Local Uniqueness and Refined Spike Profiles of Ground States for Two-Dimensional Attractive Bose-Einstein Condensates

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Abstract

We consider ground states of two-dimensional Bose-Einstein condensates in a trap with attractive interactions, which can be described equivalently by positive minimizers of the L^2-critical constraint Gross-Pitaevskii energy functional. It is known that ground states exist if and only if $a < a^* := \|w\|_2^2$, where a denotes the interaction strength and w is the unique positive solution of $\Delta w - w + w^3 = 0$ in $\mathbb{R}^2$. In this paper, we prove the local uniqueness and refined spike profiles of ground states as $a \nearrow a^*$, provided that the trapping potential $h(x)$ is homogeneous and $H(y) = \int_{\mathbb{R}^2} h(x+y)w^2(x)dx$ admits a unique and non-degenerate critical point.

Keywords: Bose-Einstein condensation; spike profiles; local uniqueness; Pohozaev identity.

1 Introduction

The phenomenon of Bose-Einstein condensation (BEC) has been investigated intensively since its first realization in cold atomic gases, see [1, 5] and references therein. In these experiments, a large number of (bosonic) atoms are confined to a trap and cooled to very low temperatures. Condensation of a large fraction of particles into the same one-particle state is observed below a critical temperature. These Bose-Einstein condensates display various interesting quantum phenomena, such as the critical-mass collapse, the superfluidity and the appearance of quantized vortices in rotating traps (e.g.[5]). Specially, if the force between the atoms in the condensates is attractive, the system collapses as soon as the particle number increases beyond a critical value, see, e.g., [23] or [5, Sec. III.B].

Bose-Einstein condensates (BEC) of a dilute gas with attractive interactions in $\mathbb{R}^2$ can be described ([2, 5, 10]) by the following Gross-Pitaevskii (GP) energy functional

$$E_a(u) := \int_{\mathbb{R}^2} \Big(|\nabla u|^2 + V(x)|u|^2\Big)dx - \frac{a}{2}\int_{\mathbb{R}^2} |u|^4 dx, \tag{1.1}$$

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where $a > 0$ describes the strength of the attractive interactions, and $V(x) \geq 0$ denotes the trapping potential satisfying $\lim_{|x|\to\infty} V(x) = \infty$. As addressed recently in [10, 11], ground states of attractive BEC in $\mathbb{R}^2$ can be described by the constraint minimizers of the GP energy

$$e(a) := \inf_{\{u\in\mathcal{H},\, \|u\|_2^2=1\}} E_a(u), \tag{1.2}$$

where the space $\mathcal{H}$ is defined by

$$\mathcal{H} := \Big\{u \in H^1(\mathbb{R}^2) : \int_{\mathbb{R}^2} V(x)|u(x)|^2 dx < \infty\Big\}. \tag{1.3}$$

The minimization problem $e(a)$ was analyzed recently in [2, 10, 11, 12, 26] and references therein. Existing results show that $e(a)$ is an L^2-critical constraint variational problem. Actually, it was shown in [2, 10] that $e(a)$ admits minimizers if and only if $a < a^* := \|w\|_2^2$, where $w = w(|x|)$ is the unique (up to translations) radial positive solution (cf. [7, 19, 14]) of the following nonlinear scalar field equation

$$\Delta w - w + w^3 = 0 \ \text{ in } \ \mathbb{R}^2, \ \text{ where } \ w \in H^1(\mathbb{R}^2)\,. \tag{1.4}$$

It turns out that the existence and nonexistence of minimizers for $e(a)$ are well connected with the following Gagliardo-Nirenberg inequality

$$\int_{\mathbb{R}^2} |u(x)|^4 dx \leq \frac{2}{\|w\|_2^2} \int_{\mathbb{R}^2} |\nabla u(x)|^2 dx \int_{\mathbb{R}^2} |u(x)|^2 dx, \ \ \forall u \in H^1(\mathbb{R}^2)\,, \tag{1.5}$$

where the equality is attained at w (cf. [25]).

Since $E_a(u) \geq E_a(|u|)$ for any $u \in \mathcal{H}$, any minimizer u_a of $e(a)$ must be either non-negative or non-positive, and it satisfies the Euler-Lagrange equation

$$-\Delta u_a + V(x)u_a = \mu_a u_a + a u_a^3 \quad \text{in } \mathbb{R}^2, \tag{1.6}$$

where $\mu_a \in \mathbb{R}$ is a suitable Lagrange multiplier. Thus, by applying the maximum principle to the equation (1.6), any minimizer u_a of $e(a)$ is further either negative or positive. Therefore, without loss of generality one can restrict the minimizations of $e(a)$ to positive functions. *In this paper positive minimizers of* $e(a)$ *are called ground states of attractive BEC.* Applying energy estimates and blow-up analysis, the spike profiles of positive minimizers for $e(a)$ as $a \nearrow a^*$ were recently discussed in [10, 11, 12] under different types of potentials $V(x)$, see our Proposition 2.1 for some related results. In spite of these facts, it remains open to discuss the refined spike profiles of positive minimizers. On the other hand, the local uniqueness of positive minimizers for $e(a)$ as *a.e.* $a \nearrow a^*$ was also proved [11] by the ODE argument, for the case where $V(r) = V(|x|)$ is radially symmetric and satisfies $V'(r) \geq 0$, see Corollary 1.1 in [11] for details. Here the locality of uniqueness means that a is near a^*. It is therefore natural to ask whether such local uniqueness still holds for the case where $V(x)$ is not radially symmetric. We should remark that all these results mentioned above were obtained mainly by analyzing the variational structures of the minimization problem $e(a)$, instead of discussing the PDE properties of the associated elliptic equation (1.6).

By investigating thoroughly the associated equation (1.6), the main purpose of this paper is to derive the refined spike profiles of positive minimizers for $e(a)$ as $a \nearrow a^*$, and extend the above local uniqueness to the cases of non-symmetric potentials $V(x)$ as well. Throughout the whole paper, we shall consider the trapping potential $V(x)$ satisfying $\lim_{|x|\to\infty} V(x) = \infty$ in the class of homogeneous functions, for which we define

Definition 1.1. $h(x) \geq 0$ in $\mathbb{R}^2$ is homogeneous of degree $p \in \mathbb{R}^+$ (about the origin), if there exists some $p > 0$ such that

$$h(tx) = t^p h(x) \text{ in } \mathbb{R}^2 \text{ for any } t > 0. \tag{1.7}$$

Following [9, Remark 3.2], the above definition implies that the homogeneous function $h(x) \in C(\mathbb{R}^2)$ of degree $p > 0$ satisfies

$$0 \leq h(x) \leq C|x|^p \text{ in } \mathbb{R}^2, \tag{1.8}$$

where $C > 0$ denotes the maximum of $h(x)$ on $\partial B_1(0)$. Moreover, since we assume that $\lim_{|x|\to\infty} h(x) = \infty$, $x = 0$ is the unique minimum point of $h(x)$. Additionally, we often need to assume that $V(x) = h(x) \in C^2(\mathbb{R}^2)$ satisfies

$$y_0 \text{ is the unique critical point of } H(y) = \int_{\mathbb{R}^2} h(x+y)w^2(x)dx. \tag{1.9}$$

The following example shows that for some non-symmetric potentials $h(x)$, $H(y)$ admits a unique critical point y_0, where y_0 satisfies $y_0 \neq 0$ and is *non-degenerate* in the sense that

$$\det\Big(\frac{\partial^2 H(y_0)}{\partial x_i \partial x_j}\Big) \neq 0, \text{ where } i, j = 1, 2. \tag{1.10}$$

Example 1.1. Suppose that the potential $h(x)$ satisfies

$$h(x) = |x|^p\big[1 + \delta h_0(\theta)\big] \geq 0, \text{ where } p \geq 2 \text{ and } \delta \in \mathbb{R}, \tag{1.11}$$

where $h_0(\theta) \in C^2([0, 2\pi])$ satisfies

$$\Big(\int_0^{2\pi} h_0(\theta)\cos\theta d\theta\Big)^2 + \Big(\int_0^{2\pi} h_0(\theta)\sin\theta d\theta\Big)^2 > 0. \tag{1.12}$$

One can check from (1.12) that if $|\delta| \geq 0$ is small enough, then $H(y)$ admits a unique critical point $y_0 = -\delta\hat{y}_0 \in \mathbb{R}^2$, where $\hat{y}_0$ satisfies

$$\hat{y}_0 \sim \Big(C_1\int_0^{2\pi} h_0(\theta)\cos\theta d\theta,\, C_2\int_0^{2\pi} h_0(\theta)\sin\theta d\theta\Big) \neq (0,0) \text{ as } \delta \to 0 \tag{1.13}$$

for some positive constants C_1 and C_2 depending only on w and p. Furthermore, if $|\delta| \geq 0$ is small enough, then $\det\Big(\frac{\partial^2 H(y_0)}{\partial x_i \partial x_j}\Big) > 0$, which implies that the unique critical point y_0 of $H(y)$ is non-degenerate.

Our first main result is concerned with the following local uniqueness as $a \nearrow a^*$, which holds for some non-symmetric homogeneous potentials $h(x)$ in view of Example 1.1.

Theorem 1.1. *Suppose $V(x) = h(x) \in C^2(\mathbb{R}^2)$ is homogeneous of degree $p \geq 2$, where $\lim_{|x|\to\infty} h(x) = \infty$, and satisfies*

$$y_0 \text{ is the unique and non-degenerate critical point of } H(y) = \int_{\mathbb{R}^2} h(x+y)w^2(x)dx. \tag{1.14}$$

Then there exists a unique positive minimizer for $e(a)$ as $a \nearrow a^*$.

The local uniqueness of Theorem 1.1 means that positive minimizers of $e(a)$ must be unique as a is near a^*. It is possible to extend Theorem 1.1 to more general potentials $V(x) = g(x)h(x)$ for a class of functions $g(x)$, which is however beyond the discussion ranges of the present paper. We also remark that the proof of Theorem 1.1 is more involved for the case where $y_0 \neq 0$ occurs in (1.14). Our proof of such local uniqueness is motivated by [3, 6, 9]. Roughly speaking, as derived in Proposition 2.1 we shall first obtain some fundamental estimates on the spike behavior of positive minimizers. Under the non-degeneracy assumption of (1.14), the local uniqueness is then proved in Subsection 2.1 by establishing various types of local Pohozaev identities.

The proof of Theorem 1.1 shows that if one considers the local uniqueness of Theorem 1.1 in other dimensional cases, where $\mathbb{R}^2$ is replaced by $\mathbb{R}^d$ and u^4 is replaced by $u^{2+\frac{4}{d}}$ for $d \neq 2$, the fundamental estimates of Proposition 2.1 are not enough. Therefore, in the following we address the refined spike behavior of positive minimizers under the assumption (1.14). To introduce our second main result, for convenience we next denote

$$\lambda_0 = \Big(\frac{p}{2}\int_{\mathbb{R}^2} h(x+y_0)w^2(x)dx\Big)^{\frac{1}{2+p}}, \tag{1.15}$$

where $y_0 \in \mathbb{R}^2$ is given by (1.14), and

$$\psi(x) = \varphi(x) - \frac{C^*}{2}\big[w(x) + x\cdot\nabla w(x)\big],$$

where $\varphi(x) \in C^2(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2)$ is the unique solution of

$$\nabla\varphi(0) = 0 \ \text{ and } \ \big[-\Delta + (1-3w^2)\big]\varphi(x) = -\frac{2w^3}{\int_{\mathbb{R}^2} w^4} - \frac{2h(x+y_0)w}{p\int_{\mathbb{R}^2} h(x+y_0)w^2} \ \text{ in } \mathbb{R}^2, \tag{1.16}$$

and the nonzero constant C^* is given by

$$C^* = \frac{2}{2+p}\Big(2\int_{\mathbb{R}^2} w\psi_3 + \int_{\mathbb{R}^2}\varphi^2\Big)$$

with $\psi_3 \in C^2(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2)$ being the unique solution of (3.29). Using above notations, we shall derive the following theorem.

Theorem 1.2. *Suppose $V(x) = h(x) \in C^2(\mathbb{R}^2)$ is homogeneous of degree $p \geq 2$, where $\lim_{|x|\to\infty} h(x) = \infty$, and satisfies (1.14) for some $y_0 \in \mathbb{R}^2$. If u_a is a positive minimizer of $e(a)$ as $a \nearrow a^*$, then we have*

$$\begin{aligned} u_a(x) = \frac{\lambda_0}{\|w\|_2}\Big\{&\frac{1}{(a^*-a)^{\frac{1}{2+p}}} w\Big(\frac{\lambda_0(x-x_a)}{(a^*-a)^{\frac{1}{2+p}}}\Big) + (a^*-a)^{\frac{1+p}{2+p}}\psi\Big(\frac{\lambda_0(x-x_a)}{(a^*-a)^{\frac{1}{2+p}}}\Big) \\ &+ (a^*-a)^{\frac{3+2p}{2+p}}\phi_0\Big(\frac{\lambda_0(x-x_a)}{(a^*-a)^{\frac{1}{2+p}}}\Big)\Big\} + o\big((a^*-a)^{\frac{3+2p}{2+p}}\big) \ \text{ as } a \nearrow a^* \end{aligned} \tag{1.17}$$

uniformly in $\mathbb{R}^2$ for some function $\phi_0 \in C^2(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2)$, where x_a is the unique maximum point of u_a satisfying

$$\Big|\frac{\lambda x_a}{(a^*-a)^{\frac{1}{2+p}}} - y_0\Big| = (a^*-a)O(|y^0|) \ \text{ as } a \nearrow a^* \tag{1.18}$$

for some $y^0 \in \mathbb{R}^2$.

Theorem 1.2 is derived directly from Theorem 1.1 and Theorem 3.6 in Section 3 with more details, where $\phi_0 \in C^2(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2)$ is given explicitly. In Section 4 we shall extend the refined spike behavior of Theorem 1.2 to more general potentials $V(x) = g(x)h(x)$, where $h(-x) = h(x)$ is homogeneous and satisfies (1.14) and $0 \le C \le g(x) \le \frac{1}{C}$ holds in $\mathbb{R}^2$, see Theorem 4.4 for details. To establish Theorem 1.2 and Theorem 4.4, our Proposition 2.1 shows that the arguments of [10, 11, 12] give the leading expansion terms of the minimizer u_a and the associated Lagrange multiplier μ_a satisfying (1.6) as well. In order to get (1.17) for the rest terms of u_a, the difficulty is to obtain the more precise estimate of μ_a, which is overcome by the very delicate analysis of the associated equation (1.6), together with the constraint condition of u_a.

This paper is organized as follows: In Section 2 we shall prove Theorem 1.1 on the local uniqueness of positive minimizers. Section 3 is concerned with proving Theorem 1.2 on the refined spike profiles of positive minimizers for $e(a)$ as $a \nearrow a^*$. The main aim of Section 4 is to derive Theorem 4.4, which extends the refined spike behavior of Theorem 1.2 to more general potentials $V(x) = g(x)h(x)$. We shall leave the proof of Lemma 3.4 to Appendix A.

2 Local Uniqueness of Positive Minimizers

This section is devoted to the proof of Theorem 1.1 on the local uniqueness of positive minimizers. Towards this purpose, we need some estimates of positive minimizers for $e(a)$ as $a \nearrow a^*$, which hold essentially for more general potential $V(x) \in C^2(\mathbb{R}^2)$ satisfying

$$V(x) = g(x)h(x), \text{ where } 0 < C \le g(x) \le \tfrac{1}{C} \text{ in } \mathbb{R}^2 \text{ and } h(x) \text{ is homogeneous of degree } p \ge 2. \tag{2.1}$$

For convenience, we always denote $\{u_k\}$ to be a positive minimizer sequence of $e(a_k)$ with $a_k \nearrow a^*$ as $k \to \infty$, and define

$$\lambda = \left(\frac{pg(0)}{2}\int_{\mathbb{R}^2} h(x+y_0)w^2(x)dx\right)^{\frac{1}{2+p}}, \tag{2.2}$$

where $V(x) = g(x)h(x)$ is assumed to satisfy (2.1) with $p \ge 2$ and $y_0 \in \mathbb{R}^2$ is given by (1.9). Recall from (1.4) that $w(|x|)$ satisfies

$$\int_{\mathbb{R}^2} |\nabla w|^2 dx = \int_{\mathbb{R}^2} |w|^2 dx = \frac{1}{2}\int_{\mathbb{R}^2} |w|^4 dx, \tag{2.3}$$

see also Lemma 8.1.2 in [4]. Moreover, it follows from [7, Prop. 4.1] that w admits the following exponential decay

$$w(x)\,,\ |\nabla w(x)| = O(|x|^{-\frac{1}{2}}e^{-|x|}) \quad \text{as } |x| \to \infty. \tag{2.4}$$

Proposition 2.1. *Suppose $V(x) = g(x)h(x) \in C^2(\mathbb{R}^2)$ satisfies $\lim_{|x|\to\infty} V(x) = \infty$ and (2.1), and assume (1.9) holds for some $y_0 \in \mathbb{R}^2$. Then there exist a subsequence, still denoted by $\{a_k\}$, of $\{a_k\}$ and $\{x_k\} \subset \mathbb{R}^2$ such that*

(I). The subsequence $\{u_k\}$ satisfies

$$(a^* - a_k)^{\frac{1}{2+p}} u_k\left(x_k + x(a^* - a_k)^{\frac{1}{2+p}}\right) \to \frac{\lambda w(\lambda x)}{\|w\|_2} \quad \text{as } k \to \infty \tag{2.5}$$

uniformly in $\mathbb{R}^2$, and x_k is the unique maximum point of u_k satisfying

$$\lim_{k\to\infty} \frac{\lambda x_k}{(a^* - a_k)^{\frac{1}{2+p}}} = y_0, \tag{2.6}$$

where $y_0 \in \mathbb{R}^2$ is the same as that of (1.9). Moreover, u_k satisfies

$$(a^* - a_k)^{\frac{1}{2+p}} u_k \left(x_k + x(a^* - a_k)^{\frac{1}{2+p}} \right) \le C e^{-\frac{\lambda}{2}|x|} \ \ in \ \mathbb{R}^2, \tag{2.7}$$

where the constant $C > 0$ is independent of k.

(II). The energy $e(a_k)$ satisfies

$$\lim_{k\to\infty} \frac{e(a_k)}{(a^* - a_k)^{p/(2+p)}} = \frac{\lambda^2}{a^*} \frac{p+2}{p}. \tag{2.8}$$

Proof. Since the proof of Proposition 2.1 is similar to those in [10, 11, 12], which handle (1.1) with different potentials $V(x)$, we shall briefly sketch the structure of the proof.

If $V(x) \in C^2(\mathbb{R}^2)$ satisfies (2.1) with $p \ge 2$, we note that $h(x) \ge 0$ satisfies (1.8). Take the test function

$$u_\tau(x) = A_\tau \frac{\tau}{\|w\|_2} \varphi(x) w(\tau x),$$

where the nonnegative cut-off function $\varphi \in C_0^\infty(\mathbb{R}^2)$ satisfies $0 \le \varphi(x) \le 1$ in $\mathbb{R}^2$, and $A_\tau > 0$ is chosen so that $\int_{\mathbb{R}^2} u_\tau(x)^2 dx = 1$. The same proof of Lemma 3 in [10] then yields that

$$e(a) \le C(a^* - a)^{\frac{p}{p+2}} \ \ \text{for} \ \ 0 \le a < a^*, \tag{2.9}$$

where the constant $C > 0$ is independent of a. By (2.9), we can follow Lemma 4 in [10] to derive that there exists a positive constant K, independent of a, such that

$$\int_{\mathbb{R}^2} |u_a(x)|^4 dx \le \frac{1}{K} (a^* - a)^{-\frac{2}{p+2}} \ \ \text{for} \ \ 0 \le a < a^*, \tag{2.10}$$

where $u_a > 0$ is any minimizer of $e(a)$. Applying (2.9) and (2.10), a proof similar to that of Theorem 2.1 in [12] then gives that there exist two positive constants $m < M$, independent of a, such that

$$m(a^* - a)^{\frac{p}{p+2}} \le e(a) \le M(a^* - a)^{\frac{p}{p+2}} \ \ \text{for} \ \ 0 \le a < a^*. \tag{2.11}$$

Based on (2.11), similar to Theorems 1.2 and 1.3 in [12], one can further deduce that there exist a subsequence (still denoted by $\{a_k\}$) of $\{a_k\}$ and $\{x_k\} \subset \mathbb{R}^2$, where $a_k \nearrow a^*$ as $k \to \infty$, such that (2.7) and (2.8) hold, and

$$(a^* - a_k)^{\frac{1}{2+p}} u_k \left(x_k + x(a^* - a_k)^{\frac{1}{2+p}} \right) \to \frac{\lambda w(\lambda x)}{\|w\|_2} \ \ \text{strongly in} \ \ H^1(\mathbb{R}^2) \tag{2.12}$$

as $k \to \infty$, where x_k is the unique maximum point of u_k. Finally, since w decays exponentially, the standard elliptic regularity theory applied to (2.12) yields that (2.5) holds uniformly in $\mathbb{R}^2$ (e.g. Lemma 4.9 in [18] for similar arguments).

