Vortex Helices for Inhomogeneous Gross-Pitaevskii Equation in Three Dimensional Spaces

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Abstract

We construct traveling wave solutions with a stationary or traveling vortex helix to the inhomogeneous Gross-Pitaevskii equation

$$i\Psi_t = \epsilon^2 \triangle \Psi + \Big(W(y) - |\Psi|^2\Big)\Psi,$$

where the unknown function Ψ is defined as $\Psi : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{C}$, ϵ is a small positive parameter and W is a real smooth potential with symmetries.

1 Introduction

In the present paper, we consider the existence of solutions with vortex helices to the nonlinear Schrödinger type problem

$$i\Psi_t = \epsilon^2 \triangle \Psi + \Big(W(y) - |\Psi|^2\Big)\Psi, \tag{1.1}$$

where the unknown function $\Psi : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{C}$, $\triangle = \partial^2_{y_1} + \partial^2_{y_2} + \partial^2_{y_3}$ is the Laplace operator in $\mathbb{R}^3$, ϵ is a small positive parameter and W is a smooth real potential. The equation (1.1), called the Gross-Pitaevskii equation [74], is a well-known mathematical model to describe Bose-Einstein condensates.

Interest in quantized vortices has grown in the past few years due to the experimental verification of the existence of Bose-Einstein condensates (cf.[9], [34]). Vortices in Bose-Einstein condensates are quantized. Their size, origin, and significance are quite different from those in normal fluids, as they exemplify superfluid properties (cf.[35], [10], [11]). In addition to the simpler two-dimensional point vortices, two types of individual topological defects in three-dimensional Bose-Einstein condensates have attracted the attention of the scientific community in recent years: vortex lines [87, 84, 43] and vortex rings. Quantized vortex rings with cores have been proven to exist when charged particles are accelerated through superfluid helium [76].

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The achievements of quantized vortices in a trapped Bose-Einstein condensate [93], [67], [66] have suggested the possibility of producing vortex rings with ultracold atoms. The existence and dynamics of vortex rings in a trapped Bose-Einstein condensate have been studied by several authors [8], [48], [49], [37], [78], [42], [80], [47]. Vortex rings and their two-dimensional analog(vortex-antivortex pair) have attracted much attention by playing an important role in the study of complex quantized structures such as superfluid turbulence [11], [10], [55], [46]. The reader can refer to the review papers [38], [40] and [11] for more details on quantized vortices in the physical sciences.

In the present paper, we are concerned with the construction of vortices by rigorous mathematical methods. We review some results on vortex structures.

1.1 The vortex structures for homogeneous cases

For the steady state, (1.1) becomes the problem

$$\epsilon^2\triangle\Psi + \Big(W(y) - |\Psi|^2\Big)\Psi = 0, \tag{1.2}$$

where the unknown function Ψ is defined as $\Psi : \mathbb{R}^3 \to \mathbb{C}$, ϵ is a small positive parameter and W is a smooth potential. The study of the problem (1.2) in the homogeneous case, i.e. $W \equiv 1$, on a bounded domain with a suitable boundary condition began with [14] by F. Bethuel, H. Brezis, F. Helein in 1994, see also the book by K. Hoffmann and Q. Tang[45]. Since then, many references addressed the existence, asymptotic behavior and dynamical behavior of solutions. We refer to the books [2] and [81] for references and background. Regarding the construction of solutions, we mention two works which are relevant to the present one. F. Pacard and T. Riviere derived a non-variational method to construct solutions with coexisting degrees of +1 and -1 in [72]. The proof is based on an analysis of the linearized operator around an approximation. M. Del Pino, M. Kowalczyk and M. Musso [29] derived a reduction method for the general existence of vortex solutions under Neumann (or Dirichlet) boundary conditions. The reader can refer to [56]-[58], [60], [86], [95], [25]-[26], [50]-[53], [82] and the references therein.

Traveling wave solutions are believed to play an important role in the full dynamics of (1.1). More precisely, when $W \equiv 1$, these are solutions of the form

$$\Psi(y,t) \,=\, \tilde{u}\big(y_1,\, y_2,\, y_3 - \epsilon\, c\, t\big).$$

Then, by a suitable rescaling, $\tilde{u}$ is a solution of the nonlinear elliptic problem

$$-\,i\,c\,\frac{\partial \tilde{u}}{\partial \tilde{y}_3} \,=\, \triangle\tilde{u} \,+\, \Big(1 - |\tilde{u}|^2\Big)\tilde{u}. \tag{1.3}$$

In the two-dimensional plane, F. Bethuel and J. Saut constructed a traveling wave with two vortices of degree ±1 in [18]. In higher dimensions, by minimizing the energy, F. Bethuel, G. Orlandi and D. Smets constructed solutions with a vortex ring [17]. See [23] for another proof by Mountain Pass Lemma and the extension of results in [16]. The reader can refer to the review paper [15] by F. Bethuel, P. Gravejat and J. Saut and the references therein. See [24] for the existence of vortex helices.

1.2 The pinning phenomena in inhomogeneous cases

Before stating our assumptions and main result, we review some references on the pinning phenomena of vortices.

We start the review by mentioning the pinning phenomena in superconductors, as described by the well known Ginzburg-Landau model, which is relevant to the topics for Gross-Pitaevskii equations. When a superconductor of type II is placed in an external magnetic field, the field penetrates the superconductor in thin tubes of magnetic flux called magnetic vortices. This will cause the dissipation of energy due to creeping or the flow of magnetic vortices[88]. In the superconductor application, it is of importance to pin vortices at fixed locations preventing their motion. Various mechanisms have been advances by physicists, engineers and mathematicians. These methods include introducing impurities into the superconducting material sample or changing the thickness of the superconducting material sample so as to derive various variants of the original Ginzburg-Landau mode of superconductivity.

We first mention the results for the modified Ginzburg-Landau equations for a superconductor with impurities

$$\begin{aligned} -\triangle_A \Psi + \lambda(|\Psi|^2 - 1)\Psi + W(x)\Psi &= 0 \quad \text{in} \quad \mathbb{R}^2 \\ \bigtriangledown \times \bigtriangledown \times A + \mathrm{Im}(\bar{\Psi} \bigtriangledown_A \Psi) &= 0, \end{aligned}$$

where $W : \mathbb{R}^2 \to \mathbb{R}$ is a potential of impurities, $\bigtriangledown_A = \bigtriangledown - iA$ is the covariant gradient and $\triangle_A = \bigtriangledown_A \cdot \bigtriangledown_A$. For a vector field A, $\bigtriangledown \times A = \partial_1 A_2 - \partial_2 A_1$. Numerical evidence shows that fundamental magnetic vortices(degrees of ± 1) of the same degree are attracted to maxima of $W(x)$ and can be found in works by Chapman, Du and Gunzburger [21], Du, Gunzburger and Peterson [36]. Strauss and Sigal [85] have derived the effective dynamics of the magnetic vortex in a local potential. Gustafson and Ting [44] have shown dynamic stability/instability of single pinned fundamental vortices. Pakylak, Ting and Wei show the pinning phenomena of multi-vortices in [73]. Ting [89] studied the effective dynamics of multi-vortices in the external potentials of different strengths.

As an extreme case of impurities, the presence of point defect or normal inclusion in some disjoint, smooth connected regions contained in the superconductor sample will also cause the pinning phenomena. Let $\mathcal{D} \subset \mathbb{R}^2$ be a smooth simply connected domain. For functions $\Psi \in H^1(\mathcal{D}; \mathbb{C})$, $A \in H^1(\mathcal{D}; \mathbb{R}^2)$, N. Andre, P. Bauman and D. Phillips considered the minimizers of the energy in [1]

$$\begin{aligned} E_\epsilon(\Psi, A) \quad \equiv \quad & \int_{\mathcal{D}} \left\{ \frac{1}{2} \big| (\nabla - iA)\Psi \big|^2 + \frac{1}{4\epsilon^2} \big(|\Psi|^2 - a(x) \big)^2 \right\} \mathrm{d}x \\ & + \int_{\mathcal{D}} \frac{1}{2} \big(\nabla \times A - h_{ex} \mathbf{e}_3 \big)^2 \mathrm{d}x. \end{aligned}$$

The domain $\mathcal{D}$ represents the cross-section of an infinite cylindrical body with $\mathbf{e}_3$ as its generator. The body is subjected to an applied magnetic field, $h_{ex}\mathbf{e}_3$ where $h_{ex} \geq 0$ is constant. If the smooth function a is nonnegative and is allowed to vanish at finitely many points, the local minimizers exhibit vortex pinning at the zeros of a. Later on, for functions $\Psi \in H^1(\mathcal{D};\mathbb{C})$, $A \in H^1(\mathcal{D};\mathbb{R}^2)$, S. Alama and L. Bronsard consider the minimizers of the energy in [5]

$$\begin{aligned} E_\epsilon(\Psi, A) \quad &\equiv \quad \int_{\mathcal{D}} \left\{ \frac{1}{2}\left|(\nabla - iA)\Psi\right|^2 + \frac{1}{4\epsilon^2}\Big[\left(|\Psi|^2 - a(x)\right)^2 - (a^-)^2 \Big] \right\} \mathrm{d}x \\ &\quad + \int_{\mathcal{D}} \frac{1}{2}\left(\nabla \times A - h_{ex}\right)^2 \mathrm{d}x \end{aligned}$$

where h_{ex} is a constant applied field. They assume that

$$\begin{aligned} a \in C^2(\mathcal{D}), \qquad \{x \in \bar{\mathcal{D}} : a(x) \leq 0\} \;=\; \overline{\cup_{m=1}^n \omega_m}\,, \\ \nabla a(x) \neq 0 \quad \text{for all} \quad x \in \partial w_m, \quad m = 1, \ldots, n, \end{aligned}$$

with finitely many smooth, simply connected domains $\omega_m \subset\subset \mathcal{D}$. For bounded applied fields (independent of ϵ), they showed that the normal regions acted as "giant vortices" acquiring large vorticity for large (fixed) applied field h_{ex}. Note that these configurations cannot have any vortices in the sense of zeros of Ψ in $\Omega = \mathcal{D} - \cup_{m=1}^n \omega_m$. Nevertheless, they do exhibit vorticity around the holes ω_m due to the nontrivial topology of the domain Ω. For $h_{ex} = O(|\log \epsilon|)$, the pinning effect of the holes eventually breaks down and free vortices begin to appear in the superconducting region $a(x) > 0$ at a point set, which is determined by solving an elliptic boundary-value problem. The reader can refer to [6] and [3].

Work has also been done on non-magnetic vortices ($A = 0$) with pinning (see [7], [13]). For example, in the model for the variance of the thickness of the superconducting material sample considered by [7], a weight function $p(x)$ is introduced into the energy

$$\mathcal{E}_\lambda = \frac{1}{2}\int_\Omega \Big[p|\bigtriangledown \Psi|^2 + \lambda(1 - |\Psi|^2)^2 \Big], \tag{1.4}$$

with a bounded domain and $\lambda \to \infty$. They show that non-magnetic vortices are localized near minima of $p(x)$ in the first part of [7]. In the second part of [7], they also analyzed the "interaction energy" between vortices approaching the same limit site by deriving estimates of the mutual distances between these vortices. In fact, they showed that the mutual distance between vortices(approaching the same limit site) is of order $O(1/\sqrt{|\log \lambda|})$. See also the paper by Lin and Du [59].

In 2006, experimentalists succeeded in creating a rotating optical lattice potential with square geometry, which they applied to a Bose-Einstein condensate with a vortex lattice [90]. They observed the pinning of vortices at the potential minima for sufficient optical strength and confirmed the theoretical prediction by Reijnders and Duine [77]. See also the papers [3] and [68] for pinning phenomena of vortices in single and multi-component Bose-Einstein condensates.

Note that the above results we mentioned are two-dimensional cases and the location of the vortices (as in the sense of zero of the order parameter) was determined by the properties of the potential. However, we are concerned with the existence of vortex lines to (1.1) in three-dimensional space.

To the leading order, the vortex lines in the Ginzburg-Landau theory move in the binormal direction with curvature-dependent velocity [75]. Moreover, the motion of vortex lines in quantum mechanics is essentially determined by four factors [19]: the shape of the vortex line, the shape of the back ground condensate wave function, the interaction between vortex lines and possible external forces. By formal asymptotic expansion, A. Svidzinsky and A. Fetter [87] gave a complete description of qualitative features of dynamics of a single vortex line in a trapped Bose-Einstein condensate in the Thomas-Fermi limit. To be specific, we shall consider a trapping potential $W(y) = m(\omega_\perp^2 r^2 + \omega_z^2 z^2)/2$ in the cylindrical coordinates (r, θ, z), with aspect ration defined by $\lambda = \omega_z/\omega_\perp$. In Thomas-Fermi limit, the density profile of the condensate is given by the positive part of

$$\rho(y) = \rho_0\,(1 - r^2/R_\perp^2 - z^2/R_z^2),$$

where $R_\perp = \sqrt{2\mu/m\omega_\perp^2}$ and $R_z = \sqrt{2\mu/m\omega_z^2}$ are the radial and axial Thomas-Fermi radii of the trapped Bose-Einstein condensates respectively; μ is the chemical potential and $\rho_0 = \mu m/4\pi\hbar^2 a$ is the central particle density. Then the velocity of a vortex line at y in nonrotating trap is given by (cf. (38) in [87])

$$\mathcal{V} = \Lambda(\xi, k)\left(\frac{T \times \bigtriangledown W(y)}{\mu\rho(y)/\rho_0} + kB\right) \tag{1.5}$$

where T and B are tangent vector and binormal of the vortex line. In the above,

$$\Lambda(\xi, k) = (-\hbar/2m)\log\left(\xi\sqrt{R_\perp^{-2} + k^2/8}\right)$$

and k is the curvature of the vortex line. For more details, the reader can refer to [87] and the references therein.

Note that there are two important cases of vortex lines: vortex rings and vortex helices. Recently, by rigourous mathematical methods, T. Lin, J. Wei and J. Yang [63] construct solutions with a single stationary (and also a traveling) vortex ring for (1.1) with inhomogeneous trap potential. More precisely, from (1.5) we see that the shape parameter(the curvature), the wave function and the gradient of the potential will determine the limit site of the stationary vortex lines, i.e. the potential will pin the vortex rings. We will call the role of the gradient of the potential as **the effect at first order of the potential**. In [91], the authors studied the role of the factor of interaction between vortex rings by adding one more vortex ring. It was found that the interaction between vortex rings will be balanced by the second derivative of the potential. We will call the role of the second derivatives of the potential as **the effect at second order of the potential**. Hence, it is natural to construct the vortex helices, which is not torsion free.

1.3 Main results: the existence of single vortex helix

In the present paper, we are concerned with the construction of vortex helices by rigorous mathematical method. We are looking for solutions to problem (1.1) in the form

$$\Psi(\tilde{y}, t) \;=\; e^{i\nu_\epsilon t}\,\tilde{u}\Big(\tilde{y}_1,\, \tilde{y}_2,\, \tilde{y}_3 - \kappa\,\epsilon^2|\log\epsilon|t\Big),$$

which has a vortex helix traveling along the y_3 direction with velocity

$$C = \kappa\, \epsilon^2 \log \frac{1}{\epsilon}. \tag{1.6}$$

Here ϵ is any small positive real number. κ and ν_ϵ are two constants to be determined later (cf. (1.12), (1.16) and (1.23)). Then $\tilde{u}$ is a solution of the nonlinear elliptic problem

$$-\, i\epsilon^2 |\log \epsilon| \kappa\, \frac{\partial \tilde{u}}{\partial \tilde{y}_3} \;=\; \epsilon^2 \triangle\, \tilde{u} \;+\; \Big(\nu_\epsilon + W(\tilde{y}) - |\tilde{u}|^2\Big)\tilde{u}. \tag{1.7}$$

Note that we need the trapping potential W of the form

$$W(\tilde{y}_1, \tilde{y}_2, \tilde{y}_3) \;=\; W(\tilde{y}_1, \tilde{y}_2, \tilde{y}_3 + \kappa\epsilon^2 |\log \epsilon| t), \tag{1.8}$$

see the assumption (**A1**) below.

1.3.1 A traveling helix

We first consider (1.7) for the case $\kappa \neq 0$ provided that the real function W in (1.7) has the following three properties.

(A1): *W is a smooth function in the form*

$$W(\tilde{y}_1, \tilde{y}_2, \tilde{y}_3) = W\big(\tilde{r}\big) \quad \textit{with } \tilde{r} = \sqrt{\tilde{y}_1^2 + \tilde{y}_2^2}\,.$$

(A2): *There is a number $\hat{r}_1$ such that*

$$\frac{\partial W}{\partial \tilde{r}}\Big|_{\tilde{r}=\hat{r}_1} + \frac{d}{\hat{r}_1} \neq 0 \quad \textit{and} \quad \frac{\partial W}{\partial \tilde{r}}\Big|_{\tilde{r}=\hat{r}_1} < 0. \tag{1.9}$$

Here d is a positive constant defined by (cf. (7.3))

$$d \;\equiv\; \frac{1}{\pi}\int_{\mathbb{R}^2} w(|\, s\,|) w'(|\, s\,|) \frac{1}{|\, s\,|}\, \mathrm{d}s > 0, \tag{1.10}$$

where w is defined by (2.1). We also assume that W is non-degenerate at $\hat{r}_1$ in the sense that

$$\frac{\partial^2 W}{\partial \tilde{r}^2}\Big|_{\tilde{r}=\hat{r}_1} - \frac{d}{\hat{r}_1^2} \;\neq\; 0. \tag{1.11}$$

Then we set the parameter κ by the relation (cf. (7.5))

$$\frac{\partial W}{\partial \tilde{r}}\Big|_{\tilde{r}=\hat{r}_1} \;+\; \frac{d}{\hat{r}_1} \;=\; \frac{\kappa\, d}{2\gamma}, \tag{1.12}$$

where γ is a geometric parameter of the vortex helix given in (3.7).

We assume that the vortex helix is directed along the curve in the form

$$\alpha \in \mathbb{R} \longmapsto (\hat{r}_{1\epsilon} \cos \alpha,\ \hat{r}_{1\epsilon} \sin \alpha,\ \hat{\lambda}\alpha) \in \mathbb{R}^3, \tag{1.13}$$

where $\hat{r}_{1\epsilon} = \hat{r}_1 + \hat{f}$ with the parameter $\hat{f}$ of order $O(\epsilon)$ to be determined in the reduction procedure, $\hat{\lambda}$ is any nonzero constant.

(A3): *There also exists a number $\hat{r}_{2\epsilon}$ with $\hat{r}_{2\epsilon} - \hat{r}_{1\epsilon} = \tau_0 + O(\epsilon)$ for a universal positive constant τ_0 independent of ϵ in such a way that*

$$
\begin{aligned}
&1 + \Big(W(\tilde{r}) - W(\hat{r}_{1\epsilon}) \Big) > 0 \quad \text{if } \tilde{r} \in \big(0,\, \hat{r}_{2\epsilon}\big), \\
&1 + \Big(W(\tilde{r}) - W(\hat{r}_{1\epsilon}) \Big) < 0 \quad \text{if } \tilde{r} \in \big(\hat{r}_{2\epsilon},\, +\infty\big), \\
&1 + \Big(W(\tilde{r}) - W(\hat{r}_{1\epsilon}) \Big) = 0 \quad \text{if } \tilde{r} = 0, \hat{r}_{2\epsilon}.
\end{aligned}
\tag{1.14}
$$

Moreover, we also assume that

$$
\begin{aligned}
1 + \Big[W(\tilde{r}) - W(\hat{r}_{1\epsilon}) \Big] &= c_0 \tilde{r}^4 + O(\tilde{r}^5), \quad \text{if } \tilde{r} \in \big(0,\, \hat{r}_0\big), \\
1 + \Big[W(\tilde{r}) - W(\hat{r}_{1\epsilon}) \Big] &\geq c_1, \quad \text{if } \tilde{r} \in \big(\hat{r}_0,\, \hat{r}_{2\epsilon} - \tau_1\big), \\
1 + \Big[W(\tilde{r}) - W(\hat{r}_{1\epsilon}) \Big] &\leq -c_2, \quad \text{if } \tilde{r} \in (\hat{r}_{2\epsilon} + \tau_2,\, +\infty), \\
\frac{\partial W}{\partial \tilde{r}}\Big|_{\tilde{r} = \hat{r}_{2\epsilon}} < 0, &\qquad \frac{\partial^2 W}{\partial \tilde{r}^2}\Big|_{\tilde{r} = \hat{r}_{2\epsilon}} \leq 0,
\end{aligned}
\tag{1.15}
$$

for some positive constants $\hat{r}_0$, c_0, c_1, c_2, τ_1 and τ_2 with $\tau_1 < \tau_0/100$ and $\hat{r}_0 < \hat{r}_1$.

Some explanation is in order to explain the physical and mathematical motivation of the assumptions in (**A1**)-(**A3**).

Remark 1.1.

- *We will need the symmetries in (**A1**) to transform (1.18) into a two-dimensional case in Section 3 in such a way that we can use the mathematical method from [62]. Moreover, we will use these symmetries to determine the locations of the vortex helices, see Remark 3.1.*

- *To determine the density function(i.e. the absolute value $|u|$ of a solution u) with decay by the classical Thomas-Fermi approach in outer region of vortices, we impose the conditions in (1.14).*

- *The first condition in (1.15) will help us deal with the singularity caused by the skew motions described in the formulation of problem in Section 3.1, see Remark 4.1. There are solutions with vortex rings to the homogeneous cases such as Gross-Pitaevskii equation [24] and Klein-Gordon equation with Ginzburg-Landau type nonlinearity [94]. It is an interesting problem to construct helicoidally symmetric solutions with vortex helices to these two equations.*

- *It is also worth mentioning that we assume that W satisfies (1.15) in the region $\tilde{r} > \hat{r}_{2\epsilon} + \tau_2$. This is because it is a vortexless region and we do not care about the effect of the potential W there. Moreover, the assumptions in (1.15) will be helpful for dealing with the problem in mathematical aspect and then determining the density function with decay at infinity, see part 5 of the proof of Lemma 6.1.* □

By setting

$$\nu_\epsilon = 1 - W(\hat{r}_{1\epsilon}), \tag{1.16}$$

to problem (1.7) and then defining $V(\tilde{r}) = W(\tilde{r}) - W(\hat{r}_{1\epsilon})$, we shall consider the following problem

$$\epsilon^2 \triangle \tilde{u} + \Big(1 + V(\tilde{r}) - |\tilde{u}|^2\Big)\tilde{u} + i\epsilon^2 |\log\epsilon| \kappa \frac{\partial \tilde{u}}{\partial \tilde{y}_3} = 0. \tag{1.17}$$

Here is the first result.

Theorem 1.2. *For ϵ sufficiently small, there exists an axially symmetric solution of problem (1.17) with the form $\tilde{u} = \tilde{u}(|\tilde{y}'|, \tilde{y}_3) \in C^\infty(\mathbb{R}^3, \mathbb{C})$ possessing a traveling vortex helix of degree $+1$*

$$\theta \in \mathbb{R} \mapsto \big(\hat{r}_{1\epsilon}\cos\theta,\, \hat{r}_{1\epsilon}\sin\theta,\, \hat{\lambda}\theta\big) \in \mathbb{R}^3,$$

where $\hat{r}_{1\epsilon} \sim \hat{r}_1$. $\tilde{u}$ is also invariant under the screw motion expressed in cylinder coordinates

$$\Sigma :\ (r, \theta, s_3) \mapsto (r, \theta + \alpha, s_3 + \hat{\lambda}\alpha), \quad \forall\, \alpha \in \mathbb{R}.$$

More precisely, the solution $\tilde{u}$ posses the following asymptotic profile

$$\tilde{u}(\tilde{y}_1, \tilde{y}_2, \tilde{y}_3) \approx \begin{cases} w\Big(\frac{\tilde{\ell}}{\epsilon}\Big)\, e^{i\varphi_0^+}, & \tilde{y} \in \mathcal{D}_2 = \{\tilde{\ell} < \tau_0/100\}, \\ \sqrt{1 + V(\tilde{r})}\, e^{i\varphi_0^+}, & \tilde{y} \in \mathcal{D}_1 = \{\tilde{r} < \hat{r}_{2\epsilon} - \epsilon^{\frac{2}{3}-\lambda}\} \setminus \mathcal{D}_2, \\ \delta_\epsilon^{1/3} q\Big(\delta_\epsilon^{1/3}\frac{\tilde{r} - \hat{r}_{2\epsilon}}{\epsilon}\Big)\, e^{i\varphi_0^+}, & \tilde{y} \in \mathcal{D}_3 = \{\tilde{r} > \hat{r}_{2\epsilon} - \epsilon^{\frac{2}{3}-\lambda}\}, \end{cases}$$

where we have denoted

$$\tilde{\ell} = \sqrt{\tilde{y}_1^2 + \tilde{y}_2^2 + \tilde{y}_3^2 - \hat{r}_{1\epsilon}^2}, \qquad \tilde{r} = \sqrt{\tilde{y}_1^2 + \tilde{y}_2^2}\,, \qquad \delta_\epsilon = -\epsilon \frac{\partial V}{\partial \tilde{r}}\Big|_{\tilde{r} = \hat{r}_{2\epsilon}} > 0,$$

and $\varphi_0^+(\tilde{y}_1, \tilde{y}_2, \tilde{y}_3) = \varphi_0^+(\tilde{r}, \tilde{y}_3)$ is the angle argument of the vector $(\tilde{r} - \hat{r}_{1\epsilon}, \tilde{y}_3)$ in the $(\tilde{r}, \tilde{y}_3)$ plane. Here q is the function defined by Lemma 2.4. □

Remark 1.3.

- *Due to the assumption* **(A3)**, *in the region $\mathcal{D}_1$ we use the classical Thomas-Fermi approximation to describe the wave function. The reader can refer to then monograph [74] for more discussions. For the asymptotic behavior of u in $\mathcal{D}_3$, there are also some formal expansions in physical works such as [64] and [39]. Here we use q in Lemma 2.4 to describe the profile beyond the Thomas-Fermi approximation.*

- *The results in Theorems 1.2 and 1.4 can be extended to higher dimensions for the existence of solutions with vortex helix submanifolds in $\mathbb{R}^N$ with the odd integer $N \geq 5$, see Remark 3.1 in [92].*

□

1.3.2 A stationary helix

We now consider the stationary case $\kappa = 0$, i.e.

$$\epsilon^2 \triangle \tilde{u} + \Big(\nu_\epsilon + W(\tilde{y}) - |\tilde{u}|^2\Big)\tilde{u} = 0, \tag{1.18}$$

by assuming that the real function W has the following properties **(P1)-(P3)**. In other words, the vortex helix will be completely pinned at a fixed site due to the role of the potential.
(P1): *W is a symmetric function with the form*

$$W(\tilde{y}_1, \tilde{y}_2, \tilde{y}_3) = W\big(\tilde{r}\big) \quad \textit{with } \tilde{r} = \sqrt{\tilde{y}_1^2 + \tilde{y}_2^2}\,.$$

(P2): *There is a point $\tilde{r}_1$ such that the following solvability condition holds*

$$\frac{\partial W}{\partial \tilde{r}}\Big|_{\tilde{r}=\tilde{r}_1} + \frac{d}{\tilde{r}_1} = 0. \tag{1.19}$$

Here d is a positive constant defined by (1.10). We also assume that W is non-degenerate at $\tilde{r}_1$ in the sense that

$$\frac{\partial^2 W}{\partial \tilde{r}^2}\Big|_{\tilde{r}=\tilde{r}_1} - \frac{d}{\tilde{r}_1^2} \neq 0. \tag{1.20}$$

We assume that the vortex helix is characterized by the curve

$$\alpha \in \mathbb{R} \longmapsto (\tilde{r}_{1\epsilon}\cos\alpha,\ \tilde{r}_{1\epsilon}\sin\alpha,\ \tilde{\lambda}\alpha) \in \mathbb{R}^3,$$

where $\tilde{r}_{1\epsilon} = \tilde{r}_1 + \tilde{f}$ with the parameter $\tilde{f}$ of order $O(\epsilon)$ to be determined in the reduction procedure.
(P3): *There exists a number $\tilde{r}_{2\epsilon}$ with $\tilde{r}_{2\epsilon} - \tilde{r}_{1\epsilon} = \tilde{\tau}_0 + O(\epsilon)$ such that the following conditions*

$$\begin{aligned}
&1 + \Big(W(\tilde{r}) - W(\tilde{r}_{1\epsilon})\Big) \geq 0 \quad \textit{if } \tilde{r} \in \big(0,\, \tilde{r}_{2\epsilon}\big),\\
&1 + \Big(W(\tilde{r}) - W(\tilde{r}_{1\epsilon})\Big) \leq 0 \quad \textit{if } \tilde{r} \in \big(\tilde{r}_{2\epsilon},\, +\infty\big),\\
&1 + \Big(W(\tilde{r}) - W(\tilde{r}_{1\epsilon})\Big) = 0 \quad \textit{if } \tilde{r} = 0, \tilde{r}_{2\epsilon},
\end{aligned} \tag{1.21}$$

with a universal positive constant $\tilde{\tau}_0$ independent of ϵ. Moreover, we also assume that

$$\begin{aligned}
1 + \Big[W(\tilde{r}) - W(\tilde{r}_{1\epsilon})\Big] &= \tilde{c}_0\tilde{r}^4 + O(\tilde{r}^5), \quad \textit{if } \tilde{r} \in \big(0,\, \tilde{\tau}_0\big),\\
\frac{\partial W}{\partial \tilde{r}}\Big|_{\tilde{r}=\tilde{r}_{2\epsilon}} &< 0, \quad \frac{\partial^2 W}{\partial \tilde{r}^2}\Big|_{\tilde{r}=\tilde{r}_{2\epsilon}} \leq 0,\\
1 + \Big[W(\tilde{r}) - W(\tilde{r}_{1\epsilon})\Big] &\geq \tilde{c}_1, \quad \textit{if } \tilde{r} \in \big(\tilde{r}_0,\, \tilde{r}_{2\epsilon} - \tilde{\tau}_1\big),\\
1 + \Big[W(\tilde{r}) - W(\tilde{r}_{1\epsilon})\Big] &\leq -\tilde{c}_2, \quad \textit{if } \tilde{r} \in (\tilde{r}_{2\epsilon} + \tilde{\tau}_2, +\infty),
\end{aligned} \tag{1.22}$$

for some positive constants $\tilde{r}_0$, $\tilde{c}_0$, $\tilde{c}_1$, $\tilde{c}_2$, $\tilde{\tau}_1$ and $\tilde{\tau}_2$ with $\tilde{\tau}_1 < \tilde{\tau}_0/100$ and $\tilde{r}_0 < \tilde{r}_1$.

