R^{4-a}(s)ds \\ &+ t^{-\nu}R^{-2-a}\left[R^2\mathbf{1}_{|x|\le R} + |x|^{-4}e^{-\frac{|x|^2}{16t}}R^6\mathbf{1}_{|x|>R}\right] \\ \lesssim\ & t^{-\epsilon}w_o(x,t), \end{aligned}$$

where

$$w_o(x,t) := t^{-\nu}R^{-a_1}\left(\mathbf{1}_{|x|\le\sqrt{t}} + t|x|^{-2}\mathbf{1}_{|x|>\sqrt{t}}\right),$$

and we have used [35, Lemma A.1, Lemma A.2]. Then

$$\left|\mathcal{T}_6^{\text{out}}\left[\Lambda_1\left[\phi,\mu,\xi\right]\right]\right| \le C_o w_o(x,t)/2$$

for a sufficiently large constant $C_o \ge 2$. For this reason, we define the norm

$$\|f\|_{\text{out}} := \sup_{t\ge t_0,\ x\in\mathbb{R}^6} \left(w_o(x,t)\right)^{-1}|f(x,t)|,$$

and we solve the outer problem (3.5) in

$$B_{\text{out}} := \{f \mid \|f\|_{\text{out}} \le C_o\}.$$

For $\Lambda_2[\phi,\mu,\xi]$ in (3.3), we have

$$|\Lambda_2[\phi,\mu,\xi]| \lesssim t^{-2\nu-\epsilon}R^{7-a}\langle y\rangle^{-7}\mathbf{1}_{|x|\le C_\mu R} \lesssim t^{-\nu}R^{-2-a}\mathbf{1}_{|x|\le C_\mu R}.$$

By (3.1) and (3.21), one has

$$\begin{aligned}
&\left|\left(\mu^{-3}\dot{\mu}Z_7(y)+\mu^{-3}\dot{\xi}\cdot(\nabla U)(y)\right)\eta(\tilde{y})(1-\eta_R)\right| + |\mathcal{E}_\eta| + \left|\mu^{-4}U^2(y)\left(\eta^2(\tilde{y})-\eta(\tilde{y})\right)\right| \\
&+\left|2\mu^{-2}U(y)\eta(\tilde{y})\Psi_0(1-\eta_R)\right| \\
&\lesssim t^{-\nu}\langle y\rangle^{-4}\mathbf{1}_{\mu R\le|x-\xi|\le 2\sqrt{t}} + t^{-1}\langle y\rangle^{-4}\mathbf{1}_{\sqrt{t}\le|x-\xi|\le 2\sqrt{t}} \\
&\lesssim t^{-\nu}|x|^{-4}\mathbf{1}_{C_\mu^{-1}R\le|x|\le 4\sqrt{t}} + t^{-3}\mathbf{1}_{\sqrt{t}/2\le|x|\le 4\sqrt{t}},
\end{aligned}$$

and if $\nu<2$, we obtain

$$\begin{aligned}
&\mathcal{T}_6^{\text{out}}\left[t^{-\nu}|x|^{-4}\mathbf{1}_{C_\mu^{-1}R\le|x|\le 4\sqrt{t}}\right] \\
&\lesssim t^{-3}e^{-\frac{|x|^2}{16t}}\int_{t_0}^{\frac{t}{2}} s^{1-\nu}ds + t^{-\nu}\left(R^{-2}\mathbf{1}_{|x|\le R} + |x|^{-2}\mathbf{1}_{R<|x|\le\sqrt{t}} + t|x|^{-4}e^{-\frac{|x|^2}{16t}}\mathbf{1}_{|x|>\sqrt{t}}\right) \\
&\lesssim t^{-\epsilon}\omega_o(x,t), \\
&\mathcal{T}_6^{\text{out}}\left[t^{-3}\mathbf{1}_{\sqrt{t}/2\le|x|\le 4\sqrt{t}}\right] \lesssim t^{-2}\left(\mathbf{1}_{|x|\le\sqrt{t}} + t|x|^{-2}\mathbf{1}_{|x|>\sqrt{t}}\right) \lesssim t^{-\epsilon}\omega_o(x,t)
\end{aligned}$$

for some $\epsilon>0$.

