ENTIRE SOLUTIONS AND GLOBAL BIFURCATIONS FOR A BIHARMONIC EQUATION WITH SINGULAR NONLINEARITY

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Abstract. We study the structure of solutions of the pinned boundary value problem

$$\Delta^2 u = \frac{\lambda}{(1-u)^p} \text{ in } B, \quad u = \Delta u = 0 \text{ on } \partial B \tag{0.1}$$

where $p > 1$ and Δ^2 is the biharmonic operator and $B \subset \mathbb{R}^N$ is the unit ball. We show that there are infinitely many turning points of the branch of the radial solutions of (0.1). The analysis of touch-down solutions depends on classification of the radial solutions of the equation

$$-\Delta^2 u = u^{-p} \quad \text{in } \mathbb{R}^N. \tag{0.2}$$

We will see that the results in higher dimensional case are similar to the three dimensional case.

1. Introduction

We continue to investigate existence, uniqueness, asymptotic behavior and further qualitative properties of radial solutions of the biharmonic equation

$$-\Delta^2 u = u^{-p} \quad \text{in } \mathbb{R}^N \tag{1.1}$$

where $p > 1$. The motivation for studying (1.1) is to understand the structure of solutions of the Navier boundary value problem

$$T\Delta u - D\Delta^2 u = \lambda (L+u)^{-2} \text{ in } \Omega, \quad u = \Delta u = 0 \text{ on } \partial\Omega \tag{1.2}$$

where $T, D, L > 0$, $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain. Problem (1.2) models the deflection of charged plates in electrostatic actuators (Lin and Yang [22]). Here $\lambda = aV^2$ where V is the electric voltage and a is constant. Associated with (1.2) is the following energy functional

$$E(u) = \int_\Omega \left\{ \frac{T}{2}|\nabla u|^2 + \frac{D}{2}|\Delta u|^2 - \frac{\lambda}{L+u} \right\} \tag{1.3}$$

where $P = \int_\Omega \frac{T}{2}|\nabla u|^2 dx$ is the stretching energy, $Q = \int_\Omega \frac{D}{2}|\Delta u|^2 dx$ corresponds to the bending energy, and $W = -\int_\Omega \frac{\lambda}{L+u(x)} dx$ is the electric potential energy.

1991 *Mathematics Subject Classification.* Primary 35B45; Secondary 35J40.

Key words and phrases. Global branch, entire solutions, biharmonic equations, singular nonlinearity infinitely many turning points.

Lin and Yang ([22]) considered two kinds of boundary conditions: pinned boundary condition

$$u = \Delta u = 0 \quad \text{on } \partial\Omega$$

and clamped boundary condition

$$u = \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega.$$

For the pinned boundary condition, they found that there exists $0 < \lambda_c < \infty$ such that for $\lambda \in (0, \lambda_c)$, (1.2) has a maximal regular solution u_λ, which can be obtained from an iterative scheme. (By a regular solution u_λ of (1.2), we mean that $u_\lambda \in C^4(\Omega) \cap C^3(\overline{\Omega})$ satisfies (1.2).) For $\lambda > \lambda_c$, (1.2) does not have any regular solution. Moreover, if $\lambda', \lambda'' \in (0, \lambda_c)$ and $\lambda' < \lambda''$, then the corresponding maximal solutions $u_{\lambda'}$ and $u_{\lambda''}$ satisfy

$$u_{\lambda'} > u_{\lambda''} \quad \text{in } \Omega.$$

The number λ_c, which determines the pull-in voltage, is called the pull-in threshold. It is known from [22] that, for $\lambda \in (0, \lambda_c)$, $\min_\Omega (L + u_\lambda) > 0$. Let $\Sigma_\lambda = \{x \in \Omega : \ L + u_\lambda(x) = 0\}$ be the singular set of (1.2). An interesting question is to study the limit of u_λ as $\lambda \to \lambda_c$. The monotonicity of u_λ with respect to λ implies that there is a well-defined function U so that

$$U(x) = \lim_{\lambda \to \lambda_c^-} u_\lambda(x); \quad -L \leq U(x) < 0, \quad x \in \Omega.$$

However $U(x)$ may touch down to $-L$ and cease to be a regular solution to (1.2). For the one-dimensional case, Lin and Yang showed that U is a regular solution, that is, the set $\Sigma_{\lambda_c} = \emptyset$.

In our previous paper [15], we showed that *for $N = 2$ or 3, U is a regular solution*. Moreover, we also showed that there is a *unique solution of (1.2) at $\lambda = \lambda_c$*. For two-dimensional convex domains, we also established the existence of *a second solution for every $\lambda \in (0, \lambda_c)$*. This shows that at least in two-dimensional domains, problem (1.2) behaviors *subcritically*. (Numerical computations as well as asymptotic behavior as $D \to 0$ are done in [23].)

In our another paper [16] we established the result: *when $N = 3$, for λ small, the maximal solution of (1.2) is unique. There exists $\lambda_* < \lambda_c$ such that the solution branch has infinitely many turning points for λ near λ_*.* This shows that problem (1.2) behaviors *supercritically* in $\mathbb{R}^3$.

We remark that problem (1.2) can find the applications in thin film problems, see [1], [2], [3], [18], [19], [20], [21]. When $D = 0$, problem (1.2) can also find the applications in MEMS devices, see, [4], [5], [8], [9], [10], [11], [12], [13], [14], [17], [25], [26].

The applications of problem (1.2) can be found in [16]. By a change $v = -u$, we see that v satisfies

$$(1.4) \qquad -T\Delta v + D\Delta^2 v = \frac{\lambda}{(L-v)^2}, \quad 0 < v < L \text{ in } \Omega, \quad v = \Delta v = 0 \text{ on } \partial\Omega.$$

As mentioned in [16], for simplicity, we only study the problem

$$(1.5) \qquad \Delta^2 v = \frac{\lambda}{(1-v)^2}, \quad 0 < v < 1 \text{ in } \Omega, \quad v = \Delta v = 0 \quad \text{on } \partial\Omega.$$

All the results obtained in this paper are still true for (1.4).

The purpose of this paper is to investigate the problem (1.1) in higher dimensions $N \geq 4$. First, we study the properties of entire radial solutions of (1.1) with $N \geq 4$. We seek solutions u of (1.1) which only depend on $|x|$. Due to the homogeneity, (1.1) is invariant under a suitable rescaling. This means that existence of a solution immediately implies the existence of infinitely many solutions, each one being characterized by its value at the origin. To ensure smoothness of the solution, one needs to require that $u'(0) = u'''(0) = 0$. We see that solutions of (1.1) can be determined only by fixing a priori value of $u''(0)$. In this paper, the proofs are performed with a shooting method which uses as a free parameter the "shooting concavity", namely the initial second derivative $u''(0)$. We consider the general case of (1.1) with a nonlinearity u^{-p} and $p > 1$.