By setting

$$\nu_\epsilon = 1 - W(\tilde{r}_{1\epsilon}), \tag{1.23}$$

to problem (1.18) and then defining $\tilde{V}(\tilde{r}) = W(\tilde{r}) - W(\tilde{r}_{1\epsilon})$, we shall consider the following problem

$$\epsilon^2 \triangle \tilde{u} + \Big(1 + \tilde{V}(\tilde{r}) - |\tilde{u}|^2\Big)\tilde{u} = 0. \tag{1.24}$$

The second result reads:

Theorem 1.4. *For ϵ sufficiently small, there exists an axially symmetric solution to problem (1.24) in the form $\tilde{u} = \tilde{u}(|\tilde{y}'|, \tilde{y}_3) \in C^\infty(\mathbb{R}^3, \mathbb{C})$ with a stationary vortex helix of degree $+1$*

$$\theta \in \mathbb{R} \mapsto \big(\tilde{r}_{1\epsilon}\cos\theta,\ \tilde{r}_{1\epsilon}\sin\theta,\ \tilde{\lambda}\theta\big) \in \mathbb{R}^3,$$

where $\tilde{r}_{1\epsilon} \sim \tilde{r}_1$. $\tilde{u}$ is also invariant under the screw motion expressed in cylinder coordinates

$$\Sigma:\ (r, \theta, s_3) \mapsto (r, \theta + \alpha, s_3 + \tilde{\lambda}\alpha), \quad \forall\, \alpha \in \mathbb{R}.$$

The profile of $\tilde{u}$ is the same as the solution in Theorem 1.2. □

Some words are in order to explain the methods for the results. By using the screw invariance of solutions, we transform the problem to a two-dimensional case (3.8) with boundary condition (3.10) on the infinite strip $\mathfrak{S}$ (cf. (3.9)) and then show the existence of solutions. Problem (3.8) is degenerate when $x_1 = 0$ due to the terms

$$\frac{1}{x_1}\frac{\partial u}{\partial x_1} \quad \text{and} \quad \gamma^{-2}\Big(1 + \frac{\lambda^2}{x_1^2}\Big)\frac{\partial^2 u}{\partial x_2^2}.$$

The new ingredient is the second term, which does not appear in [63] and [91] for the existence of vortex rings. Here, by using the first condition imposed in **(A3)** we shall make careful analysis to deal with the problem near the origin, see Remark 4.1.

The remaining part of this paper is devoted to the complete proof of Theorem 1.2 by the reduction method, see [29] and also [62]. The proof to Theorem 1.4 is similar and we omit it here. The organization of the paper is as follows: in Section 2, we review some preliminary results. Section 3 is devoted the formulation of the problem and outline of the proof. For the convenience of readers, we collect notation in the end of Section 3.1. More details of the proof of Theorem 1.2 will be given in Sections 4-7.

2 Preliminaries

By (ℓ, φ) designating the usual polar coordinates $s_1 = \ell\cos\varphi$, $s_2 = \ell\sin\varphi$, we introduce the standard vortex block solution

$$U_0(s_1, s_2) = w(\ell)e^{i\varphi}, \tag{2.1}$$

with degree $+1$ in the whole plane, where $w(\ell)$ is the unique solution of the problem

$$w'' + \frac{1}{\ell}w' - \frac{1}{\ell^2}w + (1-|w|^2)w = 0 \quad \text{for } \ell \in (0,+\infty), \quad w(0) = 0, \quad w(+\infty) = 1. \tag{2.2}$$

The properties of the function w are stated in the following lemma.

Lemma 2.1. *There hold the following properties:*
(1) $w(0) = 0, \quad w'(0) > 0, \quad 0 < w(\ell) < 1, \quad w'(\ell) > 0$ *for all* $\ell > 0$*,*
(2) $w(\ell) = 1 - \frac{1}{2\ell^2} + O(\frac{1}{\ell^4})$ *for large* ℓ*,*
(3) $w(\ell) = k\ell - \frac{k}{8}\ell^3 + O(\ell^5)$ *for* ℓ *close to* 0*, where* k *is a positive constant.*
(4) Define $T = \frac{\mathrm{d}w}{\mathrm{d}\ell} - \frac{w}{\ell}$*, then* $T < 0$ *in* $(0,+\infty)$*.*

Proof. Partial proof of this lemma can be found in [22] and the references therein. □

We introduce the bilinear form

$$\mathcal{B}(\phi,\phi) = \int_{\mathbb{R}^2} |\bigtriangledown \phi|^2 \; - \int_{\mathbb{R}^2} (1-w^2)|\phi|^2 \; + \; 2\int_{\mathbb{R}^2} |\mathrm{Re}(\bar{U}_0\phi)|^2, \tag{2.3}$$

defined in the natural space $\mathcal{H}$ of all locally-H^1 functions with

$$||\phi||_{\mathcal{H}} = \int_{\mathbb{R}^2} |\bigtriangledown \phi|^2 \; - \int_{\mathbb{R}^2} (1-w^2)|\phi|^2 \; + \; 2\int_{\mathbb{R}^2} |\mathrm{Re}(\bar{U}_0\phi)|^2 < +\infty. \tag{2.4}$$

Let us consider, for a given ϕ, its associated ψ defined by the relation

$$\phi = iU_0\psi. \tag{2.5}$$

Then we decompose ψ by the form

$$\psi = \psi_0(\ell) + \sum_{m\geq 1}\Big[\psi_m^1 + \psi_m^2\Big], \tag{2.6}$$

where we have denoted

$$\begin{aligned}
\psi_0 &= \psi_{01}(\ell) + i\psi_{02}(\ell),\\
\psi_m^1 &= \psi_{m1}^1(\ell)\cos(m\vartheta) + i\psi_{m2}^1(\ell)\sin(m\vartheta),\\
\psi_m^2 &= \psi_{m1}^2(\ell)\sin(m\vartheta) + i\psi_{m2}^2(\ell)\cos(m\vartheta).
\end{aligned}$$

This bilinear form is non-negative, as it follows from various results in [14, 20, 69, 70, 83], see also [28, 71]. The nondegeneracy of U_0 is contained in the following lemma, whose proof can be found in the appendix of [29].

Lemma 2.2. *There exists a constant* $C > 0$ *such that if* $\phi \in \mathcal{H}$ *decomposes like in (2.5)-(2.6) with* $\psi_0 \equiv 0$*, and satisfies the orthogonality conditions*

$$\mathrm{Re}\int_{B(0,1/2)} \bar{\phi}\,\frac{\partial U_0}{\partial s_l} = 0, \quad l = 1,2,$$

then there holds

$$\mathcal{B}(\phi,\phi) \geq C\int_{\mathbb{R}^2} \frac{|\phi|^2}{1+|\;s\;|^2}.$$

□

The linear operator L_0 corresponding to the bilinear form $\mathcal{B}$ can be defined by

$$L_0(\phi) = \Big(\frac{\partial^2}{\partial s_1^2} + \frac{\partial^2}{\partial s_2^2}\Big)\phi + (1-|w|^2)\phi - 2\mathrm{Re}\big(\bar{U}_0\phi\big)U_0.$$

The nondegeneracy of U_0 can be also stated as following lemma, whose proof can be found in [28].

Lemma 2.3. *Suppose that $L_0[\phi]=0$ with $\phi\in\mathcal{H}$, then*

$$\phi = c_1\frac{\partial U_0}{\partial s_1} + c_2\frac{\partial U_0}{\partial s_2}, \tag{2.7}$$

for some real constants c_1, c_2. $\square$

To construct a approximate solution in Section 3, we also prepare the following lemma [63].

Lemma 2.4. *There exists a unique solution q to the following problem*

$$q'' - q\big(\ell + q^2\big) = 0 \quad on\ \mathbb{R}, \tag{2.8}$$

such that the following properties hold:

$$\begin{aligned} &q(\ell) > 0 \quad \text{for all } \ell\in\mathbb{R}, &&q'(\ell)<0 \quad \text{for any } \ell>0,\\ &q(\ell)\sim\sqrt{-\ell} \quad \text{as } \ell\to-\infty, &&q(\ell)\sim\exp\left(-\ell^{3/2}\right) \quad \text{as } \ell\to+\infty. \end{aligned}$$

3 Formulation of the problem and outline of the proof

As stated in Section 1, we will only give the proof to Theorem 1.2. By using the symmetry, we will first transform (1.17) into a two dimensional case in the form (3.8) with conditions (3.13) and then give an outline of the proof.

3.1 The formulation of problem and notation

Making rescaling $\tilde{y} = \epsilon\check{y}$, problem (1.17) takes the form

$$\triangle u + \Big(1 + V(\epsilon\check{y}) - |u|^2\Big)u + i\epsilon|\log\epsilon|\kappa\frac{\partial u}{\partial\check{y}_3} = 0. \tag{3.1}$$

Introduce a new coordinates $(r,\theta,\check{y}_3)\in(0,+\infty)\times(0,2\pi]\times\mathbb{R}$ in the form

$$\check{y}_1 = r\cos\theta, \quad \check{y}_2 = r\sin\theta, \quad \check{y}_3=\check{y}_3. \tag{3.2}$$

Then problem (3.1) takes the form

$$\Big(\frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial\check{y}_3^2} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{1}{r}\frac{\partial}{\partial r}\Big)u + \Big(1+V(\epsilon r) - |u|^2\Big)u + i\epsilon|\log\epsilon|\kappa\frac{\partial u}{\partial\check{y}_3} = 0. \tag{3.3}$$

For problem (3.3), we want to find a solution u which has a vortex helix directed along the curve in the form

$$\theta \in \mathbb{R} \mapsto (r_{1\epsilon} \cos\theta, r_{1\epsilon} \sin\theta, \lambda\theta) \in \mathbb{R}^3,$$

with two parameters

$$r_{1\epsilon} = \hat{r}_{1\epsilon}/\epsilon, \quad \lambda = \hat{\lambda}/\epsilon. \tag{3.4}$$

Moreover, u is also invariant under the screw motion

$$\Sigma: \ (r, \theta, \breve{y}_3) \mapsto (r, \theta + \alpha, \breve{y}_3 + \lambda\alpha), \quad \forall\, \alpha \in \mathbb{R}. \tag{3.5}$$

Then u is a solution to

$$\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \Big(1 + \frac{\lambda^2}{r^2}\Big)\frac{\partial^2}{\partial \breve{y}_3^2}\right] u + \Big(1 + V(\epsilon r) - |u|^2\Big)u + i\epsilon|\log\epsilon|\,\kappa\frac{\partial u}{\partial \breve{y}_3} = 0, \tag{3.6}$$

which will be defined on the region $\big\{(r, \breve{y}_3) \in [0, \infty) \times (-\lambda\pi, \lambda\pi)\big\}$. Recall the parameters in (3.4) and then set

$$\sigma = \frac{\hat{\lambda}}{\hat{r}_{1\epsilon}}, \quad \gamma = \sqrt{1+\sigma^2}, \quad \delta = \frac{1}{\sqrt{|\hat{r}_{1\epsilon}|^2 + |\hat{\lambda}|^2}} = \frac{1}{\gamma\,\hat{r}_{1\epsilon}}. \tag{3.7}$$

Hence we consider a two-dimensional problem

$$\left[\frac{\partial^2}{\partial x_1^2} + \frac{1}{x_1}\frac{\partial}{\partial x_1} + \gamma^{-2}\Big(1 + \frac{\lambda^2}{x_1^2}\Big)\frac{\partial^2}{\partial x_2^2}\right] u + \Big(1 + V(\epsilon|x_1|) - |u|^2\Big)u + i\epsilon|\log\epsilon|\,\frac{\kappa}{\gamma}\,\frac{\partial u}{\partial x_2} = 0. \tag{3.8}$$

By using the symmetries, in the sequel, we shall consider the problem on the region

$$\mathfrak{S} = \big\{z = x_1 + ix_2 \,:\, x_1 \in \mathbb{R},\ x_2 \in (-\lambda\pi/\gamma, \lambda\pi/\gamma)\big\}, \tag{3.9}$$

and then impose the boundary conditions

$$\begin{gathered} |u(z)| \to 0 \quad \text{as } |x_1| \to +\infty, \\ \frac{\partial u}{\partial x_1}(0, x_2) = 0, \quad \forall\, x_2 \in (-\lambda\pi/\gamma, \lambda\pi/\gamma), \\ u(x_1, -\lambda\pi/\gamma) = u(x_1, \lambda\pi/\gamma), \quad \forall\, x_1 \in \mathbb{R}, \\ u_{x_2}(x_1, -\lambda\pi/\gamma) = u_{x_2}(x_1, \lambda\pi/\gamma), \quad \forall\, x_1 \in \mathbb{R}. \end{gathered} \tag{3.10}$$

Before finishing this section, some words are in order to explain the strategies of solving problem (3.8) with boundary conditions in (3.10). It is easy to see that problem(3.8) is invariant under the following two transformations

$$u(z) \to \overline{u(\bar{z})}, \quad u(z) \to u(-\bar{z}). \tag{3.11}$$

Thus we impose the following symmetry on the solution u

$$\Pi := \Big\{ u(z) = \overline{u(\bar{z})},\ u(z) = u(-\bar{z}) \Big\}. \tag{3.12}$$

This symmetry will play an important role in our analysis. As a conclusion, if we write

$$u(x_1, x_2) = u_1(x_1, x_2) + i u_2(x_1, x_2),$$

then u_1 and u_2 enjoy the following conditions:

$$\begin{aligned}
&|u(x_1, x_2)| \to 0 \quad \text{as } |x_1| \to +\infty,\\
u_1(x_1, x_2) &= u_1(-x_1, x_2), \qquad u_1(x_1, x_2) = u_1(x_1, -x_2),\\
u_2(x_1, x_2) &= u_2(-x_1, x_2), \qquad u_2(x_1, x_2) = -u_2(x_1, -x_2),\\
\frac{\partial u_1}{\partial x_1}(0, x_2) &= 0, \qquad \frac{\partial u_2}{\partial x_1}(0, x_2) = 0,\\
u_1(x_1, -\lambda\pi/\gamma) &= u_1(x_1, \lambda\pi/\gamma), \qquad \frac{\partial u_2}{\partial x_2}(x_1, -\lambda\pi/\gamma) = \frac{\partial u_2}{\partial x_2}(x_1, \lambda\pi/\gamma),\\
\frac{\partial u_1}{\partial x_2}(x_1, -\lambda\pi/\gamma) = \frac{\partial u_1}{\partial x_2}(x_1, \lambda\pi/\gamma) &= 0, \qquad u_2(x_1, -\lambda\pi/\gamma) = u_2(x_1, \lambda\pi/\gamma) = 0.
\end{aligned} \tag{3.13}$$

Problem (3.8) becomes a two-dimensional problem with conditions in (3.13). The key point is then to construct a solution with a vortex of degree $+1$ at ξ_+ and its antipair of degree -1 at ξ_-, where $\xi_\pm$ are defined in (3.16). Additional to the computations for standard vortices in two dimensional case, there are two extra derivative terms

$$\frac{1}{x_1}\frac{\partial u}{\partial x_1} \quad \text{and} \quad \gamma^{-2}\Big(1 + \frac{\lambda^2}{x_1^2}\Big)\frac{\partial^2 u}{\partial x_2^2}.$$

These will lead us to further improve the approximate solution to satisfy the conditions in (3.13). Note that the potential V in (3.8) possesses the following properties due to the assumptions (**A**1)-(**A**3):

$$\begin{aligned}
&\frac{\partial V}{\partial \tilde{r}}\Big|_{\tilde{r}=0} = 0, \qquad 0 \le 1 + V(\tilde{r}) = c_0\tilde{r}^4 + O(\tilde{r}^5) \quad \text{if } \tilde{r} \in \big(0, \hat{r}_0\big),\\
&\frac{\partial V}{\partial \tilde{r}}\Big|_{\tilde{r}=\hat{r}_1} + \frac{d}{\hat{r}_1} = \frac{\kappa\, d}{2}, \qquad \frac{\partial^2 V}{\partial \tilde{r}^2}\Big|_{\tilde{r}=\hat{r}_1} - \frac{d}{|\hat{r}_1|^2} \ne 0,\\
&1 + V(\hat{r}_{1\epsilon}) = 1, \qquad 1 + V(\hat{r}_{2\epsilon}) = 0, \qquad \frac{\partial W}{\partial \tilde{r}}\Big|_{\tilde{r}=\hat{r}_{2\epsilon}} < 0, \qquad \frac{\partial^2 W}{\partial \tilde{r}^2}\Big|_{\tilde{r}=\hat{r}_{2\epsilon}} \le 0,\\
&1 + V(\tilde{r}) \ge c_1 \quad \text{if } \tilde{r} \in \big(\hat{r}_0, \hat{r}_{2\epsilon} - \tau_1\big), \qquad 1 + V(\tilde{r}) \le -c_2 \quad \text{if } \tilde{r} \in (\hat{r}_{2\epsilon} + \tau_2, +\infty).
\end{aligned} \tag{3.14}$$

Notation: *We have used $x = (x_1, x_2) = (r, \check{y}_3/\gamma)$ and also write $\ell = |x|$. We may write $z = x_1 + ix_2$. In this rescaled coordinates, we write*

$$r_{0\epsilon} \equiv \hat{r}_0/\epsilon, \qquad r_{1\epsilon} \equiv \hat{r}_1/\epsilon + f \equiv \hat{r}_{1\epsilon}/\epsilon \quad \textit{with } f = \hat{f}/\epsilon, \qquad r_{2\epsilon} \equiv \hat{r}_{2\epsilon}/\epsilon, \tag{3.15}$$

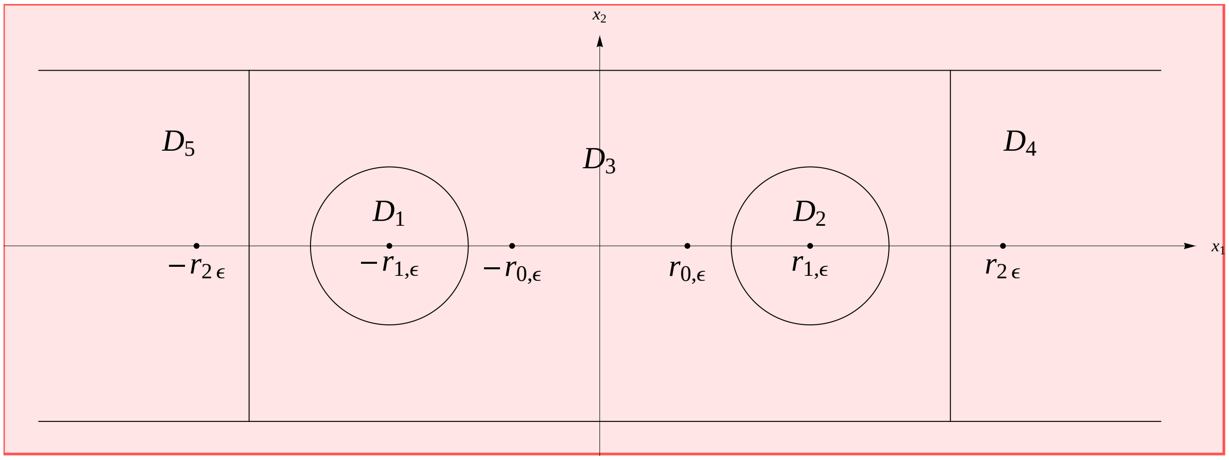

Figure 1: Decompositions of the Domain $\mathfrak{S}$

where the constants $\hat{f}$, $\hat{r}_{1\epsilon}$ *and* $\hat{r}_{2\epsilon}$, $\hat{r}_0$ *are defined in (1.13), (1.14) and (1.15). By setting,* $\xi_+ = (r_{1\epsilon}, 0)$ *and* $\xi_- = (-r_{1\epsilon}, 0)$, *we introduce the translated variable*

$$s = x - \xi_+ \quad or \quad \tilde{s} = x - \xi_-, \tag{3.16}$$

in a small neighborhood of the vortices. For any given $(x_1, x_2) \in \mathbb{R}^2$, *let* $\varphi_0^+(x_1, x_2)$ *and* $\varphi_0^-(x_1, x_2)$ *be respectively the angle arguments of the vectors* $(x_1 - r_{1\epsilon}, x_2)$ *and* $(x_1 + r_{1\epsilon}, x_2)$ *in the* (x_1, x_2) *plane. We also let*

$$\ell_1(x_1, x_2) = \sqrt{(x_1 + r_{1\epsilon})^2 + x_2^2}\,, \qquad \ell_2(x_1, x_2) = \sqrt{(x_1 - r_{1\epsilon})^2 + x_2^2}\,, \tag{3.17}$$

be the distance functions between the point (x_1, x_2) *and the pair of vortices of degree* ± 1 *at the points* ξ_+ *and* ξ_-. *We decompose the region* $\mathfrak{S}$ *in (3.9) into different parts* D_1, D_2, D_3, D_4 *and* D_5 *in the following forms, see Figure 1*

$$\begin{aligned}
D_1 &\equiv \Big\{(x_1, x_2) \in \mathfrak{S} : \ \ell_1 < \epsilon^{-\lambda_1}\Big\}, \\
D_2 &\equiv \Big\{(x_1, x_2) \in \mathfrak{S} : \ \ell_2 < \epsilon^{-\lambda_1}\Big\}, \\
D_3 &\equiv \Big\{(x_1, x_2) \in \mathfrak{S} : \ |x_1| < r_{2\epsilon} - \epsilon^{-\lambda_2}\Big\} \setminus \big(D_2 \cup D_1\big), \\
D_4 &\equiv \Big\{\ (x_1, x_2) \in \mathfrak{S} : \ x_1 > r_{2\epsilon} - \epsilon^{-\lambda_2}\ \Big\}, \\
D_5 &\equiv \Big\{\ (x_1, x_2) \in \mathfrak{S} : \ x_1 < -r_{2\epsilon} + \epsilon^{-\lambda_2}\ \Big\}.
\end{aligned} \tag{3.18}$$

Here λ_1 *and* λ_2 *are two constants, independent of* ϵ, *with* $0 < \lambda_1,\ \lambda_2 < 1/3$, *see (4.22) for the choice of* λ_1. *To the end of construction of vortex pairs locating at* ξ_+ *and* ξ_-, *we write locally*

$$\frac{\partial^2 u}{\partial x_1^2} + \Big(1 + \frac{\lambda^2}{x_1^2}\Big)\gamma^{-2}\frac{\partial^2 u}{\partial x_2^2} = \Big(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\Big)u + \gamma^{-2}\left[\frac{\lambda^2}{x_1^2} - \frac{\lambda^2}{|r_{1\epsilon}|^2}\right]\frac{\partial^2 u}{\partial x_2^2}. \tag{3.19}$$

Finally, we decompose the operator as

$$\mathbb{S}[u] = \mathbb{S}_0[u] + \mathbb{S}_1[u] + \mathbb{S}_2[u] + \mathbb{S}_3[u] + \mathbb{S}_4[u],$$

with the explicit form

$$\begin{aligned} &\mathbb{S}_0[u] := \Big(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\Big)u, \qquad \mathbb{S}_1[u] := \Big(1 + V(\epsilon|x_1|) - |u|^2\Big)u, \\ &\mathbb{S}_2[u] := \gamma^{-2}\left[\frac{\lambda^2}{x_1^2} - \frac{\lambda^2}{|r_{1\epsilon}|^2}\right]\frac{\partial^2 u}{\partial x_2^2}, \qquad \mathbb{S}_3[u] := \frac{1}{x_1}\frac{\partial u}{\partial x_1}, \qquad \mathbb{S}_4[u] := i\epsilon\,\frac{\kappa}{\gamma}\,|\log\epsilon|\,\frac{\partial u}{\partial x_2}. \end{aligned} \tag{3.20}$$

We will use these notation without any further statement in the sequel.

3.2 Outline of the Proof

Step 1. To construct a solution to (3.8)-(3.10) and prove Theorem 1.2, the first step is to construct an approximate solution, denoted by $\mathcal{U}_2$ in (4.52), possessing a pair of vortices with degree ± 1 locating at $\xi_+ = (r_{1\epsilon}, 0)$ and $\xi_- = (-r_{1\epsilon}, 0)$. The heuristic method is to find suitable approximations in different regions and then patch them together. So we decompose the plane into different regions D_1, D_2, D_3, D_4 and D_5 as in (3.18), see Figure 1. Note that the components of D_1 and D_2 center at ξ_- and ξ_+.

The first approximation $\mathcal{U}_1$ is a solution which has a profile of a pair of standard vortices in $D_1 \cup D_2$, which possess the degrees ± 1 and centers ξ_+ and ξ_-, see (4.1). Then in D_3 we set $\mathcal{U}_1$ by Thomas-Fermi approximation in the form (4.2) and make an extension to the regions D_4 and D_5. In fact, by some type of rescaling, in D_4 and D_5 we use q in Lemma 2.4 as a bridge when $|x|$ crossing $r_{2\epsilon}$ and then reduce the norm of the approximate solution to zero as $|x|$ tends to ∞.

Now there are two types of singularities caused by the phase term of standard vortices and the Thomas-Fermi approximation, which will be described in Section 4.1. In fact, to cancel the singularities caused by $\mathbb{S}_2[\varphi_0]$ and $\mathbb{S}_3[\varphi_0]$ with the standard phase φ_0 in (4.1), we will add one more correction term φ_d in (4.33) to the phase component. Finally we get the approximate solution $\mathcal{U}_2$ in (4.52), which has the symmetry

$$\mathcal{U}_2(x_1, x_2) = \overline{\mathcal{U}_2(x_1, -x_2)}, \qquad \mathcal{U}_2(x_1, x_2) = \mathcal{U}_2(-x_1, x_2). \tag{3.21}$$

These are done in subsections 4.1 and 4.2. The Section 4.3 is devoted to estimation of the errors in suitable weighted norms. The reader can refer to the papers [29] and [62].

Step 2. To get explicit information of the linearized problem, we then also divide further D_2 and D_4 into small parts in (5.10), see Figure 2. In Section 5, we then express the error and formulate the problem in suitable local forms in different regions by the method in [29]. More precisely, for the perturbation $\psi = \psi_1 + i\psi_2$ with conditions in (5.5), we take the solution u in the form (5.3). The key points that we shall mention are the roles of local forms of the linearized problem for further deriving of the linear resolution theory in Section 6.

- In D_3, the linear operators have approximate forms, (cf. (5.32))

$$\tilde{L}_3(\psi_1) \equiv \Big(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{1}{x_1}\frac{\partial}{\partial x_1}\Big)\psi_1 + \frac{2}{\beta_1}\nabla\beta_1\cdot\nabla\psi_1,$$
$$\bar{L}_3(\psi_2) \equiv \Big(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{1}{x_1}\frac{\partial}{\partial x_1}\Big)\psi_2 - 2|\mathcal{U}_2|^2\psi_2 + \frac{2}{\beta_1}\nabla\beta_1\cdot\nabla\psi_2.$$

The type of the linear operator $\tilde{L}_3$ was handled in [62], while $\bar{L}_3$ is a good operator since $|\mathcal{U}_2|$ stays uniformly away from 0 in D_3 by the assumption (**A3**), see (5.30).

- In the vortex core regions $D_{1,1}$ and $D_{2,1}$, we use a type of symmetry (3.21) to deal with the kernel of the linear operator related to the standard vortex.

- In $D_{4,1}$, the lowest approximations of the linear operators are, (cf. (5.35))

$$L_{41*}(\psi_1) = \Big(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\Big)\psi_1 - \big(\ell + q^2(\ell)\big)\psi_1\,,$$
$$L_{41**}(\psi_2) = \Big(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\Big)\psi_2 - \big(\ell + 3q^2(\ell)\big)\psi_2\,.$$

By Lemma 2.4, the facts that $L_{41*}(q) = 0$ and $L_{41**}(-q') = 0$ with $-q' > 0$ and $q > 0$ on $\mathbb{R}$ will give the application of maximum principle.

- The linear operators in the region $D_{4,2}$ can be approximated by a good linear operator of the form, (cf.(5.40))

$$L_{42*}[\tilde{\psi}] \equiv \Big(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\Big)\tilde{\psi} + \big(1 + V\big)\tilde{\psi},$$

with $\big(1 + V\big) < -c_2 < 0$ by the assumption (**A3**). For more details, the reader can refer to the proof of Lemma 6.1.

Step 3. After deriving the linear resolution theory by Lemmas 6.1 and 6.2, and then solving the nonlinear projected problem (5.42) in Section 6, as the standard reduction method we adjust the parameter $\tilde{f}$ to get a solution with a vortex helix in Theorem 1.2. It is showed in Section 7 that this is equivalent to solve the following algebraic equation, (cf. (7.5))

$$\begin{aligned}\mathcal{C}(\hat{f}) \,:=\, &-2\pi\epsilon\left[\frac{\partial V}{\partial \tilde{r}}\Big|_{\tilde{r}=\hat{r}_1+\hat{f}}\log\frac{1}{\epsilon} + \frac{d}{\hat{r}_1+\hat{f}}\log\frac{\hat{r}_1+\hat{f}}{\epsilon} - \frac{\kappa\, d}{2\gamma}\log\frac{1}{\epsilon}\right] + O(\epsilon)\\ &= 0,\end{aligned}$$

where $O(\epsilon)$ is a continuous function of the parameter $\hat{f}$. By the solvability condition (1.9) and the non-degeneracy condition (1.11), we can find a zero of $\mathcal{C}\big(\hat{f}\big)$ at some small $\hat{f}$ with the help of the simple mean-value theorem.

Remark 3.1. *Similarly, to prove Theorem 1.4, we need to solve the equation,*

$$\mathcal{C}(\tilde f) := -2\,\epsilon\pi\left[\frac{\partial\tilde V}{\partial\tilde r}\Big|_{\tilde r=\tilde r_1+\tilde f}\log\frac{1}{\epsilon}+\frac{d}{\tilde r_1+\tilde f}\log\frac{\tilde r_1+\tilde f}{\epsilon}\right]+O(\epsilon)$$
$$=0,$$

where $O(\epsilon)$ is a continuous function of the parameter $\tilde f$. By simple mean-value theorem and the solvability condition (1.19) and the non-degeneracy condition (1.20), we can find a zero of $\mathcal{C}(\tilde f)$ at some small $\tilde f$. □

4 Approximate solutions

As we stated in Section 3, in the rest part of the present paper, we shall solve the two-dimensional problem (3.8) with conditions in (3.10) by finding a solution with a vortex of degree $+1$ at ξ_+ and its antipair of degree -1 at ξ_-. The main objective of this section is to construct a good approximate solution and evaluate its error.

4.1 First approximate solution

Recalling the definition of the standard vortex of degree $+1$ in (2.1) and notation in Section 3.1, the construction of the first approximate solution $\mathcal{U}_1$ can be roughly done as follows:

(1) If $(x_1,x_2)\in D_2\cup D_1$, we choose $\mathcal{U}_1$ by

$$\mathcal{U}_1(x_1,x_2) = U_2(x_1,x_2) \equiv \tilde\rho e^{i\varphi_0}, \tag{4.1}$$

where $\tilde\rho = w(\ell_2)w(\ell_1)$ and the phase term φ_0 is defined by $\varphi_0=\varphi_0^+ - \varphi_0^-$.

(2) If $(x_1,x_2)\in D_3$,, we write

$$\mathcal{U}_1(x_1,x_2) = U_3(x_1,x_2) \equiv \sqrt{1+V(\epsilon|x_1|)}\,e^{i\varphi_0}. \tag{4.2}$$

The choice of U_3 here is well defined due to the assumption (**A3**), see also (3.14). It is the standard Thomas-Fermi approximation, see [74].

(3) By the assumption (**A3**), there exists a small positive ϵ_0 such that for $0<\epsilon<\epsilon_0$, we can set

$$\delta_\epsilon := -\epsilon\frac{\partial V}{\partial\tilde r}\Big|_{\tilde r=\hat r_{2\epsilon}} > 0. \tag{4.3}$$

Let q be the unique solution given by Lemma 2.4. Choose

$$U_4(x_1,x_2)=\hat q(x_1)e^{i\varphi_0}\ \text{on } D_4,\qquad U_5(x_1,x_2)=\hat q(-x_1)e^{i\varphi_0}\ \text{on } D_5, \tag{4.4}$$

where the function $\hat q$ is given by

$$\hat q(x_1)=\delta_\epsilon^{1/3}q\Big(\delta_\epsilon^{1/3}(x_1-r_{2\epsilon})\Big). \tag{4.5}$$

It is obvious that the approximation on $\mathfrak{S}$ will vanish at infinity.