We finally follow (1.9) and (2.5) to derive the estimate (2.6). Following (2.5), we define

$$\bar{u}_k(x) := \frac{\sqrt{a^*} \varepsilon_k}{\lambda} u_k \Big(\frac{\varepsilon_k}{\lambda} x + x_k \Big), \ \ \text{where} \ \ \varepsilon_k := (a^* - a_k)^{\frac{1}{2+p}} > 0,$$

so that $\bar{u}_k(x) \to w(x)$ uniformly in $\mathbb{R}^2$ as $k \to \infty$. We then derive from (1.5) that

$$
\begin{aligned}
e(a_k) = E_{a_k}(u_k) &= \frac{\lambda^2}{a^* \varepsilon_k^2}\Big[\int_{\mathbb{R}^2} |\nabla \bar{u}_k(x)|^2 dx - \frac{1}{2}\int_{\mathbb{R}^2} \bar{u}_k^4(x) dx\Big] + \frac{\lambda^2 \varepsilon_k^p}{2(a^*)^2}\int_{\mathbb{R}^2} \bar{u}_k^4(x) dx \\
&\quad + \frac{1}{a^*}\int_{\mathbb{R}^2} V\big(\frac{\varepsilon_k}{\lambda}x + x_k\big)\bar{u}_k^2(x) dx \qquad (2.13) \\
&\ge \frac{\lambda^2 \varepsilon_k^p}{2(a^*)^2}\int_{\mathbb{R}^2} \bar{u}_k^4(x) dx + \frac{1}{a^*}\Big(\frac{\varepsilon_k}{\lambda}\Big)^p \int_{\mathbb{R}^2} g\big(\frac{\varepsilon_k}{\lambda}x + x_k\big) h\big(x + \frac{\lambda x_k}{\varepsilon_k}\big)\bar{u}_k^2(x) dx,
\end{aligned}
$$

which then implies from (2.5) that $|\frac{\lambda x_k}{\varepsilon_k}|$ is bounded uniformly in k. Therefore, there exist a subsequence (still denoted by $\{\frac{\lambda x_k}{\varepsilon_k}\}$) of $\{\frac{\lambda x_k}{\varepsilon_k}\}$ and $y^0 \in \mathbb{R}^2$ such that

$$
\frac{\lambda x_k}{\varepsilon_k} \to y^0 \text{ as } k \to \infty.
$$

Note that

$$
\begin{aligned}
&\liminf_{k\to\infty} \int_{\mathbb{R}^2} g\big(\frac{\varepsilon_k}{\lambda}x + x_k\big) h\big(x + \frac{\lambda x_k}{\varepsilon_k}\big)\bar{u}_k^2(x) dx \\
&\ge \liminf_{k\to\infty} \int_{B_{\frac{1}{\sqrt{\varepsilon_k}}}(0)} g\big(\frac{\varepsilon_k}{\lambda}x + x_k\big) h\big(x + \frac{\lambda x_k}{\varepsilon_k}\big)\bar{u}_k^2(x) dx \qquad (2.14) \\
&= g(0)\int_{\mathbb{R}^2} h(x + y^0) w^2(x) dx.
\end{aligned}
$$

Since u_k gives the least energy of $e(a_k)$ and the assumption (1.9) implies that y_0 is essentially the unique global minimum point of $H(y) = \int_{\mathbb{R}^2} h(x+y)w^2(x)dx$, we conclude from (2.13) and (2.14) that $y^0 = y_0$, which thus implies that (2.6) holds, and the proof is therefore complete. □

2.1 Proof of local uniqueness

Following Proposition 2.1, this subsection is focussed on the proof of Theorem 1.1, and in the whole subsection we always assume that $V(x) = h(x) \in C^2(\mathbb{R}^2)$ is homogeneous of degree $p \ge 2$ and satisfies (1.14) and $\lim_{|x|\to\infty} h(x) = \infty$. Our proof is stimulated by [3, 6, 9]. We first define the linearized operator $\mathcal{L}$ by

$$
\mathcal{L} := -\Delta + (1 - 3w^2) \text{ in } \mathbb{R}^2,
$$

where $w = w(|x|) > 0$ is the unique positive solution of (1.4) and w satisfies the exponential decay (2.4). Recall from [14, 20] that

$$
ker(\mathcal{L}) = span\Big\{\frac{\partial w}{\partial x_1}, \frac{\partial w}{\partial x_2}\Big\}. \qquad (2.15)
$$

For any positive minimizer u_k of $e(a_k)$, where $a_k \nearrow a^*$ as $k \to \infty$, one can note that u_k solves the Euler-Lagrange equation

$$
-\Delta u_k(x) + V(x)u_k(x) = \mu_k u_k(x) + a_k u_k^3(x) \text{ in } \mathbb{R}^2, \qquad (2.16)
$$

where $\mu_k \in \mathbb{R}$ is a suitable Lagrange multiplier and satisfies

$$
\mu_k = e(a_k) - \frac{a_k}{2}\int_{\mathbb{R}^2} u_k^4(x) dx. \qquad (2.17)
$$

Moreover, under the more general assumption (2.1), one can derive from (2.3) and (2.5) that u_k satisfies

$$\int_{\mathbb{R}^2} u_k^4(x)dx = (a^* - a_k)^{-\frac{2}{2+p}}\Big[\frac{2\lambda^2}{a^*} + o(1)\Big] \ \text{ as } \ k \to \infty. \tag{2.18}$$

It then follows from (2.3), (2.17) and (2.18) that μ_k satisfies

$$\frac{\mu_k \varepsilon_k^2}{\lambda^2} \to -1 \ \text{ as } \ k \to +\infty, \tag{2.19}$$

where we denote

$$\varepsilon_k := (a^* - a_k)^{\frac{1}{2+p}} > 0.$$

Set

$$\bar{u}_k(x) := \frac{\sqrt{a^*}\varepsilon_k}{\lambda} u_k\Big(\frac{\varepsilon_k}{\lambda}x + x_k\Big),$$

so that Proposition 2.1 gives $\bar{u}_k(x) \to w(x)$ uniformly in $\mathbb{R}^2$ as $k \to \infty$. Note from (2.16) that $\bar{u}_k$ satisfies

$$-\Delta \bar{u}_k(x) + \Big(\frac{\varepsilon_k}{\lambda}\Big)^2 V\Big(\frac{\varepsilon_k}{\lambda}x + x_k\Big)\bar{u}_k(x) = \frac{\mu_k \varepsilon_k^2}{\lambda^2}\bar{u}_k(x) + \frac{a_k}{a^*}\bar{u}_k^3(x) \ \text{ in } \ \mathbb{R}^2. \tag{2.20}$$

Moreover, by the exponential decay (2.7), there exist $C_0 > 0$ and $R > 0$ such that

$$|\bar{u}_k(x)| \le C_0 e^{-\frac{|x|}{2}} \ \text{ for } \ |x| > R, \tag{2.21}$$

which then implies that

$$\Big|\Big(\frac{\varepsilon_k}{\lambda}\Big)^2 V\Big(\frac{\varepsilon_k}{\lambda}x + x_k\Big)\bar{u}_k(x)\Big| \le CC_0 e^{-\frac{|x|}{4}} \ \text{ for } \ |x| > R,$$

if $V(x)$ satisfies (2.1) with $p \ge 2$. Therefore, under the assumption (2.1), applying the local elliptic estimates (see (3.15) in [8]) to (2.20) yields that

$$|\nabla \bar{u}_k(x)| \le Ce^{-\frac{|x|}{4}} \ \text{ as } \ |x| \to \infty, \tag{2.22}$$

where the estimates (2.19) and (2.21) are also used. In the following, we shall follow Proposition 2.1 and (2.22) to derive Theorem 1.1 on the local uniqueness of positive minimizers as $a \nearrow a^*$.

Proof of Theorem 1.1. Suppose that there exist two different positive minimizers $u_{1,k}$ and $u_{2,k}$ of $e(a_k)$ with $a_k \nearrow a^*$ as $k \to \infty$. Let $x_{1,k}$ and $x_{2,k}$ be the unique local maximum point of $u_{1,k}$ and $u_{2,k}$, respectively. Following (2.16), $u_{i,k}$ then solves the Euler-Lagrange equation

$$-\Delta u_{i,k}(x) + h(x)u_{i,k}(x) = \mu_{i,k}u_{i,k}(x) + a_k u_{i,k}^3(x) \quad \text{in } \ \mathbb{R}^2, \quad i = 1, 2, \tag{2.23}$$

where $V(x) = h(x)$ and $\mu_{i,k} \in \mathbb{R}$ is a suitable Lagrange multiplier. Define

$$\bar{u}_{i,k}(x) := \frac{\sqrt{a^*}\varepsilon_k}{\lambda} u_{i,k}(x), \quad \text{where } \ i = 1, 2. \tag{2.24}$$

Proposition 2.1 then implies that $\bar{u}_{i,k}\big(\frac{\varepsilon_k}{\lambda}x+x_{2,k}\big)\to w(x)$ uniformly in $\mathbb{R}^2$, and $\bar{u}_{i,k}$ satisfies the equation

$$-\varepsilon_k^2\Delta\bar{u}_{i,k}(x)+\varepsilon_k^2 h(x)\bar{u}_{i,k}(x)=\mu_{i,k}\varepsilon_k^2\bar{u}_{i,k}(x)+\frac{\lambda^2 a_k}{a^*}\bar{u}_{i,k}^3(x)\quad\text{in }\ \mathbb{R}^2,\quad i=1,2. \tag{2.25}$$

Because $u_{1,k}\not\equiv u_{2,k}$, we consider

$$\bar{\xi}_k(x)=\frac{u_{2,k}(x)-u_{1,k}(x)}{\|u_{2,k}-u_{1,k}\|_{L^\infty(\mathbb{R}^2)}}=\frac{\bar{u}_{2,k}(x)-\bar{u}_{1,k}(x)}{\|\bar{u}_{2,k}-\bar{u}_{1,k}\|_{L^\infty(\mathbb{R}^2)}}.$$

Then $\bar{\xi}_k$ satisfies the equation

$$-\varepsilon_k^2\Delta\bar{\xi}_k+\bar{C}_k(x)\bar{\xi}_k=\bar{g}_k(x)\quad\text{in }\ \mathbb{R}^2, \tag{2.26}$$

where the coefficient $\bar{C}_k(x)$ satisfies

$$\bar{C}_k(x):=-\mu_{1,k}\varepsilon_k^2-\frac{\lambda^2 a_k}{a^*}\big(\bar{u}_{2,k}^2+\bar{u}_{2,k}\bar{u}_{1,k}+\bar{u}_{1,k}^2\big)+\varepsilon_k^2 h(x), \tag{2.27}$$

and the nonhomogeneous term $\bar{g}_k(x)$ satisfies

$$\begin{aligned}\bar{g}_k(x):=\frac{\varepsilon_k^2\bar{u}_{2,k}(\mu_{2,k}-\mu_{1,k})}{\|\bar{u}_{2,k}-\bar{u}_{1,k}\|_{L^\infty(\mathbb{R}^2)}}&=-\frac{\lambda^4 a_k\bar{u}_{2,k}}{2(a^*)^2\varepsilon_k^2}\int_{\mathbb{R}^2}\frac{\bar{u}_{2,k}^4-\bar{u}_{1,k}^4}{\|\bar{u}_{2,k}-\bar{u}_{1,k}\|_{L^\infty(\mathbb{R}^2)}}dx\\&=-\frac{\lambda^4 a_k\bar{u}_{2,k}}{2(a^*)^2\varepsilon_k^2}\int_{\mathbb{R}^2}\bar{\xi}_k\big(\bar{u}_{2,k}^2+\bar{u}_{1,k}^2\big)\big(\bar{u}_{2,k}+\bar{u}_{1,k}\big)dx,\end{aligned} \tag{2.28}$$

due to the relation (2.17).

Motivated by [3], we first claim that for any $x_0\in\mathbb{R}^2$, there exists a small constant $\delta>0$ such that

$$\int_{\partial B_\delta(x_0)}\Big[\varepsilon_k^2|\nabla\bar{\xi}_k|^2+\frac{\lambda^2}{2}|\bar{\xi}_k|^2+\varepsilon_k^2 h(x)|\bar{\xi}_k|^2\Big]dS=O(\varepsilon_k^2)\quad\text{as }k\to\infty. \tag{2.29}$$

To prove the above claim, multiplying (2.26) by $\bar{\xi}_k$ and integrating over $\mathbb{R}^2$, we obtain that

$$\begin{aligned}&\varepsilon_k^2\int_{\mathbb{R}^2}|\nabla\bar{\xi}_k|^2-\mu_{i,k}\varepsilon_k^2\int_{\mathbb{R}^2}|\bar{\xi}_k|^2+\varepsilon_k^2\int_{\mathbb{R}^2}h(x)|\bar{\xi}_k|^2\\&=\frac{\lambda^2 a_k}{a^*}\int_{\mathbb{R}^2}\big(\bar{u}_{2,k}^2+\bar{u}_{2,k}\bar{u}_{1,k}+\bar{u}_{1,k}^2\big)|\bar{\xi}_k|^2\\&\quad-\frac{\lambda^4 a_k}{2(a^*)^2\varepsilon_k^2}\int_{\mathbb{R}^2}\bar{u}_{2,k}\bar{\xi}_k\int_{\mathbb{R}^2}\bar{\xi}_k\big(\bar{u}_{2,k}^2+\bar{u}_{1,k}^2\big)\big(\bar{u}_{2,k}+\bar{u}_{1,k}\big)\\&\le\frac{\lambda^2 a_k}{a^*}\int_{\mathbb{R}^2}\big(\bar{u}_{2,k}^2+\bar{u}_{2,k}\bar{u}_{1,k}+\bar{u}_{1,k}^2\big)+\frac{\lambda^4 a_k}{2(a^*)^2\varepsilon_k^2}\int_{\mathbb{R}^2}\bar{u}_{2,k}\int_{\mathbb{R}^2}\big(\bar{u}_{2,k}^2+\bar{u}_{1,k}^2\big)\big(\bar{u}_{2,k}+\bar{u}_{1,k}\big)\\&\le C\varepsilon_k^2\ \text{ as }\ k\to\infty,\end{aligned}$$

since $|\bar{\xi}_k|$ and $\bar{u}_{i,k}\big(\frac{\varepsilon_k}{\lambda}x+x_{2,k}\big)$ are bounded uniformly in k, and $\bar{u}_{i,k}\big(\frac{\varepsilon_k}{\lambda}x+x_{2,k}\big)$ decays exponentially as $|x|\to\infty$, $i=1,\,2$. This implies that there exists a constant $C_1>0$ such that

$$I:=\varepsilon_k^2\int_{\mathbb{R}^2}|\nabla\bar{\xi}_k|^2+\frac{\lambda^2}{2}\int_{\mathbb{R}^2}|\bar{\xi}_k|^2+\varepsilon_k^2\int_{\mathbb{R}^2}h(x)|\bar{\xi}_k|^2<C_1\varepsilon_k^2\quad\text{as }\ k\to\infty. \tag{2.30}$$

Applying Lemma 4.5 in [3], we then conclude that for any $x_0 \in \mathbb{R}^2$, there exist a small constant $\delta > 0$ and $C_2 > 0$ such that

$$\int_{\partial B_\delta(x_0)} \Big[\varepsilon_k^2 |\nabla \bar{\xi}_k|^2 + \frac{\lambda^2}{2} |\bar{\xi}_k|^2 + \varepsilon_k^2 h(x) |\bar{\xi}_k|^2\Big] dS \le C_2 I \le C_1 C_2 \varepsilon_k^2 \ \text{ as } \ k \to \infty,$$

which therefore implies the claim (2.29).

We next define

$$\xi_k(x) = \bar{\xi}_k\big(\frac{\varepsilon_k}{\lambda} x + x_{2,k}\big), \quad k = 1, 2, \cdots, \tag{2.31}$$

and

$$\tilde{u}_{i,k}(x) := \frac{\sqrt{a^*}\varepsilon_k}{\lambda} u_{i,k}\big(\frac{\varepsilon_k}{\lambda} x + x_{2,k}\big), \quad \text{where} \quad i = 1, 2,$$

so that $\tilde{u}_{i,k}(x) \to w(x)$ uniformly in $\mathbb{R}^2$ as $k \to \infty$ in view of Proposition 2.1. Under the non-degeneracy assumption (1.14), we shall carry out the proof of Theorem 1.1 by deriving a contradiction through the following three steps.

Step 1. There exist a subsequence $\{a_k\}$ and some constants b_0, b_1 and b_2 such that $\xi_k(x) \to \xi_0(x)$ in $C_{loc}(\mathbb{R}^2)$ as $k \to \infty$, where

$$\xi_0(x) = b_0\big(w + x \cdot \nabla w\big) + \sum_{i=1}^{2} b_i \frac{\partial w}{\partial x_i}. \tag{2.32}$$

Note that ξ_k satisfies

$$-\Delta \xi_k + C_k(x)\xi_k = g_k(x) \quad \text{in} \quad \mathbb{R}^2, \tag{2.33}$$

where the coefficient $C_k(x)$ satisfies

$$\begin{aligned} C_k(x) := &-\Big(1 - \frac{\varepsilon_k^{2+p}}{a^*}\Big)\Big[\tilde{u}_{2,k}^2(x) + \tilde{u}_{2,k}(x)\tilde{u}_{1,k}(x) + \tilde{u}_{1,k}^2(x)\Big] \\ &- \frac{\varepsilon_k^2}{\lambda^2}\mu_{1,k} + \frac{\varepsilon_k^2}{\lambda^2} h\big(\frac{\varepsilon_k x}{\lambda} + x_{2,k}\big), \end{aligned} \tag{2.34}$$

and the nonhomogeneous term $g_k(x)$ satisfies

$$\begin{aligned} g_k(x) := \frac{\tilde{u}_{2,k}}{\lambda^2} \frac{\varepsilon_k^2(\mu_{2,k} - \mu_{1,k})}{\|\tilde{u}_{2,k} - \tilde{u}_{1,k}\|_{L^\infty}} &= -\frac{\tilde{u}_{2,k}}{\lambda^2} \frac{a_k \varepsilon_k^2}{2} \int_{\mathbb{R}^2} \frac{u_{2,k}^4 - u_{1,k}^4}{\|\tilde{u}_{2,k} - \tilde{u}_{1,k}\|_{L^\infty}} dx \\ &= -\frac{a_k \tilde{u}_{2,k}}{2(a^*)^2} \int_{\mathbb{R}^2} \xi_k\big(\tilde{u}_{2,k}^2 + \tilde{u}_{1,k}^2\big)\big(\tilde{u}_{2,k} + \tilde{u}_{1,k}\big) dx. \end{aligned} \tag{2.35}$$

Here we have used (2.17) and (2.25). Since $\|\xi_k\|_{L^\infty(\mathbb{R}^2)} \le 1$, the standard elliptic regularity then implies (cf. [8]) that $\|\xi_k\|_{C^{1,\alpha}_{loc}(\mathbb{R}^2)} \le C$ for some $\alpha \in (0, 1)$, where the constant $C > 0$ is independent of k. Therefore, there exist a subsequence $\{a_k\}$ and a function $\xi_0 = \xi_0(x)$ such that $\xi_k(x) \to \xi_0(x)$ in $C_{loc}(\mathbb{R}^2)$ as $k \to \infty$. Applying Proposition 2.1, direct calculations yield from (2.17) and (2.18) that

$$C_k(x) \to 1 - 3w^2(x) \ \text{ uniformly on } \mathbb{R}^2 \ \text{ as } \ k \to \infty,$$

and

$$g_k(x) \to -\frac{2w(x)}{a^*} \int_{\mathbb{R}^2} w^3 \xi_0 \ \text{ uniformly on } \mathbb{R}^2 \ \text{ as } \ k \to \infty.$$

This implies from (2.33) that ξ_0 solves

$$\mathcal{L}\xi_0 = -\Delta\xi_0 + (1-3w^2)\xi_0 = \Big(-\frac{2}{a^*}\int_{\mathbb{R}^2} w^3\xi_0\Big)w \ \text{ in } \ \mathbb{R}^2. \tag{2.36}$$

Since $\mathcal{L}(w + x\cdot\nabla w) = -2w$, we then conclude from (2.15) and (2.36) that (2.32) holds for some constants b_0, b_1 and b_2.

Step 2. The constants $b_0 = b_1 = b_2 = 0$ in (2.32).

We first derive the following Pohozaev-type identity

$$b_0\int_{\mathbb{R}^2}\frac{\partial h(x+y_0)}{\partial x_j}\big(x\cdot\nabla w^2\big) - \sum_{i=1}^{2} b_i\int_{\mathbb{R}^2}\frac{\partial^2 h(x+y_0)}{\partial x_j\partial x_i}w^2 = 0, \quad j=1,\,2. \tag{2.37}$$

Multiplying (2.25) by $\frac{\partial\bar u_{i,k}}{\partial x_j}$, where $i,j=1,2$, and integrating over $B_\delta(x_{2,k})$, where $\delta>0$ is small and given by (2.29), we calculate that

$$\begin{aligned}
&-\varepsilon_k^2\int_{B_\delta(x_{2,k})}\frac{\partial\bar u_{i,k}}{\partial x_j}\Delta\bar u_{i,k} + \varepsilon_k^2\int_{B_\delta(x_{2,k})}h(x)\frac{\partial\bar u_{i,k}}{\partial x_j}\bar u_{i,k}\\
&= \mu_{i,k}\varepsilon_k^2\int_{B_\delta(x_{2,k})}\frac{\partial\bar u_{i,k}}{\partial x_j}\bar u_{i,k} + \frac{\lambda^2 a_k}{a^*}\int_{B_\delta(x_{2,k})}\frac{\partial\bar u_{i,k}}{\partial x_j}\bar u_{i,k}^3\\
&= \frac{1}{2}\mu_{i,k}\varepsilon_k^2\int_{\partial B_\delta(x_{2,k})}\bar u_{i,k}^2\nu_j dS + \frac{\lambda^2 a_k}{4a^*}\int_{\partial B_\delta(x_{2,k})}\bar u_{i,k}^4\nu_j dS,
\end{aligned} \tag{2.38}$$

where $\nu = (\nu_1,\nu_2)$ denotes the outward unit normal of $\partial B_\delta(x_{2,k})$. Note that

$$\begin{aligned}
&-\varepsilon_k^2\int_{B_\delta(x_{2,k})}\frac{\partial\bar u_{i,k}}{\partial x_j}\Delta\bar u_{i,k}\\
&= -\varepsilon_k^2\int_{\partial B_\delta(x_{2,k})}\frac{\partial\bar u_{i,k}}{\partial x_j}\frac{\partial\bar u_{i,k}}{\partial\nu}dS + \varepsilon_k^2\int_{B_\delta(x_{2,k})}\nabla\bar u_{i,k}\cdot\nabla\frac{\partial\bar u_{i,k}}{\partial x_j}\\
&= -\varepsilon_k^2\int_{\partial B_\delta(x_{2,k})}\frac{\partial\bar u_{i,k}}{\partial x_j}\frac{\partial\bar u_{i,k}}{\partial\nu}dS + \frac{1}{2}\varepsilon_k^2\int_{\partial B_\delta(x_{2,k})}|\nabla\bar u_{i,k}|^2\nu_j dS,
\end{aligned}$$

and

$$\varepsilon_k^2\int_{B_\delta(x_{2,k})}h(x)\frac{\partial\bar u_{i,k}}{\partial x_j}\bar u_{i,k} = \frac{\varepsilon_k^2}{2}\int_{\partial B_\delta(x_{2,k})}h(x)\bar u_{i,k}^2\nu_j dS - \frac{\varepsilon_k^2}{2}\int_{B_\delta(x_{2,k})}\frac{\partial h(x)}{\partial x_j}\bar u_{i,k}^2.$$

We then derive from (2.38) that

$$\begin{aligned}
&\varepsilon_k^2\int_{B_\delta(x_{2,k})}\frac{\partial h(x)}{\partial x_j}\bar u_{i,k}^2\\
&= -2\varepsilon_k^2\int_{\partial B_\delta(x_{2,k})}\frac{\partial\bar u_{i,k}}{\partial x_j}\frac{\partial\bar u_{i,k}}{\partial\nu}dS + \varepsilon_k^2\int_{\partial B_\delta(x_{2,k})}|\nabla\bar u_{i,k}|^2\nu_j dS\\
&\quad+\varepsilon_k^2\int_{\partial B_\delta(x_{2,k})}h(x)\bar u_{i,k}^2\nu_j dS - \mu_{i,k}\varepsilon_k^2\int_{\partial B_\delta(x_{2,k})}\bar u_{i,k}^2\nu_j dS\\
&\quad-\frac{\lambda^2 a_k}{2a^*}\int_{\partial B_\delta(x_{2,k})}\bar u_{i,k}^4\nu_j dS.
\end{aligned} \tag{2.39}$$