We consider the following initial value problem

$$(1.6) \qquad \begin{array}{l} \Delta^2 u = -u^{-p}, \;\; u = u(r) \quad \text{in } \mathbb{R}^N, \;\; N \geq 4 \\ u(0) = 1, \quad u'(0) = u'''(0) = 0, \quad u''(0) = \gamma > 0. \end{array}$$

Our first theorem is on the classification of entire solutions to (1.6):

Theorem 1.1. *There exists a unique $\gamma^* \in (0,\infty)$ such that for $\gamma \in (0,\gamma^*)$, there is a unique $R_\gamma \in (0,\infty)$ such that $\Delta u_\gamma(R_\gamma) = 0$ and $(\Delta u_\gamma)'(r) < 0$ for $r \in (0, R_\gamma)$. The function R_γ is continuous and increasing with respect to γ and $R_\gamma \to \infty$ as $\gamma \to \gamma^*$. For $\gamma > \gamma^*$, there exists $C := C(\gamma) > 0$ such that $(\Delta u_\gamma)'(r) < 0$ for $r > 0$, $\Delta u_\gamma(r) \to C$ as $r \to \infty$ and u_γ has the growth Cr^2 near ∞. For $\gamma = \gamma^*$, we have $(\Delta u_{\gamma^*})'(r) < 0$ for $r > 0$, $\Delta u_{\gamma^*}(r) \to 0$ as $r \to \infty$. Thus $\Delta u_{\gamma^*}(r) > 0$ for $r \in (0,\infty)$ and $u'_{\gamma^*}(r) > 0$ for $r \in (0,\infty)$.*

It is easy to know that the equation in (1.6) has a singular solution

$$(1.7) \qquad U_0(r) = [-K_0]^{-\frac{1}{p+1}} r^{\frac{4}{p+1}}$$

where

(1.8)

$$K_0 = \frac{8}{(p+1)^4}\Big(N(N+2)+(N^2-6N-8)p+(16-2N-N^2)p^2-(N-2)(N-4)p^3\Big).$$

Note that we are interested in the case that $K_0 \le 0$. We see that $K_0 > 0$ for $N = 2$ and all $p > 1$; that $K_0 \le 0$ for $N = 3$ and $1 \le p \le 3$; that $K_0 < 0$ for $N \ge 4$ and all $p > 1$. This implies that the main results of [16] should be also true for $1 < p < 3$ (we only discuss the case $p = 2$ in [16]).

Theorem 1.1 implies that the equation in (1.6) has a unique entire solution u_{γ^*} with $(\Delta u_{\gamma^*})'(r) < 0$ for $r > 0$ and $\Delta u_{\gamma^*}(r) \to 0$ as $r \to \infty$. Our second theorem is on the qualitative properties of this entire solution u_{γ^*}.

Let $p_m := p(N)$ be the maximal root of the equation

$$-(N-2)(N-4)(p+1)^3+4(N^2-10N+20)(p+1)^2+48(N-4)(p+1)+128 = 0.$$

Then a simple calculation implies that $p_m > 1$ for $N \ge 4$. (A simple calculation by using the Mathlab, we obtain that $p_m = 2\sqrt{2}-1$ for $N = 4$; $p_m = 2.0704$ for $N = 5$; $p_m = 2.2361$ for $N = 6$; $p_m = 2.3541$ for $N = 7$; $p_m = 2.4415$ for $N = 8$; $p_m = 2.5086$ for $N = 9$; $p_m = 2.5616$ for $N = 10$; $p_m = 2.6043$ for $N = 11$; $p_m = 2.6396$ for $N = 12$.)

Theorem 1.2. *Let $u_{\gamma^*}(r)$ be the entire solution to (1.6) (given by Theorem 1.1). Then for $N \ge 4$ and $p > p_m$,*

$$\lim_{r\to\infty} r^{-\frac{4}{p+1}} u_{\gamma^*}(r) = (-K_0)^{-\frac{1}{p+1}}. \tag{1.9}$$

Theorem 1.3. *Let $N \ge 4$ and $p > p_m$. Then for any $a > 0$ the equation (1.1) admits a unique radial positive solution $u = u(r)$ such that $u(0) = a$ and $\Delta u(r) \to 0$ as $r \to \infty$. The solution u satisfies $u'(r) > 0$ for all $r > 0$ and*

$$\lim_{r\to\infty} r^{-\frac{4}{p+1}} u(r) = (-K_0)^{-\frac{1}{p+1}}.$$

Moreover, if $N \ge 13$ or $9 \le N \le 12$ and

$$p_m < p \le p_c \tag{1.10}$$

then $u(r) > (-K_0)^{-1/(p+1)} r^{4/(p+1)}$ for all $r > 0$ and the solutions are strictly ordered with respect to the initial value $a = u(0)$. If $N = 4$ or $5 \le N \le 12$ and

$$p > \max\{p_c, p_m\} \tag{1.11}$$

then $u(r) - (-K_0)^{-1/(p+1)} r^{4/(p+1)}$ changes sign infinitely many times.

Using the Mathlab, we obtain that $p_c = 1.0536$ for $N = 5$; $p_c = 1.2047$ for $N = 6$; $p_c = 1.4671$ for $N = 7$; $p_c = 1.8889$ for $N = 8$; $p_c = 2.5845$ for $N = 9$; $p_c = 3.8545$ for $N = 10$; $p_c = 6.7957$ for $N = 11$; $p_c = 20.2087$ for $N = 12$. Then, $p_c < p_m$ for $N = 5; 6; 7; 8$.

Let us remark that results similar to Theorem 1.2 and Theorem 1.3 have been obtained for the following biharmonic equation with power-like nonlinearity

$$(-\Delta)^2 u = u^p, u = u(r) > 0 \tag{1.12}$$

in [6] and [7]. However, our proofs are completely different and also work for (1.12).

Finally we consider the structure of radial solutions of (1.5) with

$$\Omega = B = \{x \in \mathbb{R}^N : |x| < 1\}.$$

Namely we study existence and properties of non-minimal radially symmetric solutions of the problem

$$\Delta^2 u = \lambda(1-u)^{-p} \text{ in } B, \quad u = \Delta u = 0 \text{ on } \partial B. \tag{1.13}$$

By arguments similar to those in [22], we obtain that there exists $0 < \lambda_c < \infty$ such that for $\lambda \in (0, \lambda_c]$, (1.13) has at least positive solution $u_\lambda \in C^4(B) \cap C^3(\overline{B})$ and when $\lambda > \lambda_c$, (1.13) has no any regular solution in $C^4(B) \cap C^3(\overline{B})$.