For further improvement of the approximation, it is crucial to evaluate the error of this approximation, which will be carried out in different regions as follows. Obviously, there hold the trivial formulas

$$\begin{aligned} \nabla_{x_1,x_2} w(\ell_2) \;=\; \frac{w'(\ell_2)}{\ell_2}\big(x_1 - r_{1\epsilon},\, x_2\big), \qquad \nabla_{x_1,x_2} w(\ell_1) \;=\; \frac{w'(\ell_1)}{\ell_1}\big(x_1 + r_{1\epsilon},\, x_2\big), \\ \nabla_{x_1,x_2}\varphi_0(x_1,x_2) \;=\; \left(\frac{-x_2}{(\ell_2)^2} + \frac{x_2}{(\ell_1)^2},\ \frac{x_1 - r_{1\epsilon}}{(\ell_2)^2} - \frac{x_1 + r_{1\epsilon}}{(\ell_1)^2}\right). \end{aligned} \tag{4.6}$$

We may work directly in the half space $\mathbb{R}^2_+ = \{(x_1,x_2) : x_1 > 0\}$ in the sequel because of the symmetry of the problem.

4.1.1 The error term: $\mathbb{S}[U_2]$

Firstly, we estimate the error near the vortices. Note that for $x_1 > 0$, the error between 1 and $w(\ell_1)$ is $O(\ell_1^{-2})$, which is of order ϵ^2, we may ignore $w(\ell_1)$ in the computations below. Note that

$$\begin{aligned} \mathbb{S}_0[U_2] \;=\; & \triangle[w(\ell_1)]w(\ell_2)e^{i\varphi_0} \;+\; \triangle[w(\ell_2)]w(\ell_1)e^{i\varphi_0} \;-\; U_2\big|\nabla\varphi_0\big|^2 \\ & +\; 2e^{i\varphi_0}\nabla w(\ell_2)\cdot\nabla w(\ell_1) \;+\; 2ie^{i\varphi_0}\nabla\big(w(\ell_2)w(\ell_1)\big)\cdot\nabla\varphi_0 \;+\; i\triangle[\varphi_0]U_2. \end{aligned}$$

In fact, $\triangle[\varphi_0] = 0$. Then, there holds

$$\begin{aligned} & \triangle\big[w(\ell_1)\big]\, w(\ell_2)e^{i\varphi_0} \;+\; \triangle\big[w(\ell_2)\big]\, w(\ell_1)e^{i\varphi_0} \;-\; U_2\big|\nabla\varphi_0\big|^2 \\ & = \Big[\, w''(\ell_1) \;+\; \frac{1}{\ell_1}w'(\ell_1) \;-\; \frac{1}{(\ell_1)^2}w(\ell_1)\Big]\, \frac{U_2}{w(\ell_1)} \\ & \quad +\; \Big[\, w''(\ell_2) \;+\; \frac{1}{\ell_2}w'(\ell_2) \;-\; \frac{1}{(\ell_2)^2}w(\ell_2)\Big]\, \frac{U_2}{w(\ell_2)} \\ & \quad -\; U_2\Bigg[\,\big|\nabla\varphi_0\big|^2 - \big|\nabla\varphi_0^+\big|^2 - \big|\nabla\varphi_0^-\big|^2\,\Bigg], \end{aligned}$$

where for $x_1 > 0$

$$\begin{aligned} U_2\Bigg[\,\big|\nabla\varphi_0\big|^2 - \big|\nabla\varphi_0^+\big|^2 - \big|\nabla\varphi_0^-\big|^2\,\Bigg] \;&=\; -\,2U_2\frac{x_2^2 + (x_1 - r_{1\epsilon})(x_1 + r_{1\epsilon})}{\ell_1^2\ell_2^2} \\ &=\; -\,2U_2\Big[\frac{1}{\ell_1^2} + \frac{2(x_1 - r_{1\epsilon})r_{1\epsilon}}{\ell_1^2\ell_2^2}\Big]. \end{aligned}$$

The next term in $\mathbb{S}_0[U_2]$ can be estimated as

$$2e^{i\varphi_0}\nabla w(\ell_2)\cdot\nabla w(\ell_1) \;=\; 2U_2\frac{x_2^2 + (x_1 - r_{1\epsilon})(x_1 + r_{1\epsilon})}{\ell_1\ell_2}\,\frac{w'(\ell_1)}{w(\ell_1)}\frac{w'(\ell_2)}{w(\ell_2)}.$$

Note that $\nabla w(\ell_2)\cdot\nabla\varphi_0^+ = 0$ and $\nabla w(\ell_1)\cdot\nabla\varphi_0^- = 0$. By the formulas in (4.6), we get

$$2ie^{i\varphi_0}\,\nabla\big(w(\ell_2)w(\ell_1)\big)\cdot\nabla\varphi_0$$

$$
\begin{aligned}
&= 2ie^{i\varphi_0} w(\ell_1)\nabla w(\ell_2)\cdot\nabla\varphi_0^- \,+\, 2ie^{i\varphi_0} w(\ell_2)\nabla w(\ell_1)\cdot\nabla\varphi_0^+ \\
&= -\,4iU_2\frac{x_2 r_{1\epsilon}}{\ell_1(\ell_2)^2}\,\frac{w'(\ell_1)}{w(\ell_1)} \,-\, 4iU_2\frac{x_2 r_{1\epsilon}}{\ell_2(\ell_1)^2}\,\frac{w'(\ell_2)}{w(\ell_2)}.
\end{aligned}
$$

Now, we will turn to the computation of $\mathbb{S}_1[U_2]$. In a small neighborhood of the point $\tilde r = \hat r_{1\epsilon} = \epsilon r_{1\epsilon}$, by Taylor expansion we also write $V(\epsilon|x_1|)$ as the form

$$
V(\epsilon|x_1|) \,=\, \epsilon\,\frac{\partial V}{\partial \tilde r}\Big|_{\tilde r=\epsilon r_{1\epsilon}}(x_1 - r_{1\epsilon}) \,+\, \epsilon^2\, O(|x_1 - r_{1\epsilon}|^2).
$$

It is easy to derive that, in the region D_2

$$
\begin{aligned}
\mathbb{S}_1[U_2] \,&=\, \Big(1 + V - |U_2|^2\Big)U_2 \\
&= \Big(1-|w(\ell_2)|^2\Big)U_2 \,+\, \Big(1-|w(\ell_1)|^2\Big)U_2 \,+\, \Big(|w(\ell_1)|^2 + |w(\ell_2)|^2 - |U_2|^2 - 1\Big)U_2 \\
&\quad + \epsilon\, U_2\,\frac{\partial V}{\partial \tilde r}\Big|_{\tilde r=\hat r_{1\epsilon}}(x_1 - r_{1\epsilon}) \,+\, \epsilon^2\, O(|x_1 - r_{1\epsilon}|^2)U_2,
\end{aligned}
$$

where

$$
\begin{aligned}
\Big(|w(\ell_1)|^2 + |w(\ell_2)|^2 - |U_2|^2 - 1\Big)U_2 \,&=\, -\,\big(|w(\ell_1)|^2 - 1\big)\big(|w(\ell_2)|^2 - 1\big)U_2 \\
&=\, -\,O\left[\frac{1}{(1+\ell_1^2)(1+\ell_2^2)}\right]U_2.
\end{aligned}
$$

Whence, we can write

$$
\mathbb{S}_0[U_2] \,+\, \mathbb{S}_1[U_2] \,=\, \Omega_{21},
$$

where

$$
\begin{aligned}
\Omega_{21} \,=\, &-\,2U_2\Big[\frac{1}{\ell_1^2} + \frac{2(x_1 - r_{1\epsilon})r_{1\epsilon}}{\ell_1^2\ell_2^2}\Big] \,+\, 2U_2\frac{x_2^2 + (x_1 - r_{1\epsilon})(x_1 + r_{1\epsilon})}{\ell_1\ell_2}\,\frac{w'(\ell_1)}{w(\ell_1)}\frac{w'(\ell_2)}{w(\ell_2)} \\
&-\,4iU_2\frac{x_2 r_{1\epsilon}}{\ell_1(\ell_2)^2}\,\frac{w'(\ell_1)}{w(\ell_1)} \,-\, 4iU_2\frac{x_2 r_{1\epsilon}}{\ell_2(\ell_1)^2}\,\frac{w'(\ell_2)}{w(\ell_2)} \\
&+\,\epsilon\, U_2\,\frac{\partial V}{\partial \tilde r}\Big|_{\tilde r=\hat r_{1\epsilon}}(x_1 - r_{1\epsilon}) \,+\, \epsilon^2\, O(|x_1 - r_{1\epsilon}|^2)U_2 \,-\, O\left[\frac{1}{(1+\ell_1^2)(1+\ell_2^2)}\right]U_2.
\end{aligned}
\tag{4.7}
$$

The calculation for the term $\mathbb{S}_2[U_2]$ is proceeded as

$$
\begin{aligned}
\mathbb{S}_2[U_2] \,&= \gamma^{-2}\left[\frac{\lambda^2}{x_1^2} - \frac{\lambda^2}{|r_{1\epsilon}|^2}\right]\frac{\partial^2}{\partial x_2^2}\Big[\tilde\rho e^{i\varphi_0}\Big] \\
&= \gamma^{-2}\left[\frac{\lambda^2}{x_1^2} - \frac{\lambda^2}{|r_{1\epsilon}|^2}\right]\left[\frac{\partial^2\tilde\rho}{\partial x_2^2} + 2i\frac{\partial\tilde\rho}{\partial x_2}\frac{\partial\varphi_0}{\partial x_2} - \tilde\rho\Big|\frac{\partial\varphi_0}{\partial x_2}\Big|^2\right]e^{i\varphi_0} \\
&\quad + \,i\,U_2\,\gamma^{-2}\left[\frac{\lambda^2}{x_1^2} - \frac{\lambda^2}{|r_{1\epsilon}|^2}\right]\frac{\partial^2\varphi_0}{\partial x_2^2} \\
&\equiv \Omega_{22} \,+\, i\,U_2\,\mathbb{S}_2[\varphi_0].
\end{aligned}
\tag{4.8}
$$

Here we need more analysis on the term $i\,U_2\,\mathbb{S}_2[\varphi_0]$. Note that

$$\frac{\partial^2\varphi_0}{\partial x_2^2} \;=\; \frac{\partial^2\varphi_0^+}{\partial x_2^2} \;-\; \frac{\partial^2\varphi_0^-}{\partial x_2^2} \;=\; \frac{-2(x_1-r_{1\epsilon})x_2}{\ell_2^4} \;-\; \frac{-2(x_1+r_{1\epsilon})x_2}{\ell_1^4}.$$

There also hold

$$\begin{aligned}\gamma^{-2}\left[\frac{\lambda^2}{x_1^2}-\frac{\lambda^2}{|r_{1\epsilon}|^2}\right] \;&=\; -\,\frac{2\sigma^2}{r_{1\epsilon}\,\gamma^2}(x_1-r_{1\epsilon}) \;+\; \frac{\sigma^2}{r_{1\epsilon}\,\gamma^2}\,\frac{3r_{1\epsilon}(x_1-r_{1\epsilon})^2+2(x_1-r_{1\epsilon})^3}{x_1^2}\\ &=-\,\frac{2\sigma^2}{r_{1\epsilon}\,\gamma^2}(x_1-r_{1\epsilon}) \;+\; O\big((x_1-r_{1\epsilon})^2/|r_{1\epsilon}|^2\big) \qquad \text{for } x\sim\xi_+,\end{aligned} \tag{4.9}$$

and

$$\begin{aligned}\gamma^{-2}\left[\frac{\lambda^2}{x_1^2}-\frac{\lambda^2}{|r_{1\epsilon}|^2}\right] \;&=\; \frac{2\sigma^2}{r_{1\epsilon}\,\gamma^2}(x_1+r_{1\epsilon}) \;+\; \frac{\sigma^2}{r_{1\epsilon}\,\gamma^2}\,\frac{3r_{1\epsilon}(x_1+r_{1\epsilon})^2-2(x_1+r_{1\epsilon})^3}{x_1^2}\\ &=\frac{2\sigma^2}{r_{1\epsilon}\,\gamma^2}(x_1+r_{1\epsilon}) \;+\; O\big((x_1+r_{1\epsilon})^2/|r_{1\epsilon}|^2\big) \qquad \text{for } x\sim\xi_-.\end{aligned} \tag{4.10}$$

Whence there is a singularity in the form, as $x\sim\xi_+$

$$-\frac{2\sigma^2}{r_{1\epsilon}\,\gamma^2}(x_1-r_{1\epsilon})\frac{\partial^2\varphi_0^+}{\partial x_2^2} \;=\; -\frac{2\sigma^2}{r_{1\epsilon}\,\gamma^2}\,s_1\,\frac{\partial^2\varphi_0^+}{\partial s_2^2} \;=\; \frac{4\sigma^2}{r_{1\epsilon}\,\gamma^2}\,\frac{s_1^2 s_2}{|s|^4} \tag{4.11}$$

with variable $s = x-\xi_+$. A similar singularity exists in the neighborhood of ξ_-

$$\frac{2\sigma^2}{r_{1\epsilon}\,\gamma^2}(x_1+r_{1\epsilon})(-1)\frac{\partial^2\varphi_0^-}{\partial x_2^2} \;=\; -\frac{2\sigma^2}{r_{1\epsilon}\,\gamma^2}\,\tilde{s}_1\,\frac{\partial^2\varphi_0^-}{\partial \tilde{s}_2^2} \;=\; \frac{4\sigma^2}{r_{1\epsilon}\,\gamma^2}\,\frac{\tilde{s}_1^2 \tilde{s}_2}{|\,\tilde{s}\,|^4} \tag{4.12}$$

with the variable $\tilde{s} = x-\xi_-$.

The term $\mathbb{S}_3[U_2]$ obeys the following asymptotic behavior

$$\begin{aligned}\frac{1}{x_1}\frac{\partial U_2}{\partial x_1} \;&=\; \frac{x_1+r_{1\epsilon}}{x_1\ell_1}\frac{w'(\ell_1)}{w(\ell_1)}U_2 \;+\; \frac{x_1-r_{1\epsilon}}{x_1\ell_2}\frac{w'(\ell_2)}{w(\ell_2)}U_2 \;+\; iU_2\,\frac{1}{x_1}\frac{\partial\varphi_0}{\partial x_1}\\ &=\Omega_{23} \;+\; i\,U_2\,\mathbb{S}_3[\varphi_0].\end{aligned} \tag{4.13}$$

We here also need more analysis on $i\,U_2\,\mathbb{S}_3[\varphi_0]$. By the computation

$$\frac{1}{x_1}\frac{\partial\varphi_0}{\partial x_1} \;=\; \frac{1}{x_1}\frac{\partial\varphi_0^+}{\partial x_1} \;-\; \frac{1}{x_1}\frac{\partial\varphi_0^-}{\partial x_1} \;=\; \frac{1}{x_1}\Big(\frac{-x_2}{\ell_2^2}+\frac{x_2}{\ell_1^2}\Big), \tag{4.14}$$

we find that it is a singular term. More precisely, the formulations

$$\frac{1}{x_1} \;=\; \frac{1}{r_{1\epsilon}} - \frac{x_1-r_{1\epsilon}}{r_{1\epsilon}\,x_1} \qquad \text{for } x\sim\xi_+,$$

and

$$\frac{1}{x_1} \;=\; -\frac{1}{r_{1\epsilon}} + \frac{x_1+r_{1\epsilon}}{r_{1\epsilon}\,x_1} \qquad \text{for } x\sim\xi_-,$$

will give that, in the neighborhood of ξ_+ with variable $s = x - \xi_+$, there is a singularity in the form

$$\frac{1}{x_1}\frac{\partial\varphi_0^+}{\partial x_1} = \Big[\frac{1}{r_{1\epsilon}} - \frac{s_1}{r_{1\epsilon}(r_{1\epsilon}+s_1)}\Big]\frac{\partial\varphi_0^+}{\partial s_1} \qquad \text{with} \quad \frac{1}{r_{1\epsilon}}\frac{\partial\varphi_0^+}{\partial s_1} = -\frac{1}{r_{1\epsilon}}\frac{s_2}{|s|^2}, \tag{4.15}$$

and a similar singularity exists in the neighborhood of ξ_- with the variable $\tilde{s} = x - \xi_-$

$$-\frac{1}{x_1}\frac{\partial\varphi_0^-}{\partial x_1} = \Big[-\frac{1}{r_{1\epsilon}} + \frac{\tilde{s}_1}{r_{1\epsilon}(\tilde{s}_1 - r_{1\epsilon})}\Big](-1)\frac{\partial\varphi_0^-}{\partial\tilde{s}_1} \qquad \text{with} \quad \frac{1}{r_{1\epsilon}}\frac{\partial\varphi_0^-}{\partial\tilde{s}_1} = -\frac{1}{r_{1\epsilon}}\frac{\tilde{s}_2}{|\,\tilde{s}\,|^2}. \tag{4.16}$$

For $\mathbb{S}_4[U_2]$, there holds

$$\begin{aligned}\mathbb{S}_4[U_2] &= \frac{\kappa}{\gamma}\,\epsilon|\log\epsilon|\,U_2\left[i\frac{x_2}{\ell_1}\frac{w'(\ell_1)}{w(\ell_1)} + i\frac{x_2}{\ell_2}\frac{w'(\ell_2)}{w(\ell_2)} - \frac{2\big(x_1^2 - x_2^2 - |r_{1\epsilon}|^2\big)r_{1\epsilon}}{(\ell_1\ell_2)^2}\right]\\ &\equiv \Omega_{24}.\end{aligned} \tag{4.17}$$

By combining all estimates above and using the equation (2.2), we write the error, near the vortices, in the form

$$\mathbb{S}[U_2] = F_{21} + F_{22}. \tag{4.18}$$

In the above, we have denoted the combination of the singular terms as F_{21} and also F_{22} in the form

$$F_{21} = iU_2\big(\mathbb{S}_2 + \mathbb{S}_3\big)[\varphi_0] \qquad \text{and} \qquad F_{22} = \sum_{j=1}^{4}\Omega_{2j}.$$

Whence we need a further correction to improve the approximation. On the other hand, we collect all terms in F_{22} and then get

$$\begin{aligned}\mathrm{Re}\,\frac{F_{22}}{-iU_2} = {}& 4\frac{x_2 r_{1\varepsilon}}{\ell_1(\ell_2)^2}\frac{w'(\ell_1)}{w(\ell_1)} + 4\frac{x_2 r_{1\varepsilon}}{\ell_2(\ell_1)^2}\frac{w'(\ell_2)}{w(\ell_2)} - \gamma^{-2}\left[\frac{\lambda^2}{x_1^2} - \frac{\lambda^2}{|r_{1\epsilon}|^2}\right]\frac{2}{\tilde{\rho}}\frac{\partial\tilde{\rho}}{\partial x_2}\frac{\partial\varphi_0}{\partial x_2}\\ &- \frac{\kappa}{\gamma}\,\epsilon|\log\epsilon|\left[\frac{x_2}{\ell_1}\frac{w'(\ell_1)}{w(\ell_1)} + \frac{x_2}{\ell_2}\frac{w'(\ell_2)}{w(\ell_2)}\right],\end{aligned} \tag{4.19}$$

and

$$\begin{aligned}\mathrm{Im}\,\frac{F_{22}}{-iU_2} = {}& -2\Big[\frac{1}{\ell_1^2} + \frac{2(x_1 - r_{1\varepsilon})r_{1\varepsilon}}{\ell_1^2\ell_2^2}\Big] + 2\frac{x_2^2 + (x_1 - r_{1\varepsilon})(x_1 + r_{1\varepsilon})}{\ell_1\ell_2}\frac{w'(\ell_1)}{w(\ell_1)}\frac{w'(\ell_2)}{w(\ell_2)}\\ &+ \varepsilon\frac{\partial V}{\partial\tilde{r}}\Big|_{(\varepsilon r_{1\varepsilon},0)}(x_1 - r_{1\varepsilon}) + \epsilon^2 O(|x_1 - r_{1\epsilon}|^2) + O\left[\frac{1}{(1+\ell_1^2)(1+\ell_2^2)}\right]\\ &+ \gamma^{-2}\left[\frac{\lambda^2}{x_1^2} - \frac{\lambda^2}{|r_{1\epsilon}|^2}\right]\left[\frac{1}{\tilde{\rho}}\frac{\partial^2\tilde{\rho}}{\partial x_2^2} - \Big|\frac{\partial\varphi_0}{\partial x_2}\Big|^2\right]\\ &+ \frac{x_1 + r_{1\varepsilon}}{x_1\ell_1}\frac{w'(\ell_1)}{w(\ell_1)} + \frac{x_1 - r_{1\varepsilon}}{x_1\ell_2}\frac{w'(\ell_2)}{w(\ell_2)} - \frac{\kappa}{\gamma}\,\epsilon|\log\epsilon|\,\frac{2\big(x_1^2 - x_2^2 - |r_{1\epsilon}|^2\big)r_{1\epsilon}}{(\ell_1\ell_2)^2}.\end{aligned} \tag{4.20}$$

By careful checking, we find that F_{22} is a term defined on the region D_2 with properties, for $\ell_1 > 3$ and $\ell_2 > 3$

$$\left|\mathrm{Re}\left(\frac{F_{22}}{-iU_2}\right)\right| \le \frac{O(\epsilon^{1-\sigma})}{(1+\ell_1)^3} + \frac{O(\epsilon^{1-\sigma})}{(1+\ell_2)^3},$$

$$\left|\mathrm{Im}\left(\frac{F_{22}}{-iU_2}\right)\right| \le \frac{O(\epsilon^{1-\sigma})}{(1+\ell_1)^{1+\sigma}} + \frac{O(\epsilon^{1-\sigma})}{(1+\ell_2)^{1+\sigma}},$$

and

$$\left\|\frac{F_{22}}{-iU_2}\right\|_{L^p\left(\{\ell_1<3\}\cup\{\ell_2<3\}\right)} \le C\epsilon|\log\epsilon|, \tag{4.21}$$

where σ and p are some universal constants. In fact, we have for example

$$\begin{aligned}\left|\varepsilon\frac{\partial V}{\partial\tilde r}\Big|_{(\varepsilon r_{1\varepsilon},0)}(x_1-r_{1\varepsilon})\right| &= \frac{\partial V}{\partial\tilde r}\Big|_{(\varepsilon r_{1\varepsilon},0)}\frac{\epsilon^{1-\sigma}}{(1+\ell_2)^{1+\sigma}}\,O\big(\epsilon^\sigma(1+\ell_2)^{2+\sigma}\big)\\ &= \frac{O(\epsilon^{1-\sigma})}{(1+\ell_2)^{1+\sigma}}, \quad \forall\, x\in D_2,\end{aligned}$$

by choosing

$$0<\lambda_1<1/3 \quad \text{and} \quad \frac{2\lambda_1}{1-\lambda_1}<\sigma<1. \tag{4.22}$$

4.1.2 The error term: $\mathbb{S}[U_3]$

Secondly, we compute the error for U_3 in D_3. It is easy to check that the error of U_3 is

$$\mathbb{S}[U_3] = \mathbb{S}_0[U_3] + \mathbb{S}_2[U_3] + \mathbb{S}_3[U_3] + \mathbb{S}_4[U_3].$$

The above components can be computed as follows.

The computations

$$\frac{\partial}{\partial x_1}\sqrt{1+V(\epsilon|x_1|)} = \frac{\epsilon}{2}\big(1+V\big)^{-1/2}\frac{\partial V}{\partial\tilde r},$$

$$\frac{\partial^2}{\partial x_1^2}\sqrt{1+V(\epsilon|x_1|)} = -\frac{\epsilon^2}{4}\big(1+V\big)^{-3/2}\left|\frac{\partial V}{\partial\tilde r}\right|^2 + \frac{\epsilon^2}{2}\big(1+V\big)^{-1/2}\frac{\partial^2 V}{\partial\tilde r^2},$$

give that the first term in $\mathbb{S}[U_3]$ is

$$\begin{aligned}\mathbb{S}_0[U_3] &= \triangle\Big[\sqrt{1+V}\Big]e^{i\varphi_0} + 2i\,e^{i\varphi_0}\,\nabla\sqrt{1+V}\cdot\nabla\varphi_0 - \sqrt{1+V}e^{i\varphi_0}\big|\nabla\varphi_0\big|^2\\ &= -\frac{1}{4}\epsilon^2\left|\frac{\partial V}{\partial\tilde r}\right|^2\frac{e^{i\varphi_0}}{\big(1+V\big)^{3/2}} + \frac{1}{2}\epsilon^2\frac{\partial^2 V}{\partial\tilde r^2}\frac{e^{i\varphi_0}}{\big(1+V\big)^{1/2}} - U_3\big|\nabla\varphi_0\big|^2\\ &\quad + i\,\frac{U_3}{1+V}\nabla V\cdot\nabla\varphi_0\\ &\equiv \Omega_{31} + i\,\frac{U_3}{1+V}\nabla V\cdot\nabla\varphi_0.\end{aligned}$$

Note that $1+V$ will tend to zero as x_1 approaches 0 or $r_{2\epsilon}$ due to the conditions in **(A3)**. We shall do careful analysis in this region. By recalling these conditions rewritten in (3.14), there holds if $x_1 \in (0,\, r_{2\epsilon} - \tau_1/\epsilon)$,

$$\Omega_{31} = O(\epsilon^2)e^{i\varphi_0} + i\,O\big(\epsilon\,(\ell_1)^{-1} + \epsilon\,(\ell_2)^{-1}\big)\,e^{i\varphi_0} + O\big((\ell_1)^{-2} + O(\ell_2)^{-2}\big)\,U_3\,.$$

On the other hand, if $x_1 \in (r_{2\epsilon} - \tau_1/\epsilon,\, r_{2\epsilon} - \epsilon^{-\lambda_2})$, we get

$$\Omega_{31} = O(\epsilon^{(3\lambda_2+1)/2})e^{i\varphi_0} + O\big((\ell_1)^{-2} + O(\ell_2)^{-2}\big)\,U_3 + i\,\epsilon^{(\lambda_2+1)/2}O\big(\,(\ell_1)^{-1} + \,(\ell_2)^{-1}\big)\,e^{i\varphi_0}\,,$$

where $0 < \lambda_2 < 1/3$.

Explicit computations give that

$$\begin{aligned}\mathbb{S}_2[U_3] &= -\,U_3\,\gamma^{-2}\left[\frac{\lambda^2}{x_1^2} - \frac{\lambda^2}{|r_{1\epsilon}|^2}\right]\left|\frac{\partial\varphi_0}{\partial x_2}\right|^2 + i\,U_3\,\mathbb{S}_2[\varphi_0]\\ &\equiv \Omega_{32} + i\,U_3\,\mathbb{S}_2[\varphi_0].\end{aligned}$$

We then write the term $\frac{\partial^2\varphi_0}{\partial x_2^2}$ in $\mathbb{S}_2[\varphi_0]$ in the form

$$\begin{aligned}\frac{\partial^2\varphi_0}{\partial x_2^2} &= \frac{-2(x_1 - r_{1\epsilon})x_2}{\ell_2^4} - \frac{-2(x_1 + r_{1\epsilon})x_2}{\ell_1^4}\\ &= \frac{4\,r_{1\epsilon}\,x_2}{(|r_{1\epsilon}|^2 + x_2^2)^2} - \frac{8x_2x_1^2r_{1\epsilon}(\ell_1+\ell_2)}{\ell_1^4\ell_2^4}\\ &\quad - \frac{2x_2x_1^2r_{1\epsilon}}{\ell_1^4\ell_2^4(|r_{1\epsilon}|^2+x_2^2)^2}\Big[\ell_1^4(|r_{1\epsilon}|^2 + x_2^2 + \ell_2^2) + \ell_2^4(|r_{1\epsilon}|^2 + x_2^2 + \ell_1^2)\Big]\\ &\quad + \frac{16x_2x_1^2r_{1\epsilon}^3}{\ell_1^4\ell_2^4(|r_{1\epsilon}|^2+x_2^2)}\Big[(\ell_1^2+\ell_2^2)(|r_{1\epsilon}|^2+x_2^2) + \ell_2^2\ell_1^2\Big]\\ &\equiv \frac{4\,r_{1\epsilon}\,x_2}{(|r_{1\epsilon}|^2 + x_2^2)^2} + \Theta,\end{aligned} \tag{4.23}$$

then obtain

$$\begin{aligned}i\,U_3\,\mathbb{S}_2[\varphi_0] &= i\,U_3\,\gamma^{-2}\left[\frac{\lambda^2}{x_1^2} - \frac{\lambda^2}{|r_{1\epsilon}|^2}\right]\Theta + i\,U_3\,\mathbb{S}_2[\arctan(x_2/r_{1\epsilon})]\\ &\equiv \Omega_{33} + i\,U_3\,\mathbb{S}_2[\arctan(x_2/r_{1\epsilon})],\end{aligned} \tag{4.24}$$

due to the relation

$$i\,U_3\,\mathbb{S}_2[\arctan(x_2/r_{1\epsilon})] \;=\; i\,U_3\,\gamma^{-2}\left[\frac{\lambda^2}{x_1^2}-\frac{\lambda^2}{|r_{1\epsilon}|^2}\right]\frac{4\,r_{1\epsilon}\,x_2}{(|r_{1\epsilon}|^2+x_2^2)^2}. \tag{4.25}$$

In the next section, we will also introduce a correction in the phase term to get rid of the singular term in $i\,U_3\,\mathbb{S}_2[\varphi_0]$. Hence, there holds

$$\mathbb{S}_2[U_3]=\Omega_{32}\,+\,\Omega_{33}\,+\,i\,U_3\,\mathbb{S}_2[\arctan(x_2/r_{1\epsilon})].$$

Moreover,

$$\Omega_{32}\;=O\big(\epsilon^2(\ell_1)^{-2}+\epsilon^2(\ell_2)^{-2}\big)\,e^{i\varphi_0},$$

due to the facts that

$$U_3(x_1,x_2)=\big[\,\hat{c}_0\,\epsilon^2\,x_1^2\,+\,O(|\epsilon x_1|^{5/2})\,\big]\,e^{i\varphi_0}\quad\text{for}\quad|\epsilon x_1|<\hat{r}_0,$$

and

$$\left|\frac{\partial\varphi_0}{\partial x_2}\right|^2\;=\;\left|\frac{x_1-r_{1\epsilon}}{\ell_2^2}-\frac{x_1+r_{1\epsilon}}{\ell_1^2}\right|^2\;=\;O\big((\ell_1)^{-2}+O(\ell_2)^{-2}\big)\quad\text{in }D_3.$$

Remark 4.1. *We pause here to remark that we need the assumption*

$$0\le 1\,+\,V(\tilde{r})=c_0\tilde{r}^4+O(\tilde{r}^5)\quad\textit{if }\;\tilde{r}\in\big(0,\,\hat{r}_0\big),$$

in (3.14) (see also (1.15)) to cancel the singularity caused by λ^2/x_1^2 *at* $x_1=0$. □

Similar calculations hold

$$\begin{aligned}\mathbb{S}_3[U_3]\;&=\frac{1}{2}\,\epsilon\,\frac{e^{i\varphi_0}}{(1+V)^{1/2}}\,\frac{1}{x_1}\frac{\partial V}{\partial\tilde{r}}\;+\;i\,U_3\,\mathbb{S}_3[\varphi_0]\\&\equiv\Omega_{34}\,+\,i\,U_3\,\mathbb{S}_3[\varphi_0].\end{aligned} \tag{4.26}$$

If $x_1\in(0,\,r_{2\epsilon}-\tau_1/\epsilon)$, we get

$$\Omega_{34}\;=\;O(\epsilon^2)e^{i\varphi_0}\,.$$

If $x_1\in(r_{2\epsilon}-\tau_1/\epsilon,\,r_{2\epsilon}-\epsilon^{-\lambda_2})$, we get

$$\Omega_{34}\;=\;O(\epsilon^{(3+\lambda_2)/2})e^{i\varphi_0}\,.$$

On the other hand

$$i\,U_3\,\mathbb{S}_3[\varphi_0]\;=\;i\,U_3\,\frac{1}{x_1}\frac{\partial\varphi_0}{\partial x_1}\;=\;-i\,U_3\,\frac{4x_2\,r_{1\epsilon}}{(\ell_2)^2(\ell_1)^2} \tag{4.27}$$

is not a singular term.