Following (2.39), we thus have

$$
\begin{aligned}
&\varepsilon_k^2\int_{B_\delta(x_{2,k})}\frac{\partial h(x)}{\partial x_j}\big(\bar u_{2,k}+\bar u_{1,k}\big)\bar\xi_k dx\\
=&-2\varepsilon_k^2\int_{\partial B_\delta(x_{2,k})}\Big[\frac{\partial \bar u_{2,k}}{\partial x_j}\frac{\partial\bar\xi_k}{\partial\nu}+\frac{\partial\bar\xi_k}{\partial x_j}\frac{\partial \bar u_{1,k}}{\partial\nu}\Big]dS\\
&+\varepsilon_k^2\int_{\partial B_\delta(x_{2,k})}\nabla\bar\xi_k\cdot\nabla\big(\bar u_{2,k}+\bar u_{1,k}\big)\nu_j dS\\
&+\varepsilon_k^2\int_{\partial B_\delta(x_{2,k})}h(x)\big(\bar u_{2,k}+\bar u_{1,k}\big)\bar\xi_k\nu_j dS-\mu_{1,k}\varepsilon_k^2\int_{\partial B_\delta(x_{2,k})}\big(\bar u_{2,k}+\bar u_{1,k}\big)\bar\xi_k\nu_j dS\\
&-\frac{\lambda^2 a_k}{2a^*}\int_{\partial B_\delta(x_{2,k})}\big(\bar u_{2,k}^2+\bar u_{1,k}^2\big)\big(\bar u_{2,k}+\bar u_{1,k}\big)\bar\xi_k\nu_j dS\\
&-\frac{\big(\mu_{2,k}-\mu_{1,k}\big)\varepsilon_k^2}{\|\bar u_{2,k}-\bar u_{1,k}\|_{L^\infty}}\int_{\partial B_\delta(x_{2,k})}\bar u_{2,k}^2\nu_j dS.
\end{aligned}
\tag{2.40}
$$

We now estimate the right hand side of (2.40) as follows. Applying (2.29), if $\delta>0$ is small, we then deduce that

$$
\begin{aligned}
&\varepsilon_k^2\int_{\partial B_\delta(x_{2,k})}\Big|\frac{\partial \bar u_{2,k}}{\partial x_j}\frac{\partial\bar\xi_k}{\partial\nu}\Big|dS\\
\le\ &\varepsilon_k\Big(\int_{\partial B_\delta(x_{2,k})}\Big|\frac{\partial \bar u_{2,k}}{\partial x_j}\Big|^2dS\Big)^{\frac12}\Big(\varepsilon_k^2\int_{\partial B_\delta(x_{2,k})}\Big|\frac{\partial\bar\xi_k}{\partial\nu}\Big|^2dS\Big)^{\frac12}\le C\varepsilon_k^2e^{-\frac{C\delta}{\varepsilon_k}}\ \text{ as }\ k\to\infty,
\end{aligned}
\tag{2.41}
$$

due to the fact that $\nabla\bar u_{2,k}\big(\frac{\varepsilon_k}{\lambda}x+x_{2,k}\big)$ satisfies the exponential decay (2.22), where $C>0$ is independent of k. Similarly, we have

$$
\varepsilon_k^2\int_{\partial B_\delta(x_{2,k})}\Big|\frac{\partial\bar\xi_k}{\partial x_j}\frac{\partial \bar u_{1,k}}{\partial\nu}\Big|dS\le C\varepsilon_k^2e^{-\frac{C\delta}{\varepsilon_k}}\ \text{ as }\ k\to\infty,
$$

and

$$
\varepsilon_k^2\Big|\int_{\partial B_\delta(x_{2,k})}\nabla\bar\xi_k\cdot\nabla\big(\bar u_{2,k}+\bar u_{1,k}\big)\nu_j dS\Big|\le C\varepsilon_k^2e^{-\frac{C\delta}{\varepsilon_k}}\ \text{ as }\ k\to\infty,
$$

On the other hand, since both $|\bar\xi_k|$ and $|\big(\mu_{2,k}-\mu_{1,k}\big)\varepsilon_k^2|$ are bounded uniformly in k, we also get from (2.22) that

$$
\begin{aligned}
&\Big|\varepsilon_k^2\int_{\partial B_\delta(x_{2,k})}h(x)\big(\bar u_{2,k}+\bar u_{1,k}\big)\bar\xi_k\nu_j dS-\mu_{1,k}\varepsilon_k^2\int_{\partial B_\delta(x_{2,k})}\big(\bar u_{2,k}+\bar u_{1,k}\big)\bar\xi_k\nu_j dS\\
&-\frac{\lambda^2 a_k}{2a^*}\int_{\partial B_\delta(x_{2,k})}\big(\bar u_{2,k}^2+\bar u_{1,k}^2\big)\big(\bar u_{2,k}+\bar u_{1,k}\big)\bar\xi_k\nu_j dS\\
&-\frac{\big(\mu_{2,k}-\mu_{1,k}\big)\varepsilon_k^2}{\|\bar u_{2,k}-\bar u_{1,k}\|_{L^\infty}}\int_{\partial B_\delta(x_{2,k})}\bar u_{2,k}^2\nu_j dS\Big|\\
=\ &o(e^{-\frac{C\delta}{\varepsilon_k}})\ \text{ as }\ k\to\infty,
\end{aligned}
\tag{2.42}
$$

due to the fact that (2.28) gives

$$
\frac{\big|\mu_{2,k}-\mu_{1,k}\big|\varepsilon_k^2}{\|\bar u_{2,k}-\bar u_{1,k}\|_{L^\infty}}\le\frac{\lambda^4a_k}{2(a^*)^2\varepsilon_k^2}\int_{\mathbb{R}^2}\big(\bar u_{2,k}^2+\bar u_{1,k}^2\big)\big(\bar u_{2,k}+\bar u_{1,k}\big)|\bar\xi_k|\le M,
\tag{2.43}
$$

where the constants $M > 0$ is independent of k. Because $h(x)$ is homogeneous of degree p, it then follows from (2.40) that for small $\delta > 0$,

$$\begin{aligned} o(e^{-\frac{C\delta}{\varepsilon_k}}) &= \varepsilon_k^2 \int_{B_\delta(x_{2,k})} \frac{\partial h(x)}{\partial x_j} \big[\bar{u}_{2,k}(x) + \bar{u}_{1,k}(x)\big] \bar{\xi}_k(x) dx \\ &= \frac{\varepsilon_k^3}{\lambda} \int_{B_{\frac{\lambda\delta}{\varepsilon_k}}(0)} \frac{\partial}{\partial y_j} h\big(\frac{\varepsilon_k}{\lambda} y + x_{2,k}\big) \bar{\xi}_k\big(\frac{\varepsilon_k}{\lambda} y + x_{2,k}\big) \\ &\qquad \cdot \Big[\bar{u}_{2,k}\big(\frac{\varepsilon_k}{\lambda} y + x_{2,k}\big) + \bar{u}_{1,k}\big(\frac{\varepsilon_k}{\lambda} y + x_{2,k}\big)\Big] dy \\ &= \frac{\varepsilon_k^{p+3}}{\lambda^{p+1}} \Big[\int_{B_{\frac{\lambda\delta}{\varepsilon_k}}(0)} \frac{\partial}{\partial y_j} h\big(y + \frac{\lambda x_{2,k}}{\varepsilon_k}\big) \bar{\xi}_k\big(\frac{\varepsilon_k}{\lambda} y + x_{2,k}\big) \\ &\qquad \cdot \Big[\bar{u}_{2,k}\big(\frac{\varepsilon_k}{\lambda} y + x_{2,k}\big) + \bar{u}_{1,k}\big(\frac{\varepsilon_k}{\lambda} y + x_{1,k}\big)\Big] dy + o(1) \Big] \end{aligned} \tag{2.44}$$

as $k \to \infty$. Applying (1.14), we thus derive from (2.6), (2.32) and (2.44) that

$$\begin{aligned} 0 = 2\int_{\mathbb{R}^2} \frac{\partial h(x+y_0)}{\partial x_j} w\, \xi_0 &= 2\int_{\mathbb{R}^2} \frac{\partial h(x+y_0)}{\partial x_j} w \Big[b_0\big(w + x\cdot\nabla w\big) + \sum_{i=1}^2 b_i \frac{\partial w}{\partial x_i}\Big] \\ &= b_0 \int_{\mathbb{R}^2} \frac{\partial h(x+y_0)}{\partial x_j} \big(x \cdot \nabla w^2\big) - \sum_{i=1}^2 b_i \int_{\mathbb{R}^2} \frac{\partial^2 h(x+y_0)}{\partial x_j \partial x_i} w^2, \end{aligned}$$

where $j = 1,\ 2$, which thus implies (2.37).

We next derive $b_0 = 0$. Using the integration by parts, we note that

$$\begin{aligned} &-\varepsilon_k^2 \int_{B_\delta(x_{2,k})} \big[(x - x_{2,k}) \cdot \nabla \bar{u}_{i,k}\big] \Delta \bar{u}_{i,k} \\ &= -\varepsilon_k^2 \int_{\partial B_\delta(x_{2,k})} \frac{\partial \bar{u}_{i,k}}{\partial \nu} \big[(x - x_{2,k}) \cdot \nabla \bar{u}_{i,k}\big] + \varepsilon_k^2 \int_{B_\delta(x_{2,k})} \nabla \bar{u}_{i,k} \nabla \big[(x - x_{2,k}) \cdot \nabla \bar{u}_{i,k}\big] \\ &= -\varepsilon_k^2 \int_{\partial B_\delta(x_{2,k})} \frac{\partial \bar{u}_{i,k}}{\partial \nu} \big[(x - x_{2,k}) \cdot \nabla \bar{u}_{i,k}\big] + \frac{\varepsilon_k^2}{2} \int_{\partial B_\delta(x_{2,k})} \big[(x - x_{2,k}) \cdot \nu\big] |\nabla \bar{u}_{i,k}|^2. \end{aligned} \tag{2.45}$$

Multiplying (2.25) by $(x - x_{2,k}) \cdot \nabla \bar{u}_{i,k}$, where $i = 1, 2$, and integrating over $B_\delta(x_{2,k})$, where $\delta > 0$ is small as before, we deduce that for $i = 1, 2$,

$$
\begin{aligned}
&-\varepsilon_k^2 \int_{B_\delta(x_{2,k})} \big[(x - x_{2,k}) \cdot \nabla \bar{u}_{i,k}\big] \Delta \bar{u}_{i,k} \\
=& \varepsilon_k^2 \int_{B_\delta(x_{2,k})} \big[\mu_{i,k} - h(x)\big] \bar{u}_{i,k} \big[(x - x_{2,k}) \cdot \nabla \bar{u}_{i,k}\big] \\
&+ \frac{\lambda^2 a_k}{a^*} \int_{B_\delta(x_{2,k})} \bar{u}_{i,k}^3 \big[(x - x_{2,k}) \cdot \nabla \bar{u}_{i,k}\big] \\
=& -\frac{\varepsilon_k^2}{2} \int_{B_\delta(x_{2,k})} \bar{u}_{i,k}^2 \Big\{ 2\big[\mu_{i,k} - h(x)\big] - (x - x_{2,k}) \cdot \nabla h(x) \Big\} \\
&+ \frac{\varepsilon_k^2}{2} \int_{\partial B_\delta(x_{2,k})} \bar{u}_{i,k}^2 \big[\mu_{i,k} - h(x)\big] (x - x_{2,k}) \nu dS \\
&- \frac{\lambda^2 a_k}{2a^*} \int_{B_\delta(x_{2,k})} \bar{u}_{i,k}^4 + \frac{\lambda^2 a_k}{4a^*} \int_{\partial B_\delta(x_{2,k})} \bar{u}_{i,k}^4 (x - x_{2,k}) \nu dS \\
=& -\mu_{i,k} \varepsilon_k^2 \int_{\mathbb{R}^2} \bar{u}_{i,k}^2 + \frac{2+p}{2} \varepsilon_k^2 \int_{\mathbb{R}^2} h(x) \bar{u}_{i,k}^2 - \frac{\lambda^2 a_k}{2a^*} \int_{\mathbb{R}^2} \bar{u}_{i,k}^4 + I_i,
\end{aligned}
\tag{2.46}
$$

where the lower order term I_i satisfies

$$
\begin{aligned}
I_i =& \mu_{i,k} \varepsilon_k^2 \int_{\mathbb{R}^2 \setminus B_\delta(x_{2,k})} \bar{u}_{i,k}^2 - \frac{2+p}{2} \varepsilon_k^2 \int_{\mathbb{R}^2 \setminus B_\delta(x_{2,k})} h(x) \bar{u}_{i,k}^2 \\
&+ \frac{\lambda^2 a_k}{2a^*} \int_{\mathbb{R}^2 \setminus B_\delta(x_{2,k})} \bar{u}_{i,k}^4 - \frac{1}{2} \varepsilon_k^2 \int_{B_\delta(x_{2,k})} \bar{u}_{i,k}^2 \big[x_{2,k} \cdot \nabla h(x)\big] \\
&+ \frac{\varepsilon_k^2}{2} \int_{\partial B_\delta(x_{2,k})} \bar{u}_{i,k}^2 \big[\mu_{i,k} - h(x)\big] (x - x_{2,k}) \nu dS \\
&+ \frac{\lambda^2 a_k}{4a^*} \int_{\partial B_\delta(x_{2,k})} \bar{u}_{i,k}^4 (x - x_{2,k}) \nu dS, \quad i = 1, 2.
\end{aligned}
\tag{2.47}
$$

Since it follows from (2.17) that

$$
-\mu_{i,k} \varepsilon_k^2 \int_{\mathbb{R}^2} \bar{u}_{i,k}^2 - \frac{\lambda^2 a_k}{2a^*} \int_{\mathbb{R}^2} \bar{u}_{i,k}^4 = -\frac{a^* \varepsilon_k^4}{\lambda^2} \Big[\mu_{i,k} + \frac{a_k}{2} \int_{\mathbb{R}^2} u_{i,k}^4\Big] = -\frac{a^* \varepsilon_k^4}{\lambda^2} e(a_k),
$$

we reduce from (2.45)–(2.47) that

$$
\begin{aligned}
\frac{a^* \varepsilon_k^4}{\lambda^2} e(a_k) - \frac{2+p}{2} \varepsilon_k^2 \int_{\mathbb{R}^2} h(x) \bar{u}_{i,k}^2 =& I_i + \varepsilon_k^2 \int_{\partial B_\delta(x_{2,k})} \frac{\partial \bar{u}_{i,k}}{\partial \nu} \big[(x - x_{2,k}) \cdot \nabla \bar{u}_{i,k}\big] \\
&- \frac{\varepsilon_k^2}{2} \int_{\partial B_\delta(x_{2,k})} \big[(x - x_{2,k}) \cdot \nu\big] |\nabla \bar{u}_{i,k}|^2, \quad i = 1, 2,
\end{aligned}
$$

which implies that

$$
-\frac{2+p}{2} \varepsilon_k^2 \int_{\mathbb{R}^2} h(x) \big[\bar{u}_{2,k} + \bar{u}_{1,k}\big] \bar{\xi}_k = T_k. \tag{2.48}
$$

Here the term T_k satisfies that for small $\delta > 0$,

$$
\begin{aligned}
T_k = & \frac{I_2 - I_1}{\|\bar{u}_{2,k} - \bar{u}_{1,k}\|_{L^\infty}} - \frac{\varepsilon_k^2}{2} \int_{\partial B_\delta(x_{2,k})} \big[(x - x_{2,k}) \cdot \nu\big] \big(\nabla \bar{u}_{2,k} + \nabla \bar{u}_{1,k}\big) \nabla \bar{\xi}_k \\
& + \varepsilon_k^2 \int_{\partial B_\delta(x_{2,k})} \Big\{ \big[(x - x_{2,k}) \cdot \nabla \bar{u}_{2,k}\big] \big(\nu \cdot \nabla \bar{\xi}_k\big) + \big(\nu \cdot \nabla \bar{u}_{1,k}\big) \big[(x - x_{2,k}) \cdot \nabla \bar{\xi}_k\big] \Big\} \\
= & \frac{I_2 - I_1}{\|\bar{u}_{2,k} - \bar{u}_{1,k}\|_{L^\infty}} + o(e^{-\frac{C\delta}{\varepsilon_k}}) \ \text{ as } \ k \to \infty,
\end{aligned}
\tag{2.49}
$$

due to (2.22) and (2.29), where the second equality follows by applying the argument of estimating (2.41).

Using the arguments of estimating (2.41) and (2.42), along with the exponential decay of $\bar{u}_{i,k}$, we also derive that for small $\delta > 0$,

$$
\begin{aligned}
& \frac{I_2 - I_1}{\|\bar{u}_{2,k} - \bar{u}_{1,k}\|_{L^\infty}} \\
= & \mu_{2,k} \varepsilon_k^2 \int_{\mathbb{R}^2 \backslash B_\delta(x_{2,k})} \big(\bar{u}_{2,k} + \bar{u}_{1,k}\big) \bar{\xi}_k - \frac{2+p}{2} \varepsilon_k^2 \int_{\mathbb{R}^2 \backslash B_\delta(x_{2,k})} h(x) \big(\bar{u}_{2,k} + \bar{u}_{1,k}\big) \bar{\xi}_k \\
& + \frac{\lambda^2 a_k}{2a^*} \int_{\mathbb{R}^2 \backslash B_\delta(x_{2,k})} \big(\bar{u}_{2,k}^2 + \bar{u}_{1,k}^2\big) \big(\bar{u}_{2,k} + \bar{u}_{1,k}\big) \bar{\xi}_k \\
& + \frac{\big(\mu_{2,k} - \mu_{1,k}\big) \varepsilon_k^2}{\|\bar{u}_{2,k} - \bar{u}_{1,k}\|_{L^\infty}} \int_{\mathbb{R}^2 \backslash B_\delta(x_{2,k})} \bar{u}_{1,k}^2 - \frac{1}{2} \varepsilon_k^2 \int_{B_\delta(x_{2,k})} \big[x_{2,k} \cdot \nabla h(x)\big] \big(\bar{u}_{2,k} + \bar{u}_{1,k}\big) \bar{\xi}_k \\
& + \frac{\lambda^2 a_k}{4a^*} \int_{\partial B_\delta(x_{2,k})} \big(\bar{u}_{2,k}^2 + \bar{u}_{1,k}^2\big) \big(\bar{u}_{2,k} + \bar{u}_{1,k}\big) \bar{\xi}_k (x - x_{2,k}) \nu dS \\
& - \frac{\varepsilon_k^2}{2} \int_{\partial B_\delta(x_{2,k})} \big(\bar{u}_{2,k} + \bar{u}_{1,k}\big) \bar{\xi}_k h(x) (x - x_{2,k}) \nu dS \\
& + \frac{\mu_{2,k} \varepsilon_k^2}{2} \int_{\partial B_\delta(x_{2,k})} \big(\bar{u}_{2,k} + \bar{u}_{1,k}\big) \bar{\xi}_k (x - x_{2,k}) \nu dS \\
& + \frac{\big(\mu_{2,k} - \mu_{1,k}\big) \varepsilon_k^2}{2\|\bar{u}_{2,k} - \bar{u}_{1,k}\|_{L^\infty}} \int_{\partial B_\delta(x_{2,k})} \bar{u}_{1,k}^2 (x - x_{2,k}) \nu dS \\
= & \frac{\big(\mu_{2,k} - \mu_{1,k}\big) \varepsilon_k^2}{\|\bar{u}_{2,k} - \bar{u}_{1,k}\|_{L^\infty}} \Big[\int_{\mathbb{R}^2 \backslash B_\delta(x_{2,k})} \bar{u}_{1,k}^2 + \frac{1}{2} \int_{\partial B_\delta(x_{2,k})} \bar{u}_{1,k}^2 (x - x_{2,k}) \nu dS \Big] \\
& - \frac{1}{2} \varepsilon_k^2 \int_{B_\delta(x_{2,k})} \big[x_{2,k} \cdot \nabla h(x)\big] \big(\bar{u}_{2,k} + \bar{u}_{1,k}\big) \bar{\xi}_k + o(e^{-\frac{C\delta}{\varepsilon_k}}) \ \text{ as } \ k \to \infty.
\end{aligned}
\tag{2.50}
$$

Note from (2.43) that

$$
\frac{\big(\mu_{2,k} - \mu_{1,k}\big) \varepsilon_k^2}{\|\bar{u}_{2,k} - \bar{u}_{1,k}\|_{L^\infty}} \Big[\int_{\mathbb{R}^2 \backslash B_\delta(x_{2,k})} \bar{u}_{1,k}^2 + \frac{1}{2} \int_{\partial B_\delta(x_{2,k})} \bar{u}_{1,k}^2 (x - x_{2,k}) \nu dS \Big] = O(e^{-\frac{C\delta}{\varepsilon_k}}) \tag{2.51}
$$

as $k \to \infty$, where the constant $C > 0$ is independent of k. Moreover, we follow from the first identity of (2.44) that

$$
\begin{aligned}
&\frac{1}{2}\varepsilon_k^2\int_{B_\delta(x_{2,k})}\big[x_{2,k}\cdot\nabla h(x)\big]\big(\bar u_{2,k}+\bar u_{1,k}\big)\bar\xi_k \\
&=\frac{1}{2}\varepsilon_k^2\sum_{i=1}^{2}x_{2,k}^i\int_{B_\delta(x_{2,k})}\frac{\partial h(x)}{\partial x_i}\big[\bar u_{2,k}(x)+\bar u_{1,k}(x)\big]\bar\xi_k(x)dx \\
&=o(e^{-\frac{C\delta}{\varepsilon_k}})\ \text{ as }\ k\to\infty,
\end{aligned}
\tag{2.52}
$$

where we denote $x_{2,k}=(x_{2,k}^1,x_{2,k}^2)$. Therefore, we deduce from (2.49)–(2.52) that

$$T_k=o(\varepsilon_k^{4+p})\ \text{ as }\ k\to\infty.$$

Further, we obtain from (2.48) that

$$
\begin{aligned}
o(\varepsilon_k^{4+p})&=-\frac{2+p}{2}\varepsilon_k^2\int_{\mathbb{R}^2}h(x)\Big[\bar u_{2,k}+\bar u_{1,k}\Big]\bar\xi_k \\
&=-\frac{2+p}{2\lambda^2}\varepsilon_k^4\int_{\mathbb{R}^2}h\big(\frac{\varepsilon_k}{\lambda}x+x_{2,k}\big)\Big[\bar u_{2,k}\big(\frac{\varepsilon_k}{\lambda}x+x_{2,k}\big)+\bar u_{1,k}\big(\frac{\varepsilon_k}{\lambda}x+x_{1,k}\big)\Big]\xi_k(x)dx \\
&\quad-\frac{2+p}{2\lambda^2}\varepsilon_k^4\int_{\mathbb{R}^2}h\big(\frac{\varepsilon_k}{\lambda}x+x_{2,k}\big)\Big[\bar u_{1,k}\big(\frac{\varepsilon_k}{\lambda}x+x_{2,k}\big)-\bar u_{1,k}\big(\frac{\varepsilon_k}{\lambda}x+x_{1,k}\big)\Big]\xi_k(x)dx \\
&=-\frac{2+p}{2\lambda^{2+p}}\varepsilon_k^{4+p}\int_{\mathbb{R}^2}h\big(x+\frac{\lambda x_{2,k}}{\varepsilon_k}\big)\Big[\bar u_{2,k}\big(\frac{\varepsilon_k}{\lambda}x+x_{2,k}\big)+\bar u_{1,k}\big(\frac{\varepsilon_k}{\lambda}x+x_{1,k}\big)\Big]\xi_k(x)dx \\
&\quad+O(\varepsilon_k^{4+p}|x_{2,k}-x_{1,k}|)\ \text{ as }\ k\to\infty.
\end{aligned}
$$

Since $(x+y_0)\cdot\nabla h(x+y_0)=ph(x+y_0)$, by Proposition 2.1 and Step 1, we thus obtain from (1.14) and above that

$$
\begin{aligned}
0=2\int_{\mathbb{R}^2}h(x+y_0)w\xi_0&=2b_0\int_{\mathbb{R}^2}h(x+y_0)w\big(w+x\cdot\nabla w\big)+\sum_{i=1}^{2}b_i\int_{\mathbb{R}^2}h(x+y_0)\frac{\partial w^2}{\partial x_i} \\
&=2b_0\Big[\int_{\mathbb{R}^2}h(x+y_0)w^2+\frac{1}{2}\int_{\mathbb{R}^2}h(x+y_0)\big(x\cdot\nabla w^2\big)\Big] \\
&=2b_0\Big\{\int_{\mathbb{R}^2}h(x+y_0)w^2-\frac{1}{2}\int_{\mathbb{R}^2}w^2\big[2h(x+y_0)+x\cdot\nabla h(x+y_0)\big]\Big\} \\
&=-pb_0\int_{\mathbb{R}^2}h(x+y_0)w^2+b_0\int_{\mathbb{R}^2}w^2\big[y_0\cdot\nabla h(x+y_0)\big] \\
&=-pb_0\int_{\mathbb{R}^2}h(x+y_0)w^2,
\end{aligned}
$$

which therefore implies that $b_0=0$.