Now, we put

$$\mathcal{C}_r^\lambda = \{u \in C^4(B) \cap C^3(\overline{B}) : u = u(|x|) \text{ solves (1.13)}\}$$

$$C_r = \cup_{\lambda > 0}\{\lambda\} \times \mathcal{C}^\lambda.$$

Theorem 1.4. *For $N = 4$ or $5 \leq N \leq 12$ and $p > \max\{p_c, p_m\}$, the secondary bifurcation point of $\mathcal{C}_r$ does not occur and $\mathcal{C}_r$ homeomorphic to $\mathbb{R}$ with the end points $(0, 0)$ and $(\lambda_*, 1 - |x|^{\frac{4}{p+1}})$, where*

$$\lambda_* = -K_0.$$

Moreover, $\mathcal{C}_r$ bends infinitely many times with respect to λ around λ_*.

Theorem 1.5. *For $N \geq 13$ or $9 \leq N \leq 12$ and $p_m < p \leq p_c$, $\mathcal{C}_r$ is the branch of the minimal positive solutions of (1.13) with the end points $(0, 0)$ and $(\lambda_c, 1 - |x|^{\frac{4}{p+1}})$ with $\lambda_c = -K_0$.*

2. Proof of Theorem 1.1

In this section, we prove Theorem 1.1. Since we are only interested in the radial solutions, by a shooting method, keeping $u(0)$ fixed, say $u(0) = 1$, we look for solutions u of the initial value problem over $[0, \infty)$:

$$\begin{cases} u^{(4)}(r) + \frac{2(N-1)}{r}u'''(r) + \frac{(N-1)(N-3)}{r^2}u''(r) - \frac{(N-1)(N-3)}{r^3}u'(r) = -u^{-p}(r) \\ u(0) = 1, \quad u'(0) = u'''(0) = 0, \quad u''(0) = \gamma > 0 \end{cases} \tag{2.1}$$

which is a radial version of (1.6). By standard ODE theory, we see that for each $\gamma > 0$, (2.1) has a unique solution $u_\gamma(r)$ for r near 0.

If $u = u(r)$ is a radial positive solution of (2.1), then

$$u_a := au(a^{-\frac{p+1}{4}}r) \quad (a > 0)$$

is a radial positive solution of the equation in (2.1) such that $u_a(0) = a$.

We apply a shooting method with initial second derivative as parameter. We remark that $Nu''(0) = (\Delta u)(0)$ and that by l'Hospital's rule

$$(\Delta u)'(0) = u'''(0) + (N-1)\lim_{r\to 0}\frac{ru''(r) - u'(r)}{r^2} = \frac{N+1}{2}u'''(0).$$

This means that the initial conditions in (2.1) also read as

$$u(0) = 1, \quad u'(0) = (\Delta u)'(0) = 0, \quad \Delta u(0) = N\gamma > 0. \tag{2.2}$$

For all $\gamma > 0$, (2.1)-(2.2) admit a unique local smooth solution u_γ defined on some right neighborhood of $r = 0$. Let

$$R_\gamma = \begin{cases} +\infty & \text{if } u_\gamma(r)(\Delta u_\gamma)(r) > 0, \ \forall r > 0 \\ \min\{r > 0;\ u_\gamma(r)(\Delta u_\gamma)(r) = 0\} & \text{otherwise.} \end{cases}$$

From now on we understand that u_γ is continued on $[0, R_\gamma)$. Let

$$I^+ = \{\gamma > 0;\ \ R_\gamma < \infty,\ \ u_\gamma(R_\gamma) = \infty\},$$

$$I^- = \{\gamma > 0;\ \ R_\gamma < \infty,\ \ (\Delta u_\gamma)(R_\gamma) = 0\}.$$

By arguments similar to [16] we see that $I^+ = \emptyset$.

Now we show that

$$I^- \neq \emptyset. \tag{2.3}$$

To continue our proof, we need a comparison principle, which has been observed by McKenna-Reichel [24].

Lemma 2.1. *(Comparison Principle). Assume that $f:\ (0,\infty)\to(0,\infty)$ is locally Lipschitz and strictly increasing. Let $u,v\in C^4([0,R))$ be such that*

$$(2.4)\qquad \begin{cases} \forall r\in[0,R):\ \ \Delta^2 v(r)-f(v(r))\geq \Delta^2 u(r)-f(u(r)) \\ v(0)\geq u(0),\quad v'(0)=u'(0)=0, \\ \Delta v(0)\geq \Delta u(0),\quad (\Delta v)'(0)=(\Delta u)'(0)=0. \end{cases}$$

Then we have for all $r\in[0,R)$:

$$(2.5)\qquad v(r)\geq u(r),\ \ v'(r)\geq u'(r),\ \ \Delta v(r)\geq \Delta u(r),\ \ (\Delta v)'(r)\geq(\Delta u)'(r).$$

Moreover,

(i) the initial point 0 can be replaced by any initial point $\rho>0$ and all four initial data are weakly ordered.

(ii) a strict inequality in one of the initial data at $\rho\geq 0$ or in the differential inequality on (ρ,R) implies a strict ordering of v, v', Δv, $(\Delta v)'$ and u, u', Δu, $(\Delta u)'$ on (ρ,R).

Considering the problem

$$(2.6)\qquad \Delta^2 v=\lambda(1-v)^{-p}\ \text{ in } B,\qquad v=\Delta v=0\ \text{ on } \partial B$$

where B is the unit ball of $\mathbb{R}^N$, we see from [22] that there is $0<\lambda_c<\infty$ such that for $\lambda\in(0,\lambda_c]$, (2.6) has a minimal positive solution $v_\lambda\in C^4(B)$ satisfying $0<v_\lambda<1$. The minimality of v_λ implies that $v_\lambda(x)=v_\lambda(r)$. Defining $w_\lambda=1-v_\lambda$, we see that w_λ satisfies the problem

$$-\Delta^2 w_\lambda=\lambda w_\lambda^{-p}\ \text{ in } B,\quad w_\lambda=1,\quad \Delta w_\lambda=0\ \text{ on } \partial B.$$

Setting $\xi_\lambda:=\min_B w_\lambda$, $y=\lambda^{1/4}\xi_\lambda^{-(p+1)/4}r$, and $\tilde w_\lambda=w_\lambda(r)/\xi_\lambda$, we see that $\tilde w_\lambda$ with $\tilde w_\lambda(0)=\min_B\tilde w_\lambda=1$ satisfies the problem

$$-\Delta^2\tilde w_\lambda=\tilde w_\lambda^{-p}\ \text{ in } B_\lambda,\quad \tilde w_\lambda=\frac{1}{\xi_\lambda},\quad \Delta_y\tilde w_\lambda=0\ \text{ on } \partial B_\lambda$$

where $B_\lambda=\{y\in\mathbb{R}^N:\ \ |y|<\lambda^{1/4}\xi_\lambda^{-(p+1)/4}\}$. Denote $N\gamma_\lambda=(\Delta\tilde w_\lambda)(0)$. We see that $\gamma_\lambda\in I^-$. Moreover,

$$R_{\gamma_\lambda}=\lambda^{1/4}\xi_\lambda^{-(p+1)/4}.$$

This proves (2.3).