For any $\psi_1,\psi_2\in B_{\text{out}}$, we have

$$\begin{aligned}
&\left|\mu^{-2}U(y)\eta(\tilde{y})(\psi_1-\psi_2)(1-\eta_R)\right| \lesssim \langle y\rangle^{-4}\|\psi_1-\psi_2\|_{\text{out}}w_o(x,t)\mathbf{1}_{\mu R\le|x-\xi|\le 2\sqrt{t}} \\
&\lesssim t^{-\nu}R^{-a_1}|x|^{-4}\mathbf{1}_{C_\mu^{-1}R\le|x|\le 4\sqrt{t}}\|\psi_1-\psi_2\|_{\text{out}}.
\end{aligned} \tag{3.24}$$

Then

$$\begin{aligned}
&\mathcal{T}_6^{\text{out}}\left[t^{-\nu}R^{-a_1}|x|^{-4}\mathbf{1}_{C_\mu^{-1}R\le|x|\le 4\sqrt{t}}\right] \\
&\lesssim t^{-3}e^{-\frac{|x|^2}{16t}}\int_{t_0}^{\frac{t}{2}} s^{1-\nu}R^{-a_1}(s)ds + t^{-\nu}R^{-a_1}\left(R^{-2}\mathbf{1}_{|x|\le\sqrt{t}} + t|x|^{-4}e^{-\frac{|x|^2}{16t}}\mathbf{1}_{|x|>\sqrt{t}}\right) \\
&\lesssim R^{-2}w_o(x,t).
\end{aligned} \tag{3.25}$$

The nonlinear terms $\mathcal{N}[\psi,\phi,\mu,\xi]$, in (3.4) with $\psi\in B_{\text{out}}$, can be estimated as

$$\begin{aligned}
|\mathcal{N}[\psi,\phi,\mu,\xi]| &\lesssim \left|\Psi_0+\psi+\mu^{-2}\phi(y,t)\eta_R\right|^2 \\
&\lesssim t^{-2\kappa}+t^{-2\nu}R^{-2a_1}\left(\mathbf{1}_{|x|\le\sqrt{t}}+t|x|^{-2}\mathbf{1}_{|x|>\sqrt{t}}\right) \\
&\quad + \tau^{-2\nu}R^{14-2a}\langle y\rangle^{-14}\mathbf{1}_{|y|\le 2R},
\end{aligned}$$

and as before, one can easily check that

$$\mathcal{T}_6^{\text{out}}[|\mathcal{N}[\psi,\phi,\mu,\xi]|] \lesssim t^{-\epsilon}w_o(x,t).$$

Putting all these estimates together, we conclude that for t_0 sufficiently large, $\mathcal{T}_6^{\text{out}}[\mathcal{G}[\psi,\phi,\mu,\xi]]\in B_{\text{out}}$, and it is a contraction mapping in B_{out} by similar estimates as (3.24) and (3.25). There then exists a unique solution $\psi\in B_{\text{out}}$. The gradient estimate (3.23) follows from a scaling argument and parabolic regularity estimate. ☐

In view of Proposition 3.1 for the inner problem, our next step is to find suitable μ_1, ξ such that the orthogonality conditions

$$\int_{B_{4R}}\mathcal{H}[\psi,\mu,\xi](y,t)Z_i(y)dy=0, \quad \mu=\mu_0+\mu_1, \quad i=1,\dots,7 \tag{3.26}$$

are satisfied, where $\psi = \psi[\phi, \mu_1, \xi]$ is solved in Proposition 3.2 (with μ_1, ξ fixed within proper spaces), and $\mathcal{H}[\psi, \mu, \xi]$ is defined in (3.8).