By the non-degeneracy assumption (1.14), setting $b_0=0$ into (2.37) then yields that $b_1=b_2=0$, and Step 2 is therefore proved.

Step 3. $\xi_0\equiv 0$ cannot occur.

Finally, let y_k be a point satisfying $|\xi_k(y_k)|=\|\xi_k\|_{L^\infty(\mathbb{R}^2)}=1$. By the same argument as employed in proving Lemma 3.1 in next section, applying the maximum principle to (2.33) yields that $|y_k|\le C$ uniformly in k. Therefore, we conclude that $\xi_k\to\xi_0\not\equiv 0$ uniformly on $\mathbb{R}^2$, which however contradicts to the fact that $\xi_0\equiv 0$ on $\mathbb{R}^2$. This completes the proof of Theorem 1.1. □

3 Refined Spike Profiles

In the following two sections, we shall derive the refined spike profiles of positive minimizers $u_k = u_{a_k}$ for $e(a_k)$ as $a_k \nearrow a^*$. The purpose of this section is to prove Theorem 1.2. Recall first that u_k satisfies the Euler-Lagrange equation (2.16). Under the assumptions of Proposition 2.1, for convenience, we denote

$$\varepsilon_k = (a^* - a_k)^{\frac{1}{2+p}} > 0, \ \ \alpha_k := \varepsilon_k^{2+p} > 0 \ \text{ and } \ \beta_k := 1 + \frac{\mu_k \varepsilon_k^2}{\lambda^2}, \tag{3.1}$$

where $\mu_k \in \mathbb{R}$ is the Lagrange multiplier of the equation (2.16), so that

$$\alpha_k \to 0 \ \text{ and } \ \beta_k \to 0 \ \text{ as } \ k \to \infty,$$

where (2.19) is used. In order to discuss the refined spike profiles of u_k as $k \to \infty$, the key is thus to obtain the refined estimate of μ_k (equivalently β_k) in terms of ε_k.

We next define

$$w_k(x) := \bar{u}_k(x) - w(x) := \frac{\sqrt{a^*}\varepsilon_k}{\lambda} u_k\Big(\frac{\varepsilon_k}{\lambda}x + x_k\Big) - w(x), \tag{3.2}$$

where x_k is the unique maximum point of u_k, so that $w_k(x) \to 0$ uniformly in $\mathbb{R}^2$ by Proposition 2.1. By applying (2.16), direct calculations then give that $\bar{u}_k$ satisfies

$$-\Delta \bar{u}_k(x) + \frac{\varepsilon_k^2}{\lambda^2} V\big(\frac{\varepsilon_k}{\lambda}x + x_k\big)\bar{u}_k(x) = \frac{\mu_k \varepsilon_k^2}{\lambda^2}\bar{u}_k(x) + \frac{a_k}{a^*}\bar{u}_k^3(x) \ \text{ in } \mathbb{R}^2.$$

Relating to the operator $\mathcal{L} := -\Delta + (1 - 3w^2)$ in $\mathbb{R}^2$, we also denote the linearized operator

$$\mathcal{L}_k := -\Delta + \big[1 - \big(\bar{u}_k^2 + \bar{u}_k w + w^2\big)\big] \ \text{ in } \mathbb{R}^2,$$

so that w_k satisfies

$$\begin{aligned} \mathcal{L}_k w_k(x) = -\alpha_k \Big[&\frac{1}{a^*}\bar{u}_k^3(x) + \frac{1}{\lambda^{2+p}} g\big(\frac{\varepsilon_k x}{\lambda} + x_k\big) h\big(x + \frac{\lambda x_k}{\varepsilon_k}\big)\bar{u}_k(x)\Big] \\ &+ \beta_k \bar{u}_k(x) \ \text{ in } \mathbb{R}^2, \quad \nabla w_k(0) = 0, \end{aligned} \tag{3.3}$$

where $V(x) = g(x)h(x)$ satisfies the assumptions of Proposition 2.1 and the coefficients $\alpha_k > 0$ and $\beta_k > 0$ are as in (3.1). Define

$$\begin{aligned} \mathcal{L}_k \psi_{1,k}(x) &= -\alpha_k \Big[\frac{1}{\lambda^{2+p}} g\big(\frac{\varepsilon_k x}{\lambda} + x_k\big) h\big(x + \frac{\lambda x_k}{\varepsilon_k}\big)\bar{u}_k(x) \\ &\qquad + \frac{1}{a^*}\bar{u}_k^3(x)\Big] \ \text{ in } \mathbb{R}^2, \quad \nabla \psi_{1,k}(0) = 0, \\ \mathcal{L}_k \psi_{2,k}(x) &= \beta_k \bar{u}_k(x) \ \text{ in } \mathbb{R}^2, \quad \nabla \psi_{2,k}(0) = 0. \end{aligned} \tag{3.4}$$

Note that the right hand side of (3.4) is orthogonal to the kernel of $\mathcal{L}_k$, which then implies that both $\psi_{1,k}$ and $\psi_{2,k}$ exist. One can get that the solution $w_k(x)$ of (3.3) then satisfies

$$w_k(x) := \psi_{1,k}(x) + \psi_{2,k}(x) \ \text{ in } \mathbb{R}^2. \tag{3.5}$$

We first employ Proposition 2.1 to address the following estimates of w_k as $k \to \infty$.

Lemma 3.1. *Under the assumptions of Proposition 2.1, where $V(x) = g(x)h(x)$, we have*

1. *$\psi_{1,k}(x)$ satisfies*
$$\psi_{1,k}(x) = \alpha_k \psi_1(x) + o(\alpha_k) \ \ as \ \ k \to \infty, \tag{3.6}$$
where $\psi_1(x) \in C^2(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ solves uniquely
$$\nabla \psi_1(0) = 0, \quad \mathcal{L}\psi_1(x) = -\frac{1}{a^*} w^3(x) - \frac{g(0)}{\lambda^{2+p}} h(x+y_0) w(x) \ \ in \ \mathbb{R}^2, \tag{3.7}$$
where $y_0 \in \mathbb{R}^2$ is given by (1.9).

2. *$\psi_{2,k}(x)$ satisfies*
$$\psi_{2,k}(x) = \beta_k \psi_2(x) + o(\beta_k) \ \ as \ \ k \to \infty, \tag{3.8}$$
where $\psi_2(x)$ solves uniquely
$$\nabla \psi_2(0) = 0, \quad \mathcal{L}\psi_2(x) = w(x) \ \ in \ \mathbb{R}^2, \tag{3.9}$$
i.e., $\psi_2(x) \in C^2(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ satisfies
$$\psi_2 = -\frac{1}{2}\big(w + x \cdot \nabla w\big). \tag{3.10}$$

3. *w_k satisfies*
$$w_k(x) := \alpha_k \psi_1(x) + \beta_k \psi_2(x) + o(\alpha_k + \beta_k) \ \ as \ \ k \to \infty. \tag{3.11}$$

Proof. 1. We first derive $|\psi_{1,k}| \le C\alpha_k$ in $\mathbb{R}^2$ by contradiction. On the contrary, we assume that
$$\lim_{k\to\infty} \frac{\|\psi_{1,k}\|_{L^\infty}}{\alpha_k} = \infty. \tag{3.12}$$
Set $\bar\psi_{1,k} = \frac{\psi_{1,k}}{\|\psi_{1,k}\|_{L^\infty}}$ so that $\|\bar\psi_{1,k}\|_{L^\infty} = 1$. Following (3.4), $\bar\psi_{1,k}$ then satisfies
$$\begin{aligned} &-\Delta\bar\psi_{1,k} + \big[1 - \big(\bar u_k^2 + \bar u_k w + w^2\big)\big]\bar\psi_{1,k} \\ &= -\frac{\alpha_k}{\|\psi_{1,k}\|_\infty}\Big[\frac{1}{\lambda^{2+p}} g\big(\frac{\varepsilon_k x}{\lambda} + x_k\big) h\big(x + \frac{\lambda x_k}{\varepsilon_k}\big)\bar u_k(x) + \frac{1}{a^*}\bar u_k^3(x)\Big] \ \text{ in } \mathbb{R}^2. \end{aligned} \tag{3.13}$$
Let y_k be the global maximum point of $\bar\psi_{1,k}$ so that $\bar\psi_{1,k}(y_k) = \max_{y\in\mathbb{R}^2} \frac{\psi_{1,k}(y)}{\|\psi_{1,k}\|_{L^\infty}} = 1$. Since both $\bar u_k$ and w decay exponentially in view of (2.7), using the maximum principle we derive from (3.13) that $|y_k| \le C$ uniformly in k.

On the other hand, applying the usual elliptic regularity theory, there exists a subsequence, still denoted by $\{\bar\psi_{1,k}\}$, of $\{\bar\psi_{1,k}\}$ such that $\bar\psi_{1,k} \to \bar\psi_1$ weakly in $H^1(\mathbb{R}^2)$ and strongly in $L^q_{loc}(\mathbb{R}^2)$ for all $q \in [2,\infty)$. Here $\bar\psi_1$ satisfies
$$\nabla\bar\psi_1(0) = 0, \quad \mathcal{L}\bar\psi_1(x) = 0 \ \text{ in } \mathbb{R}^2,$$
which implies that $\bar\psi_1 = \sum_{i=1}^2 c_i \frac{\partial w}{\partial y_i}$. Since $\nabla\bar\psi_1(0) = 0$, we obtain that $c_1 = c_2 = 0$. Thus, we have $\bar\psi_1(y) \equiv 0$ in $\mathbb{R}^2$, which however contradicts to the fact that $1 = \bar\psi_{1,k}(y_k) \to \bar\psi_1(\bar y_0)$ for some $\bar y_0 \in \mathbb{R}^2$ by passing to a subsequence if necessary. Therefore, we have $|\psi_{1,k}| \le C\alpha_k$ in $\mathbb{R}^2$.

We next set $\phi_{1,k}(x) = \psi_{1,k}(x) - \alpha_k\psi_1(x)$, where $\psi_1(x) \in C^2(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ is a solution of (3.7). Then either $\phi_{1,k}(x) = O(\alpha_k)$ or $\phi_{1,k}(x) = o(\alpha_k)$ as $k \to \infty$, and $\phi_{1,k}$ satisfies
$$\nabla\phi_{1,k}(0) = 0, \quad -\Delta\phi_{1,k} + \big[1 - \big(\bar u_k^2 + \bar u_k w + w^2\big)\big]\phi_{1,k} = -\alpha_k f_k(x) \ \text{ in } \mathbb{R}^2,$$

where $f_k(x)$ satisfies

$$f_k(x) = \big(2w^2 - \bar{u}_k^2 - \bar{u}_k w\big)\psi_1(x) + \frac{1}{a^*}\big(\bar{u}_k^3(x) - w^3(x)\big)$$
$$+\frac{1}{\lambda^{2+p}}\Big[g\big(\frac{\varepsilon_k x}{\lambda} + x_k\big)h\big(x + \frac{\lambda x_k}{\varepsilon_k}\big)\bar{u}_k(x) - g(0)h(x+y_0)w(x)\Big].$$

One can note that $f_k(x) \to 0$ uniformly as $k \to \infty$. Therefore, applying the previous argument yields necessarily that $\phi_{1,k}(x) = o(\alpha_k)$ as $k \to \infty$, and the proof of (3.6) is then complete. Also, the property (2.15) gives the uniqueness of solutions for (3.7).

2. Since the proof of (3.8) is very similar to that of (3.6), we omit the details. Further, the property (2.15) gives the uniqueness of ψ_2. Also, one can check directly that $-(w + x\cdot\nabla w)/2$ is a solution of (3.9), which therefore implies that (3.10) holds.

3. The expansion (3.11) now follows immediately from (3.5), (3.6) and (3.8), and the proof is therefore complete. □

3.1 Proof of Theorem 1.2

The main aim of this subsection is to prove Theorem 1.2 on the refined spike behavior of positive minimizers. In this whole subsection, we assume that the potential $V(x) = h(x) \in C^2(\mathbb{R}^2)$ satisfies $\lim_{|x|\to\infty} h(x) = \infty$ and (1.14), where $h(x)$ is homogeneous of degree $p \geq 2$. Following (3.1), from now on we denote for simplicity that

$$o\big([\alpha_k + \beta_k]^2\big) = o\big(\alpha_k^2\big) + o\big(\alpha_k\beta_k\big) + o\big(\beta_k^2\big) \text{ as } k \to \infty, \tag{3.14}$$

where α_k and β_k are defined in (3.1). We first use Lemma 3.1 to establish the following lemmas.

Lemma 3.2. *Suppose that $V(x) = h(x) \in C^2(\mathbb{R}^2)$ satisfies $\lim_{|x|\to\infty} h(x) = \infty$ and (1.14) for some $y_0 \in \mathbb{R}^2$, where $h(x)$ is homogeneous of degree $p \geq 2$. Then there exists an $x_0 \in \mathbb{R}^2$ such that the unique maximum point x_k of u_k satisfies*

$$\Big|\alpha_k\Big(\frac{\lambda x_k}{\varepsilon_k} - y_0\Big) - \alpha_k\beta_k\frac{y_0}{2}\Big| = \alpha_k^2 O(|x_0|) + o\big([\alpha_k + \beta_k]^2\big) \text{ as } k \to \infty. \tag{3.15}$$

Proof. Multiplying (3.7) and (3.9) by $\frac{\partial w}{\partial x_1}$ and then integrating over $\mathbb{R}^2$, respectively, we obtain from (1.14) and (2.15) that

$$\int_{\mathbb{R}^2} \frac{\partial w}{\partial x_1}\mathcal{L}w_k = \int_{\mathbb{R}^2} \frac{\partial w}{\partial x_1} w = \int_{\mathbb{R}^2} \frac{\partial w}{\partial x_1} h(x+y_0)w = 0, \tag{3.16}$$

where y_0 is given by the assumption (1.14). Similarly, we derive from (3.3) and (3.11) that

$$\begin{aligned}\int_{\mathbb{R}^2} \frac{\partial w}{\partial x_1}\mathcal{L}_k w_k &= \beta_k \int_{\mathbb{R}^2} \frac{\partial w}{\partial x_1}\bar{u}_k - \alpha_k \int_{\mathbb{R}^2} \frac{\partial w}{\partial x_1}\Big[\frac{1}{a^*}\bar{u}_k^3 + \frac{\bar{u}_k}{\lambda^{2+p}}h\big(x + \frac{\lambda x_k}{\varepsilon_k}\big)\Big] \\ &= \alpha_k\beta_k \int_{\mathbb{R}^2} \frac{\partial w}{\partial x_1}\psi_1 + o(\alpha_k\beta_k + \beta_k^2) - I_1,\end{aligned} \tag{3.17}$$

where the identity $\int_{\mathbb{R}^2} \frac{\partial w}{\partial x_1}\psi_2 = 0$ is used, since $\frac{\partial w}{\partial x_1}\psi_2$ is odd in x_1 by the radial symmetry of ψ_2. We obtain from (1.14) and (3.16) that

$$
\begin{aligned}
I_1 &= \alpha_k \int_{\mathbb{R}^2} \frac{\partial w}{\partial x_1} \Big[\frac{1}{a^*}\bar{u}_k^3 + \frac{\bar{u}_k}{\lambda^{2+p}} h\big(x + \frac{\lambda x_k}{\varepsilon_k}\big)\Big] \\
&= \alpha_k \int_{\mathbb{R}^2} \frac{\partial w}{\partial x_1} \Big\{\frac{1}{a^*}(\bar{u}_k^3 - w^3) + \frac{1}{\lambda^{2+p}}\Big[h\big(x + \frac{\lambda x_k}{\varepsilon_k}\big)\bar{u}_k - h(x+y_0)w\Big]\Big\} \\
&= \frac{\alpha_k}{a^*} \int_{\mathbb{R}^2} \frac{\partial w}{\partial x_1} w_k\big(3w^2 + 3ww_k + w_k^2\big) \\
&\quad + \frac{\alpha_k}{\lambda^{2+p}} \int_{\mathbb{R}^2} \frac{\partial w}{\partial x_1} \Big[h\big(x + \frac{\lambda x_k}{\varepsilon_k}\big)\bar{u}_k - h(x+y_0)w\Big] \\
&= \frac{3\alpha_k^2}{a^*} \int_{\mathbb{R}^2} \frac{\partial w}{\partial x_1} w^2\psi_1 + o(\alpha_k^2 + \alpha_k\beta_k) + I_2,
\end{aligned}
\tag{3.18}
$$

where we have used the identity $\int_{\mathbb{R}^2} \frac{\partial w}{\partial x_1} w^2\psi_2 = 0$, since $\frac{\partial w}{\partial x_1} w^2\psi_2$ is odd in x_1 by the radial symmetry of ψ_2. Further, applying (3.11) and (3.16) yields that

$$
\begin{aligned}
\frac{\lambda^{2+p}}{\alpha_k} I_2 &= \int_{\mathbb{R}^2} \frac{\partial w}{\partial x_1} \Big\{h\big(x + \frac{\lambda x_k}{\varepsilon_k}\big)\big[\bar{u}_k - w\big] + \big[h\big(x + \frac{\lambda x_k}{\varepsilon_k}\big) - h(x+y_0)\big]w\Big\} \\
&= \int_{\mathbb{R}^2} \frac{\partial w}{\partial x_1} h(x+y_0)w_k + o(\alpha_k + \beta_k) \\
&\quad + \int_{\mathbb{R}^2} \frac{\partial w}{\partial x_1} \Big[\Big(\frac{\lambda x_k}{\varepsilon_k} - y_0\Big)\cdot \nabla h(x+y_0)\Big]w + o\Big(\Big|\frac{\lambda x_k}{\varepsilon_k} - y_0\Big|\Big) \\
&= \alpha_k \int_{\mathbb{R}^2} \frac{\partial w}{\partial x_1} h(x+y_0)\psi_1 + \beta_k \int_{\mathbb{R}^2} \frac{\partial w}{\partial x_1} h(x+y_0)\psi_2 \\
&\quad + \int_{\mathbb{R}^2} \frac{\partial w}{\partial x_1} \Big[\Big(\frac{\lambda x_k}{\varepsilon_k} - y_0\Big)\cdot \nabla h(x+y_0)\Big]w + o\Big(\alpha_k + \big|\frac{\lambda x_k}{\varepsilon_k} - y_0\big| + \beta_k\Big),
\end{aligned}
\tag{3.19}
$$

where (2.6) is used for the second identity. We thus get that

$$
\begin{aligned}
I_1 &= \alpha_k^2\Big[\frac{3}{a^*}\int_{\mathbb{R}^2} \frac{\partial w}{\partial x_1} w^2\psi_1 + \frac{1}{\lambda^{2+p}}\int_{\mathbb{R}^2} \frac{\partial w}{\partial x_1} h(x+y_0)\psi_1\Big] \\
&\quad + \frac{\alpha_k\beta_k}{\lambda^{2+p}} \int_{\mathbb{R}^2} \frac{\partial w}{\partial x_1} h(x+y_0)\psi_2 + \frac{\alpha_k}{\lambda^{2+p}} \int_{\mathbb{R}^2} \frac{\partial w}{\partial x_1} \Big[\Big(\frac{\lambda x_k}{\varepsilon_k} - y_0\Big)\cdot \nabla h(x+y_0)\Big]w \\
&\quad + o\Big(\alpha_k\big|\frac{\lambda x_k}{\varepsilon_k} - y_0\big| + [\alpha_k + \beta_k]^2\Big).
\end{aligned}
\tag{3.20}
$$

On the other hand, we obtain from (3.16) that

$$
\begin{aligned}
\int_{\mathbb{R}^2} \frac{\partial w}{\partial x_1} \mathcal{L}_k w_k &= \int_{\mathbb{R}^2} \frac{\partial w}{\partial x_1} \mathcal{L} w_k + \int_{\mathbb{R}^2} \frac{\partial w}{\partial x_1} (\mathcal{L}_k - \mathcal{L}) w_k \\
&= -\int_{\mathbb{R}^2} \frac{\partial w}{\partial x_1} w_k^2(3w + w_k) \\
&= -3\alpha_k^2 \int_{\mathbb{R}^2} \frac{\partial w}{\partial x_1} w\psi_1^2 - 6\alpha_k\beta_k \int_{\mathbb{R}^2} \frac{\partial w}{\partial x_1} w\psi_1\psi_2 + o(\alpha_k^2 + \alpha_k\beta_k).
\end{aligned}
\tag{3.21}
$$

Combining (3.17), (3.21) and (3.20), we now conclude from (1.14) and (3.11) that

$$\begin{aligned}
&\frac{\alpha_k}{\lambda^{2+p}}\int_{\mathbb{R}^2}\frac{\partial w}{\partial x_1}\Big[\Big(\frac{\lambda x_k}{\varepsilon_k}-y_0\Big)\cdot\nabla h(x+y_0)\Big]w\\
&=\alpha_k\beta_k\Big[\int_{\mathbb{R}^2}\frac{\partial w}{\partial x_1}\psi_1+6\int_{\mathbb{R}^2}\frac{\partial w}{\partial x_1}w\psi_1\psi_2-\frac{1}{\lambda^{2+p}}\int_{\mathbb{R}^2}\frac{\partial w}{\partial x_1}h(x+y_0)\psi_2\Big]\\
&-\alpha_k^2\Big[\frac{3}{a^*}\int_{\mathbb{R}^2}\frac{\partial w}{\partial x_1}w^2\psi_1+\frac{1}{\lambda^{2+p}}\int_{\mathbb{R}^2}\frac{\partial w}{\partial x_1}h(x+y_0)\psi_1-3\int_{\mathbb{R}^2}\frac{\partial w}{\partial x_1}w\psi_1^2\Big]\\
&+o([\alpha_k+\beta_k]^2).
\end{aligned}\tag{3.22}$$

We *claim* that the coefficient I_3 of the term $\alpha_k\beta_k$ in (3.22) satisfies

$$\begin{aligned}
I_3:&=\int_{\mathbb{R}^2}\frac{\partial w}{\partial x_1}\psi_1+6\int_{\mathbb{R}^2}\frac{\partial w}{\partial x_1}w\psi_1\psi_2-\frac{1}{\lambda^{2+p}}\int_{\mathbb{R}^2}\frac{\partial w}{\partial x_1}h(x+y_0)\psi_2\\
&=\frac{1}{2\lambda^{2+p}}\int_{\mathbb{R}^2}w\Big[y_0\cdot\nabla h(x+y_0)\Big]\frac{\partial w}{\partial x_1}.
\end{aligned}\tag{3.23}$$

If (3.23) holds, we then derive from (3.22) that there exists some $x_0=(x_{10},x_{20})\in\mathbb{R}^2$ such that

$$\begin{aligned}
&\frac{1}{2\lambda^{2+p}}\int_{\mathbb{R}^2}\frac{\partial w^2}{\partial x_j}\Big[\alpha_k\Big(\frac{\lambda x_k}{\varepsilon_k}-y_0\Big)-\alpha_k\beta_k\frac{y_0}{2}\Big]\cdot\nabla h(x+y_0)\\
&=\alpha_k^2O(|x_{j0}|)+o([\alpha_k+\beta_k]^2),\quad j=1,2.
\end{aligned}\tag{3.24}$$

By the non-degeneracy assumption of (1.14), we further conclude from (3.24) that (3.15) holds for some $x_0\in\mathbb{R}^2$, and the lemma is therefore proved.