Define $\gamma^*=\sup I^-$. We will show that $\gamma^*<\infty$. Indeed, for $\epsilon>0$ sufficiently small (e.g. $\epsilon<2(p-1)/(p+1)$) and $b>0$ sufficiently large, it follows from Lemma 3.5 of [24] that the function $v_\epsilon(r)=(1+b^2r^2)^{1-\frac{\epsilon}{2}}$ satisfies

$$\Delta^2 v+v^{-p}\leq 0\ \text{ on } (0,\infty).$$

Now we construct a subsolution to the equation in (2.1) with the growth $O(r^2)$. Let $V(r)=1+r^2+v_\epsilon(r)$. We see that

$$\Delta^2 V+V^{-p}\le \Delta^2 v_\epsilon+v_\epsilon^{-p}\le 0 \quad \text{on } (0,\infty).$$

We easily see that $\Delta V(r)>0$ for $r\in(0,\infty)$ and $\Delta V(r)\to 2N$ as $r\to\infty$. Setting $\tilde\gamma=V''(0)$, we see from Lemma 2.1 that the solution $u_{\tilde\gamma}\ge V$ and $\Delta u_{\tilde\gamma}\ge \Delta V$ on $(0,\infty)$. On the other hand, the function $\overline{V}(r)=Ar^2+D$ $(A>0,\ D>0)$ is a supersolution to the equation in (2.1), thus by choosing A and D sufficiently large and applying Lemma 2.1, we see that $u_{\tilde\gamma}\le\overline{V}$ on $(0,\infty)$. Thus, $u_{\tilde\gamma}$ is a solution of (2.1) with growth $O(r^2)$ near ∞. The comparison principle implies that $\gamma^*<\tilde\gamma$. Now we use Lemma 2.1 to show that R_γ is an increasing function of γ. For any $\gamma_1,\gamma_2\in I^-$ and $\gamma_1>\gamma_2$, by Lemma 2.1, we see that $u_{\gamma_1}(r)>u_{\gamma_2}(r)$ and $\Delta u_{\gamma_1}(r)>\Delta u_{\gamma_2}(r)$ for $r\in(0,\min\{R_{\gamma_1},R_{\gamma_2}\}]$. This clearly implies that $R_{\gamma_1}>R_{\gamma_2}$. The continuity of R_γ on γ can be obtained by standard ODE theory. We easily know that $\Delta u_{\gamma^*}(r)\to 0$ as $r\to\infty$.

Now we show that u_{γ^*} is the unique solution to (2.1) with $\Delta u(r)\to 0$ as $r\to\infty$. On the contrary, there are $\gamma^{**}>\gamma^*$ such that $\Delta u_{\gamma^{**}}(r)\to 0$, $\Delta u_{\gamma^*}(r)\to 0$ as $r\to\infty$. Then it follows from the comparison principle that

$$u_{\gamma^{**}}>u_{\gamma^*} \quad \text{on } (0,\infty).$$

It follows from the equations of $u_{\gamma^{**}}$ and u_{γ^*} that

$$(r^{N-1}(\Delta(u_{\gamma^{**}}-u_{\gamma^*}))'(r))'=-(u_{\gamma^{**}}^{-p}-u_{\gamma^*}^{-p})>0 \quad \text{on } (0,\infty)$$

and this implies that

$$(\Delta(u_{\gamma^{**}}-u_{\gamma^*}))'(r)>0 \quad \text{on } (0,\infty).$$

This contradicts the fact that $\Delta(u_{\gamma^{**}}-u_{\gamma^*})(r)\to 0$ as $r\to\infty$. This implies that u_{γ^*} is the unique solution of (2.1) satisfying $\Delta u(r)\to 0$ as $r\to\infty$. Since $\Delta u_{\gamma^*}(r)>0$ for $r\in(0,\infty)$, we see that $(r^{N-1}u'_{\gamma^*}(r))'>0$ for $r\in(0,\infty)$. Integrating it on $(0,r)$ and noting $u'_{\gamma^*}(0)=0$, we see that $u'_{\gamma^*}(r)>0$ for $r\in(0,\infty)$. This completes the proof of Theorem 1.1. □

3. Properties of entire solutions: proof of Theorem 1.2

Let u_{γ^*} be given by Theorem 1.1. We prove Theorem 1.2 in this section.

We use some ideas from [7]. To this end, we use the Emden-Fowler transformation:

$$u_{\gamma^*}(r)=r^{\frac{4}{p+1}}v(t), \qquad t=\log r\ (r>0). \tag{3.1}$$

Therefore, after the change (3.1), the equation in (2.1) may be rewritten as

$$v^{(4)}(t)+K_3v'''(t)+K_2v''(t)+K_1v'(t)+K_0v(t)=-v^{-p}(t),\quad t\in\mathbb{R} \tag{3.2}$$

where the constants $K_i=K_i(N,p)$ $(i=0,\dots,3)$ are given by

$$K_0=\frac{8}{(p+1)^4}\Big[-(N-2)(N-4)(p+1)^3+2(N^2-10N+20)(p+1)^2+16(N-4)(p+1)+32\Big]$$

which is given in (1.8).

$$K_1=\frac{2}{(p+1)^3}\Big[-(N-2)(N-4)(p+1)^3+4(N^2-10N+20)(p+1)^2+48(N-4)(p+1)+128\Big]$$

$$K_2=\frac{1}{(p+1)^2}\Big[(N^2-10N+20)(p+1)^2+24(N-4)(p+1)+96\Big]$$

$$K_3=\frac{2}{p+1}\Big[(N-4)(p+1)+8\Big].$$

(Note that $K_1<0$ for $4\le N\le 12$ and $p>p_m$.) This implies that the entire solution of (2.1) corresponds to a solution of (3.2). For $\gamma>\gamma^*$, the solution u_γ has a growth $O(r^2)$, this corresponds to $v(t)\to\infty$ as $t\to\infty$. We claim that $u_{\gamma^*}(r)$ corresponds to the solution v of (3.2) satisfying $\lim_{t\to\infty}v(t)=(-K_0)^{-1/(p+1)}$.