Proposition 3.3. *For t_0 sufficiently large, there exists a solution $(\mu_1, \xi) = (\mu_1[\phi], \xi[\phi])$ for (3.26) satisfying*

$$|\mu_1| \lesssim t^{-\epsilon}, \quad |\dot{\mu}_1| \lesssim t^{-\nu-\epsilon}, \quad |\xi| \lesssim t^{-\epsilon}, \quad |\dot{\xi}| \lesssim t^{-\nu-\epsilon}. \tag{3.27}$$

Proof. From (3.8), we can write (3.26) as

$$\dot{\mu} = -2\mu \left(\int_{B_{4R}} Z_7^2(y)dy \right)^{-1} \int_{B_{4R}} \left(\Psi_0(\mu y + \xi, t) + \psi(\mu y + \xi, t)\right) U(y) Z_7(y) dy, \tag{3.28}$$

$$\dot{\xi} = \vec{\mathcal{S}}[\mu_1, \xi] := \left(\mathcal{S}_1[\mu_1, \xi], \ldots, \mathcal{S}_6[\mu_1, \xi]\right) \tag{3.29}$$

$$\mathcal{S}_i[\mu_1, \xi] := -2\mu \left(\int_{B_{4R}} Z_i^2(y)dy \right)^{-1} \int_{B_{4R}} \Big[\Psi_0(\mu y + \xi, t) - \Psi_0(0, t) + \psi(\mu y + \xi, t) - \psi(0, t)\Big] U(y) Z_i(y) dy,$$

for $i = 1, 2, \ldots, 6$. By $\mu = \mu_0 + \mu_1$ with μ_0 chosen in (3.15), we rewrite (3.28) as

$$\dot{\mu}_1 + \beta(t)\mu_1 = \mathcal{F}[\mu_1, \xi](t), \tag{3.30}$$

where

$$\beta(t) := 2 \left(\int_{B_{4R}} Z_7^2(y)dy \right)^{-1} \Psi_0(0, t) \int_{B_{4R}} U(y) Z_7(y) dy,$$

and

$$\mathcal{F}[\mu_1, \xi](t) := -2 \left(\int_{B_{4R}} Z_7^2(y)dy \right)^{-1} \Bigg[\mu \int_{B_{4R}} \psi(\mu y + \xi, t) U(y) Z_7(y) dy + \mu \int_{B_{4R}} \left(\Psi_0(\mu y + \xi, t) - \Psi_0(0, t)\right) U(y) Z_7(y) dy\Bigg].$$

Solving the system (3.29)-(3.30) is reduced to finding fixed point $(\dot{\mu}_1, \dot{\xi})$ of

$$\begin{aligned} \dot{\mu}_1 &= \mathcal{S}_7[\mu_1, \xi] := \frac{d}{dt}\left(\int_\infty^t \mathcal{F}[\mu_1, \xi](s) e^{\int_t^s \beta(a)da} ds \right) = -\beta(t) \int_\infty^t \mathcal{F}[\mu_1, \xi](s) e^{\int_t^s \beta(a)da} ds + \mathcal{F}[\mu_1, \xi](t), \\ \mu_1 &= \mu_1[\dot{\mu}_1](t) := \int_\infty^t \dot{\mu}_1(a) da, \quad \dot{\xi} = \vec{\mathcal{S}}[\mu_1, \xi], \quad \xi = \xi[\dot{\xi}](t) := \int_\infty^t \dot{\xi}(a) da, \end{aligned} \tag{3.31}$$

and we will work in the space

$$B_{\dot{\mu}_1} := \left\{ f \in C[t_0, \infty) \mid \|f\|_{\dot{\mu}_1} \le 1 \right\}, \quad B_{\dot{\xi}} = \left\{ \vec{f} = (f_1, \ldots, f_5) \in C[t_0, \infty) \mid \|\vec{f}\|_{\dot{\xi}} \le 1 \right\}$$

with the norm

$$\|f\|_{\dot{\mu}_1} := \sup_{t \ge t_0} t^{\nu+\epsilon} |f(t)|, \quad \|\vec{f}\|_{\dot{\xi}} := \sup_{t \ge t_0} t^{\nu+\epsilon} |\vec{f}(t)|. \tag{3.32}$$