To complete the proof of the lemma, the rest is to prove the claim (3.23). Indeed, using the integration by parts, we derive from (3.10) that

$$\begin{aligned}
A:&=\int_{\mathbb{R}^2}\frac{\partial w}{\partial x_1}\psi_1+6\int_{\mathbb{R}^2}\frac{\partial w}{\partial x_1}w\psi_1\psi_2\\
&=\int_{\mathbb{R}^2}\frac{\partial w}{\partial x_1}\psi_1-3\int_{\mathbb{R}^2}\frac{\partial w}{\partial x_1}w^2\psi_1-\frac{3}{2}\int_{\mathbb{R}^2}\frac{\partial w}{\partial x_1}\psi_1(x\cdot\nabla w^2)\\
&=\int_{\mathbb{R}^2}\frac{\partial w}{\partial x_1}\psi_1-3\int_{\mathbb{R}^2}\frac{\partial w}{\partial x_1}w^2\psi_1\\
&\quad+\frac{3}{2}\int_{\mathbb{R}^2}w^2\Big[2\frac{\partial w}{\partial x_1}\psi_1+x\cdot\nabla\Big(\frac{\partial w}{\partial x_1}\psi_1\Big)\Big]\\
&=\int_{\mathbb{R}^2}\frac{\partial w}{\partial x_1}\psi_1+\frac{3}{2}\int_{\mathbb{R}^2}w^2\Big[\frac{\partial w}{\partial x_1}(x\cdot\nabla\psi_1)+\psi_1x\cdot\nabla\Big(\frac{\partial w}{\partial x_1}\Big)\Big].
\end{aligned}$$

Since $(x+y_0)\cdot\nabla h(x+y_0)=ph(x+y_0)$, we obtain from (1.14), (3.10) and (3.16) that

$$\begin{aligned}
B:&=-\frac{1}{\lambda^{2+p}}\int_{\mathbb{R}^2}\frac{\partial w}{\partial x_1}h(x+y_0)\psi_2\\
&=\frac{1}{2\lambda^{2+p}}\int_{\mathbb{R}^2}\frac{\partial w}{\partial x_1}h(x+y_0)(w+x\cdot\nabla w)\\
&=-\frac{1}{2\lambda^{2+p}}\int_{\mathbb{R}^2}w\Big[2h(x+y_0)\frac{\partial w}{\partial x_1}+x\cdot\nabla\Big(\frac{\partial w}{\partial x_1}h(x+y_0)\Big)\Big]\\
&=-\frac{1}{2\lambda^{2+p}}\int_{\mathbb{R}^2}w\Big\{\big[x\cdot\nabla h(x+y_0)\big]\frac{\partial w}{\partial x_1}+h(x+y_0)x\cdot\nabla\Big(\frac{\partial w}{\partial x_1}\Big)\Big\}\\
&=-\frac{1}{2\lambda^{2+p}}\int_{\mathbb{R}^2}wh(x+y_0)\Big[x\cdot\nabla\Big(\frac{\partial w}{\partial x_1}\Big)\Big]+\frac{1}{2\lambda^{2+p}}\int_{\mathbb{R}^2}w\Big[y_0\cdot\nabla h(x+y_0)\Big]\frac{\partial w}{\partial x_1}.
\end{aligned}$$

By above calculations, we then get from (3.23) that

$$
\begin{aligned}
I_3 = A + B = &\int_{\mathbb{R}^2} \frac{\partial w}{\partial x_1}\psi_1 + \frac{3}{2}\int_{\mathbb{R}^2} w^2 \frac{\partial w}{\partial x_1}(x\cdot\nabla\psi_1)\\
&+\frac{1}{2}\int_{\mathbb{R}^2}\Big[3w^2\psi_1 - \frac{wh(x+y_0)}{\lambda^{2+p}}\Big]\Big[x\cdot\nabla\Big(\frac{\partial w}{\partial x_1}\Big)\Big]\\
&+\frac{1}{2\lambda^{2+p}}\int_{\mathbb{R}^2} w\Big[y_0\cdot\nabla h(x+y_0)\Big]\frac{\partial w}{\partial x_1}\\
:= I_4 + &\frac{1}{2\lambda^{2+p}}\int_{\mathbb{R}^2} w\Big[y_0\cdot\nabla h(x+y_0)\Big]\frac{\partial w}{\partial x_1}.
\end{aligned}
\tag{3.25}
$$

Applying the integration by parts, we derive from (3.7) that

$$
\begin{aligned}
&\int_{\mathbb{R}^2} \frac{\partial w}{\partial x_1}\psi_1 + \frac{1}{2}\int_{\mathbb{R}^2}\Big[3w^2\psi_1 - \frac{wh(x+y_0)}{\lambda^{2+p}}\Big]\Big[x\cdot\nabla\Big(\frac{\partial w}{\partial x_1}\Big)\Big]\\
=&\int_{\mathbb{R}^2} \frac{\partial w}{\partial x_1}\psi_1 + \frac{1}{2}\int_{\mathbb{R}^2}\Big[3w^2\psi_1 - \Big(\frac{w^3}{a^*} + \frac{wh(x+y_0)}{\lambda^{2+p}}\Big)\Big]\Big[x\cdot\nabla\Big(\frac{\partial w}{\partial x_1}\Big)\Big]\\
=&\int_{\mathbb{R}^2} \frac{\partial w}{\partial x_1}\psi_1 + \frac{1}{2}\int_{\mathbb{R}^2}\big(-\Delta\psi_1+\psi_1\big)\Big[x\cdot\nabla\Big(\frac{\partial w}{\partial x_1}\Big)\Big]\\
=&\int_{\mathbb{R}^2} \frac{\partial w}{\partial x_1}\psi_1 + \frac{1}{2}\int_{\mathbb{R}^2}\big(-\Delta\psi_1\big)\Big[x\cdot\nabla\Big(\frac{\partial w}{\partial x_1}\Big)\Big] - \frac{1}{2}\int_{\mathbb{R}^2}\frac{\partial w}{\partial x_1}\big[2\psi_1 + x\cdot\nabla\psi_1\big]\\
=&\frac{1}{2}\int_{\mathbb{R}^2}\big(-\Delta\psi_1\big)\Big[x\cdot\nabla\Big(\frac{\partial w}{\partial x_1}\Big)\Big] - \frac{1}{2}\int_{\mathbb{R}^2}\frac{\partial w}{\partial x_1}\big(x\cdot\nabla\psi_1\big),
\end{aligned}
$$

which then gives from (3.25) that

$$
\begin{aligned}
-2I_4 =& \int_{\mathbb{R}^2}\Delta\psi_1\Big[x\cdot\nabla\Big(\frac{\partial w}{\partial x_1}\Big)\Big] + \int_{\mathbb{R}^2}\frac{\partial}{\partial x_1}\Big[w - w^3\Big]\big(x\cdot\nabla\psi_1\big)\\
=& \int_{\mathbb{R}^2}\Delta\psi_1\Big[x\cdot\nabla\Big(\frac{\partial w}{\partial x_1}\Big)\Big] + \int_{\mathbb{R}^2}\frac{\partial\Delta w}{\partial x_1}\big(x\cdot\nabla\psi_1\big).
\end{aligned}
\tag{3.26}
$$

To further simplify I_4, we next rewrite ψ_1 as $\psi_1(x) = \psi_1(r,\theta)$, where $x = r(\cos\theta, \sin\theta)$

and (r,θ) is the polar coordinate in $\mathbb{R}^2$. We then follow from (3.7) and (3.26) that

$$
\begin{aligned}
-2I_4 =& \int_0^\infty \int_0^{2\pi} \Big\{\big[r(\psi_1)_r\big]_r + \frac{1}{r}(\psi_1)_{\theta\theta}\Big\} r\frac{\partial}{\partial r}(w'\cos\theta)d\theta dr \\
&+ \int_0^\infty \int_0^{2\pi} \frac{\partial}{\partial r}\Big(w'' + \frac{w'}{r}\Big)\cos\theta\, r^2 (\psi_1)_r d\theta dr \\
=& -\int_0^\infty \int_0^{2\pi} r(\psi_1)_r (rw'')'\cos\theta d\theta dr \\
&+ \int_0^\infty \int_0^{2\pi} r(\psi_1)_r \Big[r\frac{\partial}{\partial r}\Big(w'' + \frac{w'}{r}\Big)\Big]\cos\theta d\theta dr \\
&+ \int_0^\infty \int_0^{2\pi} (\psi_1)_{\theta\theta} w''\cos\theta d\theta dr \\
=& -\int_0^\infty \int_0^{2\pi} r(\psi_1)_r \Big\{(rw'')' - \Big[r\frac{\partial}{\partial r}\Big(w'' + \frac{w'}{r}\Big)\Big]\Big\}\cos\theta d\theta dr \\
&- \int_0^\infty \int_0^{2\pi} \psi_1 w''\cos\theta d\theta dr \\
=& -\int_0^\infty \int_0^{2\pi} (\psi_1)_r w'\cos\theta d\theta dr - \int_0^\infty \int_0^{2\pi} \psi_1 w''\cos\theta d\theta dr = 0,
\end{aligned}
\tag{3.27}
$$

i.e., $I_4 = 0$, which therefore implies that the claim (3.23) holds by applying (3.25). □

Remark 3.1. *Whether the point $x_0 \in \mathbb{R}^2$ in Lemma 3.2 is the origin or not is determined completely by the fact that whether the coefficient I_5 of the term α_k^2 in (3.22) is zero or not, where I_5 satisfies*

$$I_5 := \frac{3}{a^*}\int_{\mathbb{R}^2} \frac{\partial w}{\partial x_1} w^2\psi_1 + \frac{1}{\lambda^{2+p}}\int_{\mathbb{R}^2} \frac{\partial w}{\partial x_1} h(x+y_0)\psi_1 - 3\int_{\mathbb{R}^2} \frac{\partial w}{\partial x_1} w\psi_1^2.$$

If $h(x)$ is not even in x, it however seems difficult to derive that whether $I_5 = 0$ or not.

Lemma 3.3. *Suppose that $V(x) = h(x) \in C^2(\mathbb{R}^2)$ satisfies $\lim_{|x|\to\infty} h(x) = \infty$ and (1.14) for some $y_0 \in \mathbb{R}^2$, where $h(x)$ is homogeneous of degree $p \geq 2$. Then we have*

$$w_k := \alpha_k\psi_1 + \beta_k\psi_2 + \alpha_k^2\psi_3 + \beta_k^2\psi_4 + \alpha_k\beta_k\psi_5 + o([\alpha_k+\beta_k]^2) \quad \text{as} \quad k \to \infty, \tag{3.28}$$

where $\psi_1(x)$, $\psi_2(x) \in C^2(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ are given in Lemma 3.1 with $g(0) = 1$, and $\psi_i(x) \in C^2(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$, $i = 3,4,5$, solves uniquely

$$\nabla\psi_i(0) = 0 \;\; \text{and} \;\; \mathcal{L}\psi_i(x) = f_i(x) \;\; \text{in} \;\; \mathbb{R}^2, \;\; i = 3,4,5, \tag{3.29}$$

and $f_i(x)$ satisfies for some $y^0 \in \mathbb{R}^2$,

$$f_i(x) = \begin{cases} 3w\psi_1^2 - \Big(\dfrac{3w^2}{a^*} + \dfrac{h(x+y_0)}{\lambda^{2+p}}\Big)\psi_1 - \dfrac{w}{\lambda^{1+p}}\big[y^0\cdot\nabla h(x+y_0)\big], & \text{if} \;\; i = 3; \\ 3w\psi_2^2 + \psi_2, & \text{if} \;\; i = 4; \\ 6w\psi_1\psi_2 + \psi_1 - \Big(\dfrac{3w^2}{a^*} + \dfrac{h(x+y_0)}{\lambda^{2+p}}\Big)\psi_2 & \\ -\dfrac{w}{2\lambda^{2+p}}\Big[y_0\cdot\nabla h(x+y_0)\Big], & \text{if} \;\; i = 5; \end{cases} \tag{3.30}$$

where $y_0 \in \mathbb{R}^2$ is given by (1.14). Moreover, ψ_4 is radially symmetric.

Proof. Following Lemma 3.1(3), set

$$v_k = w_k - \alpha_k\psi_1 - \beta_k\psi_2,$$

so that

$$\begin{aligned}\mathcal{L}_k w_k &= \mathcal{L}_k(v_k + \alpha_k\psi_1 + \beta_k\psi_2)\\ &= \mathcal{L}_k v_k + \alpha_k(\mathcal{L}_k - \mathcal{L})\psi_1 + \beta_k(\mathcal{L}_k - \mathcal{L})\psi_2 + \alpha_k\mathcal{L}\psi_1 + \beta_k\mathcal{L}\psi_2\\ &= \mathcal{L}_k v_k - w_k(\alpha_k\psi_1 + \beta_k\psi_2)(3w + w_k) - \alpha_k\Big[\frac{w^3}{a^*} + \frac{h(x+y_0)w}{\lambda^{2+p}}\Big] + \beta_k w.\end{aligned}\tag{3.31}$$

Applying (3.3), we then have

$$\begin{aligned}\mathcal{L}_k v_k &= \mathcal{L}_k w_k + w_k(\alpha_k\psi_1 + \beta_k\psi_2)(3w + w_k) + \alpha_k\Big[\frac{w^3}{a^*} + \frac{h(x+y_0)w}{\lambda^{2+p}}\Big] - \beta_k w\\ &= w_k(\alpha_k\psi_1 + \beta_k\psi_2)(3w + w_k) + \beta_k(\bar{u}_k - w)\\ &\quad -\alpha_k\Big\{\frac{1}{a^*}(\bar{u}_k^3 - w^3) + \frac{1}{\lambda^{2+p}}\Big[h\big(x + \frac{\lambda x_k}{\varepsilon_k}\big)\bar{u}_k - h(x+y_0)w\Big]\Big\}\\ &= w_k(\alpha_k\psi_1 + \beta_k\psi_2)(3w + w_k) + \beta_k w_k - I_6,\end{aligned}\tag{3.32}$$

where I_6 satisfies

$$\begin{aligned}I_6 &= \frac{\alpha_k}{a^*}w_k(3w^2 + 3ww_k + w_k^2)\\ &\quad + \frac{\alpha_k}{\lambda^{2+p}}\Big\{h(x+y_0)(\bar{u}_k - w) + \Big[h\big(x + \frac{\lambda x_k}{\varepsilon_k}\big) - h(x+y_0)\Big]\bar{u}_k\Big\}\\ &= \frac{\alpha_k}{a^*}w_k(3w^2 + 3ww_k + w_k^2) + \frac{\alpha_k}{\lambda^{2+p}}h(x+y_0)w_k\\ &\quad + \frac{\alpha_k}{\lambda^{2+p}}\Big[\Big(\frac{\lambda x_k}{\varepsilon_k} - y_0\Big)\cdot\nabla h(x+y_0)\Big]\bar{u}_k + o\big([\alpha_k + \beta_k]^2\big)\\ &= \alpha_k w_k\Big(\frac{3w^2}{a^*} + \frac{h(x+y_0)}{\lambda^{2+p}}\Big) + \frac{\alpha_k}{a^*}w_k^2(3w + w_k)\\ &\quad + \frac{\alpha_k}{\lambda^{2+p}}\Big[\Big(\frac{\lambda x_k}{\varepsilon_k} - y_0\Big)\cdot\nabla h(x+y_0)\Big]\bar{u}_k + o\big([\alpha_k + \beta_k]^2\big),\end{aligned}$$

where Lemma 3.2 is used in the second equality. By Lemma 3.2 again, there exists $y^0 \in \mathbb{R}^2$ such that

$$\Big|\alpha_k\Big(\frac{\lambda x_k}{\varepsilon_k} - y_0\Big) - \alpha_k\beta_k\frac{y_0}{2} - \alpha_k^2 y^0\Big| = o\big([\alpha_k + \beta_k]^2\big) \text{ as } k \to \infty.$$

We thus obtain from above that

$$\begin{aligned}\mathcal{L}_k v_k &= w_k(\alpha_k\psi_1 + \beta_k\psi_2)(3w + w_k) + \beta_k w_k - \alpha_k w_k\Big(\frac{3w^2}{a^*} + \frac{h(x+y_0)}{\lambda^{2+p}}\Big)\\ &\quad -\frac{\alpha_k}{\lambda^{2+p}}\Big[\Big(\frac{\lambda x_k}{\varepsilon_k} - y_0\Big)\cdot\nabla h(x+y_0)\Big]\bar{u}_k - \frac{\alpha_k}{a^*}w_k^2(3w + w_k) + o([\alpha_k + \beta_k]^2)\\ &= \alpha_k^2\Big\{3w\psi_1^2 - \Big(\frac{3w^2}{a^*} + \frac{h(x+y_0)}{\lambda^{2+p}}\Big)\psi_1 - \frac{w}{\lambda^{1+p}}\Big[y^0\cdot\nabla h(x+y_0)\Big]\Big\}\\ &\quad +\alpha_k\beta_k\Big\{6w\psi_1\psi_2 + \psi_1 - \Big(\frac{3w^2}{a^*} + \frac{h(x+y_0)}{\lambda^{2+p}}\Big)\psi_2 - \frac{1}{2\lambda^{2+p}}w\Big[y_0\cdot\nabla h(x+y_0)\Big]\Big\}\\ &\quad +\beta_k^2(3w\psi_2^2 + \psi_2) + o([\alpha_k + \beta_k]^2) \quad \text{in } \mathbb{R}^2.\end{aligned}\tag{3.33}$$

Following (3.33), the same argument of proving Lemma 3.1 then gives (3.28). Finally, since $f_4(x)$ is radially symmetric, there exists a radial solution ψ_4. Further, the property (2.15) gives the uniqueness of ψ_4. Therefore, ψ_4 must be radially symmetric, and the proof is complete. □

Lemma 3.4. *Suppose that $V(x) = h(x) \in C^2(\mathbb{R}^2)$ satisfies $\lim_{|x|\to\infty} h(x) = \infty$ and (1.14) for some $y_0 \in \mathbb{R}^2$, where $h(x)$ is homogeneous of degree $p \geq 2$. Then we have*

$$\int_{\mathbb{R}^2} w\psi_1 = 0, \quad \int_{\mathbb{R}^2} w\psi_2 = 0, \tag{3.34}$$

and

$$I = \int_{\mathbb{R}^2} \big(2w\psi_4 + \psi_2^2\big) = 0. \tag{3.35}$$

However, we have

$$II = 2\int_{\mathbb{R}^2} w\psi_5 + 2\int_{\mathbb{R}^2} \psi_1\psi_2 = -\frac{2+p}{2} < 0. \tag{3.36}$$

Here $\psi_1(x), \cdots, \psi_5(x) \in C^2(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ are given in Lemma 3.1 with $g(0) = 1$ and Lemma 3.3.

Since the proof of Lemma 3.4 is very involved, we leave it to the appendix. Following above lemmas, we are now ready to derive the comparison relation between β_k and α_k.

Proposition 3.5. *Suppose that $V(x) = h(x) \in C^2(\mathbb{R}^2)$ satisfies $\lim_{|x|\to\infty} h(x) = \infty$ and (1.14) for some $y_0 \in \mathbb{R}^2$, where $h(x)$ is homogeneous of degree $p \geq 2$. Then we have*

$$\beta_k = C^*\alpha_k \ \text{ as } \ k \to \infty, \tag{3.37}$$

where the constant C^* satisfies

$$C^* = \frac{2}{2+p}\Big(2\int_{\mathbb{R}^2} w\psi_3 + \int_{\mathbb{R}^2} \psi_1^2\Big) \neq 0. \tag{3.38}$$

Moreover, w_k satisfies

$$w_k := \big[\psi_1 + C^*\psi_2\big]\alpha_k + \big[\psi_3 + (C^*)^2\psi_4 + C^*\psi_5\big]\alpha_k^2 + o(\alpha_k^2) \ \text{ as } \ k \to \infty, \tag{3.39}$$

Here $\psi_1(x), \cdots, \psi_5(x) \in C^2(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ are given in Lemma 3.1 with $g(0) = 1$ and Lemma 3.3.