Note that (3.2) admits the constant solution $v_0=(-K_0)^{-1/(p+1)}$, which, by (3.1) corresponds to the singular solution $U_0(r)=(-K_0)^{-1/(p+1)}r^{4/(p+1)}$ of (2.1).

We now write (3.2) as a system in $\mathbb{R}^4$. By (3.1) we have

$$u'_{\gamma^*}(r)=0\ \text{ is equivalent to } v'(t)=-\tfrac{4}{p+1}v(t).$$

This fact suggests us to define

$$\begin{aligned} w_1(t)&=v(t)\\ w_2(t)&=v'(t)+\frac{4}{p+1}v(t)\\ w_3(t)&=v''(t)+\frac{4}{p+1}v'(t)\\ w_4(t)&=v'''(t)+\frac{4}{p+1}v''(t)\end{aligned}$$

so that (3.2) becomes

$$\begin{cases} w_1'(t)=-\frac{4}{p+1}w_1(t)+w_2(t)\\ w_2'(t)=w_3(t)\\ w_3'(t)=w_4(t)\\ w_4'(t)=C_2w_2(t)+C_3w_3(t)+C_4w_4(t)-w_1^{-p}(t)\end{cases} \tag{3.3}$$

where

$$C_m=-\Sigma_{i=m-1}^{4}\frac{K_i4^{i+1-m}}{(-1)^{i+1-m}(p+1)^{i+1-m}}\quad\text{for } m=1,2,3,4 \text{ with } K_4=1. \tag{3.4}$$

This gives that $C_1 = 0$ so that the term $C_1 w_1(t)$ does not appear in the last equation of (3.3). Moreover, we have the explicit formulae:

$$\begin{aligned} C_2 &= -\frac{2}{(p+1)^3}\Big[-(N-2)(N-4)(p+1)^3 + 2(N^2-10N+20)(p+1)^2 \\ &\qquad +16(N-4)(p+1)+32\Big] = -\frac{p+1}{4}K_0 \end{aligned}$$

$$C_3 = -\frac{1}{(p+1)^2}\Big[(N^2-10N+20)(p+1)^2 + 16(N-4)(p+1)+48\Big]$$

$$C_4 = -\frac{2}{p+1}\Big[(N-4)(p+1)+6\Big].$$

System (3.3) has one stationary point (corresponding to v_t)

$$P\Big((-K_0)^{-\frac{1}{p+1}},\ \frac{4}{p+1}(-K_0)^{-\frac{1}{p+1}}, 0, 0\Big).$$

Around this "singular point" P the linearized matrix of the system (3.3) is given by

$$M_P = \begin{pmatrix} -\frac{4}{p+1} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -pK_0 & C_2 & C_3 & C_4 \end{pmatrix}$$

The corresponding characteristic polynomial is

$$\nu \mapsto \nu^4 + K_3\nu^3 + K_2\nu^2 + K_1\nu + (p+1)K_0$$

and the eigenvalues are given by

$$\nu_1 = -\frac{N_1 + \sqrt{N_2 + 4\sqrt{N_3}}}{2(p+1)}, \quad \nu_2 = -\frac{N_1 - \sqrt{N_2 + 4\sqrt{N_3}}}{2(p+1)} \tag{3.5}$$

$$\nu_3 = -\frac{N_1 + \sqrt{N_2 - 4\sqrt{N_3}}}{2(p+1)}, \quad \nu_4 = -\frac{N_1 - \sqrt{N_2 - 4\sqrt{N_3}}}{2(p+1)} \tag{3.6}$$

where

$$N_1 := (N-4)(p+1)+8, \quad N_2 := (N^2-4N+8)(p+1)^2$$

$$\begin{aligned} N_3 : &= (9N-34)(N-2)(p+1)^4 - 8(3N-8)(N-6)(p+1)^3 \\ &\qquad +(16N^2-288N+832)(p+1)^2 + 128(N-6)(p+1) + 256. \end{aligned}$$

Proposition 3.1. *Assume $p > 1$.*

(i) For any $N \geq 4$, we have $\nu_1, \nu_2 \in \mathbb{R}$ and $\nu_2 > 0 > \nu_1$,

(ii) For $N = 4$, we have $\nu_3, \nu_4 \notin \mathbb{R}$ and $\Re(\nu_3) = \Re(\nu_4) < 0$,

(iii) For any $5 \leq N \leq 12$ there exists $p_c > 1$ such that

if $1 < p < p_c$, then $\nu_3, \nu_4 \in \mathbb{R}$ and $\nu_3 < \nu_4 < 0$,

if $p = p_c$, then $\nu_3, \nu_4 \in \mathbb{R}$ and $\nu_3 = \nu_4 < 0$

if $p > p_c$, then $\nu_3, \nu_4 \notin \mathbb{R}$ and $\Re(\nu_3) = \Re(\nu_4) < 0$.

The number p_c is the unique value of $p > 1$ such that

$$\begin{aligned}-(N-4)(N^3 - 4N^2 - 128N + 256)(p+1)^4 - 128(3N-8)(N-6)(p+1)^3 \\ +256(N^2 - 18N + 52)(p+1)^2 + 2048(N-6)(p+1) + 4096 = 0.\end{aligned}$$

(iv) For $N \geq 13$, we have $\nu_3, \nu_4 \in \mathbb{R}$ and $\nu_3 < \nu_4 < 0$.

Proof. We first observe that $N_1 > 0$ and

$$\begin{aligned} N_2 - N_1^2 &= (N^2 - 4N + 8)(p+1)^2 - (N-4)^2(p+1)^2 - 16(N-4)(p+1) - 64 \\ &= 4(N-2)(p+1)^2 - 16(N-4)(p+1) - 64 \\ &= 4[(N-2)(p+1)^2 - 4(N-4)(p+1) - 16]. \end{aligned}$$

Since the equation

$$(N-2)x^2 - 4(N-4)x - 16 = 0$$

has two roots:

$$x_1 = \frac{2}{N-2}[(N-4) + \sqrt{N^2 - 4N + 8}], \quad x_2 = \frac{2}{N-2}[N - 4 - \sqrt{N^2 - 4N + 8}],$$

we see that

$$N_2 - N_1^2 > 0 \quad \text{for } p + 1 > x_1. \tag{3.7}$$

Next, we show that for $p > 1$

$$N_3 > \frac{(N_2 - N_1^2)}{16}. \tag{3.8}$$

Indeed,

$$\begin{aligned} N_3 - \frac{(N_2 - N_1^2)}{16} &= 8(N-2)(N-4)(p+1)^4 \\ &\quad -16(N^2 - 10N + 20)(p+1)^3 - 128(N-4)(p+1)^4 - 256(p+1) \\ &= (p+1)[8(N-2)(N-4)(p+1)^3 - 16(N^2 - 10N + 20)(p+1)^2 \\ &\quad -128(N-4)(p+1) - 256]. \end{aligned}$$