The last term in $\mathbb{S}[U_3]$ can be estimated by

$$\begin{aligned}\mathbb{S}_4[U_3] \;&=\; -\,\frac{\kappa}{\gamma}\,\epsilon|\log\epsilon|\,U_3\frac{2\big(x_1^2-x_2^2-|r_{1\epsilon}|^2\big)r_{1\epsilon}}{(\ell_1\ell_2)^2}\\ &\equiv \Omega_{35}.\end{aligned}$$

In summary, we write

$$\mathbb{S}[U_3] \;\equiv\; F_{31} \;+\; F_{32}. \tag{4.28}$$

In the above, we have denoted the combination of the terms as F_{31} and also F_{32} in the form

$$F_{31} \;=\; i\,U_3\,\frac{\nabla V\cdot\nabla\varphi_0}{1+V} \;+\; i\,U_3\,\mathbb{S}_2[\arctan(x_2/r_{1\epsilon})] \;+\; i\,U_3\,\mathbb{S}_3[\varphi_0],$$

and

$$F_{32} \;=\; \sum_{j=1}^{5}\Omega_{3j}.$$

By careful checking, we find that F_{32} is a term defined on the region D_3 with properties,

$$\begin{aligned}\left|\mathrm{Re}\left(\frac{F_{32}}{ie^{i\varphi_0}}\right)\right| \;&\leq\; \frac{O(\epsilon^{1-\sigma})}{(1+\ell_1)^3} \;+\; \frac{O(\epsilon^{1-\sigma})}{(1+\ell_2)^3},\\ \left|\mathrm{Im}\left(\frac{F_{32}}{ie^{i\varphi_0}}\right)\right| \;&\leq\; \frac{O(\epsilon^{1-\sigma})}{(1+\ell_1)^{1+\sigma}} \;+\; \frac{O(\epsilon^{1-\sigma})}{(1+\ell_2)^{1+\sigma}}.\end{aligned} \tag{4.29}$$

4.1.3 The error terms: $\mathbb{S}[U_4]$ and $\mathbb{S}[U_5]$

Finally, the errors on D_4 and D_5. We begin with the error of U_4

$$\mathbb{S}[U_4] \;=\mathbb{S}_0[U_4] \;+\; \mathbb{S}_1[U_4] \;+\; \mathbb{S}_2[U_4] \;+\; \mathbb{S}_3[U_4] \;+\; \mathbb{S}_4[U_4],$$

where

$$\mathbb{S}_0[U_4] \;=\delta_\epsilon\, q''\Big(\delta_\epsilon^{1/3}(x_1-r_{2\epsilon})\Big)\,e^{i\varphi_0} \;+\; 2ie^{i\varphi_0}\nabla\hat{q}\cdot\nabla\varphi_0 \;-\; \hat{q}\,e^{i\varphi_0}\big|\nabla\varphi_0\big|^2.$$

By the conditions in $(\mathbf{A3})$

$$\begin{aligned}1+V(\epsilon|x_1|) \;&=\; 1+V(\epsilon r_{2\epsilon})+\frac{\partial V}{\partial\tilde{r}}\Big|_{\tilde{r}=\epsilon r_{2\epsilon}}\epsilon(x_1-r_{2\epsilon})+O\big(\epsilon^2(x_1-r_{2\epsilon})^2\big)\\ &=\; -\delta_\epsilon(x_1-r_{2\epsilon})+O\big(\epsilon^2(x_1-r_{2\epsilon})^2\big).\end{aligned} \tag{4.30}$$

We also write $\mathbb{S}_1[U_4]$ of the form

$$\begin{aligned}\mathbb{S}_1[U_4] \;&= \delta_\epsilon\Big[-\delta_\epsilon^{1/3}(x_1-r_{2\epsilon})-q^2\Big(\delta_\epsilon^{1/3}(x_1-r_{2\epsilon})\Big)\Big]q\Big(\delta_\epsilon^{1/3}(x_1-r_{2\epsilon})\Big)e^{i\varphi_0}\\ &\quad+\;\Big[(1+V)+\delta_\epsilon(x_1-r_{2\epsilon})\Big]\hat{q}e^{i\varphi_0}.\end{aligned}$$

The equation of q in Lemma 2.4 implies that there holds

$$\begin{aligned}\mathbb{S}_0[U_4]\;+\;\mathbb{S}_1[U_4]\;&=\;2ie^{i\varphi_0}\,\delta_\epsilon^{2/3}\,q'\Big(\delta_\epsilon^{1/3}(x_1-r_{2\epsilon})\Big)\frac{\partial\varphi_0}{\partial x_1}\;-\;\hat{q}\,e^{i\varphi_0}\big|\nabla\varphi_0\big|^2\\ &\quad+\;\Big[(1+V)-\delta_\epsilon(x_1-r_{2\epsilon})\Big]\,\hat{q}e^{i\varphi_0}\\ &\equiv\;\Omega_{41}.\end{aligned}$$

We proceed with the calculations

$$\begin{aligned}\mathbb{S}_2[U_4]\;&\equiv\;U_4\,\gamma^{-2}\left[\frac{\lambda^2}{x_1^2}-\frac{\lambda^2}{|r_{1\epsilon}|^2}\right]\left(-\left|\frac{\partial\varphi_0}{\partial x_2}\right|^2+\;i\frac{\partial^2\varphi_0}{\partial x_2^2}\right)\\ &=\;-\;U_4\,\gamma^{-2}\left[\frac{\lambda^2}{x_1^2}-\frac{\lambda^2}{|r_{1\epsilon}|^2}\right]\left|\frac{\partial\varphi_0}{\partial x_2}\right|^2\;+\;i\,U_4\,\mathbb{S}_2[\varphi_0]\\ &\equiv\;\Omega_{42}\;+\;i\,U_4\,\mathbb{S}_2[\varphi_0],\end{aligned}$$

where

$$\left|\frac{\partial\varphi_0}{\partial x_2}\right|^2\;=\;\left|\frac{x_1-r_{1\epsilon}}{\ell_2^2}-\frac{x_1+r_{1\epsilon}}{\ell_1^2}\right|^2\;=\;O(1/\ell_2^2)\quad\text{ in }D_4.$$

Similar calculations hold

$$\begin{aligned}\mathbb{S}_3[U_4]\;&=\;\frac{1}{x_1}\,\delta_\epsilon^{2/3}\,q'\Big(\delta_\epsilon^{1/3}(x_1-r_{2\epsilon})\Big)e^{i\varphi_0}\;+\;i\,U_4\,\mathbb{S}_3[\varphi_0]\\ &\equiv\;\Omega_{43}\;+\;i\,U_4\,\mathbb{S}_3[\varphi_0].\end{aligned}$$

The last term is

$$\begin{aligned}\mathbb{S}_4[U_4]\;&=\;-\;\frac{\kappa}{\gamma}\,\epsilon|\log\epsilon|\,U_4\frac{2\big(x_1^2-x_2^2-|r_{1\epsilon}|^2\big)r_{1\epsilon}}{(\ell_1\ell_2)^2}\\ &\equiv\;\Omega_{44}.\end{aligned}$$

In summary, we write

$$\mathbb{S}[U_4]\;\equiv\;F_{41}\;+\;F_{42},\tag{4.31}$$

where

$$F_{41}\;=\;i\,U_4\,\mathbb{S}_2[\varphi_0]\;+\;i\,U_4\,\mathbb{S}_3[\varphi_0],\qquad F_{42}\;=\;\sum_{j=1}^{4}\Omega_{4j}.$$

By careful checking, we find that F_{42} is a term defined on the region D_4 with properties,

$$\begin{aligned} \left|\mathrm{Re}\left(\frac{F_{42}}{ie^{i\varphi_0}}\right)\right| \,\leq\, \frac{O(\epsilon^{1-\sigma})}{(1+\ell_1)^3} \,+\, \frac{O(\epsilon^{1-\sigma})}{(1+\ell_2)^3}, \\ \left|\mathrm{Im}\left(\frac{F_{42}}{ie^{i\varphi_0}}\right)\right| \,\leq\, \frac{O(\epsilon^{1-\sigma})}{(1+\ell_1)^{1+\sigma}} \,+\, \frac{O(\epsilon^{1-\sigma})}{(1+\ell_2)^{1+\sigma}}. \end{aligned} \tag{4.32}$$

Similar estimates hold for U_5 on D_5.

4.2 Further improvement of the approximation

To get rid of the singularities given in previous section, we formally add a correction term to the phase component of the approximation in the form

$$\varphi_d(z) \,=\, \varphi_s(z) \,+\, \varphi_1(z) \,+\, \varphi_2(z). \tag{4.33}$$

The components will be found explicitly in the sequel.

Recall the constants $\hat{r}_1$, τ_0 ad τ_1 given in Section 1.3.1, and set

$$\tau = \min\{\,\hat{r}_1,\, \tau_0,\, \tau_1\,\}. \tag{4.34}$$

By defining the smooth cut-off function $\eta_\epsilon(x_1) = \tilde{\eta}(\epsilon|x_1|)$ where $\tilde{\eta}$ is in the form

$$\tilde{\eta}(\vartheta) = 1 \quad \text{for} \quad |\vartheta| \leq \tau/10, \qquad \tilde{\eta}(\vartheta) = 0 \quad \text{for} \quad |\vartheta| \geq \tau/5, \tag{4.35}$$

we add the last component

$$\varphi_2 \,=\, -\,\eta_\epsilon(x_1)\arctan(x_2/r_{1\epsilon}), \tag{4.36}$$

in the above formula to cancel the singular term in (4.25). Note that

$$\begin{aligned} \Delta\varphi_2 \,&=\, -\,\eta_\epsilon(x_1)\,\frac{\partial^2}{\partial x_2^2}\arctan(x_2/r_{1\epsilon}) \,+\, O(\epsilon^2) \\ &=\, -\,\eta_\epsilon(x_1)\,\frac{4\,r_{1\epsilon}\,x_2}{(r_{1\epsilon}^2+x_2^2)^2} \,+\, O(\epsilon^2). \end{aligned} \tag{4.37}$$

To cancel the singularities in (4.11) and (4.15) rewritten in the form

$$\frac{s_2}{r_{1\epsilon}|s|^2} - \frac{4\sigma^2}{r_{1\epsilon}\gamma^2}\frac{s_1^2 s_2}{|s|^4} \,=\, \frac{s_2}{r_{1\epsilon}\gamma^2|s|^2} - \frac{\sigma^2(3s_1^2-s_2^2)s_2}{r_{1\epsilon}\gamma^2|s|^4},$$

we want to find a function $\Phi(s_1, s_2)$ by solving the problem in the translated coordinates (s_1, s_2)

$$\frac{\partial^2\Phi}{\partial s_1^2} + \frac{\partial^2\Phi}{\partial s_2^2} \,=\, \frac{s_2}{r_{1\epsilon}\gamma^2|s|^2} - \frac{\sigma^2(3s_1^2-s_2^2)s_2}{r_{1\epsilon}\gamma^2|s|^4} \quad \text{in } \mathbb{R}^2. \tag{4.38}$$

In fact, we can solve this problem by separation of variables and then obtain

$$\Phi(s_1, s_2) \;=\; \frac{1}{4r_{1\epsilon}\gamma^2} s_2 \log|s|^2 \;-\; \frac{\sigma^2}{2r_{1\epsilon}\gamma^2}\frac{s_2^3}{|s|^2} \;+\; \frac{3\sigma^2}{8r_{1\epsilon}\gamma^2} s_2. \tag{4.39}$$

After setting χ a smooth cut-off function such that

$$\chi(\vartheta) = 1 \quad \text{for } \vartheta < \tau/10, \qquad \chi(\vartheta) = 0 \quad \text{for} \quad \vartheta > \tau/5,$$

the singular part is defined by

$$\begin{aligned} \varphi_s(z) =\; & \chi(\varepsilon\ell_1)\Big(\frac{x_2}{4r_{1\epsilon}\,\gamma^2} \log\frac{\ell_1^2}{\ell_2^2} \;-\; \frac{\sigma^2}{2r_{1\epsilon}\,\gamma^2}\frac{x_2^3}{\ell_1^2} \;+\; \frac{3\sigma^2}{8r_{1\epsilon}\,\gamma^2}x_2\Big) \\ & +\; \chi(\varepsilon\ell_2)\Big(\frac{x_2}{4r_{1\epsilon}\,\gamma^2} \log\frac{\ell_2^2}{\ell_1^2} \;-\; \frac{\sigma^2}{2r_{1\epsilon}\,\gamma^2}\frac{x_2^3}{\ell_2^2} \;+\; \frac{3\sigma^2}{8r_{1\epsilon}\,\gamma^2}x_2\Big). \end{aligned} \tag{4.40}$$

For later use, we compute:

$$\begin{aligned} \frac{\partial\varphi_s}{\partial x_1} =\; & \chi'(\varepsilon\ell_1)\,\varepsilon\,\frac{x_1 - r_{1\epsilon}}{\ell_1}\left(\frac{x_2}{4r_{1\epsilon}\,\gamma^2}\log\frac{\ell_1^2}{\ell_2^2} - \frac{\sigma^2}{2r_{1\epsilon}\,\gamma^2}\frac{x_2^3}{\ell_1^2} + \frac{3\sigma^2}{8r_{1\epsilon}\,\gamma^2}x_2\right) \\ & +\; \chi(\varepsilon\ell_1)\left(\frac{x_2\,(x_1 - r_{1\epsilon})}{2r_{1\epsilon}\,\gamma^2\,\ell_1^2} - \frac{x_2\,(x_1 + r_{1\epsilon})}{2r_{1\epsilon}\,\gamma^2\,\ell_2^2} + \frac{\sigma^2}{r_{1\epsilon}\,\gamma^2}\frac{x_2^3\,(x_1 - r_{1\epsilon})}{\ell_1^4}\right) \\ & +\; \chi'(\varepsilon\ell_2)\,\varepsilon\,\frac{x_1 + r_{1\epsilon}}{\ell_2}\left(\frac{x_2}{4r_{1\epsilon}\,\gamma^2}\log\frac{\ell_2^2}{\ell_1^2} - \frac{\sigma^2}{2r_{1\epsilon}\,\gamma^2}\frac{x_2^3}{\ell_2^2} + \frac{3\sigma^2}{8r_{1\epsilon}\,\gamma^2}x_2\right) \\ & +\; \chi(\varepsilon\ell_2)\left(\frac{x_2\,(x_1 + r_{1\epsilon})}{2r_{1\epsilon}\,\gamma^2\,\ell_2^2} - \frac{x_2\,(x_1 - r_{1\epsilon})}{2r_{1\epsilon}\,\gamma^2\,\ell_1^2} + \frac{\sigma^2}{r_{1\epsilon}\,\gamma^2}\frac{x_2^3\,(x_1 + r_{1\epsilon})}{\ell_2^4}\right), \end{aligned} \tag{4.41}$$

$$\begin{aligned} \frac{\partial\varphi_s}{\partial x_2} =\; & \chi'(\varepsilon\ell_1)\,\varepsilon\,\frac{x_2}{\ell_1}\left(\frac{x_2}{4r_{1\epsilon}\,\gamma^2}\log\frac{\ell_1^2}{\ell_2^2} - \frac{\sigma^2}{2r_{1\epsilon}\,\gamma^2}\frac{x_2^3}{\ell_1^2} + \frac{3\sigma^2}{8r_{1\epsilon}\,\gamma^2}x_2\right) \\ & +\; \chi(\varepsilon\ell_1)\left[\frac{1}{4r_{1\epsilon}\gamma^2}\log\frac{\ell_1^2}{\ell_2^2} + \frac{2x_2^2}{4r_{1\epsilon}\gamma^2}\Big(\frac{1}{\ell_1^2} - \frac{1}{\ell_2^2}\Big) - \frac{\sigma^2}{2r_{1\epsilon}\gamma^2}\frac{3x_2^2}{\ell_1^2} \right. \\ & \qquad\qquad \left. +\; \frac{\sigma^2}{2r_{1\epsilon}\gamma^2}\frac{2x_2^4}{\ell_1^4} + \frac{3\sigma^2}{8r_{1\epsilon}\gamma^2}\right] \\ & +\; \chi'(\varepsilon\ell_2)\,\varepsilon\,\frac{x_2}{\ell_2}\left(\frac{x_2}{4r_{1\epsilon}\,\gamma^2}\log\ell_2^2 - \frac{\sigma^2}{2r_{1\epsilon}\,\gamma^2}\frac{x_2^3}{\ell_2^2} + \frac{3\sigma^2}{8r_{1\epsilon}\,\gamma^2}x_2\right) \\ & +\; \chi(\varepsilon\ell_2)\left[\frac{1}{4r_{1\epsilon}\gamma^2}\log\frac{\ell_2^2}{\ell_1^2} + \frac{2x_2^2}{4r_{1\epsilon}\gamma^2}\Big(\frac{1}{\ell_2^2} - \frac{1}{\ell_1^2}\Big) - \frac{\sigma^2}{2r_{1\epsilon}\gamma^2}\frac{3x_2^2}{\ell_2^2} \right. \\ & \qquad\qquad \left. +\; \frac{\sigma^2}{2r_{1\epsilon}\gamma^2}\frac{2x_2^4}{\ell_2^4} + \frac{3\sigma^2}{8r_{1\epsilon}\gamma^2}\right]. \end{aligned} \tag{4.42}$$

Hence, we obtain

$$\begin{aligned}\nabla\varphi_s &= \chi\big(\varepsilon\ell_1\big)\left[\frac{1}{2r_{1\epsilon}\gamma^2}\Big(0,\,\log\frac{\ell_1}{\ell_2}\Big)+O(\varepsilon)\right]\,+\,\chi'\big(\varepsilon\ell_1\big)O(\varepsilon|\log\varepsilon|)\\ &\qquad+\,\chi\big(\varepsilon\ell_2\big)\left[\frac{1}{2r_{1\epsilon}\gamma^2}\Big(0,\,\log\frac{\ell_2}{\ell_1}\Big)+O(\varepsilon)\right]\,+\,\chi'\big(\varepsilon\ell_2\big)O(\varepsilon|\log\varepsilon|).\end{aligned}\tag{4.43}$$

Note that the function φ_s is continuous but $\nabla\varphi_s$ is not. The singularity of φ_s comes from its derivatives.

While by recalling the operators $\mathbb{S}_2$, $\mathbb{S}_3$ in (3.20), the region $\mathfrak{S}$ in (3.9) and its decompositions in (3.18), we find the term $\varphi_1(z)$ by solving the problem

$$\begin{aligned}&\Big[\Delta+\mathbb{S}_2+\mathbb{S}_3+\eta_3\frac{1}{1+V}\nabla V\cdot\nabla+2\nabla\eta_3\cdot\nabla\Big]\varphi_1\\ &\qquad=\,-\Big[\Delta+\mathbb{S}_2+\mathbb{S}_3+\eta_3\frac{1}{1+V}\nabla V\cdot\nabla+2\nabla\eta_3\cdot\nabla\Big]\big(\varphi_0+\varphi_s+\varphi_2\big)\quad\text{in}\quad\mathfrak{S},\end{aligned}\tag{4.44}$$

$$\varphi_1\,=\,-\,\varphi_0\,-\,\varphi_s\,-\,\varphi_2\quad\text{on}\quad\partial\mathfrak{S}.\tag{4.45}$$

We derive the estimate of φ_1 by computing the right hand side of (4.44).

We begin with the computation on the ball $B_{\tau/10\epsilon}(\xi_+)$. Recall the formulas (4.9), (4.11), (4.14), (4.15). For $z\in B_{\tau/10\epsilon}(\xi_+)$, there holds

$$\begin{aligned}&\Big(\Delta+\mathbb{S}_2+\mathbb{S}_3\Big)\varphi_0+\Delta\varphi_s\\ &=\,\Delta\varphi_s\,+\,\Big(\mathbb{S}_2+\mathbb{S}_3\Big)\varphi_0\\ &=\,-\frac{x_1-r_{1\epsilon}}{r_{1\epsilon}\,x_1}\frac{\partial\varphi_0^+}{\partial x_1}\,+\,\frac{\sigma^2}{\gamma^2}\,\frac{3r_{1\epsilon}(x_1-r_{1\epsilon})^2+2(x_1-r_{1\epsilon})^3}{r_{1\epsilon}\,x_1^2}\,\frac{\partial^2\varphi_0^+}{\partial x_2^2}\,+\,O(\epsilon^2)\\ &=\,\frac{x_1-r_{1\epsilon}}{r_{1\epsilon}\,x_1}\frac{x_2}{\ell_2^2}\,-\,\frac{2\sigma^2}{\gamma^2}\,\frac{3r_{1\epsilon}(x_1-r_{1\epsilon})^2+2(x_1-r_{1\epsilon})^3}{r_{1\epsilon}\,x_1^2}\,\frac{(x_1-r_{1\epsilon})x_2}{\ell_2^4}\,+\,O(\epsilon^2)\\ &=\,O(\epsilon^2),\end{aligned}$$

due to $\Delta\varphi_0=0$. Similarly, by recalling (4.9) and (4.42), we obtain for $z\in B_{\tau/10\epsilon}(\xi_+)$

$$\begin{aligned}\mathbb{S}_2[\varphi_s]\,&=\,\left[-\,\frac{2\sigma^2}{r_{1\epsilon}\,\gamma^2}(x_1-r_{1\epsilon})+O\Big((x_1-r_{1\epsilon})^2/|r_{1\epsilon}|^2\Big)\right]\\ &\qquad\times\left\{\frac{(1-3\sigma^2)x_2}{2r_{1\epsilon}\,\gamma^2\ell_2^2}-\frac{(-1+7\sigma^2)x_2^3}{r_{1\epsilon}\,\gamma^2\ell_2^4}-\frac{4\sigma^2x_2^5}{r_{1\epsilon}\,\gamma^2\ell_1^6}\right\}\,+\,O(\epsilon^2)\\ &=O(\epsilon^2),\end{aligned}$$

and by recalling (4.41)

$$
\begin{aligned}
\mathbb{S}_3[\varphi_s] \;&=\; \frac{1}{x_1}\frac{1}{2r_{1\epsilon}\,\gamma^2}\Big[\,\frac{x_2(x_1-r_{1\epsilon})}{\ell_2^2}-\frac{x_2(x_1+r_{1\epsilon})}{\ell_1^2}\,\Big]\\
&\quad+\;\frac{1}{x_1}\frac{\sigma^2}{r_{1\epsilon}\,\gamma^2}\Big[\,\frac{x_2^3(x_1-r_{1\epsilon})}{\ell_2^4}-\frac{x_2^3(x_1+r_{1\epsilon})}{\ell_1^4}\,\Big]\;+\;O(\epsilon^2)\\
&=O(\epsilon^2).
\end{aligned}
$$

We also get for $z\in B_{\tau/10\epsilon}(\xi_+)$

$$
\Big[\eta_3\frac{1}{1+V}\nabla V\cdot\nabla+2\nabla\eta_3\cdot\nabla\Big](\varphi_0+\varphi_s)=0,
$$

due to $\eta_3=0$ in this region, and also

$$
\Big[\Delta+\mathbb{S}_2+\mathbb{S}_3+\eta_3\frac{1}{1+V}\nabla V\cdot\nabla+2\nabla\eta_3\cdot\nabla\Big]\varphi_2\;=0,
$$

due to $\varphi_2=0$ in this region. For $z\in B_{\tau/10\epsilon}(\xi_-)$, similar estimates can be checked.

By recalling (4.23), (4.25) and (4.27) and (4.37), we do the computation on the region

$$
\mathfrak{S}_0=\big\{\,x\in\mathfrak{S}\,:\,|x_1|\le\tau/10\epsilon\,\big\}.
$$

There also hold $\varphi_s=0$ and $\eta_3=1$, and so by the expression of φ_2 in (4.36)

$$
\begin{aligned}
&\Big[\Delta+\mathbb{S}_2+\mathbb{S}_3+\eta_3\frac{1}{1+V}\nabla V\cdot\nabla+2\nabla\eta_3\cdot\nabla\Big]\big(\varphi_0+\varphi_s+\varphi_2\big)\\
&=\;\Delta\varphi_2\;+\;\mathbb{S}_3[\varphi_0]\;+\;\mathbb{S}_2[\varphi_0+\varphi_2]\;+\;\frac{1}{1+V}\nabla V\cdot\nabla\varphi_0\\
&=\;-\frac{4\,r_{1\epsilon}\,x_2}{(r_{1\epsilon}^2+x_2^2)^2}\;-\;\frac{4\,r_{1\epsilon}\,x_2}{\ell_2^2\ell_1^2}\;+\;O(\epsilon^2)\\
&\quad+\;\gamma^{-2}\left[\frac{\lambda^2}{x_1^2}-\frac{\lambda^2}{r_{1\epsilon}^2}\right]\Bigg\{\;-\;\frac{8x_2\,x_1^2\,r_{1\epsilon}(\ell_2+\ell_1)}{\ell_2^4\ell_1^4}\;-\;\frac{16x_2\,x_1^2\,r_{1\epsilon}^3}{\ell_2^4\ell_1^4(r_{1\epsilon}^2+x_2^2)}\Big((\ell_2^2+\ell_1^2)(r_{1\epsilon}^2+x_2^2)+\ell_1^2\ell_2^2\Big)\\
&\qquad\qquad-\;\frac{2x_2\,x_1^2\,r_{1\epsilon}}{\ell_2^4\ell_1^4(r_{1\epsilon}^2+x_2^2)^2}\Big(\ell_2^4(r_{1\epsilon}^2+x_2^2+\ell_1^2)+\ell_1^4(r_{1\epsilon}^2+x_2^2+\ell_2^2)\Big)\Bigg\}\\
&\quad+\;\frac{\epsilon}{1+V}\frac{\partial V}{\partial\tilde r}\frac{4\,r_{1\epsilon}\,x_2\,x_1}{\ell_2^2\ell_1^2}.
\end{aligned}
$$

In the region

$$
\Big(B_{\tau/5\epsilon}(\xi_+)\setminus B_{\tau/10\epsilon}(\xi_+)\Big)\cup\Big(B_{\tau/5\epsilon}(\xi_-)\setminus B_{\tau/10\epsilon}(\xi_-)\Big)\cup\big\{\,(x_1,x_2)\,:\,\tau/10\epsilon<|x_1|<\tau/5\epsilon\,\big\},
$$

similar estimates can be derived.

For z in the region

$$
\mathfrak{S}\;\setminus\;\Big(B_{\tau/5\epsilon}(\xi_+)\cup B_{\tau/5\epsilon}(\xi_-)\cup\big\{\,(x_1,x_2)\,:\,|x_1|\le\tau/5\epsilon\,\big\}\Big),
$$

we have $\varphi_s = \varphi_2 = 0$ and then

$$
\begin{aligned}
& \Big[\Delta + \mathbb{S}_2 + \mathbb{S}_3 + \eta_3 \frac{1}{1+V} \nabla V \cdot \nabla + 2\nabla\eta_3 \cdot \nabla\Big]\big(\varphi_0 + \varphi_s + \varphi_2\big) \\
= \ & \Big[\mathbb{S}_2 + \mathbb{S}_3 + \eta_3 \frac{1}{1+V} \nabla V \cdot \nabla + 2\nabla\eta_3 \cdot \nabla\Big]\varphi_0 \\
= \ & \gamma^{-2}\left[\frac{\lambda^2}{x_1^2} - \frac{\lambda^2}{|r_{1\epsilon}|^2}\right]\frac{\partial^2\varphi_0}{\partial x_2^2} \ + \ \frac{1}{x_1}\frac{\partial\varphi_0}{\partial x_1} \ + \ \eta_3\frac{1}{1+V}\nabla V \cdot \nabla\varphi_0 \ + \ 2\nabla\eta_3 \cdot \nabla\varphi_0 \\
= \ & -\gamma^{-2}\left[\frac{\lambda^2}{x_1^2} - \frac{\lambda^2}{|r_{1\epsilon}|^2}\right]\left(\frac{2x_2(x_1 - r_{1\epsilon})}{\ell_2^4} - \frac{2x_2(x_1 + r_{1\epsilon})}{\ell_1^4}\right) - \frac{4x_2 r_{1\epsilon}}{\ell_2^2\ell_1^2} \\
& - \ \epsilon\eta_3\frac{1}{1+V}\frac{\partial V}{\partial\tilde{r}}\frac{4x_1x_2r_{1\epsilon}}{\ell_2^2\ell_1^2} \ + \ 2\frac{\partial\eta_3}{\partial x_1}\frac{4x_1x_2r_{1\epsilon}}{\ell_2^2\ell_1^2},
\end{aligned}
\tag{4.46}
$$

due to $\Delta\varphi_0 = 0$.

Whence, going back to the original variable $(r, \check{y}_3)$ in (3.2) and letting $\hat{\varphi}(r, \check{y}_3) = \varphi_1(z)$ we see that

$$
\left|\Delta_{r,\check{y}_3}\hat{\varphi} \ + \ \mathbb{S}_2[\hat{\varphi}] \ + \ \mathbb{S}_3[\hat{\varphi}] \ + \ \frac{1}{1+V}\nabla V \cdot \nabla\hat{\varphi} + 2\nabla\eta_3 \cdot \nabla\hat{\varphi}\right| \le \frac{C}{\Big(\sqrt{1 + r^2 + |\check{y}_3|^2}\Big)^3}. \tag{4.47}
$$

Thus we can choose φ_1 such that

$$
\hat{\varphi} = O\Big(\frac{1}{\sqrt{1 + r^2 + |\check{y}_3|^2}}\Big).
$$

The regular term φ_1 is C^1 in the original variable $(r, \check{y}_3)$.

We observe also that by our definition, the function

$$
\varphi := \varphi_0 + \varphi_d = \varphi_0 + \varphi_s + \varphi_1 + \varphi_2, \tag{4.48}
$$

satisfies

$$
\begin{aligned}
\Big[\Delta + \mathbb{S}_2 + \mathbb{S}_3 + \eta_\epsilon\frac{1}{1+V}\nabla V \cdot \nabla + 2\nabla\eta_\epsilon \cdot \nabla\Big]\varphi \ &= \ 0 \quad \text{on} \quad \mathfrak{S}, \\
\varphi \ &= \ 0 \quad \text{on} \quad \partial\mathfrak{S}.
\end{aligned}
\tag{4.49}
$$

From the decomposition of φ_d, we see that the singular term contains $x_2 \log|z - \xi_+|$ which becomes dominant when we calculate the speed.