Proof. Note from (3.2) that w_k satisfies

$$\int_{\mathbb{R}^2} w^2 = \int_{\mathbb{R}^2} \bar{u}_k^2 = \int_{\mathbb{R}^2} \big(w + w_k\big)^2, \ \text{i.e.} \ \ 2\int_{\mathbb{R}^2} ww_k + \int_{\mathbb{R}^2} w_k^2 = 0. \tag{3.40}$$

Applying (3.40), we then derive from Lemma 3.3 that

$$\begin{aligned}
0 &= 2\int_{\mathbb{R}^2} ww_k + \int_{\mathbb{R}^2} w_k^2 \\
&= 2\int_{\mathbb{R}^2} w(\alpha_k\psi_1 + \beta_k\psi_2 + \alpha_k^2\psi_3 + \beta_k^2\psi_4 + \alpha_k\beta_k\psi_5) \\
&\quad + \int_{\mathbb{R}^2} (\alpha_k\psi_1 + \beta_k\psi_2 + \alpha_k^2\psi_3 + \beta_k^2\psi_4 + \alpha_k\beta_k\psi_5)^2 + o([\alpha_k+\beta_k]^2) \\
&= \alpha_k\Big(2\int_{\mathbb{R}^2} w\psi_1\Big) + \beta_k\Big(2\int_{\mathbb{R}^2} w\psi_2\Big) + \beta_k^2\Big(2\int_{\mathbb{R}^2} w\psi_4 + \int_{\mathbb{R}^2}\psi_2^2\Big) \\
&\quad + \alpha_k\beta_k\Big(2\int_{\mathbb{R}^2} w\psi_5 + 2\int_{\mathbb{R}^2}\psi_1\psi_2\Big) + \alpha_k^2\Big(2\int_{\mathbb{R}^2} w\psi_3 + \int_{\mathbb{R}^2}\psi_1^2\Big) + o([\alpha_k+\beta_k]^2) \\
&= -\frac{2+p}{2}\alpha_k\beta_k + \alpha_k^2\Big(2\int_{\mathbb{R}^2} w\psi_3 + \int_{\mathbb{R}^2}\psi_1^2\Big) + o([\alpha_k+\beta_k]^2),
\end{aligned} \tag{3.41}$$

where Lemma 3.4 is used in the last equality. It then follows from (3.41) that

$$2\int_{\mathbb{R}^2} w\psi_3 + \int_{\mathbb{R}^2}\psi_1^2 \neq 0,$$

and moreover,

$$-\frac{2+p}{2}\beta_k + \alpha_k\Big(2\int_{\mathbb{R}^2} w\psi_3 + \int_{\mathbb{R}^2}\psi_1^2\Big) = 0, \quad i.e., \ \beta_k = C^*\alpha_k,$$

where $C^* \neq 0$ is as in (3.38). Finally, the expansion (3.39) follows directly from (3.37) and Lemma 3.3, and we are done. □

We remark from (3.1) and Proposition 3.5 that the Lagrange multiplier $\mu_k \in \mathbb{R}$ of the Euler-Lagrange equation (2.16) satisfies

$$\mu_k = -\frac{\lambda}{\varepsilon_k^2} + \lambda^2 C^*\varepsilon_k^p + o(\varepsilon_k^p) \ \text{ as } \ k \to \infty, \tag{3.42}$$

where $\lambda > 0$ is defined by (2.2) with $g(0) = 1$, and $C^* \neq 0$ is given by (3.38). Moreover, following above results we finally conclude the following refined spike profiles.

Theorem 3.6. *Suppose that $V(x) = h(x) \in C^2(\mathbb{R}^2)$ satisfies $\lim_{|x|\to\infty} h(x) = \infty$ and (1.14) for some $y_0 \in \mathbb{R}^2$, where $h(x)$ is homogeneous of degree $p \geq 2$. If u_a is a positive minimizer of $e(a)$ for $a < a^*$. Then for any sequence $\{a_k\}$ with $a_k \nearrow a^*$ as $k \to \infty$, there exist a subsequence, still denoted by $\{a_k\}$, of $\{a_k\}$ and $\{x_k\} \subset \mathbb{R}^2$ such that the subsequence solution $u_k = u_{a_k}$ satisfies for $\varepsilon_k := (a^* - a_k)^{\frac{1}{2+p}}$,*

$$\begin{aligned}
u_k(x) = \frac{\lambda}{\|w\|_2}\Big\{&\frac{1}{\varepsilon_k} w\Big(\frac{\lambda(x-x_k)}{\varepsilon_k}\Big) + \varepsilon_k^{1+p}\Big[\psi_1 + C^*\psi_2\Big]\Big(\frac{\lambda(x-x_k)}{\varepsilon_k}\Big) \\
&+ \varepsilon_k^{3+2p}\Big[\psi_3 + (C^*)^2\psi_4 + C^*\psi_5\Big]\Big(\frac{\lambda(x-x_k)}{\varepsilon_k}\Big)\Big\} + o(\varepsilon_k^{3+2p}) \ \text{ as } \ k \to \infty
\end{aligned} \tag{3.43}$$

uniformly in $\mathbb{R}^2$, where the unique maximum point x_k of u_k satisfies

$$\Big|\frac{\lambda x_k}{\varepsilon_k} - y_0\Big| = \varepsilon_k^{2+p} O(|y^0|) \ \text{ as } \ k \to \infty \tag{3.44}$$

for some $y^0 \in \mathbb{R}^2$, and $C^* \neq 0$ is given by (3.38). Here $\psi_1(x), \cdots, \psi_5(x) \in C^2(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ are given in Lemma 3.1 with $g(0) = 1$ and Lemma 3.3.

Proof. The refined spike profile (3.43) follows immediately from (3.2) and (3.39). Also, Lemma 3.2 and (3.37) yield that the estimate (3.44) holds. □

Proof of Theorem 1.2. Since the local uniqueness of Theorem 1.1 implies that Theorem 3.6 holds for the whole sequence $\{a_k\}$, Theorem 1.2 is proved. □

4 Refined Spike Profiles: $V(x) = g(x)h(x)$

The main purpose of this section is to derive Theorem 4.4 which extends the refined spike behavior of Theorem 1.2 to more general potentials $V(x) = g(x)h(x) \in C^2(\mathbb{R}^2)$, where $V(x)$ satisfies $\lim_{|x|\to\infty} V(x) = \infty$ and

(V). $h(-x) = h(x)$ satisfies (1.14) and is homogeneous of degree $p \geq 2$, $g(x) \in C^m(\mathbb{R}^2)$ for some $2 \leq m \in \mathbb{N} \cup \{+\infty\}$ satisfies $0 < C \leq g(x) \leq \frac{1}{C}$ in $\mathbb{R}^2$ and $G(x) := g(x) - g(0)$,

$$\mathcal{D}^\alpha G(0) = 0 \text{ for all } |\alpha| \leq m-1, \text{ and } \mathcal{D}^\alpha G(0) \neq 0 \text{ for some } |\alpha| = m.$$

Here it takes $m = +\infty$ if $g(x) \equiv 1$.

Remark 4.1. *The property $h(-x) = h(x)$ in the above assumption (V) implies that $y_0 = 0$ must occur in (1.14).*

For the above type of potentials $V(x)$, suppose $\{u_k\}$ is a positive minimizer sequence of $e(a_k)$ with $a_k \nearrow a^*$ as $k \to \infty$, and let w_k be defined by (3.2), where x_k is the unique maximum point of u_k. Then Lemma 3.1 still holds in this case, where $\alpha_k > 0$ and $\beta_k > 0$ are defined in (3.1). Similar to Lemma 3.2, we start with the following estimates.

Lemma 4.1. *Suppose $V(x) = g(x)h(x) \in C^2(\mathbb{R}^2)$ satisfies $\lim_{|x|\to\infty} V(x) = \infty$ and the assumption (V) for $p \geq 2$ and $2 \leq m \in \mathbb{N} \cup \{+\infty\}$. Then the unique maximum point x_k of u_k satisfies the following estimates:*

1. *If m is even, then we have*

$$\frac{\lambda\alpha_k|x_k|}{\varepsilon_k} = o\big([\alpha_k + \beta_k]^2 + \alpha_k\varepsilon_k^m\big) \text{ as } k \to \infty. \tag{4.1}$$

2. *If m is odd, then we have*

$$\frac{\lambda\alpha_k|x_k|}{\varepsilon_k} = O(\alpha_k\varepsilon_k^m|x_0|) + o\big([\alpha_k + \beta_k]^2 + \alpha_k\varepsilon_k^m\big) \text{ as } k \to \infty, \tag{4.2}$$

where $x_0 \in \mathbb{R}^2$ satisfies

$$g(0)\int_{\mathbb{R}^2} \frac{\partial w}{\partial x_1}\big[x_0 \cdot \nabla h(x)\big]w + \frac{1}{\lambda^m}\sum_{|\alpha|=m}\int_{\mathbb{R}^2}\frac{\partial w}{\partial x_1}\Big[\frac{x^\alpha}{\alpha!}\mathcal{D}^\alpha g(0)\Big]h(x)w = 0. \tag{4.3}$$

Proof. Recall that $\psi_1(x)$ and $\psi_2(x)$ are given in Lemma 3.1. Since $h(-x) = h(x)$, we have $\psi_i(-x) = \psi_i(x)$ for $i = 1, 2$ and thus

$$\int_{\mathbb{R}^2} \frac{\partial w}{\partial x_1}\psi_1 = \int_{\mathbb{R}^2} \frac{\partial w}{\partial x_1} w\psi_1^2 = \int_{\mathbb{R}^2} \frac{\partial w}{\partial x_1} w\psi_1\psi_2 = 0. \tag{4.4}$$

Since (1.14) holds with $y_0 = 0$ as shown in Remark 4.1, the same calculations of (3.17)–(3.18) then yield that

$$
\begin{aligned}
o(\alpha_k^2 + \alpha_k\beta_k) &= \int_{\mathbb{R}^2} \frac{\partial w}{\partial x_1} \mathcal{L}_k w_k \\
&= o(\alpha_k\beta_k + \beta_k^2) - \alpha_k \int_{\mathbb{R}^2} \frac{\partial w}{\partial x_1} \Big[\frac{1}{a^*} \bar{u}_k^3 + \frac{\bar{u}_k}{\lambda^{2+p}} g\big(\frac{\varepsilon_k x}{\lambda} + x_k\big) h\big(x + \frac{\lambda x_k}{\varepsilon_k}\big) \Big] \\
&= o(\alpha_k\beta_k + \beta_k^2) - \frac{\alpha_k}{a^*} \int_{\mathbb{R}^2} \frac{\partial w}{\partial x_1} (\bar{u}_k^3 - w^3) \\
&\quad - \frac{\alpha_k}{\lambda^{2+p}} \int_{\mathbb{R}^2} \frac{\partial w}{\partial x_1} \Big[g\big(\frac{\varepsilon_k x}{\lambda} + x_k\big) h\big(x + \frac{\lambda x_k}{\varepsilon_k}\big) \bar{u}_k - g(0) h(x) w \Big] \\
&= o(\alpha_k\beta_k + \beta_k^2) - \frac{\alpha_k}{\lambda^{2+p}} \int_{\mathbb{R}^2} \frac{\partial w}{\partial x_1} \Big[g\big(\frac{\varepsilon_k x}{\lambda} + x_k\big) h\big(x + \frac{\lambda x_k}{\varepsilon_k}\big) \bar{u}_k - g(0) h(x) w \Big] \\
&= o(\alpha_k\beta_k + \beta_k^2) - I_1,
\end{aligned}
\tag{4.5}
$$

where the first equality follows from (3.21) and (4.4). Similar to (3.19), we deduce from (1.14) with $y_0 = 0$ that

$$
\begin{aligned}
\frac{\lambda^{2+p}}{\alpha_k} I_1 &= \int_{\mathbb{R}^2} \frac{\partial w}{\partial x_1} \Big\{ g(0) h\big(x + \frac{\lambda x_k}{\varepsilon_k}\big) \big[\bar{u}_k - w\big] + g(0) \big[h\big(x + \frac{\lambda x_k}{\varepsilon_k}\big) - h(x) \big] w \Big\} \\
&\quad + \int_{\mathbb{R}^2} \frac{\partial w}{\partial x_1} \Big[g\big(\frac{\varepsilon_k x}{\lambda} + x_k\big) - g(0) \Big] h\big(x + \frac{\lambda x_k}{\varepsilon_k}\big) \bar{u}_k \\
&= o(\alpha_k + \big|\tfrac{\lambda x_k}{\varepsilon_k}\big| + \beta_k) + g(0) \int_{\mathbb{R}^2} \frac{\partial w}{\partial x_1} \Big(\frac{\lambda x_k}{\varepsilon_k} \cdot \nabla h(x) \Big) w \\
&\quad + \Big(\frac{\varepsilon_k}{\lambda}\Big)^m \sum_{|\alpha|=m} \int_{\mathbb{R}^2} \frac{\partial w}{\partial x_1} \Big[\frac{1}{\alpha!} \Big(x + \frac{\lambda x_k}{\varepsilon_k} \Big)^\alpha \mathcal{D}^\alpha g(0) \Big] h\big(x + \frac{\lambda x_k}{\varepsilon_k}\big) \bar{u}_k + o(\varepsilon_k^m),
\end{aligned}
$$

which then implies that

$$
\begin{aligned}
I_1 &= \frac{\alpha_k}{\lambda^{2+p}} \Big(\frac{\varepsilon_k}{\lambda}\Big)^m \sum_{|\alpha|=m} \int_{\mathbb{R}^2} \frac{\partial w}{\partial x_1} \Big[\frac{1}{\alpha!} \Big(x + \frac{\lambda x_k}{\varepsilon_k} \Big)^\alpha \mathcal{D}^\alpha g(0) \Big] h\big(x + \frac{\lambda x_k}{\varepsilon_k}\big) \bar{u}_k \\
&\quad + \frac{\alpha_k}{\lambda^{2+p}} g(0) \int_{\mathbb{R}^2} \frac{\partial w}{\partial x_1} \Big(\frac{\lambda x_k}{\varepsilon_k} \cdot \nabla h(x) \Big) w + o(\alpha_k^2 + \alpha_k\beta_k + \big|\frac{\lambda x_k}{\varepsilon_k}\big| + \alpha_k\varepsilon_k^m).
\end{aligned}
\tag{4.6}
$$

Combining (4.5) and (4.6), we then conclude from the estimate (3.11) that

$$
\begin{aligned}
&\frac{\alpha_k}{\lambda^{2+p}} g(0) \int_{\mathbb{R}^2} \frac{\partial w}{\partial x_1} \Big(\frac{\lambda x_k}{\varepsilon_k} \cdot \nabla h(x) \Big) w \\
&= -\frac{\alpha_k}{\lambda^{2+p}} \Big(\frac{\varepsilon_k}{\lambda}\Big)^m \sum_{|\alpha|=m} \int_{\mathbb{R}^2} \frac{\partial w}{\partial x_1} \Big[\frac{x^\alpha}{\alpha!} \mathcal{D}^\alpha g(0) \Big] h(x) w + o\Big([\alpha_k + \beta_k]^2 + \alpha_k\varepsilon_k^m \Big).
\end{aligned}
\tag{4.7}
$$

If m is even, one can note that

$$
\sum_{|\alpha|=m} \int_{\mathbb{R}^2} \frac{\partial w}{\partial x_1} \Big[\frac{x^\alpha}{\alpha!} \mathcal{D}^\alpha g(0) \Big] h(x) w = 0,
$$

and it then follows from (4.7) and (1.14) with $y_0 = 0$ that (4.1) holds. If m is odd, we then derive from (4.7) that both (4.2) and (4.3) hold. □

Lemma 4.2. *Suppose $V(x) = g(x)h(x) \in C^2(\mathbb{R}^2)$ satisfies $\lim_{|x|\to\infty} V(x) = \infty$ and the assumption (V) for $p \geq 2$ and $2 \leq m \in \mathbb{N} \cup \{+\infty\}$. Let $\psi_1(x)$ and $\psi_2(x)$ be given in Lemma 3.1 with $y_0 = 0$. Then w_k satisfies*

$$\begin{aligned} w_k :&= \alpha_k\psi_1 + \beta_k\psi_2 + \alpha_k^2\psi_3 + \beta_k^2\psi_4 \\ &\quad + \alpha_k\varepsilon_k^m\phi + \alpha_k\beta_k\psi_5 + o\big([\alpha_k+\beta_k]^2 + \alpha_k\varepsilon_k^m\big) \quad \text{as} \quad k \to \infty, \end{aligned} \tag{4.8}$$

where $\psi_i(x) \in C^2(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$, $i = 3, 4, 5$, solves uniquely

$$\nabla\psi_i(0) = 0 \quad \text{and} \quad \mathcal{L}\psi_i(x) = g_i(x) \quad \text{in} \ \mathbb{R}^2, \quad i = 3,4,5, \tag{4.9}$$

and $g_i(x)$ satisfies

$$g_i(x) = \begin{cases} 3w\psi_1^2 - \Big(\dfrac{3w^2}{a^*} + \dfrac{g(0)h(x)}{\lambda^{2+p}}\Big)\psi_1, & \text{if} \ \ i = 3; \\ 3w\psi_2^2 + \psi_2, & \text{if} \ \ i = 4; \\ 6w\psi_1\psi_2 + \psi_1 - \Big(\dfrac{3w^2}{a^*} + \dfrac{g(0)h(x)}{\lambda^{2+p}}\Big)\psi_2, & \text{if} \ \ i = 5. \end{cases} \tag{4.10}$$

Here $\phi \in C^2(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ solves uniquely

$$\begin{aligned} \mathcal{L}\phi(x) = -\frac{1}{\lambda^{2+p}}\Big\{&\big[x_0 \cdot \nabla h(x)\big]g(0)w \\ &+ \frac{1}{\lambda^m}\sum_{|\alpha|=m}\Big[\frac{x^\alpha}{\alpha!}\mathcal{D}^\alpha g(0)\Big]h(x)w\Big\} \quad \text{in} \ \mathbb{R}^2, \quad \text{and} \ \nabla\phi(0) = 0, \end{aligned} \tag{4.11}$$

where $x_0 = 0$ holds for the case where m is even, and $x_0 \in \mathbb{R}^2$ satisfies (4.3) for the case where m is odd.

Proof. Following Lemma 3.1(3), we set

$$v_k = w_k - \alpha_k\psi_1 - \beta_k\psi_2.$$

Similar to (3.32), we then have

$$\begin{aligned} \mathcal{L}_k v_k &= w_k(\alpha_k\psi_1 + \beta_k\psi_2)(3w + w_k) + \beta_k w_k - \frac{\alpha_k}{a^*}(\bar{u}_k^3 - w^3) \\ &\quad - \frac{\alpha_k}{\lambda^{2+p}}\Big[g\big(\frac{\varepsilon_k x}{\lambda} + x_k\big)h\big(x + \frac{\lambda x_k}{\varepsilon_k}\big)\bar{u}_k - g(0)h(x)w\Big] \\ &= w_k(\alpha_k\psi_1 + \beta_k\psi_2)(3w + w_k) + \beta_k w_k \\ &\quad - \frac{\alpha_k}{a^*}w_k(3w^2 + 3ww_k + w_k^2) - I_2, \end{aligned} \tag{4.12}$$

where I_2 satisfies

$$\begin{aligned}
I_2 = & \frac{\alpha_k}{\lambda^{2+p}}\Big\{\Big[g\big(\frac{\varepsilon_k x}{\lambda}+x_k\big)-g(0)\Big]h\big(x+\frac{\lambda x_k}{\varepsilon_k}\big)\bar u_k \\
& +g(0)\Big[h\big(x+\frac{\lambda x_k}{\varepsilon_k}\big)-h(x)\Big]\bar u_k+g(0)h(x)\big(\bar u_k-w\big)\Big\} \\
= & \frac{\alpha_k}{\lambda^{2+p}}\Big\{\Big(\frac{\varepsilon_k}{\lambda}\Big)^m\sum_{|\alpha|=m}\Big[\frac{1}{\alpha!}\Big(x+\frac{\lambda x_k}{\varepsilon_k}\Big)^\alpha\mathcal{D}^\alpha g(0)\Big]h\big(x+\frac{\lambda x_k}{\varepsilon_k}\big)\bar u_k \\
& +g(0)\Big(\frac{\lambda x_k}{\varepsilon_k}\cdot\nabla h(x)\Big)\bar u_k+g(0)h(x)w_k\Big\}+o\Big(\frac{\alpha_k x_k}{\varepsilon_k}+\alpha_k\varepsilon_k^m\Big) \\
= & \alpha_k w_k\frac{g(0)h(x)}{\lambda^{2+p}}+\frac{\alpha_k}{\lambda^{2+p}}\Big(\frac{\lambda x_k}{\varepsilon_k}\cdot\nabla h(x)\Big)g(0)\bar u_k \\
& +\frac{\alpha_k\varepsilon_k^m}{\lambda^{2+p+m}}\sum_{|\alpha|=m}\Big[\frac{1}{\alpha!}\Big(x+\frac{\lambda x_k}{\varepsilon_k}\Big)^\alpha\mathcal{D}^\alpha g(0)\Big]h\big(x+\frac{\lambda x_k}{\varepsilon_k}\big)\bar u_k+o\big(\alpha_k\varepsilon_k^m\big),
\end{aligned} \tag{4.13}$$

where Lemma 4.1 is used in the last equality. Applying Lemma 4.1 again, we then obtain from (4.12) and (4.13) that

$$\begin{aligned}
\mathcal{L}_k v_k = & w_k(\alpha_k\psi_1+\beta_k\psi_2)(3w+w_k)+\beta_k w_k-\frac{\alpha_k}{a^*}w_k^2(3w+w_k) \\
& -\alpha_k w_k\Big[\frac{3w^2}{a^*}+\frac{g(0)h(x)}{\lambda^{2+p}}\Big]-\frac{\alpha_k}{\lambda^{2+p}}\Big(\frac{\lambda x_k}{\varepsilon_k}\cdot\nabla h(x)\Big)g(0)\bar u_k \\
& -\frac{\alpha_k\varepsilon_k^m}{\lambda^{2+p+m}}\sum_{|\alpha|=m}\Big[\frac{1}{\alpha!}\Big(x+\frac{\lambda x_k}{\varepsilon_k}\Big)^\alpha\mathcal{D}^\alpha g(0)\Big]h\big(x+\frac{\lambda x_k}{\varepsilon_k}\big)\bar u_k+o\big(\alpha_k\varepsilon_k^m\big) \\
= & \alpha_k^2\Big[3w\psi_1^2-\Big(\frac{3w^2}{a^*}+\frac{g(0)h(x)}{\lambda^{2+p}}\Big)\psi_1\Big] \\
& +\alpha_k\beta_k\Big[6w\psi_1\psi_2+\psi_1-\Big(\frac{3w^2}{a^*}+\frac{g(0)h(x)}{\lambda^{2+p}}\Big)\psi_2\Big] \\
& -\frac{\alpha_k\varepsilon_k^m}{\lambda^{2+p}}\Big\{\big[x_0\cdot\nabla h(x)\big]g(0)w+\frac{1}{\lambda^m}\sum_{|\alpha|=m}\Big[\frac{x^\alpha}{\alpha!}\mathcal{D}^\alpha g(0)\Big]h(x)w\Big\} \\
& +\beta_k^2(3w\psi_2^2+\psi_2)+o\big([\alpha_k+\beta_k]^2+\alpha_k\varepsilon_k^m\big)\quad\text{in }\ \mathbb{R}^2,
\end{aligned} \tag{4.14}$$

where $x_0=0$ holds for the case where m is even, and $x_0\in\mathbb{R}^2$ satisfies (4.3) for the case where m is odd. Following (4.14), the same argument of proving Lemma 3.1 then gives (4.8), and the proof is therefore complete. □

Proposition 4.3. *Suppose $V(x)=g(x)h(x)\in C^2(\mathbb{R}^2)$ satisfies $\lim_{|x|\to\infty}V(x)=\infty$ and the assumption (V) for $p\geq 2$ and $2\leq m\in\mathbb{N}\cup\{+\infty\}$. Let $\psi_1(x),\cdots,\psi_5(x)\in C^2(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2)$ be given in Lemma 3.1 with $y_0=0$ and Lemma 4.2, and ϕ is given by (4.11).*

1. *If $m>2+p$, then*
$$\beta_k=C^*\alpha_k, \tag{4.15}$$
and w_k satisfies
$$w_k:=\big[\psi_1+C^*\psi_2\big]\alpha_k+\big[\psi_3+(C^*)^2\psi_4+C^*\psi_5\big]\alpha_k^2+o(\alpha_k^2)\ \text{ as }\ k\to\infty, \tag{4.16}$$
where the constant C^* satisfies
$$C^*:=\frac{2}{2+p}\Big(2\int_{\mathbb{R}^2}w\psi_3+\int_{\mathbb{R}^2}\psi_1^2\Big)\neq 0. \tag{4.17}$$

2. If $1 \le m \le 2+p$ and m is odd, then $\beta_k = C^*\alpha_k$ and w_k satisfies

$$\begin{aligned} w_k := & \big[\psi_1 + C^*\psi_2\big]\alpha_k + \phi\, \alpha_k \varepsilon_k^m \\ & + \big[\psi_3 + (C^*)^2\psi_4 + C^*\psi_5\big]\alpha_k^2 + o(\alpha_k \varepsilon_k^m) \;\; as \;\; k \to \infty, \end{aligned} \tag{4.18}$$

where the constant $C^* \neq 0$ is given by (4.17).