For $N = 4$,

$$N_3 - \frac{(N_2 - N_1^2)}{16} = 64(p+1)^2 - 256 > 0 \ \text{ for } p > 1.$$

For $N \geq 5$ and $p > 1$,

$$\begin{aligned}
N_3 - \frac{(N_2 - N_1^2)}{16} &> (p+1)\{[16(N-2)(N-4) - 16(N^2 - 10N + 20)](p+1)^2 \\
&\quad -128(N-4)(p+1) - 256\} \\
&= (p+1)[(64N - 192)(p+1)^2 - 128(N-4)(p+1) - 256] \\
&= 64(p+1)[(N-3)(p+1)^2 - 2(N-4)(p+1) - 4] > 0
\end{aligned}$$

note that the equation

$$(N-3)x^2 - 2(N-4)x - 4 = 0$$

has two roots:

$$x_1 = 2, \quad x_2 = -\frac{2}{N-3}.$$

In particular, (3.8) implies that $N_3 > 0$. We see from (3.7) that

$$N_2 > N_1^2 \quad \text{for } p > \tfrac{2}{N-2}[(N-4) + \sqrt{N^2 - 4N + 8}] - 1$$

$$N_2 \leq N_1^2 \quad \text{for } p \in (1, \tfrac{2}{N-2}[(N-4) + \sqrt{N^2 - 4N + 8}] - 1].$$

For $N_1^2 \geq N_2$, we see from (3.8) that $\sqrt{N_2 + 4\sqrt{N_3}} > N_1$. For $N_2 > N_1^2$, we see from (3.8) that $4\sqrt{N_3} > N_2 - N_1^2$. This implies $4\sqrt{N_3} + N_2 > N_2 > N_1^2$ and thus $\sqrt{N_2 + 4\sqrt{N_3}} > N_1$. These implies that (i) of Proposition 3.1 holds for $p > 1$.

We now discuss the properties of ν_3 and ν_4. We introduce the function

$$\begin{aligned}
N_4(N,p) : &= 16N_3 - N_2^2 = -(N-4)(N^3 - 4N^2 - 128N + 256)(p+1)^4 \\
&\quad -128(3N-8)(N-6)(p+1)^3 \\
&\quad +256(N^2 - 18N + 52)(p+1)^2 + 2048(N-6)(p+1) + 4096.
\end{aligned}$$

For $N = 4$, we see that

$$\begin{aligned}
N_4(4,p) &= 1024(p+1)^3 + 1024(p+1)^2 - 4096(p+1) + 4096 \\
&= 1024(p+1)^2 p - 4096p \\
&> 4096p - 4096p = 0.
\end{aligned}$$

This implies that (ii) of Proposition 3.1 holds.

For any $5 \leq N \leq 12$, we see that the first coefficient of $N_4(N,p)$ is positive. Moreover,

$$\begin{aligned}
\frac{\partial^2 N_4}{\partial p^2} &= -12(N-4)(N^3 - 4N^2 - 128N + 256)(p+1)^2 \\
&\quad -768(3N-8)(N-6)(p+1) + 512(N^2 - 18N + 52)
\end{aligned}$$

and simple calculations imply that $\frac{\partial^2 N_4}{\partial p^2} > 0$ for $p > 1$ and $N_4(N,1) < 0$ for any fixed $N \in [5,12]$. Thus, $N_4(N,p)$ is a convex function of p for any fixed $N \in [5,12]$. Therefore there is a unique $p_c \in (1,\infty)$ such that for each fixed $N \in [5,12]$,

$$N_4(N,p) < 0 \quad \text{for } 1 < p < p_c, \qquad N_4(N,p) \geq 0 \quad \text{for } p \geq p_c.$$

This implies that (iii) of Proposition 3.1 holds.

For $N \geq 13$, we see that the first coefficient of $N_4(N,p)$ is negative. Moreover, for $p > 1$,

$$\begin{aligned}
\frac{\partial^2 N_4}{\partial p^2} &= -12(N-4)(N^3-4N^2-128N+256)(p+1)^2 \\
&\quad -768(3N-8)(N-6)(p+1) + 512(N^2-18N+52) \\
&< -1536(3N^2-26N+48) + 512(N^2-18N+52) \\
&= -4096N^2 + 30720N - 47104 \\
&< -53248N + 30720N - 47104 \\
&= -22528N - 47104 < 0.
\end{aligned}$$

We also know that $N_4(N,1) < 0$ for $N \geq 13$. On the other hand, we see that for $p > 1$ and $N \geq 13$,

$$\begin{aligned}
\frac{\partial N_4}{\partial p} &= -4(N-4)(N^3-4N^2-128N+256)(p+1)^3 - 384(3N-8)(N-6)(p+1)^2 \\
&\quad +512(N^2-18N+52)(p+1) + 2046(N-6) \\
&< -768(3N-8)(N-6)(p+1) + 512(N^2-18N+52)(p+1) + 2046(N-6) \\
&= [-1792N^2 + 10752N - 10240](p+1) + 2046(N-6) \\
&< -(12544N + 10240)(p+1) + 2046(N-6) < 0.
\end{aligned}$$

Therefore, $N_4(N,p) < 0$ for $N \geq 13$ and $p > 1$. This implies that (iv) of Proposition 3.1 holds and completes the proof of this proposition. $\square$

Proposition 3.1 implies that P has three dimensional stable manifold and a one dimensional unstable manifold for $N \geq 4$ and $p > 1$.

Let u be the unique entire solution of (1.1) with $\Delta u(r) \to 0$ as $r \to +\infty$. Let v be defined according to (3.1) so that it solves (3.2), and $\mathbf{w}(t) = (w_1(t), w_2(t), w_3(t), w_4(t))$ be the vector solution of the corresponding first order system (3.3). Then we see from $\Delta u(r) \to 0$ as $r \to +\infty$ that

(3.9)
$$e^{-\frac{2(p-1)}{p+1}t}\Big[v''(t) + \Big(N-2+\frac{8}{p+1}\Big)v'(t) + \frac{4}{p+1}\Big(N-2+\frac{4}{p+1}\Big)v(t)\Big] \to 0 \quad \text{as } t \to \infty.$$

Proposition 3.2. *We have*

$$\lim_{t\to\infty} \mathbf{w}(t) = P.$$

In particular, the trajectory $\mathbf{w}$ *is on the stable manifold of* P.

To prove this proposition, we first prove some useful lemmas.