By defining smooth cut-off functions as follows

$$
\tilde{\eta}_2(s) = \begin{cases} 1, & |\, s \,| \le 1, \\ 0, & |\, s \,| \ge 2, \end{cases} \qquad \tilde{\eta}_4(s) = \begin{cases} 1, & s \ge -1, \\ 0, & s \le -2, \end{cases} \qquad \tilde{\eta}_5(s) = \begin{cases} 1, & s \le 1, \\ 0, & s \ge 2, \end{cases} \tag{4.50}
$$

we choose the cut-off functions by

$$
\begin{aligned}
\eta_2\big(\epsilon x_1, \epsilon x_2\big) \ &= \ \tilde{\eta}_2\big(\epsilon^{\lambda_1}\ell_1/30\big) \ + \ \tilde{\eta}_2\big(\epsilon^{\lambda_1}\ell_2/30\big), \\
\eta_4\big(\epsilon x_1, \epsilon x_2\big) \ &= \ \tilde{\eta}_4\big(\epsilon^{\lambda_2}(x_1 - r_{2\epsilon})\big), \\
\eta_5\big(\epsilon x_1, \epsilon x_2\big) \ &= \ \tilde{\eta}_5\big(\epsilon^{\lambda_2}(x_1 + r_{2\epsilon})\big), \\
\eta_3\big(\epsilon x_1, \epsilon x_2\big) \ &= \ 1 - \eta_2 - \eta_4 - \eta_5.
\end{aligned}
\tag{4.51}
$$

We then choose the final approximate solution to (3.8) by, for $(x_1, x_2) \in \mathbb{R}^2$,

$$\begin{aligned} \mathcal{U}_2(x_1, x_2) \, = & \sqrt{1 + V(\epsilon|x_1|)}\, \eta_3\, e^{i\varphi} \, + \, w(\ell_2)w(\ell_1)\, \eta_2\, e^{i\varphi} \\ & + \, \hat{q}(x_1)\, \eta_4\, e^{i\varphi} \, + \, \hat{q}(-x_1)\, \eta_5\, e^{i\varphi}. \end{aligned} \tag{4.52}$$

By recalling the definition of U_2, U_3, U_4 and U_5 in (4.1), (4.2) and (4.4), we also write the approximation as

$$\mathcal{U}_2 \, = U_2\, \eta_2\, e^{i\varphi_d} \, + \, U_3\, \eta_3\, e^{i\varphi_d} \, + \, U_4\, \eta_4\, e^{i\varphi_d} \, + \, U_5\, \eta_5\, e^{i\varphi_d}. \tag{4.53}$$

4.3 Estimates of the error

We shall check that $\mathcal{U}_2$ is a good approximate solution in the sense that it satisfies the conditions in (3.13) and has a small error. It is easy to show that

$$\mathcal{U}_2(z) \, = \, \overline{\mathcal{U}_2(\bar{z})}, \qquad \mathcal{U}_2(z) \, = \, \mathcal{U}_2(-\bar{z}), \qquad \frac{\partial \mathcal{U}_2}{\partial x_1}(0, x_2) \, = \, 0.$$

Recall the boundary condition in (4.49). It is obvious that

$$\begin{aligned} \mathrm{Im}\mathcal{U}_2 \, = & \Big[\sqrt{1 + V(\epsilon|x_1|)}\, \eta_3 \, + \, w(\ell_2)w(\ell_1)\, \eta_2 \, + \, \hat{q}(x_1)\, \eta_4 \, + \, \hat{q}(-x_1)\, \eta_5 \Big] \sin\varphi \\ = & \, 0 \quad \text{on} \quad \partial\mathfrak{S}, \end{aligned} \tag{4.54}$$

and

$$\begin{aligned} \frac{\partial\, \mathrm{Re}\mathcal{U}_2}{\partial x_2} \, = & \, \frac{\partial}{\partial x_2} \bigg\{ \Big[\sqrt{1 + V(\epsilon|x_1|)}\, \eta_3 \, + \, w(\ell_2)w(\ell_1)\, \eta_2 \, + \, \hat{q}(x_1)\, \eta_4 \, + \, \hat{q}(-x_1)\, \eta_5 \Big] \cos\varphi \bigg\} \\ = & \Big[\sqrt{1 + V(\epsilon|x_1|)}\, \eta_3 \, + \, \hat{q}(x_1)\, \eta_4 \, + \, \hat{q}(-x_1)\, \eta_5 \Big] \sin\varphi\, \frac{\partial\varphi}{\partial x_2} \quad \text{on} \quad \partial\mathfrak{S}, \end{aligned} \tag{4.55}$$

which is of order $O(\epsilon^2)$ due the fact

$$\frac{\partial\varphi}{\partial x_2} = \frac{\partial\varphi_0}{\partial x_2} \, + \, \frac{\partial\varphi_d}{\partial x_2}$$

It can be checked that $\mathcal{U}_2$ satisfies the conditions in (3.13) except

$$\frac{\partial \mathcal{U}_2}{\partial x_2} \, = \, 0 \quad \text{on} \quad \partial\mathfrak{S}. \tag{4.56}$$

For the computation of errors, we work directly in the half space $\mathbb{R}^2_+ = \{(x_1, x_2) : \, x_1 > 0\}$ in the sequel because of the symmetry of the problem. Recalling the definitions of the operators in (3.20), let us start to compute the error $\mathbb{E} = \mathbb{S}[\mathcal{U}_2]$ in the form

$$
\begin{aligned}
\mathbb{E} \,=\, & \mathbb{S}[U_2]\,\eta_2\,e^{i\varphi_d} \,+\, U_2\,\big(\mathbb{S}_0+\mathbb{S}_2+\mathbb{S}_3+\mathbb{S}_4\big)\Big[\eta_2 e^{i\varphi_d}\Big] \\
& +\, 2\nabla U_2\cdot\nabla\Big(\eta_2 e^{i\varphi_d}\Big) \,+\, 2\gamma^{-2}\left[\frac{\lambda^2}{x_1^2}-\frac{\lambda^2}{|r_{1\epsilon}|^2}\right]\frac{\partial U_2}{\partial x_2}\frac{\partial\big(\eta_2 e^{i\varphi_d}\big)}{\partial x_2} \\
& +\, \mathbb{S}[U_3]\,\eta_3\,e^{i\varphi_d} \,+\, U_3\,\big(\mathbb{S}_0+\mathbb{S}_2+\mathbb{S}_3+\mathbb{S}_4\big)\Big[\eta_3 e^{i\varphi_d}\Big] \\
& +\, 2\nabla U_3\cdot\nabla\Big(\eta_3 e^{i\varphi_d}\Big) \,+\, 2\gamma^{-2}\left[\frac{\lambda^2}{x_1^2}-\frac{\lambda^2}{|r_{1\epsilon}|^2}\right]\frac{\partial U_3}{\partial x_2}\frac{\partial\big(\eta_3 e^{i\varphi_d}\big)}{\partial x_2} \\
& +\, \mathbb{S}[U_4]\,\eta_4\,e^{i\varphi_d} \,+\, U_4\,\big(\mathbb{S}_0+\mathbb{S}_2+\mathbb{S}_3+\mathbb{S}_4\big)\Big[\eta_4 e^{i\varphi_d}\Big] \\
& +\, 2\nabla U_4\cdot\nabla\Big(\eta_4 e^{i\varphi_d}\Big) \,+\, 2\gamma^{-2}\left[\frac{\lambda^2}{x_1^2}-\frac{\lambda^2}{|r_{1\epsilon}|^2}\right]\frac{\partial U_4}{\partial x_2}\frac{\partial\big(\eta_4 e^{i\varphi_d}\big)}{\partial x_2} \\
& +\, \mathbb{S}[U_5]\,\eta_5\,e^{i\varphi_d} \,+\, U_5\,\big(\mathbb{S}_0+\mathbb{S}_2+\mathbb{S}_3+\mathbb{S}_4\big)\Big[\eta_5 e^{i\varphi_d}\Big] \\
& +\, 2\nabla U_5\cdot\nabla\Big(\eta_5 e^{i\varphi_d}\Big) \,+\, 2\gamma^{-2}\left[\frac{\lambda^2}{x_1^2}-\frac{\lambda^2}{|r_{1\epsilon}|^2}\right]\frac{\partial U_5}{\partial x_2}\frac{\partial\big(\eta_5 e^{i\varphi_d}\big)}{\partial x_2} \,+\, \mathbb{N},
\end{aligned}
\tag{4.57}
$$

where the nonlinear term $\mathbb{N}$ is defined by

$$
\mathbb{N} \,=\, \eta_2|U_2|^2U_2e^{i\varphi_d} \,+\, \eta_3|U_3|^2U_3e^{i\varphi_d} \,+\, \eta_4|U_4|^2U_4e^{i\varphi_d} \,+\, \eta_5|U_5|^2U_5e^{i\varphi_d} \,-\, |\mathcal{U}_2|^2\mathcal{U}_2. \tag{4.58}
$$

The main components in the above formula can be estimated as follows. It will be shown that the singular terms in $\mathbb{S}[U_2]$ will be canceled by the relation

$$
\Delta\varphi_d \,+\, \mathbb{S}_2[\varphi_d] \,+\, \mathbb{S}_3[\varphi_d] \,=\, -\,\mathbb{S}_2[\varphi_0] \,-\, \mathbb{S}_3[\varphi_0]. \tag{4.59}
$$

Here we have used the relation $\Delta\varphi_0 = 0$ and the equation in (4.49).

Recall F_{21} and F_{22} in (4.18). Using the equation (4.59), the singular term F_{21} in $\mathbb{S}[U_2]$ is canceled and we then get

$$
\begin{aligned}
& \mathbb{S}[U_2]\,\eta_2\,e^{i\varphi_d} \,+\, U_2\,\big(\mathbb{S}_0+\mathbb{S}_2+\mathbb{S}_3+\mathbb{S}_4\big)[\eta_2 e^{i\varphi_d}] \\
& = \mathbb{S}[U_2]\,\eta_2\,e^{i\varphi_d} \,+\, i\,U_2\,\eta_2 e^{i\varphi_d}\,\big(\mathbb{S}_0+\mathbb{S}_2+\mathbb{S}_3+\mathbb{S}_4\big)[\varphi_d] \,+\, 2iU_2\nabla\eta_2\cdot\nabla\varphi_d \\
& \quad -\, U_2\,\eta_2\,e^{i\varphi_d}\,\big|\nabla\varphi_d\big|^2 \,+\, \gamma^{-2}\left[\frac{\lambda^2}{x_1^2}-\frac{\lambda^2}{|r_{1\epsilon}|^2}\right]\left(2iU_2\frac{\partial\eta_2}{\partial x_2}\cdot\frac{\partial\varphi_d}{\partial x_2} \,-\, U_2\,\eta_2\,e^{i\varphi_d}\left|\frac{\partial\varphi_d}{\partial x_2}\right|^2\right) \\
& \quad +\, U_2\,e^{i\varphi_d}\,\big(\mathbb{S}_0+\mathbb{S}_2+\mathbb{S}_3+\mathbb{S}_4\big)[\eta_2] \\
& = F_{22}\,\eta_2\,e^{i\varphi_d} \,+\, \epsilon^2 O(|\ell_2|^2).
\end{aligned}
$$

The formulas in (4.41)-(4.42) imply that

$$
2\nabla U_2\cdot\nabla\Big(\eta_2 e^{i\varphi_d}\Big) \,=\, 2\eta_2 U_2 e^{i\varphi_d}\frac{4x_1x_2\,r_{1\epsilon}}{(\ell_1\ell_2)^2}\,\frac{x_2\big[x_1^2-x_2^2-r_{1\epsilon}^2\big]}{\ell_2^2\ell_1^2}
$$

$$
\begin{aligned}
&+\ 2i\eta_2 U_2 e^{i\varphi_d}\,\frac{x_1-r_{1\epsilon}}{\ell_2}\,\frac{w'(\ell_2)}{w(\ell_2)}\,\frac{x_2\big[x_1^2-x_2^2-r_{1\epsilon}^2\big]}{\ell_2^2\ell_1^2}\\
&+\ 2i\eta_2 U_2 e^{i\varphi_d}\,\frac{x_2}{\ell_2}\,\frac{w'(\ell_2)}{w(\ell_2)}\,\frac{2x_1x_2^2}{\ell_2^2\ell_1^2}\\
&-\ 4\eta_2 U_2 e^{i\varphi_d}\,\frac{r_{1\epsilon}\big[x_1^2-x_2^2-r_{1\epsilon}^2\big]}{(\ell_1\ell_2)^2}\,\frac{1}{4r_{1\epsilon}}\log\frac{\ell_2^2}{\ell_1^2}\\
&+\ 2i\eta_2 U_2 e^{i\varphi_d}\,\frac{x_2}{\ell_2}\,\frac{w'(\ell_2)}{w(\ell_2)}\,\frac{1}{4r_{1\epsilon}}\log\frac{\ell_2^2}{\ell_1^2}\ +\ O(\epsilon)\\
=\ & 2\eta_2 U_2 e^{i\varphi_d}\,\frac{(x_1+r_{1\epsilon})(x_1-r_{1\epsilon})}{(\ell_1\ell_2)^2}\log r_{1\epsilon}\\
&-\ i\eta_2 U_2 e^{i\varphi_d}\,\frac{x_2}{\ell_2}\,\frac{w'(\ell_2)}{w(\ell_2)}\,\frac{1}{r_{1\epsilon}}\log r_{1\epsilon}\ +\ O(\epsilon\log\ell_2).
\end{aligned}
$$

Similar estimate holds for

$$
\gamma^{-2}\left[\frac{\lambda^2}{x_1^2}-\frac{\lambda^2}{|r_{1\epsilon}|^2}\right]\frac{\partial U_2}{\partial x_2}\frac{\partial(\eta_2 e^{i\varphi_d})}{\partial x_2}\quad \text{and}\quad U_2\,\mathbb{S}_4\Big[\eta_2 e^{i\varphi_d}\Big].
$$

It is worth to mention that, in the vortex-core region

$$
D_2\ =\ \Big\{(x_1,x_2):\ell_2<\epsilon^{-\lambda_1}\Big\},
$$

we estimate the error by

$$
\begin{aligned}
\mathbb{E}\ =\ & U_2\eta_2 e^{i\varphi_d}\left[\frac{x_1-r_{1\epsilon}}{x_1\ell_2}\frac{w'(\ell_2)}{w(\ell_2)}\ +\ \epsilon(x_1-r_{1\epsilon})\frac{\partial V}{\partial\tilde r}\Big|_{\tilde r=\epsilon r_{1\epsilon}}\right]\\
&+\ \eta_2 U_2 e^{i\varphi_d}\,\frac{2(x_1+r_{1\epsilon})(x_1-r_{1\epsilon})}{(\ell_1\ell_2)^2}\log r_{1\epsilon}\ +\ O(\epsilon^2\ell_2^2)\\
&-\ i\eta_2 U_2 e^{i\varphi_d}\,\frac{x_2}{\ell_2}\,\frac{w'(\ell_2)}{w(\ell_2)}\,\frac{1}{r_{1\epsilon}}\log r_{1\epsilon}\ +\ O(\epsilon\log\ell_2).
\end{aligned}
\tag{4.60}
$$

The singularity of the last formula will play an important role in the final reduction step.

We then consider the error in the region

$$
D_3\ =\ \Big\{(x_1,x_2):|x_1|<r_{2\epsilon}-\epsilon^{-\lambda_2}\Big\}\setminus\big(D_2\cup D_1\big).
$$

From the relation (4.59), there holds

$$
\begin{aligned}
&\mathbb{S}[U_3]\,\eta_3\,e^{i\varphi_d}\ +\ U_3\,\big(\mathbb{S}_0+\mathbb{S}_2+\mathbb{S}_3+\mathbb{S}_4\big)[\eta_3 e^{i\varphi_d}]\\
&\ =\ \mathbb{S}[U_3]\,\eta_3\,e^{i\varphi_d}\ +\ i\,U_3\,\eta_3 e^{i\varphi_d}\,\big(\mathbb{S}_0+\mathbb{S}_2+\mathbb{S}_3+\mathbb{S}_4\big)[\varphi_d]\ +\ 2iU_3\nabla\eta_3\cdot\nabla\varphi_d\\
&\quad -\ U_3\,\eta_3\,e^{i\varphi_d}\,\big|\nabla\varphi_d\big|^2\ +\ \gamma^{-2}\left[\frac{\lambda^2}{x_1^2}-\frac{\lambda^2}{|r_{1\epsilon}|^2}\right]\left(2iU_3\frac{\partial\eta_3}{\partial x_2}\cdot\frac{\partial\varphi_d}{\partial x_2}\ -\ U_3\,\eta_3\,e^{i\varphi_d}\,\Big|\frac{\partial\varphi_d}{\partial x_2}\Big|^2\right)\\
&\quad +\ U_3\,e^{i\varphi_d}\,\big(\mathbb{S}_0+\mathbb{S}_2+\mathbb{S}_3+\mathbb{S}_4\big)[\eta_3]
\end{aligned}
$$

$$= F_{32}\,\eta_3\, e^{i\varphi_d} \,+\, \epsilon^2 O(|\ell_2|^2).$$

In this region, $|\bigtriangledown \varphi_0| = O(\epsilon)$ and $|\bigtriangledown \varphi_d| = O(\epsilon)$. Whence, by using (4.28), we obtain

$$\eta_3 e^{i\varphi_d}\mathbb{S}[U_3] \,+\, U_3\left(\mathbb{S}_0 + \mathbb{S}_2 + \mathbb{S}_3\right)[\eta_3 e^{i\varphi_d}] = \eta_3 e^{i\varphi_d} U_3 O(\epsilon^2).$$

Using the formulas (4.41)-(4.42), we obtain

$$\begin{aligned}
2\nabla U_3\cdot\nabla\Big(\eta_3 e^{i\varphi_d}\Big) \;=\;& 2\eta_3 i U_3 e^{i\varphi_d}\left[\frac{1}{2}\big(1+V\big)^{-1}\epsilon\,\frac{\partial V}{\partial \tilde{r}} \,-\, i\frac{4x_1x_2\, r_{1\epsilon}}{\ell_2^2\ell_1^2}\right]\\
&\times\left(\frac{x_2}{4r_{1\epsilon}}\,\epsilon\eta'\big(\epsilon\ell_2\big)\frac{x_1-r_{1\epsilon}}{\ell_2}\log\frac{\ell_2^2}{\ell_1^2}\right.\\
&\left. +\;\eta\big(\epsilon\ell_2\big)\frac{x_2\big(x_1^2-x_2^2-(r_{1\epsilon})^2\big)}{\ell_2^2\ell_1^2} \,+\, O(\epsilon)\right)\\
&+\;2\eta_3 U_3 e^{i\varphi_d}\left[\frac{1}{2}\big(1+V\big)^{-1}\epsilon\,\frac{\partial V}{\partial \tilde{y}_3} - \frac{2r_{1\epsilon}\big[{x_1}^2-x_2^2-r_{1\epsilon}^2\big]}{\ell_2^2\ell_1^2}\right]\\
&\times\left[\frac{x_2}{4r_{1\epsilon}}\,\epsilon\eta'\big(\epsilon\ell_2\big)\frac{x_2}{\ell_2}\log\frac{\ell_2^2}{\ell_1^2} \,+\, \frac{1}{4r_{1\epsilon}}\,\eta\big(\epsilon\ell_2\big)\log\frac{\ell_2^2}{\ell_1^2}\right.\\
&\left. +\;\eta\big(\epsilon\ell_2\big)\frac{2x_1x_2^2}{\ell_2^2\ell_1^2} \,+\, O(\epsilon)\right]\\
&+\;2U_3 e^{i\varphi_d}\left[\frac{1}{2}\big(1+V\big)^{-1}\epsilon\,\frac{\partial V}{\partial \tilde{r}} \,-\, i\frac{4x_1x_2\, r_{1\epsilon}}{\ell_2^2\ell_1^2}\right]\times\frac{\partial\eta_3}{\partial x_1}\\
&+\;2U_3 e^{i\varphi_d}\left[\frac{1}{2}\big(1+V\big)^{-1}\epsilon\,\frac{\partial V}{\partial \tilde{y}_3} - \frac{2r_{1\epsilon}\big[{x_1}^2-x_2^2-r_{1\epsilon}^2\big]}{\ell_2^2\ell_1^2}\right]\times\frac{\partial\eta_3}{\partial x_2}\\
=\;& \eta_3 U_3 e^{i\varphi_d} O(\epsilon^2).
\end{aligned}$$

Similar estimate holds for

$$\gamma^{-2}\left[\frac{\lambda^2}{x_1^2} - \frac{\lambda^2}{|r_{1\epsilon}|^2}\right]\frac{\partial U_3}{\partial x_2}\frac{\partial(\eta_3 e^{i\varphi_d})}{\partial x_2}\quad\text{and}\quad U_3\,\mathbb{S}_4\Big[\eta_3 e^{i\varphi_d}\Big].$$

Whence we conclude that, in D_3, the error is estimated by

$$\mathbb{E} \;=\; (\eta_3 e^{i\varphi_d} U_3) O(\epsilon^2).$$

In the region

$$D_4 \;=\; \Big\{(x_1,x_2) : x_1 > r_{2\epsilon} - \epsilon^{-\lambda_2}\Big\},$$

by using the equation (4.59) and the similar computations as before, we now obtain

$$\eta_4\, e^{i\varphi_d}\mathbb{S}[U_4] \,+\, U_4\left(\mathbb{S}_0 + \mathbb{S}_2 + \mathbb{S}_3\right)[\eta_4 e^{i\varphi_d}] \;=\; \eta_4 e^{i\varphi_d} F_{42} \,+\, \eta_4 e^{i\varphi} O(\epsilon^2) \;=\; \eta_4 e^{i\varphi} O(\epsilon^2).$$

In the above, we have use the relation (4.49). Hence, there holds

$$\mathbb{E} \,=\, \eta_4 e^{i\varphi} O(\epsilon^2).$$

Similar estimate holds on the region $D_5 \,=\, \Big\{ (x_1,x_2) : x_1 < -r_{2\epsilon} + \epsilon^{-\lambda_2} \Big\}$.

For a complex function $h = h_1 + ih_2$ with real functions h_1, h_2, define a norm of the form

$$\begin{aligned} ||h||_{**} \equiv & \sum_{j=1}^{2} ||h||_{L^p(\ell_j<3)} \,+\, \sum_{j=1}^{2} ||h_j||_{L^p(D_4\cup D_5)} \\ & + \sum_{j=1}^{2} \bigg[\Big\| \, \ell_j^{2+\sigma} \Big(\frac{h_1}{-i\,\mathcal{U}_2} \Big) \, \Big\|_{L^\infty(\tilde{D}\cup D_3)} \,+\, \Big\| \, \ell_j^{1+\sigma} \Big(\frac{h_2}{-i\,\mathcal{U}_2} \Big) \, \Big\|_{L^\infty(\tilde{D}\cup D_3)} \bigg], \end{aligned} \tag{4.61}$$

where we have denoted

$$\tilde{D} = \big(D_1 \cup D_2 \big) \setminus \big\{ \ell_1 < 3 \;\text{ or }\; \ell_2 < 3 \big\}, \tag{4.62}$$

for ℓ_1 and ℓ_2 defined in (3.17). As a conclusion, we have the following lemma.

Lemma 4.2. *There holds*

$$\big\| \mathbb{E} \big\|_{L^p(\{\ell_1<3\}\cup\{\ell_2<3\})} \,\leq\, C\epsilon |\log \epsilon|.$$

As a consequence, there also holds

$$||\mathbb{E}||_{**} \leq C\epsilon^{1-\sigma},$$

for some $\sigma \in (0,1)$ independent of ϵ. □

5 Suitable decompositions of the perturbations

We look for a solution $u = u(x_1, x_2)$ to problem (3.8) with additional conditions in (3.13) in the form of small perturbation of $\mathcal{U}_2$. Define cut-off functions

$$\begin{aligned} \chi(x_1,x_2) \,&= \tilde{\eta}(\ell_2) \,+\, \tilde{\eta}(\ell_1), \\ \chi_{2\epsilon}(x_1,x_2) \,&= \tilde{\eta}(\epsilon^{\lambda_1}\ell_1/2) \,+\, \tilde{\eta}(\epsilon^{\lambda_1}\ell_2/2), \\ \chi_{8\epsilon}(x_1,x_2) \,&= \tilde{\eta}(\epsilon^{\lambda_1}\ell_1/8) \,+\, \tilde{\eta}(\epsilon^{\lambda_1}\ell_2/8), \\ \chi_{100\epsilon}(x_1,x_2) \,&= \tilde{\eta}(\epsilon^{\lambda_1}\ell_1/100) \,+\, \tilde{\eta}(\epsilon^{\lambda_1}\ell_2/100), \end{aligned} \tag{5.1}$$

with $\tilde{\eta}$ in (4.35). Recalling (4.50)-(4.53) and setting the components of the approximation $\mathcal{U}_2$ as

$$\begin{aligned} v_2(x_1,x_2) \,&=\, \eta_2\, U_2\, e^{i\varphi_d}, \quad v_3(x_1,x_2) \,=\, \eta_3\, U_3\, e^{i\varphi_d}, \\ v_4(x_1,x_2) \,&=\, \eta_4\, U_4\, e^{i\varphi_d}, \quad v_5(x_1,x_2) \,=\, \eta_5\, U_5\, e^{i\varphi_d}, \end{aligned} \tag{5.2}$$

in such a way that

$$\mathcal{U}_2 \,=\, v_2 \,+\, v_3 \,+\, v_4 \,+\, v_5,$$

we want to choose the ansatz of the form

$$u = \Big[\chi\big(v_2 + iv_2\psi\big) + \big(1-\chi\big)v_2 e^{i\psi}\Big] + \Big[(v_3+v_4+v_5) + i(1-\chi_{2\epsilon})e^{i\varphi}\psi\Big]. \tag{5.3}$$

where φ is defined in (4.48) and ψ is an unknown perturbation term. The above nonlinear decomposition of the perturbation in the vortex core region was first introduced in [29].

To find the perturbation term, the main objective of this section is to write the equation for the perturbation as a linear one with a right hand side given by a lower order nonlinear term. The conditions imposed on u in (3.10) and (3.12) can be transmitted to ψ

$$\begin{aligned} \psi(z) = \psi(-\bar{z}), &\qquad \psi(z) = -\overline{\psi(\bar{z})}, \\ \frac{\partial\psi}{\partial x_1}(0,x_2) = 0, &\qquad \psi(x_1,-\lambda\pi/\gamma) = \psi(x_1,\lambda\pi/\gamma), \\ \Big[\frac{\partial\mathcal{U}_2}{\partial x_2} + i\mathcal{U}_2\,\psi_{x_2}\Big]\Big|_{(x_1,-\lambda\pi/\gamma)} &= \Big[\frac{\partial\mathcal{U}_2}{\partial x_2} + i\mathcal{U}_2\,\psi_{x_2}\Big]\Big|_{(x_1,\lambda\pi/\gamma)}. \end{aligned} \tag{5.4}$$

More precisely, by the computations in (4.54) and (4.55), for $\psi = \psi_1 + i\psi_2$, there hold the conditions

$$\begin{aligned} \psi_1(x_1,x_2) = \psi_1(-x_1,x_2), &\qquad \psi_1(x_1,x_2) = -\psi_1(x_1,-x_2), \\ \psi_2(x_1,x_2) = \psi_2(-x_1,x_2), &\qquad \psi_2(x_1,x_2) = \psi_2(x_1,-x_2), \\ \frac{\partial\psi_1}{\partial x_1}(0,x_2) = 0, &\qquad \frac{\partial\psi_2}{\partial x_1}(0,x_2) = 0, \\ \psi_1(x_1,-\lambda\pi/\gamma) = \psi_1(x_1,\lambda\pi/\gamma) = 0, &\qquad \frac{\partial\psi_1}{\partial x_2}(x_1,-\lambda\pi/\gamma) = \frac{\partial\psi_1}{\partial x_2}(x_1,\lambda\pi/\gamma), \\ \frac{\partial\psi_2}{\partial x_2}(x_1,-\lambda\pi/\gamma) = \frac{1}{\tilde{\rho}}\frac{\partial\tilde{\rho}}{\partial x_2}(x_1,-\lambda\pi/\gamma), &\qquad \frac{\partial\psi_2}{\partial x_2}(x_1,\lambda\pi/\gamma) = \frac{1}{\tilde{\rho}}\frac{\partial\tilde{\rho}}{\partial x_2}(x_1,\lambda\pi/\gamma), \\ \psi_2(x_1,-\lambda\pi/\gamma) = \psi_2(x_1,\lambda\pi/\gamma). & \end{aligned} \tag{5.5}$$

Let us observe that

$$\begin{aligned} u &= \Big[v_2 + iv_2\psi + \big(1-\chi\big)v_2\big(e^{i\psi} - 1 - i\psi\big)\Big] + \Big(v_3+v_4+v_5 + i(1-\chi_{2\epsilon})e^{i\varphi}\psi\Big) \\ &= \mathcal{U}_2 + iv_2\psi + i(1-\chi_{2\epsilon})e^{i\varphi}\psi + \Gamma, \end{aligned}$$

where we have denoted

$$\Gamma = \big(1-\chi\big)v_2\big(e^{i\psi} - 1 - i\psi\big). \tag{5.6}$$

A direct computation shows that u satisfies (3.8) if and only if ψ satisfies the following equation

$$\begin{aligned} & iv_2\triangle\psi + iv_2\mathbb{S}_2[\psi] + iv_2\mathbb{S}_3[\psi] + iv_2\mathbb{S}_4[\psi] + 2i\bigtriangledown v_2\cdot\bigtriangledown\psi \\ &+ 2i\gamma^{-2}\left[\frac{\lambda^2}{x_1^2} - \frac{\lambda^2}{|r_{1\epsilon}|^2}\right]\frac{\partial v_2}{\partial x_2}\frac{\partial\psi}{\partial x_2} + i\Big[1 + V - |\mathcal{U}_2|^2\Big]v_2\psi - 2\mathrm{Re}\big(\bar{\mathcal{U}}_2 iv_2\psi\big)\mathcal{U}_2 \\ &+ i\triangle v_2\psi + i\mathbb{S}_2[v_2]\psi + i\mathbb{S}_3[v_2]\psi + i\mathbb{S}_4[v_2]\psi \end{aligned} \tag{5.7}$$

$$
\begin{aligned}
&+\, i(1-\chi_{2\epsilon})e^{i\varphi}\triangle\psi \,+\, i(1-\chi_{2\epsilon})e^{i\varphi}\mathbb{S}_2[\psi] \,+\, i(1-\chi_{2\epsilon})e^{i\varphi}\mathbb{S}_3[\psi] \,+\, i(1-\chi_{2\epsilon})e^{i\varphi}\mathbb{S}_4[\psi]\\
&+\, 2i\,\nabla\left((1-\chi_{2\epsilon})e^{i\varphi}\right)\cdot\nabla\psi \,+\, 2i\gamma^{-2}\left[\frac{\lambda^2}{x_1^2}-\frac{\lambda^2}{|r_{1\epsilon}|^2}\right]\frac{\partial(1-\chi_{2\epsilon})e^{i\varphi}}{\partial x_2}\frac{\partial\psi}{\partial x_2}\\
&+\,(1-\chi_{2\epsilon})e^{i\varphi}\Big[\,1+V-|\mathcal{U}_2|^2\,\Big]\psi \,-\, 2(1-\chi_{2\epsilon})\mathrm{Re}\big(\bar{\mathcal{U}}_2e^{i\varphi}\psi\big)\mathcal{U}_2\\
&+\, i\triangle\Big[\,(1-\chi_{2\epsilon})e^{i\varphi}\,\Big]\psi \,+\, i\mathbb{S}_2[(1-\chi_{2\epsilon})e^{i\varphi}]\psi \,+\, i\mathbb{S}_3[(1-\chi_{2\epsilon})e^{i\varphi}]\psi \,+\, i\mathbb{S}_4[(1-\chi_{2\epsilon})e^{i\varphi}]\psi\\
&=\,-E\,+\,N\quad\text{in } \mathfrak{S}.
\end{aligned}
$$

In the last formula, the error term E is defined as $\mathbb{E}$ in (4.57) and N is the nonlinear operator defined by

$$
\begin{aligned}
N \,=\, &-\triangle\Gamma \,-\, i\,\mathbb{S}_2[\Gamma] \,-\, i\,\mathbb{S}_3[\Gamma] \,-\, i\,\mathbb{S}_4[\Gamma] \,-\, \big(1+V-|\mathcal{U}_2|^2\big)\Gamma\\
&+\,\Big[2\mathrm{Re}\big(i\bar{\mathcal{U}}_2v_2\psi\big)+2\mathrm{Re}\big(i\bar{\mathcal{U}}_2(1-\chi_{2\epsilon})e^{i\varphi}\psi\big)\Big]\times\Big(iv_2\psi+i(1-\chi_{2\epsilon})e^{i\varphi}\psi+\Gamma\Big)\\
&+\,\Big[\,2\mathrm{Re}(\bar{\mathcal{U}}_2\Gamma)+\big|iv_2\psi+i(1-\chi_{2\epsilon})e^{i\varphi}\psi+\Gamma\big|^2\,\Big]\\
&\qquad\times\Big(\mathcal{U}_2+iv_2\psi+i(1-\chi_{2\epsilon})e^{i\varphi}\psi+\Gamma\Big).
\end{aligned}
$$

Note that E and N can be written as

$$
E\,=\,\chi_{8\epsilon}E\,+\,(1-\chi_{8\epsilon})E,\qquad N\,=\,\chi_{8\epsilon}N\,+\,(1-\chi_{8\epsilon})N.
$$

Whence, we decompose (5.7) into

$$
\begin{aligned}
&iv_2\triangle\psi \,+\, iv_2\mathbb{S}_2[\psi] \,+\, iv_2\mathbb{S}_3[\psi] \,+\, iv_2\mathbb{S}_4[\psi] \,+\, 2i\,\nabla\, v_2\cdot\nabla\psi\\
&+\,2i\gamma^{-2}\left[\frac{\lambda^2}{x_1^2}-\frac{\lambda^2}{|r_{1\epsilon}|^2}\right]\frac{\partial v_2}{\partial x_2}\frac{\partial\psi}{\partial x_2} \,+\, i\Big[\,1+V-|\mathcal{U}_2|^2\,\Big]v_2\psi \,-\, 2\mathrm{Re}\big(\bar{\mathcal{U}}_2iv_2\psi\big)\mathcal{U}_2\\
&+\, i\triangle v_2\psi \,+\, i\mathbb{S}_2[v_2]\psi \,+\, i\mathbb{S}_3[v_2]\psi \,+\, i\mathbb{S}_4[v_2]\psi\\
&=\,-\chi_{8\epsilon}E\,+\,\chi_{8\epsilon}N,
\end{aligned}
\tag{5.8}
$$

and

$$
\begin{aligned}
&i(1-\chi_{2\epsilon})e^{i\varphi}\triangle\psi \,+\, i(1-\chi_{2\epsilon})e^{i\varphi}\mathbb{S}_2[\psi] \,+\, i(1-\chi_{2\epsilon})e^{i\varphi}\mathbb{S}_3[\psi] \,+\, i(1-\chi_{2\epsilon})e^{i\varphi}\mathbb{S}_4[\psi]\\
&+\, 2i\,\nabla\left((1-\chi_{2\epsilon})e^{i\varphi}\right)\cdot\nabla\psi \,+\, 2i\gamma^{-2}\left[\frac{\lambda^2}{x_1^2}-\frac{\lambda^2}{|r_{1\epsilon}|^2}\right]\frac{\partial(1-\chi_{2\epsilon})e^{i\varphi}}{\partial x_2}\frac{\partial\psi}{\partial x_2}\\
&+\, i(1-\chi_{2\epsilon})e^{i\varphi}\Big[\,1+V-|\mathcal{U}_2|^2\,\Big]\psi \,-\, 2(1-\chi_{2\epsilon})\mathrm{Re}\big(\bar{\mathcal{U}}_2ie^{i\varphi}\psi\big)\mathcal{U}_2\\
&+\, i\triangle\Big[\,(1-\chi_{2\epsilon})e^{i\varphi}\,\Big]\psi \,+\, i\mathbb{S}_2[(1-\chi_{2\epsilon})e^{i\varphi}]\psi \,+\, i\mathbb{S}_3[(1-\chi_{2\epsilon})e^{i\varphi}]\psi \,+\, i\mathbb{S}_4[(1-\chi_{2\epsilon})e^{i\varphi}]\psi\\
&=\,-(1-\chi_{8\epsilon})E\,+\,(1-\chi_{8\epsilon})N.
\end{aligned}
\tag{5.9}
$$

Note that we, here and in the sequel, follow the gluing method from [30]. The application of further analysis to obtain a resolution theory relies on suitable local forms of (5.8) and (5.9) and the properties of the corresponding linear operators, which will be done in the following.