By (2.3) and (3.16), we have

$$\beta(t) \sim -t^{-\kappa}, \ 1 < \kappa \le 3. \tag{3.33}$$

For any $(\dot{\mu}_1, \dot{\xi}) \in B_{\dot{\mu}_1} \times B_{\dot{\xi}}$, one has

$$|\mu_1| \lesssim t^{1-\nu-\epsilon}, \quad |\xi| \lesssim t^{1-\nu-\epsilon}. \tag{3.34}$$

Therefore, μ_1, $\dot{\mu}_1$, ξ, $\dot{\xi}$ satisfy the assumption (3.21) in Proposition 3.2. By (2.4) and (3.23), we obtain

$$\left|\vec{\mathcal{S}}[\mu_1, \xi]\right| \lesssim \mu \left(\|\nabla_x \Psi_0(\cdot, t)\|_{L^\infty(\mathbb{R}^6)} + \|\nabla_x \psi(\cdot, t)\|_{L^\infty(\mathbb{R}^6)} \right) (|\mu| + |\xi|) \lesssim t^{-\frac{1}{2}-\kappa} + t^{-\nu} R^{-1-a_1}. \tag{3.35}$$

Using (3.22), (2.4), (2.3) and (3.34), we get

$$\left| \mu \int_{B_{4R}} \psi(\mu y + \xi, t) U(y) Z_7(y) dy \right| \lesssim t^{-\nu} R^{-a_1},$$

$$\left| \mu \int_{B_{4R}} \left(\Psi_0(\mu y + \xi, t) - \Psi_0(0, t)\right) U(y) Z_7(y) dy \right| \lesssim t^{-\frac{1}{2}-\kappa},$$

and thus

$$|\mathcal{F}[\mu_1,\xi](t)| \lesssim t^{-\nu}R^{-a_1}. \tag{3.36}$$

Therefore, we have

$$\left|\beta(t)\int_\infty^t \mathcal{F}[\mu_1,\xi](s)e^{\int_t^s \beta(a)da}ds\right| \lesssim t^{1-\nu-\kappa}R^{-a_1}.$$

Combining (3.35) and (3.36), one has

$$|\mathcal{S}_7[\mu_1,\xi]| \lesssim t^{1-\nu-\kappa}R^{-a_1}, \quad \left|\vec{\mathcal{S}}[\mu_1,\xi]\right| \lesssim t^{-\nu}R^{-1-a_1}, \tag{3.37}$$

yielding $\left(\mathcal{S}_7,\vec{\mathcal{S}}\right)[\mu_1,\xi] \in B_{\dot\mu_1}\times B_{\dot\xi}$.

For any sequence $(\dot\mu_1^{[j]},\dot\xi^{[j]})_{j\ge 1} \subset B_{\dot\mu_1}\times B_{\dot\xi}$, denote $\mu_1^{[j]} = \int_\infty^t \dot\mu_1^{[j]}(a)da$, $\xi^{[j]} = \int_\infty^t \dot\xi^{[j]}(a)da$. We set $\tilde{\dot\mu}_1^{[j]} := \mathcal{S}_7[\mu_1^{[j]},\xi^{[j]}]$, $\tilde{\dot\xi}^{[j]} := \vec{\mathcal{S}}[\mu_1^{[j]},\xi^{[j]}]$. Similar to (3.37), we have

$$|\tilde{\dot\mu}_1^{[j]}| \le C_1 t^{1-\nu-\kappa}R^{-a_1}, \quad |\tilde{\dot\xi}^{[j]}| \le C_1 t^{-\nu}R^{-1-a_1} \quad \text{for all } \ j\ge 1 \tag{3.38}$$

for a constant $C_1>0$ independent of j.