3. If $1 \le m < 2+p$ and m is even, consider

$$\mathcal{S} = \sum_{|\alpha|=m} \int_{\mathbb{R}^2} \Big[\frac{x^\alpha}{\alpha!}\mathcal{D}^\alpha g(0)\Big] h(x) w^2. \tag{4.19}$$

Then for the case where $\mathcal{S} = 0$, we have $\beta_k = C^*\alpha_k$ and w_k satisfies (4.18), where the constant $C^* \neq 0$ is given by (4.17). However, for the case where $\mathcal{S} \neq 0$, we have

$$\beta_k = C_1^* \varepsilon_k^m, \tag{4.20}$$

and w_k satisfies

$$w_k := C_1^*\psi_2\varepsilon_k^m + \psi_1\alpha_k + (C_1^*)^2\psi_4\varepsilon_k^{2m} + o(\varepsilon_k^{\min\{2+p,2m\}}) \;\; as \;\; k \to \infty, \tag{4.21}$$

where the constant C_1^* satisfies

$$C_1^* = -\frac{m+p}{(2+p)\lambda^{2+p+m}} \sum_{|\alpha|=m} \int_{\mathbb{R}^2} \Big[\frac{x^\alpha}{\alpha!}\mathcal{D}^\alpha g(0)\Big] h(x) w^2 \neq 0. \tag{4.22}$$

4. If $m = 2+p$ is even, then

$$\beta_k = C_2^*\alpha_k, \tag{4.23}$$

and w_k satisfies

$$w_k := \big[\psi_1 + C_2^*\psi_2\big]\alpha_k + \big[\psi_3 + (C_2^*)^2\psi_4 + C_2^*\psi_5 + \phi\big]\alpha_k^2 + o(\alpha_k^2) \;\; as \;\; k \to \infty, \tag{4.24}$$

where the constant C_2^* satisfies

$$C_2^* = \frac{2}{2+p}\Big[2\int_{\mathbb{R}^2} w\psi_3 + \int_{\mathbb{R}^2} \psi_1^2 + 2\int_{\mathbb{R}^2} w\phi\Big] \neq 0. \tag{4.25}$$

Proof. The same argument of proving Lemma 3.4 with $y_0 = 0$ yields that

$$\int_{\mathbb{R}^2} w\psi_1 = 0, \quad \int_{\mathbb{R}^2} w\psi_2 = 0 \;\text{ and }\; I = \int_{\mathbb{R}^2} \big(2w\psi_4 + \psi_2^2\big) = 0, \tag{4.26}$$

and

$$II = 2\int_{\mathbb{R}^2} w\psi_5 + 2\int_{\mathbb{R}^2} \psi_1\psi_2 = -\frac{2+p}{2} < 0. \tag{4.27}$$

It thus follows from (3.40) and Lemma 4.2 that

$$
\begin{aligned}
0 &= 2\int_{\mathbb{R}^2} w w_k + \int_{\mathbb{R}^2} w_k^2 \\
&= 2\int_{\mathbb{R}^2} w\big(\alpha_k\psi_1 + \beta_k\psi_2 + \alpha_k^2\psi_3 + \beta_k^2\psi_4 + \alpha_k\varepsilon_k^m\phi + \alpha_k\beta_k\psi_5\big) \\
&\quad + \int_{\mathbb{R}^2} \big(\alpha_k\psi_1 + \beta_k\psi_2 + \alpha_k^2\psi_3 + \beta_k^2\psi_4 + \alpha_k\varepsilon_k^m\phi + \alpha_k\beta_k\psi_5\big)^2 \\
&\quad + o\big([\alpha_k+\beta_k]^2 + \alpha_k\varepsilon_k^m\big) \\
&= \alpha_k\Big(2\int_{\mathbb{R}^2} w\psi_1\Big) + \beta_k\Big(2\int_{\mathbb{R}^2} w\psi_2\Big) + \beta_k^2\Big(2\int_{\mathbb{R}^2} w\psi_4 + \int_{\mathbb{R}^2}\psi_2^2\Big) \\
&\quad + \alpha_k\beta_k\Big(2\int_{\mathbb{R}^2} w\psi_5 + 2\int_{\mathbb{R}^2}\psi_1\psi_2\Big) + \alpha_k^2\Big(2\int_{\mathbb{R}^2} w\psi_3 + \int_{\mathbb{R}^2}\psi_1^2\Big) \\
&\quad + \alpha_k\varepsilon_k^m\Big(2\int_{\mathbb{R}^2} w\phi\Big) + o\big([\alpha_k+\beta_k]^2 + \alpha_k\varepsilon_k^m\big) \\
&= -\frac{2+p}{2}\alpha_k\beta_k + \alpha_k^2\Big(2\int_{\mathbb{R}^2} w\psi_3 + \int_{\mathbb{R}^2}\psi_1^2\Big) + \alpha_k\varepsilon_k^m\Big(2\int_{\mathbb{R}^2} w\phi\Big) \\
&\quad + o\big([\alpha_k+\beta_k]^2 + \alpha_k\varepsilon_k^m\big),
\end{aligned} \tag{4.28}
$$

where (4.26) and (4.27) are used in the last equality. Following (4.28), we next carry out the proof by considering separately the following four cases:

Case 1. $m > 2+p$. In this case, it follows from (4.28) that the constant C^* defined in (4.17) is nonzero and

$$
-\frac{2+p}{2}\beta_k + \alpha_k\Big(2\int_{\mathbb{R}^2} w\psi_3 + \int_{\mathbb{R}^2}\psi_1^2\Big) = 0, \quad i.e.,\ \beta_k = C^*\alpha_k.
$$

Moreover, the expansion (4.16) follows directly from (4.15) and Lemma 4.2, and Case 1 is therefore proved.

Case 2. $1 \le m \le 2+p$ *and* m *is odd.* In this case, since m is odd and $h(-x) = h(x)$, we obtain from (3.10) and (4.11) that

$$
\begin{aligned}
2\int_{\mathbb{R}^2} w\phi &= 2\int_{\mathbb{R}^2} \phi\mathcal{L}\psi_2 = 2\int_{\mathbb{R}^2}\psi_2\mathcal{L}\phi \\
&= \frac{1}{\lambda^{2+p}}\int_{\mathbb{R}^2}\Big\{\big[x_0\cdot\nabla h(x)\big]g(0)w \\
&\quad + \frac{1}{\lambda^m}\sum_{|\alpha|=m}\Big[\frac{x^\alpha}{\alpha!}\mathcal{D}^\alpha g(0)\Big]h(x)w\Big\}\big(w + x\cdot\nabla w\big) = 0.
\end{aligned}
$$

We then derive from (4.28) that (4.17) still holds and thus $\beta_k = C^*\alpha_k$. Further, the expansion (4.18) follows directly from (4.8) and (4.15).

Case 3. $1 \le m < 2+p$ *and* m *is even.* Since m is even, then $x_0 = 0$ holds in (4.11). Further, since $x^\alpha h(x)$ is homogeneous of degree $m+p$, we then obtain from (4.11) that

$$
\begin{aligned}
2\int_{\mathbb{R}^2} w\phi &= 2\int_{\mathbb{R}^2} \phi\mathcal{L}\psi_2 = 2\int_{\mathbb{R}^2} \psi_2\mathcal{L}\phi \\
&= \frac{1}{\lambda^{2+p+m}} \sum_{|\alpha|=m} \int_{\mathbb{R}^2} \Big[\frac{x^\alpha}{\alpha!}\mathcal{D}^\alpha g(0)\Big] h(x) w\big(w + x\cdot\nabla w\big) \\
&= \frac{1}{\lambda^{2+p+m}} \sum_{|\alpha|=m} \int_{\mathbb{R}^2} \Big[\frac{x^\alpha}{\alpha!}\mathcal{D}^\alpha g(0)\Big] h(x) w^2 \\
&\quad + \frac{1}{2\lambda^{2+p+m}} \sum_{|\alpha|=m} \int_{\mathbb{R}^2} \Big[\frac{x^\alpha}{\alpha!}\mathcal{D}^\alpha g(0)\Big] h(x)\big(x\cdot\nabla w^2\big) \\
&= \frac{1}{\lambda^{2+p+m}} \sum_{|\alpha|=m} \int_{\mathbb{R}^2} \Big[\frac{x^\alpha}{\alpha!}\mathcal{D}^\alpha g(0)\Big] h(x) w^2 \\
&\quad - \frac{1}{2\lambda^{2+p+m}} \sum_{|\alpha|=m} \int_{\mathbb{R}^2} w^2 \Big\{ 2\Big[\frac{x^\alpha}{\alpha!}\mathcal{D}^\alpha g(0) h(x)\Big] \\
&\quad + x\cdot\nabla\Big[\frac{x^\alpha}{\alpha!}\mathcal{D}^\alpha g(0) h(x)\Big]\Big\} \\
&= -\frac{m+p}{2\lambda^{2+p+m}} \sum_{|\alpha|=m} \int_{\mathbb{R}^2} \Big[\frac{x^\alpha}{\alpha!}\mathcal{D}^\alpha g(0)\Big] h(x) w^2 := -\frac{m+p}{2\lambda^{2+p+m}}\mathcal{S},
\end{aligned}
\tag{4.29}
$$

where $\mathcal{S}$ is as in (4.19). Therefore, if $\mathcal{S} = 0$, then we are in the same situation as that of above Case 2, which gives that $\beta_k = C^*\alpha_k$ and w_k satisfies (4.18), where the constant $C^* \neq 0$ is given by (4.17).

We next consider the case where $\mathcal{S} \neq 0$. By applying (4.29), in this case we derive from (4.28) that

$$
-\frac{2+p}{2}\alpha_k\beta_k + \alpha_k\varepsilon_k^m\Big(2\int_{\mathbb{R}^2} w\phi\Big) = 0,
$$

which implies that $\beta_k = C_1^*\varepsilon_k^m$, where the constant $C_1^* \neq 0$ satisfies (4.22) in view of (4.29). Further, the expansion (4.21) follows directly from (4.20) and Lemma 4.2.

Case 4. $m = 2+p$ *is even.* In this case, we derive from (4.28) that

$$
-\frac{2+p}{2}\alpha_k\beta_k + \alpha_k^2\Big(2\int_{\mathbb{R}^2} w\psi_3 + \int_{\mathbb{R}^2} \psi_1^2 + 2\int_{\mathbb{R}^2} w\phi\Big) = 0,
$$

which gives that $\beta_k = C_2^*\alpha_k$, where the constant $C_2^* \neq 0$ satisfies (4.25). Further, the expansion (4.24) follows directly from (4.23) and Lemma 4.2. □

Applying directly Lemmas 4.1 and 4.2 as well as Proposition 4.3, we now conclude the following main results of this section. Recall that $\lambda > 0$ is defined by (2.2) with $y_0 = 0$, $\psi_1(x), \cdots, \psi_5(x) \in C^2(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ are given in Lemma 3.1 with $y_0 = 0$ and Lemma 4.2, and ϕ is given by (4.11).

Theorem 4.4. *Suppose* $V(x) = g(x)h(x) \in C^2(\mathbb{R}^2)$ *satisfies* $\lim_{|x|\to\infty} V(x) = \infty$ *and the assumption* (V) *for* $p \ge 2$ *and* $2 \le m \in \mathbb{N} \cup \{+\infty\}$*. Let* u_a *be a positive minimizer of (1.1) for* $a < a^*$*. Then for any sequence* $\{a_k\}$ *with* $a_k \nearrow a^*$ *as* $k \to \infty$*, there exists a subsequence, still denoted by* $\{a_k\}$*, of* $\{a_k\}$ *such that* $u_k = u_{a_k}$ *has a unique maximum point* $x_k \in \mathbb{R}^2$ *and satisfies for* $\varepsilon_k := (a^* - a_k)^{\frac{1}{2+p}}$,

1. *If $m > 2+p$, then we have*

$$u_k(x) = \frac{\lambda}{\|w\|_2}\Big\{\frac{1}{\varepsilon_k} w\Big(\frac{\lambda(x-x_k)}{\varepsilon_k}\Big) + \varepsilon_k^{1+p}\Big[\psi_1 + C^*\psi_2\Big]\Big(\frac{\lambda(x-x_k)}{\varepsilon_k}\Big)$$
$$+\varepsilon_k^{3+2p}\Big[\psi_3 + (C^*)^2\psi_4 + C^*\psi_5\Big]\Big(\frac{\lambda(x-x_k)}{\varepsilon_k}\Big)\Big\} + o(\varepsilon_k^{3+2p}) \text{ as } k\to\infty \tag{4.30}$$

uniformly in $\mathbb{R}^2$, where x_k satisfies

$$\frac{|x_k|}{\varepsilon_k} = O(\varepsilon_k^m |y^0|) + o(\varepsilon_k^{2+p}) \text{ as } k\to\infty \tag{4.31}$$

for some $y^0 \in \mathbb{R}^2$, and the constant $C^* \neq 0$ is given by (4.17). Further, if m is even, then x_k satisfies

$$\frac{|x_k|}{\varepsilon_k^{3+p}} = o(1) \text{ as } k\to\infty. \tag{4.32}$$

2. *If $1 \le m \le 2+p$ and m is odd, then we have*

$$u_k(x) = \frac{\lambda}{\|w\|_2}\Big\{\frac{1}{\varepsilon_k} w\Big(\frac{\lambda(x-x_k)}{\varepsilon_k}\Big) + \varepsilon_k^{1+p}\Big[\psi_1 + C^*\psi_2\Big]\Big(\frac{\lambda(x-x_k)}{\varepsilon_k}\Big)$$
$$+\varepsilon_k^{3+2p}\big[\psi_3 + (C^*)^2\psi_4 + C^*\psi_5\big]\Big(\frac{\lambda(x-x_k)}{\varepsilon_k}\Big) \tag{4.33}$$
$$+\varepsilon_k^{1+m+p}\phi\Big(\frac{\lambda(x-x_k)}{\varepsilon_k}\Big)\Big\} + o(\varepsilon_k^{1+m+p}) \text{ as } k\to\infty$$

uniformly in $\mathbb{R}^2$, where x_k satisfies

$$\frac{|x_k|}{\varepsilon_k^{m+1}} = O(|y^0|) \text{ as } k\to\infty. \tag{4.34}$$

for some $y^0 \in \mathbb{R}^2$, and the constant $C^* \neq 0$ is given by (4.17).

3. *If $m = 2+p$ is even, then we have*

$$u_k(x) = \frac{\lambda}{\|w\|_2}\Big\{\frac{1}{\varepsilon_k} w\Big(\frac{\lambda(x-x_k)}{\varepsilon_k}\Big) + \varepsilon_k^{1+p}\Big[\psi_1 + C_2^*\psi_2\Big]\Big(\frac{\lambda(x-x_k)}{\varepsilon_k}\Big)$$
$$+\varepsilon_k^{3+2p}\Big[\psi_3 + (C_2^*)^2\psi_4 + C_2^*\psi_5 + \phi\Big]\Big(\frac{\lambda(x-x_k)}{\varepsilon_k}\Big)\Big\} + o(\varepsilon_k^{3+2p}) \text{ as } k\to\infty \tag{4.35}$$

uniformly in $\mathbb{R}^2$, where x_k satisfies (4.32) and the constant $C_2^* \neq 0$ is defined by (4.25).

4. *If $1 \le m < 2+p$ and m is even, let the constant $\mathcal{S}$ be defined in (4.19). Then for the case where $\mathcal{S} = 0$, u_k satisfies (4.33) and x_k satisfies*

$$\frac{|x_k|}{\varepsilon_k^{m+1}} = o(1) \text{ as } k\to\infty. \tag{4.36}$$

However, for the case where $\mathcal{S} \neq 0$, u_k satisfies

$$u_k(x) = \frac{\lambda}{\|w\|_2}\Big\{\frac{1}{\varepsilon_k} w\Big(\frac{\lambda(x-x_k)}{\varepsilon_k}\Big) + \varepsilon_k^{m-1}C_1^*\psi_2\Big(\frac{\lambda(x-x_k)}{\varepsilon_k}\Big)$$
$$+\varepsilon_k^{2m-1}(C_1^*)^2\psi_4\Big(\frac{\lambda(x-x_k)}{\varepsilon_k}\Big) + \varepsilon_k^{1+p}\psi_1\Big(\frac{\lambda(x-x_k)}{\varepsilon_k}\Big)\Big\} \tag{4.37}$$
$$+o(\varepsilon_k^{\min\{2+p,2m\}-1}) \text{ as } k\to\infty$$

uniformly in $\mathbb{R}^2$, where x_k satisfies (4.36), and the constant $C_1^* \neq 0$ is defined by (4.22).

Proof. (1). If $m > 2+p$, then (4.30) follows directly from Proposition 4.3(1), and (4.31) follows from Lemma 4.1. Specially, if m is even, then Lemma 4.1 gives $y^0 = 0$, and therefore (4.31) implies (4.32).

(2). If $1 \le m \le 2+p$ and m is odd, then Proposition 4.3(2) gives (4.33). Moreover, it yields from (4.2) that x_k satisfies $\left|\frac{x_k}{\varepsilon_k}\right| = O(\varepsilon_k^m |y^0|) + o(\varepsilon_k^m)$ as $k \to \infty$, which then implies (4.34) for some $y^0 \in \mathbb{R}^2$.

(3). If $m = 2+p$ is even, then Proposition 4.3(4) gives (4.35), and we reduce from (4.1) that x_k satisfies (4.32).

(4). If $1 \le m < 2+p$ and m is even, it then follows from (4.1) that x_k always satisfies (4.36). Moreover, Proposition 4.3(3) gives that if $\mathcal{S} = 0$, then u_k satisfies (4.33); if $\mathcal{S} \neq 0$, then u_k satisfies (4.37). □

A Appendix: The Proof of Lemma 3.4

In this appendix, we shall follow Lemmas 3.1 and 3.3 to address the proof of Lemma 3.4, i.e., (3.34)–(3.36).

The proof of (3.34). Under the assumptions of Lemma 3.4, we first note that the equation (3.7) can be simplified as

$$\nabla \psi_1(0) = 0, \quad \mathcal{L}\psi_1 = -\frac{2w^3}{\int_{\mathbb{R}^2} w^4} - \frac{2h(x+y_0)w}{p\int_{\mathbb{R}^2} h(x+y_0)w^2} \text{ in } \mathbb{R}^2, \tag{A.1}$$

due to the fact that

$$a^* = \|w\|_2^2 = \frac{1}{2}\int_{\mathbb{R}^2} w^4. \tag{A.2}$$

By (1.14), (3.10) and (A.1), we then have

$$\begin{aligned}
2\int_{\mathbb{R}^2} w\psi_1 &= 2\int_{\mathbb{R}^2} \mathcal{L}\psi_2\psi_1 = 2\int_{\mathbb{R}^2} \psi_2\mathcal{L}\psi_1 \\
&= \int_{\mathbb{R}^2} \Big[\frac{2w^3}{\int_{\mathbb{R}^2} w^4} + \frac{2h(x+y_0)w}{p\int_{\mathbb{R}^2} h(x+y_0)w^2}\Big](w + x\cdot\nabla w) \\
&= 2 + \frac{2}{p} + \frac{2}{\int_{\mathbb{R}^2} w^4}\int_{\mathbb{R}^2} w^3\big(x\cdot\nabla w\big) + \frac{2}{p\int_{\mathbb{R}^2} h(x+y_0)w^2}\int_{\mathbb{R}^2} h(x+y_0)w\big(x\cdot\nabla w\big) \\
&= 2 + \frac{2}{p} + \frac{1}{2\int_{\mathbb{R}^2} w^4}\int_{\mathbb{R}^2} \big(x\cdot\nabla w^4\big) + \frac{1}{p\int_{\mathbb{R}^2} h(x+y_0)w^2}\int_{\mathbb{R}^2} h(x+y_0)\big(x\cdot\nabla w^2\big) \\
&= 2 + \frac{2}{p} - 1 - \frac{1}{p\int_{\mathbb{R}^2} h(x+y_0)w^2}\int_{\mathbb{R}^2} w^2\Big[2h(x+y_0) + \big(x\cdot\nabla h(x+y_0)\big)\Big] \\
&= 2 + \frac{2}{p} - 1 - \frac{(p+2)}{p} = 0,
\end{aligned}$$

since $(x+y_0)\cdot\nabla h(x+y_0) = ph(x+y_0)$ and $\int_{\mathbb{R}^2} w^2\big[y_0\cdot\nabla h(x+y_0)\big] = 0$. Also, we deduce from (3.10) that

$$2\int_{\mathbb{R}^2} w\psi_2 = -\int_{\mathbb{R}^2} w(w + x\cdot\nabla w) = -\int_{\mathbb{R}^2} w^2 - \frac{1}{2}\int_{\mathbb{R}^2} (x\cdot\nabla w^2) = 0,$$

which thus completes the proof of (3.34). □

The proof of (3.35). By Lemmas 3.1 and 3.3, we obtain that

$$\begin{aligned}
I &= \int_{\mathbb{R}^2} \big(2w\psi_4 + \psi_2^2\big) = \int_{\mathbb{R}^2} \psi_2^2 + 2\langle \mathcal{L}\psi_2, \psi_4\rangle \\
&= \int_{\mathbb{R}^2} \psi_2^2 + 2\langle \psi_2, \mathcal{L}\psi_4\rangle = \int_{\mathbb{R}^2} \psi_2^2 + 2\langle \psi_2, (3w\psi_2^2 + \psi_2)\rangle \\
&= 3\int_{\mathbb{R}^2} \psi_2^2 + 6\int_{\mathbb{R}^2} w\psi_2^3,
\end{aligned}$$

which implies that

$$\begin{aligned}
\frac{2I}{3\pi} &= \frac{2}{\pi}\Big[\int_{\mathbb{R}^2} \psi_2^2 + 2\int_{\mathbb{R}^2} w\psi_2^3\Big] \\
&= \int_0^\infty r(w + rw')^2 - \int_0^\infty rw(w + rw')^3 := A - B.
\end{aligned} \tag{A.3}$$