Lemma 3.3. *Let* v *be the global solution and assume* $L \in [0,\infty]$ *such that*

$$\lim_{t\to\infty} v(t) = L.$$

Then $L = (-K_0)^{-1/(p+1)}$.

Proof. We first exclude the case $L = +\infty$. By (3.9), we see that

$$v''(t)+\Big(N-2+\frac{8}{p+1}\Big)v'(t)+\frac{4}{p+1}\Big(N-2+\frac{4}{p+1}\Big)v(t) := g(t) = o(e^{\frac{2(p-1)}{p+1}t}) \ \text{ as } t\to\infty.$$

The standard ODE theory implies that

$$\begin{aligned} v(t) &= B_1 e^{-\frac{4}{p+1}t} + B_2 e^{-(N-2+\frac{4}{p+1})t} + \int_T^t \Big(e^{-(N-2+\frac{4}{p+1})(t-s)} - e^{-\frac{4}{p+1}(t-s)}\Big) g(s)ds \\ &\le B_3 e^{-\frac{4}{p+1}t} + B_4 e^{-\frac{4}{p+1}t}\int_T^t e^{\frac{4}{p+1}s} g(s)ds \\ &= o(e^{\frac{2(p-1)}{p+1}t}) \ \text{ as } t\to\infty \end{aligned}$$

where $T > 0$ is sufficiently large. On the other hand, since $v(t) \to +\infty$ as $t \to +\infty$, we see from (3.2) that

$$v^{(4)}(t) + K_3 v'''(t) + K_2 v''(t) + K_1 v'(t) + K_0 v(t) = o(1), \ \text{ as } t\to\infty. \tag{3.10}$$

The corresponding characteristic equation is

$$\rho^4 + K_3\rho^3 + K_2\rho^2 + K_1\rho + K_0 = 0$$

which has only one positive root $\rho = \frac{2(p-1)}{p+1}$. Therefore,

$$e^{-\frac{2(p-1)}{p+1}t} v(t) \to c, \quad c>0 \ \text{ as } t\to\infty.$$

This contradicts the fact that $v(t) = o(e^{\frac{2(p-1)}{p+1}t})$ obtained above.

If $L \ne (-K_0)^{-1/(p+1)}$, then $-v^{-p}(t) - K_0 v(t) \to \alpha \ne 0$ as $t\to\infty$ and for $\epsilon > 0$ sufficiently small there exists $T > 0$ such that

$$\alpha - \epsilon \le v^{(4)}(t) + K_3 v'''(t) + K_2 v''(t) + K_1 v'(t) \le \alpha + \epsilon, \quad \forall t \ge T. \tag{3.11}$$

Take $\epsilon < |\alpha|$ so that $\alpha-\epsilon$ and $\alpha+\epsilon$ have the same sign and let

$$\delta := \sup_{t\ge T} |v(t) - v(T)| < \infty.$$

Integrating (3.11) over $[T,t]$ for any $t \geq T$ yields

$$\begin{aligned} (\alpha-\epsilon)(t-T)+C-|K_1|\delta &\leq v'''(t)+K_3v''(t)+K_2v'(t) \\ &\leq (\alpha+\epsilon)(t-T)+C+|K_1|\delta, \quad \forall t \geq T, \end{aligned}$$

where $C = C(T)$ is a constant containing all the terms $v(T)$, $v'(T)$, $v''(T)$ and $v'''(T)$. Repeating twice more this procedure gives

$$\frac{\alpha-\epsilon}{6}(t-T)^3+O(t^2) \leq v'(t) \leq \frac{\alpha+\epsilon}{6}(t-T)^3+O(t^2) \text{ as } t \to \infty.$$

This contradicts the assumption that v admits a finite limit as $t \to \infty$. This completes the proof. □

If v is eventually monotonous, then Lemma 3.3 implies that (1.9) holds. So, we need to consider the case that v oscillates infinitely many times near $t = \infty$, i.e. v has an unbounded sequence of consecutive local maxima and minima. In the sequel we always restrict to this kind of solutions without explicit mention.

We define the energy function

$$E(t) = \frac{v^{1-p}}{p-1} - \frac{K_0}{2}v^2(t) - \frac{K_2}{2}(v'(t))^2 + \frac{1}{2}(v''(t))^2. \tag{3.12}$$

We prove first that on consecutive extrema of v, the energy is decreasing. For the proof of the following lemma, the sign of the coefficients K_1, K_3 in front of the odd order derivatives in equation (3.2) is absolutely crucial.

Lemma 3.4. *Assume that $p > p_m$ and $t_0 < t_1$ and that $v'(t_0) = v'(t_1) = 0$. Then*

$$E(t_0) \geq E(t_1).$$

If v is not constant, then the identity is strict.

Proof. From the equation (3.2) we find:

$$\begin{aligned} E'(t) &= -v^{-p}(t)v'(t) - K_0v(t)v'(t) - K_2v'(t)v''(t) + v''v''' \\ &= (-v^{-p} - K_0v - K_2v'')v' + v''v''' \\ &= (v^{(4)}(t) + K_3v''' + K_1v')v'(t) + v''v'''. \end{aligned}$$

Integrating by parts, this yields:

$$\begin{aligned}
E(t_1) - E(t_0) &= \int_{t_0}^{t_1} E'(s)ds = -\int_{t_0}^{t_1} v'''(s)v''(s)ds - K_3 \int_{t_0}^{t_1} |v''(s)|^2 ds \\
&\quad + K_1 \int_{t_0}^{t_1} |v'(s)|^2 ds + \int_{t_0}^{t_1} v'''(s)v''(s)ds \\
&= -K_3 \int_{t_0}^{t_1} |v''(s)|^2 ds + K_1 \int_{t_0}^{t_1} |v'(s)|^2 ds \leq 0
\end{aligned}$$

since $K_3 > 0$ and $K_1 < 0$. If v is not a constant, the inequality is strict. □

Lemma 3.5. *Assume $p > p_m$. There are $0 < \theta_1 < \theta_2$ such that*

$$\theta_1 \leq v(t) \leq \theta_2 \quad \textit{for } t \textit{ sufficiently large.} \tag{3.13}$$

Proof. Let $\{t_k\}_{k \in \mathbb{N}}$ denote the sequence of consecutive positive critical points of v, we see that there are θ_1, $\theta_2 > 0$ such that $\theta_1 \leq v(t_k) \leq \theta_2$ for all k. On the contrary, we can find a subsequence (still denoted by $\{t_k\}$) such that $v(t_k) \to 0$ or $v(t_k) \to \infty$ as $k \to \infty$. We only consider the first case, the second case is similar. By Lemma 3.4, we see that

$$E(t_1) \geq E(t_k) \quad \text{for any large } k. \tag{3.14}$$

Since $v(t_k) \to 0$ as $t \to \infty$, we easily see that $E(t_k) \to \infty$ as $k \to \infty$, this contradicts (3.14). This completes the proof. □