For any compact subset $K \subset\subset [t_0,\infty)$, by the equation (3.31) and the Hölder regularity for the outer solution ψ, for all $j\ge1$, $\tilde{\dot\mu}_1^{[j]}$ and $\tilde{\dot\xi}^{[j]}$ are uniformly Hölder continuous in K. Then up to a subsequence,

$$\tilde{\dot\mu}_1^{[j]} \to g, \quad \tilde{\dot\xi}^{[j]} \to \vec{g} \ \text{ in } \ L^\infty(K) \ \text{ as } \ j\to\infty$$

for some $g,\ \vec g \in C[t_0,\infty)$. By (3.38), we have

$$|g| \le C_1 t^{1-\nu-\kappa}R^{-a_1}, \quad |\vec g| \le C_1 t^{-\nu}R^{-1-a_1}.$$

Thus, for any $\epsilon_1>0$, there exists t_1 sufficiently large such that for all $j\ge1$,

$$\sup_{t\ge t_1} t^{\nu+\kappa-1}R^{a_1-\epsilon}(t)\left|\left(\tilde{\dot\mu}_1^{[j]}-g\right)(t)\right| + \sup_{t\ge t_1} t^{\nu}R^{1+a_1-\epsilon}(t)\left|\left(\tilde{\dot\xi}^{[j]}-\vec g\right)(t)\right| < \epsilon_1,$$

and moreover, one has

$$\lim_{j\to\infty}\left[\sup_{t_0\le t\le t_1} t^{\nu+\kappa-1}R^{a_1-\epsilon}(t)\left|\left(\tilde{\dot\mu}_1^{[j]}-g\right)(t)\right| + \sup_{t_0\le t\le t_1} t^{\nu}R^{1+a_1-\epsilon}(t)\left|\left(\tilde{\dot\xi}^{[j]}-\vec g\right)(t)\right|\right] = 0,$$

i.e., $\lim\limits_{j\to\infty}\left(\|\tilde{\dot\mu}_1^{[j]}-g\|_{\dot\mu_1} + \|\tilde{\dot\xi}^{[j]}-\vec g\|_{\dot\xi}\right)=0$, which implies that $\left(\mathcal{S}_7,\vec{\mathcal{S}}\right)[\mu_1,\xi]$ is a compact mapping on $B_{\dot\mu_1}\times B_{\dot\xi}$. Then Schauder fixed-point theorem ensures the existence of a solution $(\dot\mu_1,\dot\xi)\in B_{\dot\mu_1}\times B_{\dot\xi}$ for the system (3.31).

□

The last step is to solve the inner problem (3.10). By (3.27), (2.4), and (3.22), we have

$$\left|\mu\dot\xi\cdot(\nabla U)(y) + 2\mu^2U(y)\Big(\Psi_0(\mu y+\xi,t)-\Psi_0(0,t)+\psi(\mu y+\xi,t)\Big)\right| \lesssim \tau^{-\nu}R^{-a_1}(t(\tau))\langle y\rangle^{-4}.$$

We write $\tilde H[\phi] := \mathcal{H}\left[\psi\left[\phi,\mu_1[\phi],\xi[\phi]\right],\mu_0+\mu_1[\phi],\xi[\phi]\right]$ for simplicity. From (3.20), we have

$$|\tilde H[\phi]| \lesssim \tau^{-\nu}\langle y\rangle^{-4}.$$

We now apply Proposition 3.1 to the inner problem (3.10), and it suffices to solve the fixed-point problem

$$\phi = \mathcal{T}_{\rm in}\big[\tilde H[\phi]\big].$$

Indeed, for any $\phi\in B_{\rm in}$, given t_0 sufficiently large (so does τ_0), by Proposition 3.1, we have

$$\langle y\rangle\left|\nabla_y\mathcal{T}_{\rm in}\big[\tilde H[\phi]\big]\right| + \left|\mathcal{T}_{\rm in}\big[\tilde H[\phi]\big]\right| \lesssim \tau^{-\nu}R^{7-a}\langle y\rangle^{-7}, \quad \left|\mathcal{T}_{e_0}\big[\tilde H[\phi]\big]\right| \lesssim \tau_0^{-\nu}R^{2-a}(t(\tau_0)),$$

which implies $\mathcal{T}_{\rm in}\big[\tilde H[\phi]\big]\in B_{\rm in}$ in particular.