Here we have

$$\begin{aligned}
A = \int_0^\infty r(w + rw')^2 &= \int_0^\infty r^3(w')^2 + \int_0^\infty rw^2 + \int_0^\infty r^2 dw^2 \\
&= \int_0^\infty r^3(w')^2 - \frac{1}{2}\int_0^\infty rw^4,
\end{aligned}$$

where (A.2) is used, and

$$\begin{aligned}
B &= \int_0^\infty rw(w + rw')^3 \\
&= \Big[\int_0^\infty rw^4 + 3\int_0^\infty r^2w^3w'\Big] + 3\int_0^\infty r^3w^2w'w' + \int_0^\infty r^4w(w')^3 \\
&= -\frac{1}{2}\int_0^\infty rw^4 + 3\int_0^\infty r^3w^2w'w' + \int_0^\infty r^4w(w')^3.
\end{aligned}$$

Therefore, we get from (A.3) that

$$\frac{2I}{3\pi} = \int_0^\infty r^3(w')^2 - \int_0^\infty r^4w(w')^3 - 3\int_0^\infty r^3w^2w'w' := C + D + E. \tag{A.4}$$

To further simplify I, recall that

$$rw'' = -w' + rw - rw^3, \tag{A.5}$$

by which we then have

$$\begin{aligned}
C = \int_0^\infty r^3w'dw &= -\int_0^\infty w(r^3w')' \\
&= -\int_0^\infty w\big[3r^2w' + r^2(-w' + rw - rw^3)\big] \\
&= -\int_0^\infty w\big[2r^2w' + r^3w - r^3w^3\big] \\
&= 2\int_0^\infty rw^2 - \int_0^\infty r^3w^2 + \int_0^\infty r^3w^4.
\end{aligned}$$

Similarly, we have

$$
\begin{aligned}
D = -\frac{1}{2}\int_0^\infty r^4(w')^2 dw^2 &= \frac{1}{2}\int_0^\infty w^2\big[4r^3(w')^2 + 2r^3w'(-w' + rw - rw^3)\big] \\
&= \int_0^\infty r^3w^2(w')^2 + \frac{1}{4}\int_0^\infty r^4 dw^4 - \frac{1}{6}\int_0^\infty r^4 dw^6 \\
&= \int_0^\infty r^3w^2(w')^2 - \int_0^\infty r^3w^4 + \frac{2}{3}\int_0^\infty r^3w^6.
\end{aligned}
$$

Note from (A.5) that

$$
\begin{aligned}
-2\int_0^\infty r^3w^2(w')^2 &= -\frac{2}{3}\int_0^\infty r^3w' dw^3 = \frac{2}{3}\int_0^\infty w^3\big[3r^2w' + r^2(-w' + rw - rw^3)\big] \\
&= \frac{4}{3}\int_0^\infty w^3r^2w' + \frac{2}{3}\int_0^\infty r^3w^4 - \frac{2}{3}\int_0^\infty r^3w^6 \\
&= -\frac{2}{3}\int_0^\infty rw^4 + \frac{2}{3}\int_0^\infty r^3w^4 - \frac{2}{3}\int_0^\infty r^3w^6.
\end{aligned}
$$

We thus derive that

$$
\begin{aligned}
D + E &= -2\int_0^\infty r^3w^2(w')^2 - \int_0^\infty r^3w^4 + \frac{2}{3}\int_0^\infty r^3w^6 \\
&= -\frac{2}{3}\int_0^\infty rw^4 - \frac{1}{3}\int_0^\infty r^3w^4,
\end{aligned}
$$

by which we conclude from (A.2) and (A.4) that

$$
\frac{2I}{3\pi} = C + D + E = \frac{1}{3}\Big[2\int_0^\infty rw^2 - 3\int_0^\infty r^3w^2 + 2\int_0^\infty r^3w^4\Big]. \tag{A.6}
$$

In the following, we note that w satisfies

$$
(rw')' = rw - rw^3, \;\; r > 0. \tag{A.7}
$$

Multiplying (A.7) by r^3w' and integrating on $[0,\infty)$, we get that

$$
\begin{aligned}
\int_0^\infty r^3w'(rw')' = \int_0^\infty r^3w'[rw - rw^3] &= \frac{1}{2}\int_0^\infty r^4 dw^2 - \frac{1}{4}\int_0^\infty r^4 dw^4 \\
&= -2\int_0^\infty r^3w^2 + \int_0^\infty r^3w^4.
\end{aligned}
$$

Note also that

$$
\int_0^\infty r^3w'(rw')' = \int_0^\infty r^3(w')^2 + \frac{1}{2}\int_0^\infty r^4 d(w')^2 = -\int_0^\infty r^3(w')^2.
$$

By combining above two identities, it yields that

$$
\int_0^\infty r^3(w')^2 = 2\int_0^\infty r^3w^2 - \int_0^\infty r^3w^4. \tag{A.8}
$$

On the other hand, multiplying (A.7) by r^2w and integrating on $[0,\infty)$, we obtain that

$$\begin{aligned}\int_0^\infty r^3w^2-\int_0^\infty r^3w^4&=\int_0^\infty r^2w(rw')'=\int_0^\infty r^2ww'+\int_0^\infty r^3wdw'\\&=\int_0^\infty r^2ww'-\int_0^\infty w'(3r^2w+r^3w')\\&=-2\int_0^\infty r^2ww'-\int_0^\infty r^3(w')^2\\&=2\int_0^\infty rw^2-\int_0^\infty r^3(w')^2,\end{aligned}$$

which then implies that

$$\int_0^\infty r^3(w')^2=2\int_0^\infty rw^2-\int_0^\infty r^3w^2+\int_0^\infty r^3w^4. \tag{A.9}$$

We thus conclude from (A.8) and (A.9) that

$$2\int_0^\infty rw^2-3\int_0^\infty r^3w^2+2\int_0^\infty r^3w^4=0,$$

which therefore implies that $I=0$ in view of (A.6), i.e., (3.35) holds. □

The proof of (3.36). Following Lemmas 3.1 and 3.3 again, we get that

$$\begin{aligned}II=&\ 2\int_{\mathbb{R}^2}\psi_5\mathcal{L}\psi_2+2\int_{\mathbb{R}^2}\psi_1\psi_2=2\int_{\mathbb{R}^2}\psi_2[\mathcal{L}\psi_5+\psi_1]\\=&\ -\int_{\mathbb{R}^2}(w+x\cdot\nabla w)(6w\psi_1\psi_2+2\psi_1)\\&-\frac12\int_{\mathbb{R}^2}\Big[\frac{3w^2}{a^*}+\frac{h(x+y_0)}{\lambda^{2+p}}\Big]\big(w+x\cdot\nabla w\big)^2\\&+\frac{1}{2\lambda^{2+p}}\int_{\mathbb{R}^2}(w+x\cdot\nabla w)\big[y_0\cdot\nabla h(x+y_0)\big]w\\:=&\ A+B.\end{aligned}\tag{A.10}$$

Since $(x+y_0)\cdot\nabla h(x+y_0)=ph(x+y_0)$ holds in $\mathbb{R}^2$, we derive from (1.14) and (A.1) that

$$\begin{aligned}B=&\ -\frac12\int_{\mathbb{R}^2}\Big[\frac{3w^2}{a^*}+\frac{h(x+y_0)}{\lambda^{2+p}}\Big]\big[w^2+2w(x\cdot\nabla w)+(x\cdot\nabla w)^2\big]\\&+\frac{1}{2\lambda^{2+p}}\int_{\mathbb{R}^2}(w+x\cdot\nabla w)\big[y_0\cdot\nabla h(x+y_0)\big]w\\=&\ -\int_{\mathbb{R}^2}\Big[\frac{3w^2}{\int_{\mathbb{R}^2}w^4}+\frac{h(x+y_0)}{p\int_{\mathbb{R}^2}h(x+y_0)w^2}\Big]\big[w^2+2w(x\cdot\nabla w)\big]\\&-\frac12\int_{\mathbb{R}^2}\Big[\frac{3w^2}{a^*}+\frac{h(x+y_0)}{\lambda^{2+p}}\Big]\big(x\cdot\nabla w\big)^2+\frac{1}{2\lambda^{2+p}}\int_{\mathbb{R}^2}w\big[y_0\cdot\nabla h(x+y_0)\big](x\cdot\nabla w)\\=&\ -3-\frac1p-\frac{3}{2\int_{\mathbb{R}^2}w^4}\int_{\mathbb{R}^2}(x\cdot\nabla w^4)-\frac{1}{p\int_{\mathbb{R}^2}h(x+y_0)w^2}\int_{\mathbb{R}^2}h(x+y_0)\big(x\cdot\nabla w^2\big)\\&-\frac12\int_{\mathbb{R}^2}\Big[\frac{3w^2}{a^*}+\frac{h(x+y_0)}{\lambda^{2+p}}\Big]\big(x\cdot\nabla w\big)^2+\frac{1}{2\lambda^{2+p}}\int_{\mathbb{R}^2}w\big[y_0\cdot\nabla h(x+y_0)\big](x\cdot\nabla w)\\:=&\ -3-\frac1p+3+\frac{2+p}{p}+C_0=\frac{p+1}{p}+C_0,\end{aligned}$$

where the term C_0 satisfies

$$\begin{aligned}
C_0 &= -\frac{1}{2}\int_{\mathbb{R}^2}\Big[\frac{3w^2}{a^*}+\frac{h(x+y_0)}{\lambda^{2+p}}\Big](x\cdot\nabla w)^2+\frac{1}{2\lambda^{2+p}}\int_{\mathbb{R}^2}w\big[y_0\cdot\nabla h(x+y_0)\big](x\cdot\nabla w)\\
&= -\frac{1}{2a^*}\int_{\mathbb{R}^2}(x\cdot\nabla w)(x\cdot\nabla w^3)-\frac{1}{2\lambda^{2+p}}\int_{\mathbb{R}^2}h(x+y_0)\big(x\cdot\nabla w\big)(x\cdot\nabla w)\\
&\quad+\frac{1}{2\lambda^{2+p}}\int_{\mathbb{R}^2}w\big[y_0\cdot\nabla h(x+y_0)\big](x\cdot\nabla w)\\
&= \frac{1}{2a^*}\int_{\mathbb{R}^2}w^3\Big[2(x\cdot\nabla w)+x\cdot\nabla(x\cdot\nabla w)\Big]\\
&\quad+\frac{1}{2\lambda^{2+p}}\int_{\mathbb{R}^2}w\Big\{2h(x+y_0)(x\cdot\nabla w)+\big[x\cdot\nabla h(x+y_0)\big](x\cdot\nabla w)\\
&\qquad\qquad+h(x+y_0)\Big[x\cdot\nabla(x\cdot\nabla w)\Big]\Big\}+\frac{1}{2\lambda^{2+p}}\int_{\mathbb{R}^2}w\big[y_0\cdot\nabla h(x+y_0)\big](x\cdot\nabla w)\\
&= \frac{1}{2}\int_{\mathbb{R}^2}\Big[\frac{w^3}{a^*}+\frac{wh(x+y_0)}{\lambda^{2+p}}\Big]\big[x\cdot\nabla\big(x\cdot\nabla w\big)\big]\\
&\quad+\frac{1}{a^*}\int_{\mathbb{R}^2}w^3(x\cdot\nabla w)+\frac{2+p}{2\lambda^{2+p}}\int_{\mathbb{R}^2}wh(x+y_0)(x\cdot\nabla w)\\
&= \frac{1}{2}\int_{\mathbb{R}^2}\Big[\frac{w^3}{a^*}+\frac{wh(x+y_0)}{\lambda^{2+p}}\Big]\big[x\cdot\nabla\big(x\cdot\nabla w\big)\big]\\
&\quad+\frac{1}{2\int_{\mathbb{R}^2}w^4}\int_{\mathbb{R}^2}(x\cdot\nabla w^4)+\frac{2+p}{2p\int_{\mathbb{R}^2}h(x+y_0)w^2}\int_{\mathbb{R}^2}h(x+y_0)\big(x\cdot\nabla w^2\big)\\
&= \frac{1}{2}\int_{\mathbb{R}^2}\Big[\frac{w^3}{a^*}+\frac{wh(x+y_0)}{\lambda^{2+p}}\Big]\big[x\cdot\nabla\big(x\cdot\nabla w\big)\big]-1-\frac{(2+p)^2}{2p},
\end{aligned}$$

in view of (A.1). We thus have

$$B=\frac{1}{2}\int_{\mathbb{R}^2}\Big[\frac{w^3}{a^*}+\frac{wh(x+y_0)}{\lambda^{2+p}}\Big]\big[x\cdot\nabla\big(x\cdot\nabla w\big)\big]-\frac{p^2+4p+2}{2p}. \tag{A.11}$$

We next calculate the term A as follows. Observe that

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}^2}\psi_1 x\cdot\nabla\big(x\cdot\nabla w\big)\\
&= -\frac{1}{2}\int_{\mathbb{R}^2}\big(x\cdot\nabla w\big)\big[2\psi_1+\big(x\cdot\nabla\psi_1\big)\big]\\
&= -\int_{\mathbb{R}^2}\psi_1\big(x\cdot\nabla w\big)-\frac{1}{2}\int_{\mathbb{R}^2}\big(x\cdot\nabla\psi_1\big)\big(x\cdot\nabla w\big)\\
&= -\int_{\mathbb{R}^2}\psi_1\big(x\cdot\nabla w\big)+\frac{1}{2}\int_{\mathbb{R}^2}w\big[2\big(x\cdot\nabla\psi_1\big)+x\cdot\nabla\big(x\cdot\nabla\psi_1\big)\big]\\
&= -\int_{\mathbb{R}^2}\psi_1\big(x\cdot\nabla w\big)+\int_{\mathbb{R}^2}w\big(x\cdot\nabla\psi_1\big)+\frac{1}{2}\int_{\mathbb{R}^2}wx\cdot\nabla\big(x\cdot\nabla\psi_1\big),
\end{aligned}$$

which implies that

$$\begin{aligned}
&-\int_{\mathbb{R}^2}\psi_1\big(x\cdot\nabla w\big)+\int_{\mathbb{R}^2}w\big(x\cdot\nabla\psi_1\big)\\
&=\frac{1}{2}\int_{\mathbb{R}^2}\psi_1 x\cdot\nabla\big(x\cdot\nabla w\big)-\frac{1}{2}\int_{\mathbb{R}^2}wx\cdot\nabla\big(x\cdot\nabla\psi_1\big).
\end{aligned} \tag{A.12}$$

Using (A.12), we then derive that

$$
\begin{aligned}
A &= -\int_{\mathbb{R}^2}(w + x\cdot\nabla w)(6w\psi_1\psi_2 + 2\psi_1) \\
&= -2\int_{\mathbb{R}^2} w\psi_1 - 2\int_{\mathbb{R}^2}\psi_1(x\cdot\nabla w) + 3\int_{\mathbb{R}^2} w\psi_1(w + x\cdot\nabla w)^2 \\
&= -2\int_{\mathbb{R}^2} w\psi_1 - \int_{\mathbb{R}^2}\psi_1(x\cdot\nabla w) \\
&\quad + \int_{\mathbb{R}^2} w\big[2\psi_1 + x\cdot\nabla\psi_1\big] + 3\int_{\mathbb{R}^2} w\psi_1(w + x\cdot\nabla w)^2 \\
&= -\int_{\mathbb{R}^2}\psi_1(x\cdot\nabla w) + \int_{\mathbb{R}^2} w(x\cdot\nabla\psi_1) + D \\
&= \frac{1}{2}\int_{\mathbb{R}^2}\psi_1 x\cdot\nabla\big(x\cdot\nabla w\big) - \frac{1}{2}\int_{\mathbb{R}^2} w x\cdot\nabla\big(x\cdot\nabla\psi_1\big) + D,
\end{aligned}
\tag{A.13}
$$

where the term D satisfies

$$
\begin{aligned}
D &= 3\int_{\mathbb{R}^2} w\psi_1\Big[w^2 + 2w(x\cdot\nabla w) + (x\cdot\nabla w)^2\Big] \\
&= 3\int_{\mathbb{R}^2} w^3\psi_1 + 6\int_{\mathbb{R}^2} w^2\psi_1(x\cdot\nabla w) + \frac{3}{2}\int_{\mathbb{R}^2}\psi_1(x\cdot\nabla w)(x\cdot\nabla w^2) \\
&= 3\int_{\mathbb{R}^2} w^3\psi_1 + 6\int_{\mathbb{R}^2} w^2\psi_1(x\cdot\nabla w) \\
&\quad -\frac{3}{2}\int_{\mathbb{R}^2} w^2\Big\{2\psi_1(x\cdot\nabla w) + (x\cdot\nabla w)(x\cdot\nabla\psi_1) + \psi_1\Big[x\cdot\nabla(x\cdot\nabla w)\Big]\Big\}.
\end{aligned}
$$

Since

$$
\begin{aligned}
&-\frac{3}{2}\int_{\mathbb{R}^2} w^2(x\cdot\nabla w)(x\cdot\nabla\psi_1) \\
&= -\frac{1}{2}\int_{\mathbb{R}^2}(x\cdot\nabla\psi_1)(x\cdot\nabla w^3) \\
&= \frac{1}{2}\int_{\mathbb{R}^2} w^3\Big[x\cdot\nabla(x\cdot\nabla\psi_1) + 2(x\cdot\nabla\psi_1)\Big] \\
&= \frac{1}{2}\int_{\mathbb{R}^2} w^3 x\cdot\nabla(x\cdot\nabla\psi_1) + \int_{\mathbb{R}^2} w^3(x\cdot\nabla\psi_1) \\
&= \frac{1}{2}\int_{\mathbb{R}^2} w^3 x\cdot\nabla(x\cdot\nabla\psi_1) - \int_{\mathbb{R}^2}\psi_1\Big[2w^3 + 3w^2(x\cdot\nabla w)\Big] \\
&= \frac{1}{2}\int_{\mathbb{R}^2} w^3 x\cdot\nabla(x\cdot\nabla\psi_1) - 2\int_{\mathbb{R}^2} w^3\psi_1 - 3\int_{\mathbb{R}^2} w^2\psi_1(x\cdot\nabla w),
\end{aligned}
$$

the term D can be further simplified as

$$
D = \int_{\mathbb{R}^2} w^3\psi_1 - \frac{3}{2}\int_{\mathbb{R}^2} w^2\psi_1\big[x\cdot\nabla(x\cdot\nabla w)\big] + \frac{1}{2}\int_{\mathbb{R}^2} w^3 x\cdot\nabla(x\cdot\nabla\psi_1). \tag{A.14}
$$

Applying (A.14), we then obtain from (A.13) that

$$
\begin{aligned}
A &= \int_{\mathbb{R}^2} w^3\psi_1 + \frac{1}{2}\int_{\mathbb{R}^2}(1 - 3w^2)\psi_1\big[x\cdot\nabla(x\cdot\nabla w)\big] \\
&\quad -\frac{1}{2}\int_{\mathbb{R}^2}\Delta w\big[x\cdot\nabla(x\cdot\nabla\psi_1)\big],
\end{aligned}
\tag{A.15}
$$

since w solves the equation $w^3 - w = -\Delta w$ in $\mathbb{R}^2$.

Combining (A.11) and (A.15) now yields that

$$\begin{aligned} II = A + B = & \int_{\mathbb{R}^2} w^3\psi_1 - \frac{p^2+4p+2}{2p} \\ & + \frac{1}{2}\int_{\mathbb{R}^2}\big[x\cdot\nabla(x\cdot\nabla w)\big]\Delta\psi_1 - \frac{1}{2}\int_{\mathbb{R}^2}\big[x\cdot\nabla(x\cdot\nabla\psi_1)\big]\Delta w. \end{aligned} \tag{A.16}$$

We claim that

$$\int_{\mathbb{R}^2} w^3\psi_1 = \frac{p+1}{p}. \tag{A.17}$$

Actually, multiplying (A.1) by w and integrating on $\mathbb{R}^2$ gives that

$$\int_{\mathbb{R}^2}\nabla\psi_1\nabla w - 3\int_{\mathbb{R}^2} w^3\psi_1 = -\int_{\mathbb{R}^2}\Big[\frac{2w^4}{\int_{\mathbb{R}^2} w^4} + \frac{2h(x+y_0)w^2}{p\int_{\mathbb{R}^2} h(x+y_0)w^2}\Big] = -\frac{2(p+1)}{p},$$

due to the fact that $\int_{\mathbb{R}^2} w\psi_1 = 0$ by (3.34). On the other hand, multiplying (1.4) by ψ_1 and integrating on $\mathbb{R}^2$ gives that

$$\int_{\mathbb{R}^2}\nabla\psi_1\nabla w = -\int_{\mathbb{R}^2} w\psi_1 + \int_{\mathbb{R}^2} w^3\psi_1 = \int_{\mathbb{R}^2} w^3\psi_1.$$

The claim (A.17) then follows directly from above two identities. We next claim that

$$\int_{\mathbb{R}^2}\big[x\cdot\nabla(x\cdot\nabla w)\big]\Delta\psi_1 = \int_{\mathbb{R}^2}\big[x\cdot\nabla(x\cdot\nabla\psi_1)\big]\Delta w. \tag{A.18}$$

To prove (A.18), rewrite ψ_1 as $\psi_1(x) = \psi_1(r,\theta)$, where (r,θ) is the polar coordinate in $\mathbb{R}^2$, such that

$$\Delta\psi_1 = \big(\psi_1\big)_{rr} + \frac{1}{r}\big(\psi_1\big)_r + \frac{1}{r^2}\big(\psi_1\big)_{\theta\theta}, \quad \nabla\psi_1 = \frac{x}{r}\big(\psi_1\big)_r + \frac{x^\perp}{r^2}\big(\psi_1\big)_\theta, \tag{A.19}$$

where $x^\perp = (-x_2, x_1)$ for $x = (x_1, x_2) \in \mathbb{R}^2$. We then derive from (3.7) that

$$\begin{aligned} \int_{\mathbb{R}^2}\big[x\cdot\nabla(x\cdot\nabla w)\big]\Delta\psi_1 &= \int_0^{2\pi}\int_0^\infty r(rw')'\Big\{\big[r\big(\psi_1\big)_r\big]_r + \frac{\big(\psi_1\big)_{\theta\theta}}{r}\Big\}drd\theta \\ &= \int_0^{2\pi}\int_0^\infty r(rw')'\big[r\big(\psi_1\big)_r\big]_r drd\theta + \int_0^{2\pi}\int_0^\infty (rw')'\big(\psi_1\big)_{\theta\theta}drd\theta \\ &= \int_0^{2\pi}\int_0^\infty r(rw')'\big[r\big(\psi_1\big)_r\big]_r drd\theta, \end{aligned}$$

and

$$\int_{\mathbb{R}^2}\big[x\cdot\nabla(x\cdot\nabla\psi_1)\big]\Delta w = \int_0^{2\pi}\int_0^\infty r\big[r\big(\psi_1\big)_r\big]_r(rw')'drd\theta,$$

which thus imply that (A.18) holds. Applying (A.17) and (A.18), we therefore conclude from (A.16) that

$$II = \frac{p+1}{p} - \frac{p^2+4p+2}{2p} = -\frac{2+p}{2},$$

which gives (3.36), and the proof is complete. □

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