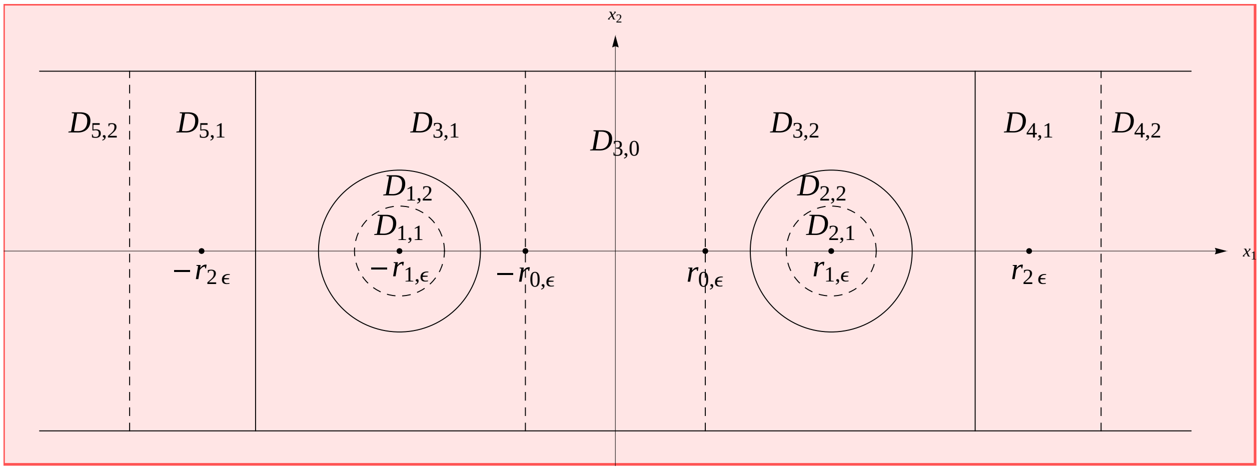

Figure 2: Further decompositions of the domain $\mathfrak{S}$

Before going further, we pause here to give some notation. By recalling the notation ℓ_1 and ℓ_2 in (3.17), and also D_1, D_2, D_3, D_4 and D_5 in (3.18) (see Figure 1), we set, see Figure 2

$$
\begin{aligned}
D_{1,1} &\equiv \Big\{(x_1,x_2)\in\mathfrak{S}:\ell_1<1\Big\}, \qquad D_{1,2} \equiv \Big\{(x_1,x_2)\in\mathfrak{S}:\ell_1<\epsilon^{-\lambda_1}\Big\}\setminus D_{1,1},\\
D_{2,1} &\equiv \Big\{(x_1,x_2)\in\mathfrak{S}:\ell_2<1\Big\}, \qquad D_{2,2} \equiv \Big\{(x_1,x_2)\in\mathfrak{S}:\ell_2<\epsilon^{-\lambda_1}\Big\}\setminus D_{2,1},\\
D_{3,0} &\equiv \Big\{\,(x_1,x_2)\in\mathfrak{S}:|x_1|<r_{0\epsilon}\,\Big\},\\
D_{3,1} &\equiv \Big\{\,(x_1,x_2)\in\mathfrak{S}:-r_{2\epsilon}+\epsilon^{-\lambda_2}<x_1<-r_{0\epsilon}\,\Big\}\setminus D_1,\\
D_{3,2} &\equiv \Big\{\,(x_1,x_2)\in\mathfrak{S}:r_{0\epsilon}<x_1<r_{2\epsilon}-\epsilon^{-\lambda_2}\,\Big\}\setminus D_2,\\
D_{4,1} &\equiv \Big\{\,(x_1,x_2)\in\mathfrak{S}:r_{2\epsilon}-\epsilon^{-\lambda_2}<x_1<r_{2\epsilon}+\frac{\tau_2}{\epsilon}\,\Big\},\\
D_{4,2} &\equiv \Big\{(x_1,x_2)\in\mathfrak{S}:x_1>r_{2\epsilon}+\frac{\tau_2}{\epsilon}\,\Big\},\\
D_{5,1} &\equiv \Big\{\,(x_1,x_2)\in\mathfrak{S}:-r_{2\epsilon}-\frac{\tau_2}{\epsilon}<x_1<-r_{2\epsilon}+\epsilon^{-\lambda_2}\,\Big\},\\
D_{5,2} &\equiv \Big\{(x_1,x_2)\in\mathfrak{S}:x_1<-r_{2\epsilon}-\frac{\tau_2}{\epsilon}\,\Big\}.
\end{aligned}
\tag{5.10}
$$

Here τ_1, τ_2, $r_{1\epsilon}$ and $r_{2\epsilon}$ are given in the assumption (**A3**).

Part 1: Recall that $\mathcal{U}_2 = v_2 = U_2 e^{i\varphi_d}$ in the sets $\{\ell_1 < 30\epsilon^{-\lambda_1}\}$ and $\{\ell_2 < 30\epsilon^{-\lambda_1}\}$. v_2 supports on $\{\ell_1 < 60\epsilon^{-\lambda_1}\}$ and $\{\ell_2 < 60\epsilon^{-\lambda_1}\}$, and $\chi_{8\epsilon}$ supports on the sets $\{\ell_1 < 16\epsilon^{-\lambda_1}\}$ and $\{\ell_2 < 16\epsilon^{-\lambda_1}\}$. We first consider (5.8) and transform it into the following form by extension method

$$
\begin{aligned}
&\triangle\psi + \Big[1-|U_2|^2\Big]\psi + 2\mathrm{Re}\big(\bar{U}_2 iU_2\psi\big) + \chi_{100\epsilon}\mathbb{S}_2[\psi] + \chi_{100\epsilon}\mathbb{S}_3[\psi] + \chi_{100\epsilon}\mathbb{S}_4[\psi]\\
&+ 2\chi_{100\epsilon}\frac{\nabla v_2}{v_2}\cdot\nabla\psi + 2\chi_{100\epsilon}\gamma^{-2}\left[\frac{\lambda^2}{x_1^2}-\frac{\lambda^2}{|r_{1\epsilon}|^2}\right]\frac{1}{v_2}\frac{\partial v_2}{\partial x_2}\frac{\partial\psi}{\partial x_2} + \chi_{100\epsilon}V\psi\\
&+ \chi_{100\epsilon}\frac{E}{v_2}\psi + \chi_{100\epsilon}\frac{(1+V-|\mathcal{U}_2|^2)}{v_2}\psi = -\chi_{8\epsilon}E + \chi_{8\epsilon}N \quad \text{in } \mathbb{R}^2,
\end{aligned}
\tag{5.11}
$$

with constraints for ψ

$$
\begin{aligned}
\psi_1(x_1,x_2) &= \psi_1(-x_1,x_2), &\quad \psi_1(x_1,x_2) &= -\psi_1(x_1,-x_2),\\
\psi_2(x_1,x_2) &= \psi_2(-x_1,x_2), &\quad \psi_2(x_1,x_2) &= \psi_2(x_1,-x_2),\\
\frac{\partial\psi_1}{\partial x_1}(0,x_2) &= 0, &\quad \frac{\partial\psi_2}{\partial x_1}(0,x_2) &= 0.
\end{aligned}
\tag{5.12}
$$

Consider the linearization of the problem on the vortex-core region $D_{1,1}$ or $D_{2,1}$. Here we only argue in the region $D_{2,1} = \big\{(x_1,x_2) : \ell_2 < 1\big\}$. It is more convenient to do this in the translated variable $(y_1,y_2) = (x_1 - r_{1\epsilon}, x_2)$ and then denote $\ell = \ell_2$ for brevity of notation. Now the term ψ is small, however possibly unbounded near the vortex. Whence, in the sequel, by setting

$$
\tilde{\phi} = iv_2\psi \quad \text{with } \psi = \psi_1 + i\psi_2, \tag{5.13}
$$

we shall require that $\tilde{\phi}$ is bounded (and smooth) near the vortices. We shall write the equation in term of a type of the function $\tilde{\phi}$ for $\ell < \delta/\epsilon$. In the region $D_{2,1}$, let us write $\mathcal{U}_2$, i.e. v_2, as the form

$$
v_2 = \beta U_0 \qquad \text{with } \beta = w(\ell_1)e^{-i\varphi_0^- + i\varphi_d}, \tag{5.14}
$$

where U_0, φ_0^- and φ_d are defined in (2.1), (3.17) and (4.33). We define the function

$$
\phi(s) = iU_0\psi \quad \text{for } |y| < \delta/\epsilon, \tag{5.15}
$$

namely

$$
\tilde{\phi} = \beta\phi. \tag{5.16}
$$

Hence, in the translated variable, the ansatz becomes in this region

$$
u = \beta(s)U_0 + \beta(s)\phi + (1-\chi)\beta(s)U_0\Big(e^{\phi/U_0} - 1 - \frac{\phi}{U_0}\Big). \tag{5.17}
$$

We also call

$$
\Gamma_{2,1} = (1-\chi)U_0\Big(e^{\phi/U_0} - 1 - \frac{\phi}{U_0}\Big).
$$

The support of this function is contained in set $|y| > 1$. In this vortex-core region, the problem, written in (s_1,s_2) coordinates, can be stated as

$$
L_{2,1}(\psi) = E_{2,1} + N_{2,1}. \tag{5.18}
$$

Let us consider the linear operator defined in the following way: for ϕ and ψ linked through formula (5.15) we set

$$
\begin{aligned}
L_{2,1}(\psi) \,=& L_0(\phi) \,+\, \gamma^{-2}\left[\frac{\lambda^2}{x_1^2} - \frac{\lambda^2}{|r_{1\epsilon}|^2}\right]\frac{\partial^2\phi}{\partial x_2^2} \,+\, \frac{1}{s_1 + r_{1\epsilon}}\frac{\partial}{\partial s_1}\phi \,+\, 2\big(1 - |\beta|^2\big)\mathrm{Re}\big(\bar{U}_0\phi\big)U_0 \\
&+ \left[\epsilon\frac{\partial V}{\partial \tilde{r}}\Big|_{\tilde{r}=r_{1\epsilon}+\vartheta s_1} + 1 - |\beta|^2\right]\phi \,+\, 2\frac{\nabla\beta}{\beta}\cdot\nabla\phi \,+\, \chi\frac{E_{2,1}}{U_0}\phi \,+\, i\epsilon\frac{\kappa}{\gamma}|\log\epsilon|\frac{\partial\phi}{\partial x_2},
\end{aligned} \tag{5.19}
$$

where ϑ is a small constant. Here we also have defined L_0 as

$$
L_0(\phi) = \Big(\frac{\partial^2}{\partial s_1^2} + \frac{\partial^2}{\partial s_2^2}\Big)\phi + (1 - |w|^2)\phi - 2\mathrm{Re}\big(\bar{U}_0\phi\big)U_0.
$$

Here, by writing the error $\mathbb{E}$ in the translated variable y, the error $E_{2,1}$ is given by

$$
E_{2,1} = \mathbb{E}/\beta. \tag{5.20}
$$

Observe that, in the region $D_{2,1}$, the error $E_{2,1}$ takes the expression

$$
\begin{aligned}
E_{2,1} \,=\,& w(\ell_2)\, e^{i\varphi_0^+}\left[\frac{x_1 - r_{1\epsilon}}{x_1\ell_2}\frac{w'(\ell_2)}{w(\ell_2)} \,+\, \epsilon\,\frac{\partial V}{\partial\tilde{r}}\Big|_{\tilde{r}=\epsilon r_{1\epsilon}}(x_1 - r_{1\epsilon})\right] \\
&+\, w(\ell_2)\, e^{i\varphi_0^+}\,\frac{2(x_1 + r_{1\epsilon})(x_1 - r_{1\epsilon})}{(\ell_1\ell_2)^2}\log r_{1\epsilon} \\
&-\, iw(\ell_2)\, e^{i\varphi_0^+}\,\frac{x_2}{\ell_2}\,\frac{w'(\ell_2)}{w(\ell_2)}\frac{1}{r_{1\epsilon}}\log r_{1\epsilon} + i\frac{\kappa}{\gamma}\,\epsilon|\log\epsilon|\,U_2\frac{x_2}{\ell_2}\,\frac{w'(\ell_2)}{w(\ell_2)} \,+\, O(\epsilon\log\ell_2),
\end{aligned} \tag{5.21}
$$

while the nonlinear term is given by

$$
\begin{aligned}
N_{2,1}(\phi) = &-\,\frac{\triangle(\beta\,\Gamma_{2,1})}{\beta} \,+\, \Big(1 + V - |U_0|^2\Big)\Gamma_{2,1} \,-\, 2|\beta|^2\,\mathrm{Re}(\bar{U}_0\phi)\big(\phi + \Gamma_{2,1}\big) \\
&-\, \Big(2|\beta|^2\mathrm{Re}\big(\bar{U}_0\Gamma_{2,1}\big) + |\beta|^2\big|\phi + \Gamma_{2,1}\big|^2\Big)\big(U_0 + \phi + \Gamma_{2,1}\big) \,+\, (\chi - 1)\frac{E_{2,1}}{U_0}\phi.
\end{aligned} \tag{5.22}
$$

Taking into account to the explicit form of the function β we get

$$
\nabla\beta = O(\epsilon), \quad \triangle\beta = O(\epsilon^2), \quad |\beta| \sim 1 + O(\epsilon^2), \tag{5.23}
$$

provided that $|y| < \delta/\epsilon$. With this in mind, we see that the linear operator is a small perturbation of L_0.

In the region $D_{2,2}$ far from the vortex core, directly from the form of the ansatz $u = \big(1 - \chi\big)\mathcal{U}_2 e^{i\psi}$, we see that, for $\ell_2 > 2$, the equation takes the simple form

$$
\begin{aligned}
L_{2,2}(\psi) \,\equiv\,& \Big(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{1}{x_1}\frac{\partial}{\partial x_1}\Big)\psi \,+\, \gamma^{-2}\left[\frac{\lambda^2}{x_1^2} - \frac{\lambda^2}{|r_{1\epsilon}|^2}\right]\frac{\partial^2\psi}{\partial x_2^2} \,+\, 2\frac{\nabla\mathcal{U}_2}{\mathcal{U}_2}\cdot\nabla\psi \\
&+\, \gamma^{-2}\left[\frac{\lambda^2}{x_1^2} - \frac{\lambda^2}{|r_{1\epsilon}|^2}\right]\frac{\partial^2\psi}{\partial x_2^2} \quad -\, 2i|\mathcal{U}_2|^2\psi_2 \,+\, i\epsilon\frac{\kappa}{\gamma}|\log\epsilon|\frac{\partial\psi}{\partial x_2} \\
=\,& E_{2,2} \,-\, i(\nabla\psi)^2 \,+\, i|\mathcal{U}_2|^2\big(1 - e^{-2\psi_2} + 2\psi_2\big),
\end{aligned}
$$

where $E_{2,2} = i\mathbb{E}/\mathcal{U}_2$. We intend next to describe in more accurate form the equation above. As before, let us also write

$$\mathcal{U}_2 = \beta U_0 \quad \text{ with } \beta = \, w(\ell_1) e^{-i\varphi_0^- + i\varphi_1}. \tag{5.24}$$

where U_0, φ_0^+ and φ_d are defined in (2.1), (3.17) and (4.33). For $\ell_2 < \frac{\delta}{\epsilon}$, there are two real functions A and B such that

$$\beta = e^{iA+B}, \tag{5.25}$$

furthermore, a direct computation shows that, in this region, there holds

$$\triangledown A = O(\epsilon), \qquad \triangle A = O(\epsilon^2), \qquad \triangledown B = O(\epsilon^3), \qquad \triangle B = O(\epsilon^4). \tag{5.26}$$

The equations become

$$\tilde{L}_{2,2}(\psi_1) = \tilde{E}_{2,2} + \tilde{N}_{2,2}, \qquad \bar{L}_{2,2}(\psi_2) = \bar{E}_{2,2} + \bar{N}_{2,2}. \tag{5.27}$$

In the above, we have denoted the linear operators by

$$\begin{aligned}
\tilde{L}_{2,2}(\psi_1) \equiv & \Big(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{1}{x_1}\frac{\partial}{\partial x_1}\Big)\psi_1 \, + \, \gamma^{-2}\left[\frac{\lambda^2}{x_1^2} - \frac{\lambda^2}{|r_{1\epsilon}|^2}\right]\frac{\partial^2 \psi_1}{\partial x_2^2} \\
& + \Big(\triangledown B + \frac{w'(\ell_2)}{w(\ell_2)}\frac{y}{\ell_2}\Big)\cdot \triangledown \psi_1 \, + \, \epsilon\frac{\kappa}{\gamma}|\log\epsilon|\frac{\partial \psi_1}{\partial x_2}, \\
\bar{L}_{2,2}(\psi_2) \equiv & \Big(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{1}{x_1}\frac{\partial}{\partial x_1}\Big)\psi_2 \, + \, \gamma^{-2}\left[\frac{\lambda^2}{x_1^2} - \frac{\lambda^2}{|r_{1\epsilon}|^2}\right]\frac{\partial^2 \psi_2}{\partial x_2^2} \, - \, 2|\mathcal{U}_2|^2\psi_2 \\
& + 2\Big(\triangledown B + \frac{w'(\ell_2)}{w(\ell_2)}\frac{y}{\ell_2}\Big)\cdot \triangledown \psi_2 \, + \, \epsilon\frac{\kappa}{\gamma}|\log\epsilon|\frac{\partial \psi_2}{\partial x_2},
\end{aligned}$$

where have used $y = (x_1 - r_{1\epsilon}, x_2)$. The nonlinear operators are

$$\begin{aligned}
\tilde{N}_{2,2} \; &= \; -2(\triangledown A + \triangledown \varphi_0^+)\cdot \triangledown \psi_2 \, + \, 2\triangledown \psi_1 \triangledown \psi_2, \\
\bar{N}_{2,2} \; &= \; -2(\triangledown A + \triangledown \varphi_0^+)\cdot \triangledown \psi_1 \, + \, |\mathcal{U}_2|^2\big(1 - e^{-2\psi_2} + 2\psi_2\big) \, + \, |\triangledown \psi_1|^2 \, - \, |\triangledown \psi_2|^2.
\end{aligned}$$

Part 2: We consider (5.9) and make an extension to get

$$\begin{aligned}
& \triangle\psi \, + \, (1-\chi_{2\epsilon})\mathbb{S}_2[\psi] \, + \, (1-\chi_{2\epsilon})\mathbb{S}_3[\psi] \, + \, (1-\chi_{2\epsilon})\mathbb{S}_4[\psi] \, + \, 2e^{-i\varphi}\triangledown\big((1-\chi_{2\epsilon})e^{i\varphi}\big)\cdot\triangledown\psi \\
& + \, 2\gamma^{-2}e^{-i\varphi}\left[\frac{\lambda^2}{x_1^2} - \frac{\lambda^2}{|r_{1\epsilon}|^2}\right]\frac{\partial(1-\chi_{2\epsilon})e^{i\varphi}}{\partial x_2}\frac{\partial\psi}{\partial x_2} \, + \, (1-\chi_{2\epsilon})\Big[1 + V - |\mathcal{U}_2|^2\Big]\psi \\
& - \, 2(1-\chi_{2\epsilon})e^{-i\varphi}\mathrm{Re}\big(\bar{\mathcal{U}}_2 i e^{i\varphi}\psi\big)\mathcal{U}_2 \, + \, e^{-i\varphi}\triangle\big[(1-\chi_{2\epsilon})e^{i\varphi}\big]\psi \, + \, e^{-i\varphi}\mathbb{S}_2\big[(1-\chi_{2\epsilon})e^{i\varphi}\big]\psi \\
& + \, e^{-i\varphi}\mathbb{S}_3\big[(1-\chi_{2\epsilon})e^{i\varphi}\big]\psi \, + \, e^{-i\varphi}\mathbb{S}_4\big[(1-\chi_{2\epsilon})e^{i\varphi}\big]\psi \\
& = \, i(1-\chi_{8\epsilon})e^{-i\varphi}E \, - \, i(1-\chi_{8\epsilon})e^{-i\varphi}N \quad \text{ in } \mathfrak{S},
\end{aligned} \tag{5.28}$$

with constraints for ψ in (5.5).

In the region D_3 far from the vortex core region, directly from the form of the ansatz $u = \mathcal{U}_2 + ie^{i\varphi}\psi$ with the approximation as

$$\mathcal{U}_2(x_1, x_2) \,=\, \sqrt{1+V}\,\eta_3\, e^{i(\varphi_0+\varphi_d)} \,+\, w(\ell_2)w(\ell_1)\,\eta_2\, e^{i(\varphi_0+\varphi_d)},$$

we see that the equation takes the simple form

$$\begin{aligned} L_3 \,\equiv\, & \Big(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{1}{x_1}\frac{\partial}{\partial x_1}\Big)\psi + 2\,\triangledown\,\varphi\cdot\triangledown\psi - 2i|\mathcal{U}_2|^2\psi_2 \,+\, \gamma^{-2}\left[\frac{\lambda^2}{x_1^2}-\frac{\lambda^2}{|r_{1\epsilon}|^2}\right]\frac{\partial^2\psi}{\partial x_2^2} \,+\, \frac{1}{x_1}\frac{\partial\psi}{\partial x_1} \\ = \,& E_3 \,-\, i(\triangledown\psi)^2 \,+\, i|\mathcal{U}_2|^2\big(1-e^{-2\psi_2}+2\psi_2\big) \,+\, \epsilon\frac{\kappa}{\gamma}|\log\epsilon|\frac{\partial\psi}{\partial x_2}, \end{aligned}$$

where $E_3 = i\,\mathbb{E}/\mathcal{U}_2$. We intend next to describe in more accurate form the equation above. Let us also write

$$\mathcal{U}_2 = e^{i\varphi}\beta_1 \quad \text{with } \beta_1 = \sqrt{1+V}\,\eta_3 \,+\, w(\ell_2)w(\ell_1)\,\eta_2.$$

For $|x| < r_{2\epsilon} - \epsilon^{-\lambda_1}$, there holds,

$$\mathcal{U}_2 = \beta_1 e^{i\varphi} = \sqrt{1+V}\, e^{i\varphi}, \tag{5.29}$$

and hence, by using the assumption (**A**3), we have

$$\begin{aligned} |\mathcal{U}_2|^2 \,&= 1+V \,>\, 1 \quad \text{for } r_{0\epsilon} < |x_1| < r_{2\epsilon} - \epsilon^{-\lambda_1}, \\ |\mathcal{U}_2|^2 \,&= 1+V \,=\, \hat{c}_0(\epsilon x_1)^4 + O(|\epsilon x_1|^5) \quad \text{for } |x_1| < r_{0\epsilon}. \end{aligned} \tag{5.30}$$

Direct computation also gives that

$$\begin{aligned} 2\frac{\triangledown\mathcal{U}_2}{\mathcal{U}_2}\cdot\triangledown\psi \,&=\, \frac{2}{\beta_1}\,\triangledown\,\beta_1\cdot\triangledown\psi_1 - 2\,\triangledown\,\varphi\cdot\triangledown\psi_2 + i\frac{2}{\beta_1}\,\triangledown\,\beta_1\cdot\triangledown\psi_2 + 2i\,\triangledown\,\varphi\cdot\triangledown\psi_1 \\ &= (A_1, 0)\cdot\triangledown\psi_1 - (A_2, B_2)\cdot\triangledown\psi_2 + i(A_1, 0)\cdot\triangledown\psi_2 + i(A_2, B_2)\cdot\triangledown\psi_1, \end{aligned}$$

where $A_1 = O(\epsilon|\log\epsilon|)$, $A_2 = O(\epsilon)$, $B_2 = O(\epsilon)$. The equations become

$$\tilde{L}_3(\psi_1) = \tilde{E}_3 \,+\, \tilde{N}_3, \qquad \bar{L}_3(\psi_2) = \bar{E}_3 \,+\, \bar{N}_3. \tag{5.31}$$

In the above, we have denoted the linear operators by

$$\begin{aligned} \tilde{L}_3(\psi_1) \,\equiv\, & \Big(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{1}{x_1}\frac{\partial}{\partial x_1}\Big)\psi_1 \,+\, \frac{2}{\beta_1}\,\triangledown\,\beta_1\cdot\triangledown\psi_1 \\ & +\, \gamma^{-2}\left[\frac{\lambda^2}{x_1^2}-\frac{\lambda^2}{|r_{1\epsilon}|^2}\right]\frac{\partial^2\psi_1}{\partial x_2^2}, \\ \bar{L}_3(\psi_2) \,\equiv\, & \Big(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{1}{x_1}\frac{\partial}{\partial x_1}\Big)\psi_2 \,-\, 2|\mathcal{U}_2|^2\psi_2 \,+\, \frac{2}{\beta_1}\,\triangledown\,\beta_1\cdot\triangledown\psi_2 \\ & +\, \gamma^{-2}\left[\frac{\lambda^2}{x_1^2}-\frac{\lambda^2}{|r_{1\epsilon}|^2}\right]\frac{\partial^2\psi_2}{\partial x_2^2}. \end{aligned}$$

The nonlinear operators are

$$\tilde{N}_3 \,=\, -\,2\bigtriangledown\varphi\cdot\bigtriangledown\psi_2 \,+\, 2\bigtriangledown\psi_1\cdot\bigtriangledown\psi_2 \,+\, \epsilon\frac{\kappa}{\gamma}|\log\epsilon|\frac{\partial\psi_1}{\partial x_2} \,+\, \frac{1}{x_1}\frac{\partial\psi_1}{\partial x_1},$$
$$\bar{N}_3 \,=\, 2\bigtriangledown\varphi\cdot\bigtriangledown\psi_1 \,+\, |\mathcal{U}_2|^2\big(1-e^{-2\psi_2}+2\psi_2\big) \,+\, |\bigtriangledown\psi_1|^2 \,-\, |\bigtriangledown\psi_2|^2 \,+\, \epsilon\frac{\kappa}{\gamma}|\log\epsilon|\frac{\partial\psi_1}{\partial x_2}.$$

For $r_{2\epsilon}-2\tau_1/\epsilon<|x|<r_{2\epsilon}-\tau_1/\epsilon$, similar estimates hold.