For any sequence $(\phi_j)_{j\geq 1} \in B_{\text{in}}$, denote $\tilde{\phi}_j := \mathcal{T}_{\text{in}}\left[\tilde{H}[\phi_j]\right]$, $\tilde{e}_j := \mathcal{T}_{e_0}\left[\tilde{H}[\phi_j]\right]$, which satisfies

$$\begin{cases} \partial_\tau \tilde{\phi}_j = \Delta_y \tilde{\phi}_j + 2U(y)\tilde{\phi}_j + \tilde{H}[\phi_j] & \text{in } \mathcal{D}_{4R} := \left\{(y,\tau) \mid \tau \in (\tau_0,\infty), \quad y \in B_{4R(t(\tau))}\right\} \\ \tilde{\phi}_j(\cdot,\tau_0) = \tilde{e}_j Z_0 & \text{in } B_{4R(t(\tau_0))}. \end{cases}$$

One sees similarly that there exists a constant C_1 independent of j such that

$$|\tilde{H}[\phi_j]| \leq C_1 \tau^{-\nu} \langle y \rangle^{-4}, \quad \langle y \rangle \left|\nabla_y \tilde{\phi}_j\right| + \left|\tilde{\phi}_j\right| \leq C_1 \tau^{-\nu} R^{7-a} \langle y \rangle^{-7}, \quad |\tilde{e}_j| \leq C_1 \tau_0^{-\nu} R^{2-a}(t(\tau_0)). \tag{3.39}$$

For any compact set $K \subset\subset \mathcal{D}_{3R} \cup (B_{3R(t_0)} \times \{\tau_0\})$, parabolic regularity theory gives that $\|\phi_j\|_{C^{1+\ell,\frac{1+\ell}{2}}(K)} \leq C_2$ with a constant C_2 independent of j and a constant $\ell \in (0,1)$. By Arzelà-Ascoli theorem, up to a subsequence, there exists a function $g \in C^1_x$ such that

$$\tilde{\phi}_j \to g, \quad \nabla_y \tilde{\phi}_j \to \nabla_y g \ \text{ in } \ L^\infty(K) \ \text{ as } \ j \to \infty.$$

By (3.39), we have

$$\langle y \rangle |\nabla_y g| + |g| \leq C_1 \tau^{-\nu} R^{7-a} \langle y \rangle^{-7} \ \text{ in } \ \mathcal{D}_{3R}.$$

The compactness of the mapping $\mathcal{T}_{\text{in}}\left[\tilde{H}[\phi]\right]$ is a consequence of modifying the constant measuring the weighted space for inner problem. By the Schauder fixed-point theorem, there exists a solution $\phi \in B_{\text{in}}$.

The desired asymptotics of global solution to (2.1) are thus captured, and based on the space that the solution lies in, it is a positive solution by maximum principle. The proof of Theorem 1.1 is complete.

4. Remarks on the cases $n = 3,\ 4$

The constructions for the lower dimensional cases $n = 3,\ 4$ consist of almost the same steps as in the previous section. But substantial modifications have to be made in the ansatz for the approximate solution. Recall that in the error of the first approximation u_1, (2.5), the spatial decay is not fast enough in lower dimensions, namely

$$Z_{n+1}(y) \notin L^2(\mathbb{R}^n) \ \text{ for } \ n = 3,\ 4.$$

Also, correction is also needed for $\mathcal{E}_\eta$ in (2.6) in the self-similar region. This suggests that modifications must be added to improve the decay of the error.

We add two global corrections ϕ_{nl} and ϕ_{ss} that solve respectively

$$\partial_t \phi_{\text{nl}} - \Delta \phi_{\text{nl}} = \mu^{-\frac{n}{2}} \dot{\mu} Z_{n+1}(y) \eta\left(\tilde{y}\right),$$
$$\partial_t \phi_{\text{ss}} - \Delta \phi_{\text{ss}} = \mathcal{E}_\eta.$$

For instance, assume that $n = 4$, the term ϕ_{nl}, expressed by the convolution form

$$\phi_{\text{nl}}(x,t) = \int_{t_0}^{t} \int_{\mathbb{R}^4} (4\pi(t-s))^{-2} e^{-\frac{|x-z|^2}{4(t-s)}} \mu^{-2}(s)\dot{\mu}(s) Z_5\left(\frac{|z|}{\mu(s)}\right) \eta\left(\frac{|z|}{\sqrt{s}}\right) dz ds,$$

has leading term in non-local form in $\dot{\mu}$

$$\phi_{\text{nl}}(x,t) \sim -2^{-\frac{1}{2}} \int_{t/2}^{t-\mu_0^2} \frac{\dot{\mu}(s)}{t-s} ds$$

while ϕ_{ss} can be approximated by a neat self-similar form. Indeed, the leading term in $\mathcal{E}_\eta$ is

$$\tilde{E} = 2^{\frac{3}{2}} \mu t^{-2} \left(2^{-1}\tilde{y}^{-1}\eta'(\tilde{y}) - \tilde{y}^{-3}\eta'(\tilde{y}) + \tilde{y}^{-2}\eta''(\tilde{y})\right), \quad \tilde{y} := \frac{x}{\sqrt{t}}.$$

We take

$$\phi_{\text{ss}} = \mu \hat{\varphi}_1, \quad \tilde{E} = \mu \hat{E},$$

then $\hat{\varphi}_1$ satisfies

$$\partial_t \hat{\varphi}_1 = \Delta \hat{\varphi}_1 + \hat{E}.$$

Writing $\hat{\varphi}_1 = t^{-1}A(\tilde{y})$ implies

$$A'' + \left(\frac{3}{\tilde{y}} + \frac{\tilde{y}}{2}\right)A' + A + h(\tilde{y}) = 0,$$
$$h(\tilde{y}) = 2^{\frac{3}{2}}\tilde{y}^{-2}\left(\eta''(\tilde{y}) - \frac{1}{\tilde{y}}\eta'(\tilde{y}) + \frac{\tilde{y}}{2}\eta'(\tilde{y})\right).$$

Then

$$\phi_{\rm ss}(x,t) = 2^{\frac{3}{2}}\mu|x|^{-2}\left(e^{-\frac{|x|^2}{4t}} - \eta(\tilde{y})\right),$$

and in particular,

$$\phi_{\rm ss}(0,t) = -2^{-\frac{1}{2}}\mu t^{-1}.$$

The terms $\phi_{\rm nl}$ and $\phi_{\rm ss}$ will create new errors, making the reduced problem of the scaling parameter $\mu(t)$ non-local also. Similar computations as in [35] and [34] eventually lead to the non-local dynamics for μ:

$$\int_{t/2}^{t-\mu^2(t)} \frac{\dot{\mu}(s)}{t-s}ds + \frac{\mu(t)}{t} \sim -v_{4,\gamma}$$

with $v_{4,\gamma}$ defined in (2.2). Above non-local equation can be solved by approximation

$$(\mu \ln t)' \sim -v_{4,\gamma},$$

yielding the trichotomy dynamics

$$\mu(t) \sim \begin{cases} (\ln t)^{-1}t^{1-\frac{\gamma}{2}}, & \gamma < 2 \\ 1, & \gamma = 2 \\ (\ln t)^{-1}, & \gamma > 2 \end{cases}.$$

However, the remainder needs to be controlled using Hölder properties of solution, which is subtle and has been dealt with in [35, Section 4] and [34, Section 5]. We omit the lengthy details here.

Remark 4.1. *For the case $n = 3$, the non-local dynamics read*

$$\int_0^t \frac{\dot{\beta}(s)}{(t-s)^{1/2}}\left(1 - e^{-\frac{M^2}{t-s}}\right)ds = h_\gamma(t),$$

where $\beta(t)$ is an explicit function of $\mu(t)$, M is a large constant, and h_γ depends on γ. The resolution of the Abel-type operator is achieved by the Laplace transform and its inverse. See [8, Section 6] and [1, Section 8].

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