In the region

$$D_{4,1}=\Big\{\,(x_1,x_2):\,r_{2\epsilon}-\tau_1/\epsilon<|x|<r_{2\epsilon}+\tau_2/\epsilon\,\Big\},$$

the approximation takes the form

$$\mathcal{U}_2 \,=\, w(\ell_2)w(\ell_1)\,\eta_2\,e^{i\varphi} \,+\, \hat{q}\,\eta_4\,e^{i\varphi}.$$

We write the ansatz as

$$u=\mathcal{U}_2 \,+\, ie^{i\varphi}\psi \,+\, \Gamma_{4,1}, \tag{5.32}$$

where $\Gamma_{4,1}$ is defined as

$$\Gamma_{4,1}=i\eta_2\Big(w(\ell_1)w(\ell_2)-1\Big)e^{i\varphi}\psi \,+\, \eta_2 w(\ell_1)w(\ell_2)e^{i\varphi}\Big(e^{i\psi}-1-i\psi\Big). \tag{5.33}$$

The equation becomes

$$\begin{aligned}
L_{4,1}[\psi] \,&\equiv\, \triangle\psi \,+\, \gamma^{-2}\left[\frac{\lambda^2}{x_1^2}-\frac{\lambda^2}{|r_{1\epsilon}|^2}\right]\frac{\partial^2\psi}{\partial x_2^2} \,+\, 2i\bigtriangledown\varphi\cdot\bigtriangledown\psi \,-\, |\bigtriangledown\varphi|^2\psi \,+\, i\triangle[\varphi]\psi\\
&\quad+\,\Big(1+V-|\mathcal{U}_2|^2\Big)\psi \,+\, 2ie^{-i\varphi}\mathrm{Re}\big(\bar{u}_2 ie^{i\varphi}\psi\big)\mathcal{U}_2 \,+\, i\epsilon\frac{\kappa}{\gamma}|\log\epsilon|\frac{\partial\psi}{\partial x_2}\\
&= E_{4,1} \,+\, N_{4,1},
\end{aligned}$$

where $E_{4,1}=ie^{-i\varphi}\mathbb{E}$. The nonlinear operator is defined by

$$\begin{aligned}
N_{4,1}(\psi) \,&=\, ie^{-i\varphi}\Big[\,\triangle\,\Gamma_{4,1}+\frac{1}{x_1}\frac{\partial}{\partial x_1}\Gamma_{4,1}+\big(1+V-|\mathcal{U}_2|^2\big)\Gamma_{4,1}\,\Big]\\
&\quad-\,ie^{-i\varphi}\Big[\,2\mathrm{Re}(\bar{u}_2\Gamma_{4,1})-|ie^{i\varphi}\psi+\Gamma_{4,1}|^2\,\Big](\mathcal{U}_2+ie^{i\varphi}\psi+\Gamma_{4,1})\\
&\quad-\,2ie^{-i\varphi}\mathrm{Re}(\bar{u}_2 ie^{i\varphi}\psi)(ie^{i\varphi}\psi+\Gamma_{4,1}).
\end{aligned}$$

More precisely, in the region $D_{4,1}$, the linear operator $L_{4,1}$ is defined as

$$\begin{aligned}
L_{4,1}[\psi]=\triangle\psi \,-\, &\big(\delta_\epsilon(\ell-r_{2\epsilon})+\hat{q}^2\big)\psi \,+\, 2ie^{-i\varphi}\mathrm{Re}\big(\bar{u}_2 ie^{i\varphi}\psi\big)\mathcal{U}_2 \,+\, \gamma^{-2}\left[\frac{\lambda^2}{x_1^2}-\frac{\lambda^2}{|r_{1\epsilon}|^2}\right]\frac{\partial^2\psi}{\partial x_2^2}\\
&+\,\Big[1+V+\delta_\epsilon(\ell-r_{2\epsilon})\Big]\psi \,+\, 2i\bigtriangledown\varphi\cdot\bigtriangledown\psi \,+\, \triangle[\varphi]\psi \,-\, |\bigtriangledown\varphi|^2\psi \,+\, i\epsilon\frac{\kappa}{\gamma}|\log\epsilon|\frac{\partial\psi}{\partial x_2}.
\end{aligned}$$

where we have used the definition of $\hat{q}$ in (4.5). We shall analyze other terms in the linear operator $L_{4,1}$. For $r_{2\epsilon}-\tau_1/\epsilon<|x|<r_{2\epsilon}+\tau_2/\epsilon$, there holds $\mathcal{U}_2=\hat{q}\,e^{i\varphi}$. It is obvious that

$$2ie^{-i\varphi}\mathrm{Re}\big(\bar{u}_2 ie^{i\varphi}\psi\big)\mathcal{U}_2\;=\;-2i\,\hat{q}^2\,\psi_2. \tag{5.34}$$

For $r_{2\epsilon}+\tau_2/\epsilon<|x|<r_{2\epsilon}+2\tau_2/\epsilon$, there holds

$$\mathcal{U}_2=w(\ell_2)w(\ell_1)\,\eta_2\,e^{i\varphi}\;+\;\hat{q}\,\eta_4\,e^{i\varphi}.$$

Whence we decompose the equation in the form

$$\begin{aligned}\tilde{L}_{4,1}[\psi_1]\;\equiv\;&\Big(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}\Big)\psi_1\;-\;\big(\delta_\epsilon(\ell-r_{2\epsilon})+\hat{q}^2\big)\psi_1\;+\;\Big[1+V+\delta_\epsilon(\ell-r_{2\epsilon})\Big]\psi_1\\&+\;\frac{1}{x_1}\frac{\partial}{\partial x_1}\psi_1\;-\;2\bigtriangledown\varphi\cdot\bigtriangledown\psi_2\;+\;\triangle[\varphi]\psi_1\;-\;|\bigtriangledown\varphi|^2\psi_1\\&+\;\epsilon\frac{\kappa}{\gamma}|\log\epsilon|\frac{\partial\psi_1}{\partial x_2}\;+\;\gamma^{-2}\left[\frac{\lambda^2}{x_1^2}-\frac{\lambda^2}{|r_{1\epsilon}|^2}\right]\frac{\partial^2\psi_1}{\partial x_2^2}\\=\;&\tilde{E}_{4,1}+\tilde{N}_{4,1},\end{aligned}$$

$$\begin{aligned}\bar{L}_{4,1}[\psi_2]\;\equiv\;&\Big(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}\Big)\psi_2\;-\;\big(\delta_\epsilon(\ell-r_{2\epsilon})+3\hat{q}^2\big)\psi_2\;+\;\Big[1+V+\delta_\epsilon(\ell-r_{2\epsilon})\Big]\psi_2\\&+\;\frac{1}{x_1}\frac{\partial}{\partial x_1}\psi_2\;+\;2\bigtriangledown\varphi\cdot\bigtriangledown\psi_1\;+\;\triangle[\varphi]\psi_2\;-\;|\bigtriangledown\varphi|^2\psi_2\\&+\;\epsilon\frac{\kappa}{\gamma}|\log\epsilon|\frac{\partial\psi_2}{\partial x_2}\;+\;\gamma^{-2}\left[\frac{\lambda^2}{x_1^2}-\frac{\lambda^2}{|r_{1\epsilon}|^2}\right]\frac{\partial^2\psi_2}{\partial x_2^2}\\=\;&\bar{E}_{4,1}+\bar{N}_{4,1}.\end{aligned}$$

If $r_{2\epsilon}-\tau_1/\epsilon<|x|<r_{2\epsilon}+\tau_2/\epsilon$, by using (4.3), we then have

$$\Xi_{4,1}\;\equiv\;1+V+\delta_\epsilon(\ell-r_{2\epsilon})\;=\;\frac{\epsilon^2}{2}\frac{\partial^2 V}{\partial\tilde{\ell}^2}(\ell-r_{2\epsilon})^2\;+\;O\big(\epsilon^3(\ell-r_{2\epsilon})^3\big).$$

The other terms with φ_0 are also lower order terms. Whence the linear operators $\tilde{L}_{4,1}$ and $\bar{L}_{4,1}$ are small perturbations of the following linear operators

$$\begin{aligned}L_{41*}[\psi_1]&\equiv\Big(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}\Big)\psi_1\;-\;\big(\delta_\epsilon(\ell-r_{2\epsilon})+\hat{q}^2\big)\psi_1,\\L_{41**}[\psi_2]&\equiv\Big(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}\Big)\psi_2\;-\;\big(\delta_\epsilon(\ell-r_{2\epsilon})+3\hat{q}^2\big)\psi_2.\end{aligned} \tag{5.35}$$

In the region $D_{4,2}$ the approximation takes the form

$$\mathcal{U}_2\;=\;\hat{q}(x_1,x_2)e^{i\varphi},$$

and the ansatz is

$$u = \mathcal{U}_2 + ie^{i\varphi}\psi.$$

The equation becomes

$$\begin{aligned}
L_{4,2}[\psi] \equiv & \triangle\psi + \gamma^{-2}\left[\frac{\lambda^2}{x_1^2} - \frac{\lambda^2}{|r_{1\epsilon}|^2}\right]\frac{\partial^2\psi}{\partial x_2^2} + \Big(1+V\Big)\psi - |\mathcal{U}_2|^2\psi + 2ie^{-i\varphi}\mathrm{Re}\big(\bar{u}_2 ie^{i\varphi}\psi\big)\mathcal{U}_2 \\
& - |\bigtriangledown\varphi|^2\psi + i\triangle[\varphi]\psi + 2i\bigtriangledown\varphi\cdot\bigtriangledown\psi + i\epsilon\frac{\kappa}{\gamma}|\log\epsilon|\frac{\partial\psi_1}{\partial x_2} \\
= & E_{4,2} + N_{4,2},
\end{aligned} \tag{5.36}$$

where $E_{4,2} = ie^{-i\varphi}\mathbb{E}$. The nonlinear operator is defined by

$$N_{4,2}(\psi) = -ie^{-i\varphi}(\mathcal{U}_2 + ie^{i\varphi}\psi)|\psi|^2 + 2i\mathrm{Re}(\bar{u}_2 ie^{i\varphi}\psi)\psi.$$

More precisely, for other term, we have

$$-|\mathcal{U}_2|^2\psi + 2ie^{-i\varphi}\mathrm{Re}\big(\bar{u}_2 ie^{i\varphi}\psi\big)\mathcal{U}_2 = -\hat{q}^2\,\psi_1 - 3i\hat{q}^2\,\psi_2.$$

The equation can be decomposed in the form

$$\begin{aligned}
\tilde{L}_{4,2}[\psi_1] \equiv & \Big(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\Big)\psi_1 + \Big(1+V\Big)\psi_1 - \hat{q}\,\psi_1 + \gamma^{-2}\left[\frac{\lambda^2}{x_1^2} - \frac{\lambda^2}{|r_{1\epsilon}|^2}\right]\frac{\partial^2\psi_1}{\partial x_2^2} \\
& + \frac{1}{x_1}\frac{\partial}{\partial x_1}\psi_2 - |\bigtriangledown\varphi|^2\psi_1 + i\triangle\varphi\,\psi_1 - 2\bigtriangledown\varphi\cdot\bigtriangledown\psi_2 + \epsilon\frac{\kappa}{\gamma}|\log\epsilon|\frac{\partial\psi_1}{\partial x_2} \\
= & \tilde{E}_{4,2} + \tilde{N}_{4,2},
\end{aligned} \tag{5.37}$$

$$\begin{aligned}
\bar{L}_{4,2}[\psi_2] \equiv & \Big(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\Big)\psi_2 + \Big(1+V\Big)\psi_2 - \hat{q}\,\psi_2 + \gamma^{-2}\left[\frac{\lambda^2}{x_1^2} - \frac{\lambda^2}{|r_{1\epsilon}|^2}\right]\frac{\partial^2\psi_2}{\partial x_2^2} \\
& + \frac{1}{x_1}\frac{\partial}{\partial x_1}\psi_2 - |\bigtriangledown\varphi|^2\psi_2 + i\triangle\varphi\,\psi_2 + 2\bigtriangledown\varphi\cdot\bigtriangledown\psi_1 + \epsilon\frac{\kappa}{\gamma}|\log\epsilon|\frac{\partial\psi_2}{\partial x_2} \\
= & \bar{E}_{4,2} + \bar{N}_{4,2}.
\end{aligned} \tag{5.38}$$

The assumption (A3) implies that, for any sufficiently small ϵ there holds

$$\Xi_{4,2} = 1 + V < -c_2 \quad \text{for } |x| > r_{2\epsilon} + \tau_2/\epsilon. \tag{5.39}$$

The other terms with φ_0 are lower order terms. From the asymptotic properties of q in Lemma 2.4, $\hat{q}\,\psi_2$ and $\hat{q}\,\psi_1$ are also lower order term. Whence the linear operators $\tilde{L}_{4,2}$ and $\bar{L}_{4,2}$ are small perturbations of the following linear operator

$$L_{42*}[\tilde{\psi}] \equiv \Big(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\Big)\tilde{\psi} + \big(1+V\big)\tilde{\psi}. \tag{5.40}$$

We now come to the conclusion of this section. Let χ be the cut-off function defined in (5.1). By recalling the definition of β in (5.14), we define

$$\Lambda \equiv \frac{\partial \mathcal{U}_2}{\partial f} \cdot \frac{\chi\big(|x-\xi_+|/\epsilon\big)+\chi\big(|x-\xi_-|/\epsilon\big)}{\beta}\,. \tag{5.41}$$

In summary, for any given f in (3.15), we want to solve the projected problem for ψ satisfying the conditions in (5.5)

$$\mathcal{L}(\psi)=\mathcal{N}(\psi)+\mathcal{E}+\mathcal{C}\Lambda, \qquad \mathrm{Re}\int_{\mathbb{R}^2}\bar{\phi}\Lambda \,=\, 0, \tag{5.42}$$

where have denoted

$$\mathcal{L}(\psi)=L_{1,j}(\psi) \quad \text{in } D_{1,j} \text{ for } j=1,2, \qquad \mathcal{L}(\psi)=L_{2,j}(\psi) \quad \text{in } D_{2,j} \text{ for } j=1,2,$$

$$\mathcal{L}(\psi)=L_3(\phi) \quad \text{in } D_3,$$

$$\mathcal{L}(\psi)=L_{4,j}(\psi) \quad \text{in } D_{4,j} \text{ for } j=1,2, \qquad \mathcal{L}(\psi)=L_{5,j}(\psi) \quad \text{in } D_{5,j} \text{ for } j=1,2,$$

with the relation

$$\phi=i\,\mathcal{U}_2\,\psi \quad \text{in } D_2. \tag{5.43}$$

As we have stated, the nonlinear operator $\mathcal{N}$ and the error term $\mathcal{E}$ also have suitable local forms in different regions.

6 The Resolution of the projected nonlinear problem

6.1 The linear resolution theory

The main objective is to consider the resolution of the linear part in previous section, which was stated in Lemma 6.2.

For that purpose, we shall firs get a priori estimates expressed in suitable norms. By recalling the norm $\|\cdot\|_{**}$ defined in (4.61), for fixed small positive numbers $0<\sigma<1$, $0<\gamma<1$, we define

$$\begin{aligned} ||\psi||_* \equiv \sum_{i=1}^{2}\Big[\,&||\phi||_{W^{2,p}(\ell_i<3)} \,+\, ||\ell_i^{\sigma}\psi_1||_{L^\infty(\tilde{D}\cup D_3)}+||\ell_i^{1+\sigma}\,\nabla\,\psi_1||_{L^\infty(\tilde{D}\cup D_3)} \\ &+\,||\ell_i^{1+\sigma}\psi_2||_{L^\infty(\tilde{D}\cup D_3)}+||\ell_i^{2+\sigma}\,\nabla\,\psi_2||_{L^\infty(\tilde{D}\cup D_3)}\Big] \\ &+\,||\psi||_{W^{2,p}(D_4\cup D_5)}, \end{aligned}$$

where we have use the relation $\phi=i\mathcal{U}_2\psi$ and the region $\tilde{D}$ is defined in (4.62). We then consider the following problem: finding ψ with the conditions in (5.5)

$$\mathcal{L}(\psi)=h \quad \text{in } \mathbb{R}^2, \quad \mathrm{Re}\int_{\mathbb{R}^2}\bar{\phi}\Lambda \,=\, 0 \;\; \text{with } \phi=i\mathcal{U}_2\psi, \tag{6.1}$$

where $\mathcal{L},\Lambda$ are defined in Section 5, (5.42), (5.41).

Lemma 6.1. *There exists a constant C, depending on γ, σ only, such that for all ϵ sufficiently small, and any solution of (6.1), we have the estimate*

$$||\psi||_* \leq ||h||_{**}.$$

Proof. We prove the result by contradiction. Suppose that there is a sequence of $\epsilon = \epsilon_n$, functions ψ^n, h_n which satisfy (6.1) with

$$||\psi^n||_* = 1, \quad ||h_n||_{**} = o(1).$$

Before any further argument, by the assumptions (5.5) for $\psi = \psi_1 + i\psi_2$, we have

$$\begin{aligned} \psi_1(x_1, -x_2) = -\psi_1(x_1, x_2), \quad \psi_1(-x_1, x_2) = \psi_1(x_1, x_2), \\ \psi_2(x_1, -x_2) = \psi_2(x_1, x_2), \quad \psi_2(-x_1, x_2) = \psi_2(x_1, x_2). \end{aligned} \tag{6.2}$$

We may just need to consider the problem in $\mathbb{R}^2_+ = \{(x_1, x_2) : x_1 > 0\}$. Then we have

$$\mathrm{Re} \int_{\mathbb{R}^2} \bar{\phi}_n \Lambda \;=\; 2\mathrm{Re} \int_{\mathbb{R}^2_+} \bar{\phi}_n \Lambda \;=\; 0, \tag{6.3}$$

for any $\phi_n = i\mathcal{U}_2 \psi^n$. To get good estimate and then derive a contradiction, we will use suitable forms of the linear operator $\mathcal{L}$ in different regions, which was stated in previous section. Hence we divide the proof into several parts.

Part 1. In the vortex-core region, we here only derive the estimates $D_{2,1}$ near ξ_+. Since $||h||_{**} = o(1), \psi^n \to \psi^0$, which satisfies

$$L_{2,1}(\psi_0) = 0, \quad ||\psi^0||_* \leq 1.$$

Whence, we get $L_0(\phi_0) = 0$. By the nondegeneracy in Lemma 2.3, we have

$$\phi_0 = c_1 \frac{\partial U_0}{\partial y_1} + c_2 \frac{\partial U_0}{\partial y_2}.$$

Observe that ϕ_0 inherits the symmetries of ϕ and hence $\phi_0 = \overline{\phi_0(x_1, -x_2)}$, while the other symmetry is not preserved under the translation $y = x - \xi_+$. Obviously, the term $\frac{\partial U_0}{\partial y_2}$ does not enjoy the above symmetry. This implies that $\phi_0 = c_1 \frac{\partial U_0}{\partial s_1}$. On the other hand, taking a limit of the orthogonality condition $\mathrm{Re} \int_{\mathbb{R}^2_+} \bar{\phi}_n \Lambda = 0$, we obtain

$$\mathrm{Re} \int_{\mathbb{R}^2} \bar{\phi}_0 \frac{U_0}{\partial y_1} = 0.$$

Thus $c_1 = 0$ and $\phi_0 = 0$. Hence, for any fixed $R > 0$, there holds

$$||\phi_1||_{L^\infty(\ell<R)} \,+\, ||\phi_2||_{L^\infty(\ell<R)} \,+\, ||\bigtriangledown \phi_1||_{L^\infty(\ell<R)} \,+\, ||\bigtriangledown \phi_2||_{L^\infty(\ell<R)} \;=\; o(1).$$

Part 2. In the outer part $D_{2,2}$, we use the following barrier function

$$\mathcal{B}(x) \;=\; \mathcal{B}_1(x) \,+\, \mathcal{B}_2(x),$$

where

$$\mathcal{B}_1(x) = |x-\xi_+|^{\varrho}\,|x_2|^{\gamma} \,+\, |x-\xi_-|^{\varrho}\,|x_2|^{\gamma}, \qquad \mathcal{B}_2(x) = C_1(1+|x|^2)^{-\sigma/2},$$

where $\varrho+\gamma=-\sigma$, $0<\sigma<\gamma<1$, and C_1 is a large number depending on σ,ϱ,γ only. Trivial computations derive that

$$\triangle\mathcal{B}_1 \,\le\, -C\big(|x-\xi_+|^2+|x-\xi_-|^2\big)^{-1-\sigma/2},$$

$$\triangle\mathcal{B}_2 + \frac{1}{x_1}\frac{\partial\mathcal{B}_2}{\partial x_1} \,\le\, -CC_1(1+|x|^2)^{-1-\sigma/2}.$$

On the other hand,

$$\frac{1}{x_1}\frac{\partial\mathcal{B}_1}{\partial x_1} \,\le\, \frac{|x_2|^{\gamma}}{x_1}\Big[\,|x-\xi_+|^{\varrho-2}(x_1-r_{1\epsilon}) \,+\, |x-\xi_-|^{\varrho-2}(x_1-r_{1\epsilon})\Big].$$

Thus for $|x-\xi_+|<c_\sigma r_{1\epsilon}$, where c_σ is small, we have

$$\frac{1}{x_1}\frac{\partial\mathcal{B}_1}{\partial x_1} \,\le\, Cc_\sigma\Big[\,|x-\xi_+|^2+|x-\xi_-|^2\,\Big]^{-1-\sigma/2}.$$

For $|x-\xi_+|>c_\sigma r_{1\epsilon}$, where c_σ is small, we have

$$\frac{1}{x_1}\frac{\partial\mathcal{B}_1}{\partial x_1} \le C(1+|x|^2)^{-1-\sigma/2}.$$

By choosing C_1 large, we have

$$\triangle\mathcal{B} + \frac{1}{x_1}\frac{\partial\mathcal{B}}{\partial x_1} \,\le\, -C\big(|x-\xi_+|^2+|x-\xi_-|^2\big)^{-1-\sigma/2}.$$

For the details of the above computations, the reader can refer to the proof of Lemma 7.2 in [62].

In the region $D_{2,2}$, we have

$$\Big(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{1}{x_1}\frac{\partial}{\partial x_1}\Big)\psi_1 \,+\, \Big(\,\triangledown B+\frac{w'(\ell_2)}{w(\ell_2)}\frac{y}{\ell_2}\Big)\cdot\triangledown\psi_1 \,+\, o(1)|\psi_1| \,=\, h_1,$$

where have used $y=(x_1-r_{1\epsilon},x_2)$. By comparison principle on the set $D_{2,2}$, we obtain

$$|\psi_1| \le C\mathcal{B}\big(||h||_{**}+o(1)\big), \quad \forall x\in D_{2,2}.$$

On the other hand, the equation for ψ_2 is

$$\Big(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{1}{x_1}\frac{\partial}{\partial x_1}\Big)\psi_2 \,-\, 2|\mathcal{U}_2|^2\psi_2 \,+\, 2\Big(\,\triangledown B+\frac{w'(\ell_2)}{w(\ell_2)}\frac{y}{\ell_2}\Big)\cdot\triangledown\psi_2 \,+\, o(1)|\psi_2| \,=\, h_2.$$

For $x\in D_{2,2}$, there holds $|\mathcal{U}_2|\sim 1$. By standard elliptic estimates we have

$$||\psi_2||_{L^\infty(\ell_i>3)} \,\le\, C||\psi_2||_{L^\infty(\ell_i=3)}(1+||\psi||_*)||h||_{**}(1+\ell_1+\ell_2)^{-1-\sigma},$$

$$|\nabla \psi_2| \le C||\psi_2||_{L^\infty(\ell_i=R)}(1+||\psi||_*)||h||_{**}(1+\ell_1+\ell_2)^{-2-\sigma}.$$

Part 3. In the outer part D_3, we still use the following barrier function

$$\mathcal{B}(x) = \mathcal{B}_1(x) + \mathcal{B}_2(x),$$

where

$$\mathcal{B}_1(x) = |x-\xi_+|^\varrho\,|x_2|^\gamma + |x-\xi_-|^\varrho\,|x_2|^\gamma, \qquad \mathcal{B}_2(x) = C_1(1+|x|^2)^{-\sigma/2},$$

where $\varrho+\gamma=-\sigma$, $0<\sigma<\gamma<1$, and C_1 is a large number depending on σ,ϱ,γ only.

In D_3, we have

$$\Big(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{1}{x_1}\frac{\partial}{\partial x_1}\Big)\psi_1 + \frac{2}{\beta_1}\nabla\beta_1\cdot\nabla\psi_1 + o(1)|\psi_1| = h_1.$$

By comparison principle on the set D_3, we obtain

$$|\psi_1| \le C\mathcal{B}\big(||h||_{**}+o(1)\big), \quad \forall\, x\in D_3.$$

On the other hand, the equation for ψ_2 is

$$\Big(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{1}{x_1}\frac{\partial}{\partial x_1}\Big)\psi_2 - 2|\mathcal{U}_2|^2\psi_2 + \frac{2}{\beta_1}\nabla\beta_1\cdot\psi_2 + o(1)|\psi_2| = h_2.$$

For $x\in D_3$, there holds $|\mathcal{U}_2|\sim 1$. By standard elliptic estimates we have

$$||\psi_2||_{L^\infty(\ell_i>3)} \le C||\psi_2||_{L^\infty(\ell_i=3)}(1+||\psi||_*)||h||_{**}(1+\ell_1+\ell_2)^{-1-\sigma},$$

$$|\nabla \psi_2| \le C||\psi_2||_{L^\infty(\ell_i=R)}(1+||\psi||_*)||h||_{**}(1+\ell_1+\ell_2)^{-2-\sigma}.$$

Part 4. In the region $D_{4,1}$, we have

$$\begin{aligned}L_{4,1}[\psi_1] \equiv{}& \Big(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}\Big)\psi_1 - \big(\delta_\epsilon(\ell-r_{2\epsilon})+\hat{q}^2\big)\psi_1 + \Big[1+V+\delta_\epsilon(\ell-r_{2\epsilon})\Big]\psi_1\\ &+ \frac{1}{x_1}\frac{\partial}{\partial x_1}\psi_1 - 2\nabla\varphi\cdot\nabla\psi_2 + \triangle[\varphi]\psi_1 - |\nabla\varphi|^2\psi_1 + o(1)|\psi_1|\\ ={}& h_1,\end{aligned}$$

$$\begin{aligned}L_{4,1}[\psi_2] \equiv{}& \Big(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}\Big)\psi_2 - \big(\delta_\epsilon(\ell-r_{2\epsilon})+3\hat{q}^2\big)\psi_2 + \Big[1+V+\delta_\epsilon(\ell-r_{2\epsilon})\Big]\psi_2\\ &+ \frac{1}{x_1}\frac{\partial}{\partial x_1}\psi_2 + 2\nabla\varphi\cdot\nabla\psi_1 + \triangle[\varphi]\psi_2 - |\nabla\varphi|^2\psi_2 + o(1)|\psi_2|\\ ={}& h_2.\end{aligned}$$

By defining a new translated variable $z = \delta_\epsilon^{1/3}(\ell - r_{2\epsilon})$, the linear operators L_{41*} and L_{41**} in (5.35) become

$$L_{41*}(\psi_{1*}) = \Big(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\Big)\psi_{1*} - \big(z + q^2(z)\big)\psi_{1*}\,,$$
$$L_{41**}(\psi_{2**}) = \Big(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\Big)\psi_{2**} - \big(z + 3q^2(z)\big)\psi_{2**}\,.$$

From Lemma 2.4, $-q'(z) > 0$ for all $z \in \mathbb{R}$, and $L_{31**}(-q') = 0$. We apply the maximum principle to $-\psi_2/q'$ and then obtain

$$|\psi_2| \le C|q'|(||h||_{**} + o(1)), \quad \forall\, x \in D_{4,1}.$$

On the other hand, $q(z) > 0$ for all $z \in \mathbb{R}$, and $L_{31*}(q) = 0$. We apply the maximum principle to ψ_1/q and then obtain

$$|\psi_1| \le Cq(||h||_{**} + o(1)), \quad \forall\, x \in D_{4,1}.$$

Part 5. In $D_{4,2}$, we consider the problem

$$\begin{aligned} L_{4,2}[\psi_1] \equiv\ & \Big(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\Big)\psi_1 + \big(1+V\big)\psi_1 - \hat{q}\,\psi_1 \\ & + \frac{1}{x_1}\frac{\partial}{\partial x_1}\psi_2 - |\nabla\varphi|^2\psi_1 + i\triangle[\varphi]\psi_1 - 2\nabla\varphi\cdot\nabla\psi_2 + o(1)|\psi_1| \\ =\ & h_1, \end{aligned}$$

$$\begin{aligned} L_{4,2}[\psi_2] \equiv & \Big(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\Big)\psi_2 + \big(1+V\big)\psi_2 - \hat{q}(|x_1|)\psi_2 \\ & + \frac{1}{x_1}\frac{\partial}{\partial x_1}\psi_2 - |\nabla\varphi|^2\psi_2 + i\triangle[\varphi]\psi_2 + 2\nabla\varphi\cdot\nabla\psi_1 + o(1)|\psi_2| \\ = & h_2. \end{aligned}$$

By using the properties of $\Xi_{4,2}$ in (5.39), i.e.

$$\Sigma_{4,2} = \big(1+V\big) < -c_2 \quad \text{in } D_{4,2},$$

we have

$$||\psi_2||_{L^\infty(\ell_i>3)} \le C||\psi_2||_{L^\infty(\ell_i=3)}(1 + ||\psi||_*)||h||_{**}(1+\ell_1+\ell_2)^{-1-\sigma},$$

$$|\nabla\psi_2| \le C||\psi_2||_{L^\infty(\ell_i=R)}(1 + ||\psi||_*)||h||_{**}(1+\ell_1+\ell_2)^{-2-\sigma}.$$

Combining all the estimates in the above, we obtain that $||\psi||_* = o(1)$, which is a contradiction. $\square$

We now consider the following linear projected problem: finding ψ with the conditions in (5.5)

$$\mathcal{L}[\psi] \,=\, h \,+\, \mathcal{C}\Lambda, \quad \mathrm{Re}\int_{\mathbb{R}^2} \bar{\phi}\Lambda \,=\, 0 \quad \text{ with } \phi = i\mathcal{U}_2\psi. \tag{6.4}$$

Lemma 6.2. *There exists a constant C, depending on γ, σ only, such that for all ϵ sufficiently small, the following holds: if $||h||_{**} < +\infty$, there exists a unique solution $(\psi_{\epsilon,f}, \mathcal{C}_{\epsilon,f}) = \mathcal{T}_{\epsilon,f}(h)$ to (6.4). Furthermore, there holds*

$$||\psi||_* \le C||h||_{**}.$$

Proof. The proof is similar to that of Proposition 4.1 in [29]. Instead of solving (6.4) in $\mathbb{R}^2$, we solve it in a bounded domain first:

$$\mathcal{L}[\psi] \,=\, h \,+\, \mathcal{C}\Lambda, \quad \mathrm{Re}\int_{\mathbb{R}^2} \bar{\phi}\Lambda \,=\, 0 \quad \text{ with } \phi = i\mathcal{U}_2\psi,$$
$$\phi = 0 \quad \text{ on } \partial B_M(0), \quad \psi \text{ satisfies the conditions in (5.5)}.$$

where $M > 10r_{1\epsilon}$. By the standard proof of a priori estimates, we also obtain the following estimates for any solution ψ_M of above problem

$$||\psi||_* \le C||h||_{**}.$$

By working with the Sobole space $H_0^1(B_M(0))$, the existence will follow by Fredholm alternatives. Now letting $M \to +\infty$, we obtain a solution with the required properties. $\square$

6.2 Solving the Projected Nonlinear Problem

We then consider the following problem: finding ψ with the conditions in (5.5)

$$\mathcal{L}[\psi] \,+\, \mathcal{N}[\psi] \,=\, \mathcal{E} \,+\, \mathcal{C}\Lambda, \quad \mathrm{Re}\int_{\mathbb{R}^2} \bar{\phi}\Lambda \,=\, 0 \quad \text{ with } \phi = i\mathcal{U}_2\psi. \tag{6.5}$$

Proposition 6.3. *There exists a constant C, depending on γ, σ only, such that for all ϵ sufficiently small, there exists a unique solution $\psi_{\epsilon,f}, c_{\epsilon,f}$ to (6.5), and*

$$||\psi||_* \le C\epsilon.$$

Furthermore, ψ is continuous in the parameter f.

Proof. Using of the operator defined by Lemma 6.2, we can write problem (6.5) as

$$\psi = \mathcal{T}_{\epsilon,f}\big(-\mathcal{N}[\psi] + \mathcal{E}\big) \equiv \mathcal{G}_\epsilon(\psi).$$

Using Lemma 4.2, we see that

$$||\mathcal{E}||_{**} \le C\epsilon^{1-\sigma}.$$

Let

$$\psi \in \mathbb{B} = \left\{ ||\psi||_* < C\epsilon^{1-\sigma} \right\},$$

then we have, using the explicit form of $\mathcal{N}(\psi)$ in Section 5

$$||\mathcal{N}(\psi)||_{**} \leq C\epsilon.$$

Whence, there holds

$$||\mathcal{G}_\epsilon(\psi)||_{**} \leq C\Big(||\mathcal{N}(\psi)||_{**} + ||\mathcal{E}||_{**}\Big) \leq C\epsilon^{1-\sigma}.$$

Similarly, we can also show that, for any $\check{\psi}, \hat{\psi} \in \mathbb{B}$

$$||\mathcal{G}_\epsilon(\check{\psi}) - \mathcal{G}_\epsilon(\hat{\psi})||_{**} \leq o(1)||\check{\psi} - \hat{\psi}||_{**}\,.$$

By contraction mapping theorem, we confirm the result of the Lemma. □

7 Reduction procedure

To find a real solution to problem (3.8)-(3.10), in this section, we solve the reduced problem by finding a suitable f such that the constant $\mathcal{C}$ in (5.42) is identical zero for any sufficiently small ϵ.

In previous section, for any given f in (3.15), we have deduced the existence of ψ with the conditions in (5.5) to the projected problem

$$\mathcal{L}(\psi) = \mathcal{N}(\psi) + \mathcal{E} + \mathcal{C}\Lambda, \quad \mathrm{Re}\int_{\mathbb{R}^2} \bar{\phi}\Lambda \,=\, 0, \tag{7.1}$$

with the relation

$$\phi = i\,\mathcal{U}_2\psi \quad \text{in } D_2.$$

Multiplying (7.1) by $\bar{\Lambda}$ and integrating, we obtain

$$\mathcal{C}\,\mathrm{Re}\int_{\mathbb{R}^2} \bar{\Lambda}\Lambda \,=\, \mathrm{Re}\int_{\mathbb{R}^2} \bar{\Lambda}\mathcal{L}(\psi) \,-\, \mathrm{Re}\int_{\mathbb{R}^2} \bar{\Lambda}\mathcal{N}(\psi) \,-\, \mathrm{Re}\int_{\mathbb{R}^2} \bar{\Lambda}\mathcal{E}. \tag{7.2}$$

Hence we can derive the estimate for $\mathcal{C}$ by computing the integrals of the right hand side.

We begin with the computation of $\mathrm{Re}\int_{\mathbb{R}^2} \bar{\Lambda}\mathcal{E}$. The term Λ has its support contained in the region $\{(x_1, x_2) : \ell_1 < \tau_0/\epsilon \text{ or } \ell_2 < \tau_0/\epsilon\}$. It is convenient to compute $\mathrm{Re}\int_{\mathbb{R}^2} \bar{\Lambda}\mathcal{E}$ on the variables (s_1, s_2). Note that, in the vortex-core region $\{(x_1, x_2) : \ell_2 \leq \tau_0/\epsilon\}$, there holds

$$\frac{\partial \mathcal{U}_2}{\partial f} \,=\, \Big[-\,\frac{w'(\ell_2)}{w(\ell_2)}\frac{x_1 - r_{1\epsilon}}{\ell_2} \,+\, i\,\frac{x_2}{\ell_2^2}\Big]\mathcal{U}_2 \,+\, O(\epsilon^2)\mathcal{U}_2,$$

which implies that

$$\Lambda \,=\, \chi\big(|x-\xi_+|/\epsilon\big)\Big[\,-\,\frac{w'(\ell_2)}{w(\ell_2)}\frac{x_1-r_{1\epsilon}}{\ell_2}\,+\,i\,\frac{x_2}{\ell_2^2}\Big]w(\ell_2)e^{i\varphi_0^+}\,+\,O(\epsilon^2).$$

By using of local form of $\mathcal{E}$ in the formula (5.21), we obtain

$$\begin{aligned}
\mathrm{Re}\int_{\mathbb{R}^2}\bar{\Lambda}\mathcal{E}\mathrm{d}x \,=\, 2\mathrm{Re}\int_{\mathbb{R}^2_+}\bar{\Lambda}\mathcal{E}\,\mathrm{d}x
&=\,-\,2\int_{\mathbb{R}^2_+}\chi\big(\ell_2/\epsilon\big)\Big[w'(\ell_2)\Big]^2\frac{(x_1-r_{1\epsilon})^2}{x_1\ell_2^2}\,\mathrm{d}x\\
&\quad-\,2\epsilon\frac{\partial V}{\partial\tilde{r}}\Big|_{\tilde{r}=\hat{r}_{1\epsilon}}\int_{\mathbb{R}^2_+}\chi\big(\ell_2/\epsilon\big)w(\ell_2)w'(\ell_2)\frac{(x_1-r_{1\epsilon})(x_1-r_{1\epsilon})}{\ell_2}\,\mathrm{d}x\\
&\quad-\,2\log r_{1\epsilon}\int_{\mathbb{R}^2_+}\chi\big(\ell_2/\epsilon\big)w(\ell_2)w'(\ell_2)\frac{2(x_1+r_{1\epsilon})(x_1-r_{1\epsilon})^2}{\ell_1^2\ell_2^3}\,\mathrm{d}x\\
&\quad-\,2\frac{1}{r_{1\epsilon}}\log r_{1\epsilon}\int_{\mathbb{R}^2_+}\chi\big(\ell_2/\epsilon\big)w(\ell_2)w'(\ell_2)\frac{x_2^2}{\ell_2^3}\,\mathrm{d}x\\
&\quad+\,2\frac{\kappa\epsilon|\log\epsilon|}{\gamma}\int_{\mathbb{R}^2_+}\chi\big(\ell_2/\epsilon\big)w(\ell_2)w'(\ell_2)\frac{x_2^2}{\ell_2^3}\,\mathrm{d}x\,+\,O(\epsilon).
\end{aligned}$$

By the translation in (3.16), we further derive that

$$\begin{aligned}
\mathrm{Re}\int_{\mathbb{R}^2}\bar{\Lambda}\mathcal{E}\mathrm{d}x \,=\, &-2\int_{\mathbb{R}^2}\chi\big(\,|\,s\,|\,/\epsilon\big)\Big[w'(|\,s\,|)\Big]^2\frac{s_1^2}{(s_1+r_{1\epsilon})\,|\,s\,|^2}\,\mathrm{d}s\\
&-\,2\epsilon\frac{\partial V}{\partial\tilde{r}}\Big|_{\tilde{r}=\hat{r}_{1\epsilon}}\int_{\mathbb{R}^2}\chi\big(\,|\,s\,|\,/\epsilon\big)w(|\,s\,|)w'(|\,s\,|)\frac{s_1^2}{|\,s\,|}\,\mathrm{d}s\\
&-\,2\log r_{1\epsilon}\int_{\mathbb{R}^2}\chi\big(\,|\,s\,|\,/\epsilon\big)w(|\,s\,|)w'(|\,s\,|)\frac{2(s_1+2r_{1\epsilon})s_1^2}{\Big[(s_1+2r_{1\epsilon})^2+s_2^2\Big]\,|\,s\,|^3}\,\mathrm{d}s\\
&-\,2\frac{1}{r_{1\epsilon}}\log r_{1\epsilon}\int_{\mathbb{R}^2}\chi\big(\,|\,s\,|\,/\epsilon\big)w(|\,s\,|)w'(|\,s\,|)\frac{s_2^2}{|\,s\,|^3}\,\mathrm{d}s\\
&+\,2\frac{\kappa\epsilon|\log\epsilon|}{\gamma}\int_{\mathbb{R}^2}\chi\big(\,|\,s\,|\,/\epsilon\big)w(|\,s\,|)w'(|\,s\,|)\frac{s_2^2}{|\,s\,|^3}\,\mathrm{d}s\,+\,O(\epsilon)\,.
\end{aligned}$$

We compute the first two terms in above formula

$$-2\int_{\mathbb{R}^2}\chi\big(\,|\,s\,|\,/\epsilon\big)\Big[w'(|\,s\,|)\Big]^2\frac{s_1^2}{(s_1+r_{1\epsilon})\,|\,s\,|^2}\,\mathrm{d}s\,=\,O(\epsilon),$$

and by the asymptotic behavior of w in Lemma 2.1

$$\begin{aligned}
&-\,2\epsilon\frac{\partial V}{\partial\tilde{r}}\Big|_{\tilde{r}=\hat{r}_{1\epsilon}}\int_{\mathbb{R}^2}\chi\big(\,|\,s\,|\,/\epsilon\big)w(|\,s\,|)w'(|\,s\,|)\frac{s_1^2}{|\,s\,|}\,\mathrm{d}s\\
&\quad=\,-2\pi\epsilon\,|\log\epsilon|\,\frac{\partial V}{\partial\tilde{r}}\Big|_{\tilde{r}=\hat{r}_{1\epsilon}}\,+\,O(\epsilon).
\end{aligned}$$

On the other hand,

$$
\begin{aligned}
& -2\log r_{1\epsilon}\int_{\mathbb{R}^2}\chi\big(\mid s\mid/\epsilon\big)w(\mid s\mid)w'(\mid s\mid)\frac{2(s_1+2r_{1\epsilon})s_1^2}{\Big[(s_1+2r_{1\epsilon})^2+s_2^2\Big]\mid s\mid^3}\,\mathrm{d}s \\
& -2\frac{1}{r_{1\epsilon}}\log r_{1\epsilon}\int_{\mathbb{R}^2}\chi\big(\mid s\mid/\epsilon\big)w(\mid s\mid)w'(\mid s\mid)\frac{s_2^2}{\mid s\mid^3}\,\mathrm{d}s \\
& \quad = -\frac{2}{r_{1\epsilon}}\log r_{1\epsilon}\int_{\mathbb{R}^2}w(\mid s\mid)w'(\mid s\mid)\frac{1}{\mid s\mid}\,\mathrm{d}s\,+\,O(\epsilon) \\
& \quad = -\,\frac{2\pi d}{r_{1\epsilon}}\log r_{1\epsilon}\,+\,O(\epsilon),
\end{aligned}
$$

where

$$
d=\frac{1}{\pi}\int_{\mathbb{R}^2}w(\mid s\mid)w'(\mid s\mid)\frac{1}{\mid s\mid}\,\mathrm{d}s>0. \tag{7.3}
$$

While the last term can be estimated by

$$
\begin{aligned}
& 2\frac{\kappa\epsilon|\log\epsilon|}{\gamma}\int_{\mathbb{R}^2}\chi\big(\mid s\mid/\epsilon\big)w(\mid s\mid)w'(\mid s\mid)\frac{s_2^2}{\mid s\mid^3}\,\mathrm{d}s \\
& =\frac{\kappa\epsilon|\log\epsilon|}{\gamma}\int_{\mathbb{R}^2}w(\mid s\mid)w'(\mid s\mid)\frac{1}{\mid s\mid}\,\mathrm{d}s\,+\,O(\epsilon) \\
& =\frac{\pi d\kappa}{\gamma}\epsilon|\log\epsilon|\,+\,O(\epsilon).
\end{aligned}
$$

Hence, there holds

$$
\mathrm{Re}\int_{\mathbb{R}^2}\bar{\Lambda}\mathcal{E}\mathrm{d}x\;=\;-2\pi\,\frac{\partial V}{\partial\tilde{r}}\Big|_{\tilde{r}=\hat{r}_{1\epsilon}}\epsilon\,|\log\epsilon|\;-\;2\pi\,\frac{d}{r_{1\epsilon}}\log r_{1\epsilon}\;+\;\frac{\pi d\,\kappa}{\gamma}\epsilon|\log\epsilon|\;+\;O(\epsilon). \tag{7.4}
$$

Using Proposition 6.3, and the expression in (5.22), we deduce that

$$
\mathrm{Re}\int_{\mathbb{R}^2}\bar{\Lambda}\mathcal{N}(\psi)\;=\;\mathrm{Re}\int_{\mathbb{R}^2}\bar{\Lambda}N_2(\psi)\;=\;O(\epsilon).
$$

On the other hand, integration by parts, we have

$$
\mathrm{Re}\int_{\mathbb{R}^2}\bar{\Lambda}\mathcal{L}(\psi)\;=\;\mathrm{Re}\int_{\mathbb{R}^2}\bar{\psi}\mathcal{L}(\bar{\Lambda})\;=\;O(\epsilon).
$$

Combining all estimates together and recalling $\hat{r}_{1\epsilon}=\hat{r}_1+\hat{f}$, we obtain the following equation

$$
\mathcal{C}(\hat{f})=-2\,\epsilon\pi\left[\frac{\partial V}{\partial\tilde{r}}\Big|_{\tilde{r}=\hat{r}_1+\hat{f}}\log\frac{1}{\epsilon}\;+\;\frac{d}{\hat{r}_1+\hat{f}}\log\frac{\hat{r}_1+\hat{f}}{\epsilon}\;-\;\frac{\kappa\,d}{2\gamma}\log\frac{1}{\epsilon}\right]\;+\;O(\epsilon), \tag{7.5}
$$

where $O(\epsilon)$ is a continuous function of the parameter $\hat{f}$. By the solvability condition (1.12) and the non-degeneracy condition(1.11), we can find a zero of $\mathcal{C}\big(\hat{f}\big)$ at some small $\hat{f}$ with the help of the simple mean-value theorem.

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References

[1] N. Andre, P. Bauman and D. Phillips, *Vortex pinning with bounded fields for the Ginzburg-Landau equation*, Ann. I. H. Poincaré, Non Lin??aire, 20 (2003), no. 4, pp. 705–729.

[2] A. Aftalion, *Vortices in Bose-Einstein condensates*, Progress in Nonlinear Differential Equations and their Applications, 67. Birkhauser Boston, Inc. Boston Ma, 2000.

[3] A. Aftalion, S. Alama and L. Bronsard, *Giant vortex and the breakdown of strong pinning in a rotating Bose-Einstein condensate*, Arch. Rational Mech. Anal., 178 (2005), no. 2, pp. 247–286.

[4] A. Aftalion and X. Blanc, *Existence of vortex-free solutions in the Painleve boundary layer of a Bose-Einstein condensate*, J. Math. Pures Appl., 83 (2004), no. 4, pp. 765–801

[5] Stan Alama and Lia Bronsard, *Pinning effects and their breakdown for a Ginzburg-Landau model with normal inclusions*, J. Math. Phys., 46 (2005), 095102.

[6] S. Alama and L. Bronsard, *Vortices and pinning effects for the Ginzburg-Landau model in multiply connected domains*, Comm. Pure Appl. Math., 59 (2006), no. 1, pp. 36–70.

[7] N. Andre and I. Shafrir, *Asymptotic behavior of minimizers for the Ginzburg-Landau functional with weight. I, II*, Arch. Rational Mech. Anal., 142 (1998), no. 1, pp. 45–73, pp. 75–98.

[8] B. Anderson, P. Haljan, C. Regal, D. Feder, L. Collins, C. Clark and E. Cornell, *Watching dark solitons decay into vortex rings in a Bose-Einstein condensate*, Phys. Rev. Lett., 86 (2001), no. 14, pp. 2926–2929.

[9] M. H. Anderson, J. R. Ensher, M. R. Matthews, C. E. Wieman and E. A. Cornell, *Observation of Bose-Einstein condensation in a dilute atomic vapor*, Science, 269 (1995), pp. 198–201.

[10] C. Barenghi, *Is the Reynolds number infinite in superfluid turbulence?*, Physica D, 237 (2008), pp. 2195–2202.

[11] C. Barenghi and R. Donnelly, *Vortex rings in classical and quantum systems*, Fluid Dyn. Res., 41 (2009), 051401.

[12] G. Baym and C. Pethick, *Ground-state properties of magnetically trapped Bose-Condensed rubidium gas*, Phys. Rev. Lett., 76 (1996), no. 1, pp. 6–9.

[13] A. Beaulieu and R. Hadiji, *Asymptotic behavior of minimizers of a Ginzburg-Landau equation with weight near their zeroes*, Asymptot. Anal., 22 (2000), no. 3-4, pp. 303–347.

[14] F. Bethuel, H. Brezis and F. Helein, *Ginzburg-Landau vortices*, Progress in Nonlinear Differential Equations and their Applications, 13. Birkhauser Boston, Inc. Boston Ma, 1994.

[15] F. Bethuel, P. Gravejat and J. Saut, *Existence and properties of travelling waves for the Gross-Pitaevskii equation*, Stationary and time dependent Gross-Pitaevskii equations, pp. 55–103, Contemp. Math., 473, Amer. Math. Soc., Providence, RI, 2008.

[16] F. Bethuel, P. Gravejat and J.-G. Saut, *Travelling waves for the Gross-Pitaevskii equation, II*, Comm. Math. Phys., 285 (2009), no. 2, pp. 567–651.

[17] F. Bethuel, G. Orlandi and D. Smets, *Vortex rings for the Gross-Pitaevskii equation*, J. Eur. Math. Soc., 6 (2004), no. 1, pp. 17–94.

[18] F. Bethuel and J.-C. Saut, *Travelling waves for the Gross-Pitaevskii equation, I*, Ann. Inst. Henri Poincaré, Physique Théorique, 70 (1999), no. 2, pp. 147–238.

[19] I. Bialynicki-Birula, Z. Bialynicka-Birula and C. Śliwa, *Motion of vortex lines in quantum mechanics*, Phys. Rev. A, 61 (2000), no. 3, 032110.

[20] H. Brezis, F. Merle and T. Rieviere, *Quantization effects for* $-\triangle u = u(1-|u|^2)$ *in* $\mathbb{R}^2$, Arch. Ration. Mech. Anal. 126 (1994), no. 1, pp. 35–58.

[21] S. J. Chapman, Q. Du, and M. D. Gunzburger, *A Ginzburg-Landau type model of superconducting/normal junctions including Josephson junctions*, Eur. J. Appl. Math., 6 (1995), pp. 97–114.

[22] X. Chen, C. M. Elliott and Q. Tang, *Shooting method for vortex solutions of a complex-valued Ginzburg-Landau equation*, Proc. Roy. Soc. Edinburgh Sect. A, 124 (1994), no. 6, pp. 1075–1088.

[23] D. Chiron, *Travelling waves for the Gross-Pitaevskii equation in dimension larger than two*, Nonlinear Anal., 58 (2004), no. 1-2, pp. 175–204.

[24] D. Chiron, *Vortex helices for the Gross-Pitaevskii equation*, J. Math. Pures Appl. 84 (2005), no. 11, pp. 1555–1647.

[25] M. del Pino and P. Felmer, *Locally energy-minimizing solution of the Ginzburg-Landau equation*, C. R. Acad. Sci. Paris Sér. I Math., 321 (1995), no. 9, pp. 1207–1211.

[26] M. del Pino and P. Felmer, *Local minimizers for the Ginzburg-Landau energy*, Math. Z., 225 (1997), no. 4, pp. 671–684.

[27] M. del Pino, P. Felmer and M. Musso, *Two-bubble solutions in the super-critical Bahri-Coron's problem*, Calc. Var. Part. Diff. Eqn., 16 (2003), no. 2, pp. 113–145.

[28] M. del Pino, P. Felmer and M. Kowalczyk, *Minimality and nondegeneracy of degree-one Ginzburg-Landau vortex as a Hardy's type inequality*, Int. Math. Res. Not., (2004), no. 30, pp. 1511–1627.

[29] M. del Pino, M. Kowalczyk and M. Musso, *Variational reduction for Ginzburg-Landau vortices*, J. Funct. Anal., 239 (2006), no. 2, pp. 497–541.

[30] M. del Pino, M. Kowalczyk and J. Wei, *Concentration on curve for nolinear Schrödinger equation*, Comm. Pure Appl. Math., 60 (2007), no. 1, pp. 113–146.

[31] M. del Pino, M. Kowalczyk and J. Wei, *The Toda system and clustering interfaces in the Allen-Cahn equation*, Arch. Ration. Mech. Anal., 190 (2008), no. 1, pp. 141–187.

[32] M. del Pino, M. Kowalczyk and J. Wei, *The Jacobi-Toda system and foliated interfaces*, Discrete Contin. Dyn. Syst., 28 (2010), no. 3, pp. 975–1006.

[33] M. del Pino, M. Kowalczyk, J. Wei and J. Yang, *Interface foliation near minimal submanifolds in Riemannian manifolds with positive Ricci curvature*, Geom. Funct. Anal., 20 (2010), no. 4, pp. 918–957.

[34] K. B. Davis, M.-O. Mewes, M. R. Andrews, N. J. van Druten, D. S. Durfee, D. M. Kurn, and W. Ketterle, *Bose-Einstein condensation in a gas of sodium atoms*, Phys. Rev. Lett., 75 (1995), no. 22, pp. 3969–3973.

[35] R. Donnelly, *Quantized vortices in Helium II.* Cambridge University Press, Cambredge, U. K. 1991.

[36] Q. Du, M. D. Gunzburger, and J. S. Peterson, *Computational simulations of type II superconductivity including pinning phenomena*, Phys. Rev. B, 51 (1995), pp. 16194–16203.

[37] D. L. Feder, M. S. Pindzola, L. A. Collins, B. I. Schneider and C. W. Clark, *Dark-soliton states of Bose-Einstein condensates in anisotropic traps*, Phys. Rev. A, 62 (2000), no. 5, 053606.

[38] A. L. Fetter, *Rotating trapped Bose-Einstein condensates*, Rev. Mod. Phys., 81 (2009), no. 2, pp. 647–691.

[39] A. Fetter and D. Feder, *Beyond the Thomas-Fermi approximation for a trapped condensed Bose-Einstein gas*, Phys. Rev. A, 58 (1998), no. 4, pp. 3185–3194.

[40] A. L. Fetter and A. A Svidzinsky, *Vortices in a trapped dilute Bose-Einstein condensate*, J. Phys.: Condens. Matter, 13 (2001), pp. 135–194.

[41] C. Gui and J. Wei, *Multiple interior peak solutions for some singularly perturbed Neumann problems*, J. Diff. Equ., 158 (1999), no.1, pp. 1–27.

[42] M. Guilleumas, D. M. Jezek, R. Mayol, M. Pi and M. Barranco, *Generating vortex rings in Bose-Einstein condensates in the line-source approximation*, Phys. Rev. A, 65 (2002), no. 5, 053609.

[43] M. Guilleumas and R. Graham, *Off-axis vortices in trapped Bose-condensed gases: Angular momentum and frequency splitting*, Phys. Rev. A, 64 (2001), no. 3, 033607.

[44] S. Gustafson and F. Ting, *Dynamic stability and instability of pinned fundamental vortices*, J. Nonlinear Science, 19 (2009), no. 4, pp. 341–374.

[45] K.-H. Hoffmann and Q. Tang, *Ginburg-Landau phase transition theory and superconductivity*, Birkhauser Verlag, Basel, 2001.

[46] T. Horng, C. Hsueh and S. Gou, *Transition to quantum turbulence in a Bose-Einstein condensate through the bending-wave instability of a single-vortex ring*, Phys. Rev. A, 77 (2008), no. 6, 063625.

[47] T. Horng, S. Gou, and T. Lin, *Bending-wave instability of a vortex ring in a trapped Bose-Einstein condensate*, Phys. Rev. A, 74 (2006), no. 4, 041603(R).

[48] B. Jackson, J. F. McCann and C. S. Adams, *Vortex rings and mutual drag in trapped Bose-Einstein condensates*, Phys. Rev. A, 60 (1999), no. 6, pp. 4882–4885.

[49] B. Jackson, J. F. McCann and C. S. Adams, *Vortex line and ring dynamics in trapped Bose-Einstein condensates*, Phys. Rev. A, 61 (1999), no. 1, 013604.

[50] S. Jimbo and Y. Morita, *Stability of nonconstant steady-state solutions to a Ginzburg-Landau equation in higher space dimensions*, Nonlinear Anal., 22 (1994), no. 6, pp. 753–770.

[51] S. Jimbo and Y. Morita, *Vortex dynamics for the Ginzburg-Landau equation with Neumann condition*, Methods Appl. Anal., 8 (2001), no. 3, pp. 451–477.

[52] S. Jimbo and Y. Morita, *Notes on the limit equation of vortex motion for the Ginzburg-Landau equation with Neumann condition*, in: Recent Topics in Mathematics Moving toward Science and Engineering, Japan J. Indust. Appl. Math., 18 (2001), no. 2, pp. 483–501.

[53] S. Jimbo, Y. Morita and J. Zhai, *Ginzburg-Landau equation and stable steady state solutions in a non-trivial domain*, Comm. Partial Differential Equations, 20 (1995), no. 11-12, pp. 2093–2112.

[54] G. Karali and C. Sourdis, *The ground state of a Gross-Pitaevskii energy with general potential in the Thomas-Fermi limit*, Archive for Rational Mechanics and Analysis, 217 (2015), no. 2, pp. 439-523.

[55] R. Kerr, *Numerical generation of a vortex ring cascade in quantum turbulence*, arXiv:1006.3911v2.

[56] F. Lin, *Solutions of Ginzburg-Landau equations and critical points of the renormalized energy*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 12 (1995), no. 5, pp. 599–622.

[57] F. Lin, *Mixed vortex-antivortex solutions of Ginzburg-Landau equations*, Arch. Ration. Mech. Anal., 133 (1995), no. 2, pp. 103–127.

[58] F. Lin, *Some dynamical properties of Ginzburg-Landau vortices*, Comm. Pure Appl. Math., 49 (1996), no. 4, pp. 323–359.

[59] F. Lin and Q. Du, *Ginzburg-Landau vortices: dynamics, pinning, and hysteresis*, SIAM J. Math. Anal., 28 (1997), no. 6, pp. 1265–1293.

[60] F. Lin and T. Lin, *Minimax solutions of the Ginzburg-Landau equations*, Selecta Math. (N.S.), 3 (1997), no. 1, pp. 99–113.

[61] F. H. Lin, W. M. Ni and J. Wei, *On the number of interior peak solutions for a singularly perturbed Neumann problem*, Comm. Pure Appl. Math., 60 (2007), no. 2, pp. 252–281.

[62] F. Lin and J. Wei, *Travelling wave solutions of Schrödinger map equation*, Comm. Pure Appl. Math., 63 (2010), no. 12, pp. 1585–1621.

[63] T. Lin, J. Wei and J. Yang, *Vortex rings for the Gross-Pitaevskii equation in* $\mathbb{R}^3$, J. Math. Pures Appl. 100 (2013), pp. 69–112.

[64] E. Lundh, C. Pethick and H. Smith, *Zero-temperature properties of a trapped Bose-condensed gas: beyond the Thomas-Fermi approximation*, Phys. Rev. A, 55 (1997), no. 3, pp. 2126–2131.

[65] D. McMahon, *Quantum Mechanics Demystified*, McGraw-Hill, 2005.

[66] K. Madison, F. Chevy, W. Wohlleben and J. Dalibard, *Vortex formation in a stirred Bose-Einstein condensate*, Phys. Rev. Lett., 84 (2000), no. 5, pp. 806–809.

[67] M. R. Matthews, B. P. Anderson, P. C. Haljan, D. S. Hall, C. E. Wieman and E. A. Cornell, *Vortices in a Bose-Einstein Condensate*, Phys. Rev. Lett., 83 (1999), no. 13, pp. 2498–2501.

[68] M. P. Mink, C. Morais Smith and R. A. Duine, *Vortex-lattice pinning in two-component Bose-Einstein condensates*, Phys. Rev. A, 79 (2009), no. 1, 013605.

[69] P. Mironescu, *On the stability of radial solutions of the Ginzburg-Landau equation*, J. Funct. Anal. 130 (1995), no. 2, pp. 334–344.

[70] P. Mironescu, *Les minimiseurs locaux pour l'??quation de Ginzburg-Landau sont ? sym??trie radiale [Local minimizers for the Ginzburg-Landau equation are radially symmetric]*, C. R. Acad. Sci. Paris S??r. I Math. 323 (1996), no. 6, pp. 593–598.

[71] Y. N. Ovchinnikov and I. M. Sigal, *Ginzburg-Landau equation. I. Static vortices.* Partial differential equations and their applications (Toronto, ON, 1995), 199-220, CRM Proc. Lecture Notes, 12, Amer. Math. Soc., Providence, RI, 1997.

[72] F. Pacard and T. Riviere, *Linear and nonlinear aspects of vortices*, Nonlinear Differential Equations Appl., vol. 39, Birkhauser Boston, Boston, MA, 2000.

[73] A. Pakylak, F. Ting and J. Wei, *Multi-vortex solutions to Ginzburg-Landau equations with external potential*, Arch. Rational Mech. Anal., 204 (2012), no. 1, pp. 313–354.

[74] C. Pethick and H. Smith, *Bose-Einstein condensation in dilute gases*, Cambridge University Press, Cambridge, 2002.

[75] L. M. Pismen and J. Rubinstein, *Motion of vortex lines in the ginzburg-Landau model*, Physica D: Nonlinear Phenomena, 47 (1991), no. 3, pp. 353–360.

[76] G. Rayfield and F. Reif, *Evidence for the creation and motion of quantized vortex rings in superfluid Helium*, Phys. Rev. Lett., 11 (1963), no. 1, pp. 305–308.

[77] J. W. Reijnders and R. A. Duine, *Pinning of vortices in a Bose-Einstein condensate by an optical lattice*, Phys. Rev. Lett., 93 (2004), no. 6, 060401.

[78] J. Ruostekoski and J. Anglin, *Creating vortex rings and three-dimensional Skyrmions in Bose-Einstein condensates*, Phys. Rev. Lett., 86 (2001), no. 18, pp. 3934–3937.

[79] C. Ryu, M. F. Andersen, P. Cladé, Vasant Natarajan, K. Helmerson and W. D. Phillips, *Observation of persistent flow of a Bose-Einstein condensate in a toroidal trap*, Phys. Rev. Lett., 99 (2007), 260401.

[80] J. Ruostekoski and Z. Dutton, *Engineering vortex rings and systems for controlled studies of vortex interactions in Bose-Einstein condensates*, Phys. Rev. A, 72 (2005), no. 6, 063626.

[81] E. Sandier and S. Serfaty, *Vortices in the Magnetic Ginzburg-Landau Model*, Progress in Nonlinear Differential Equations and Their Applications, Vol. 70, Birkhauser, Boston, 2007.

[82] S. Serfaty, *Stability in 2D Ginzburg-Landau passes to the limit*, Indiana Univ. Math. J., 54 (2005), no. 1, pp. 199–222.

[83] I. Shafrir, Remarks on solutions of $-\Delta u = (1-|u|^2)u$ in $\mathbb{R}^2$, *C. R. Acad. Sci. Paris S??r. I Math.* 318 (1994), no. 4, pp. 327–331.

[84] F. Sols, *Vortex matter in atomic Bose-Einstein condensates*, Physica C, 369 (2002), pp. 125–134.

[85] Y. Strauss and I. M. Sigal, *Effective dynamics of a magnetic vortex in a local potential*, J. Nonlinear Sci., 16 (2006), no. 2, pp. 123–157.

[86] M. Struwe, *On the asymptotic behavior of minimizers of the Ginzburg-Landau model in 2 dimensions*, Differential Integral Equations, 7 (1994), no. 5-6, pp. 1613–1624.

[87] A. Svidzinsky and A. Fetter, *Dynamics of a vortex in trapped Bose-Einstein condensate*, Phys. Rev. A, 62 (2000), no. 6, 063617.

[88] M. Tinkham, *Introduction to Superconductivity*, McGraw-Hill, New York, USA, 1975.

[89] F. Ting, *Effective dynamics of multi-vortices in an external potential for Ginzburg-Landau gradient flow*, Nonlinearity, 23 (2010), pp. 179–210.

[90] S. Tung, V. Schweikhard and E. A. Cornell, *Observation of vortex pinning in Bose-Einstein condensates*, Phys. Rev. Lett., 97 (2006), no. 7, 240402.

[91] J. Wei and J. Yang, *Vortex rings pinning for the Gross-Pitaevskii equation in three dimensional space*, SIAM Journal on Mathematical Analysis 44 (2012), No. 6, 3991-4047.

[92] J. Wei and J. Yang, *Traveling vortex helices for Schrödinger map equations*, Transactions of the American Mathematical Society, http://dx.doi.org/10.1090/tran/6379.

[93] J. E. Williams and M. J. Holland, *Preparing topological states of a Bose-Einstein condensate*, Nature(London), 401 (1999), no. 6753, pp. 568–572.

[94] J. Yang, *Vortex structures for Klein-Gordon equation with Ginzburg-Landau nonlinearity*, Discrete Contin. Dyn. Syst. 34 (2014), no. 5, pp. 2359–2388.

[95] F. Zhou and Q. Zhou, *A remark on multiplicity of solutions for the Ginzburg-Landau equation*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 16 (1999), no. 2, pp. 255–267.