Lemma 3.6. *Assume $p > p_m$. For $T > 0$ sufficiently large,*

$$\int_T^\infty |v'(s)|^2 ds + \int_T^\infty |v''(s)|^2 ds < \infty.$$

Proof. We take the same sequence $\{t_k\}_{k \in \mathbb{N}}$ as in the proof of Lemma 3.5. We assume that $T > t_1$. Then for any k:

$$-K_3 \int_{t_1}^{t_k} |v''(s)|^2 ds + K_1 \int_{t_1}^{t_k} |v'(s)|^2 ds = E(t_k) - E(t_1) \geq -E(t_1) > -\infty.$$

The statement follows by letting $k \to \infty$ and using again that $K_3 > 0$ and $K_1 < 0$. □

Lemma 3.7.

$$\int_T^\infty |v'''(s)|^2 ds < \infty.$$

Proof. Since $u_{\gamma^*}(r) > 0$ for $r \in (0,\infty)$, we see that $v'(t) + \frac{4}{p+1}v(t) > 0$ for $t \in (-\infty,\infty)$ and thus

$$-v'(t) < \frac{4}{p+1}v(t) \quad \text{for } t \in (-\infty,\infty). \tag{3.15}$$

We choose $\{t_k\}_{k\in\mathbb{N}}$ as in the previous lemmas. Now we can choose another monotonicity increasing diverging sequence $\{\tau_k\}_{k\in\mathbb{N}}$ of flex points of v such that v is decreasing there. We choose

$$\tau_k > T, \quad \tau_k \nearrow \infty$$

$$v'(\tau_k) \le 0, \quad v''(\tau_k) = 0.$$

It follows from (3.15) and Lemma 3.5 that $-v'(\tau_k) < \frac{4}{p+1}v(\tau_k)$ and thus $|v'(\tau_k)| \le 2\theta_2$ for all k. We multiply the equation (3.2) by v'' and integrate over (T,τ_k) :

$$\int_T^{\tau_k} \Big(v^{(4)}(s)+K_3v'''(s)+K_2v''(s)+K_1v'(s)+K_0v(s)\Big)v''(s)ds = -\int_T^{\tau_k} v^{-p}(s)v''(s)ds. \tag{3.16}$$

We show that all the lower order terms remain bounded, when $k\to\infty$. We see that

$$\Big|\int_T^{\tau_k} v^{-p}(s)v''(s)ds\Big| = \Big|[v^{-p}v']_T^{\tau_k} - p\int_T^{\tau_k} v^{-(p+1)}(s)|v'(s)|^2ds\Big| \le C \tag{3.17}$$

by Lemmas 3.5 and 3.6. With the same argument, one also obtains

$$\Big|\int_T^{\tau_k} v(s)v''(s)ds\Big| \le C. \tag{3.18}$$

The Hölder inequality and Lemma 3.6 imply

$$\Big|\int_T^{\tau_k} v'(s)v''(s)ds\Big| \le C. \tag{3.19}$$

By our choice of τ_k (recall that $v''(\tau_k)=0$), we obtain

$$\Big|\int_T^{\tau_k} v'''(s)v''(s)ds\Big| = \frac{1}{2}|v''(T)|^2 \le C. \tag{3.20}$$

Finally, integrating by parts, we find from (3.16)-(3.20) that

$$\int_T^{\tau_k} (v'''(s))^2ds \le \Big|\int_T^{\tau_k} v^{(4)}(s)v''(s)ds\Big| + |v'''(T)v''(T)| \le C. \tag{3.21}$$

Letting $k\to\infty$, we obtain our conclusion. □

Lemma 3.8.

$$\int_T^{\infty} |v^{(4)}(s)|^2ds < \infty.$$

Proof. In view of Lemmas 3.5-3.7 we may find a sequence $\{s_k\}$ such that

$$\lim_{k\to\infty} s_k = \infty, \quad v(s_k) = O(1), \quad \lim_{k\to\infty} v'(s_k) = \lim_{k\to\infty} v''(s_k) = \lim_{k\to\infty} v'''(s_k) = 0.$$

We multiply the equation (3.2) by $v^{(4)}$ and integrate over $[T, s_k)$:

(3.22)

$$\int_T^{s_k} (v^{(4)}(s))^2 ds = \int_T^{s_k} (-v^{-p}(s) - K_0 v(s) - K_1 v'(s) - K_2 v''(s) - K_3 v'''(s)) v^{(4)}(s) ds.$$

By using Lemmas 3.5-3.7 and arguing as in the previous proofs we obtain

$$\int_T^{s_k} v^{(4)}(s) v'''(s) ds = \left[\frac{1}{2}|v'''(s)|^2\right]_T^{s_k} = O(1);$$

$$\int_T^{s_k} v^{(4)}(s) v''(s) ds = O(1) - \int_T^{s_k} |v'''(s)|^2 ds = O(1);$$

$$\int_T^{s_k} v^{(4)}(s) v'(s) ds = O(1) - \int_T^{s_k} v'''(s) v''(s) ds = O(1);$$

$$\int_T^{s_k} v^{(4)}(s) v(s) ds = O(1) - \int_T^{s_k} v'''(s) v'(s) ds = O(1) + \int_T^{s_k} |v''(s)|^2 ds = O(1);$$

$$\begin{aligned} \int_T^{s_k} v^{(4)}(s) v^{-p}(s) ds &= O(1) + p \int_T^{s_k} v^{-(p+1)} v'''(s) v'(s) ds \\ &\le O(1) + C\Big(\int_T^{s_k} |v'''(s)|^2 ds\Big)^{1/2} \Big(\int_T^{s_k} |v'(s)|^2 ds\Big)^{1/2} \\ &\le O(1). \end{aligned}$$

Inserting all these estimates into (3.22), then claim follows. □

Lemma 3.9.

$$\int_T^{\infty} v^2(s) (v^{-(p+1)}(s) + K_0)^2 ds < \infty.$$

Proof. From the equation (3.2) we conclude

$$(v^{(4)}(s) + K_3 v'''(s) + K_2 v''(s) + K_1 v'(s))^2 = v^2(s) (v^{-(p+1)}(s) + K_0)^2.$$

The statement follows now immediately from Lemmas 3.5-3.8. □

The proof of Proposition 3.2 and Theorem 1.2 will be completed by showing:

Lemma 3.10. *Let $w = (w_1, w_2, w_3, w_4)$ be as in Proposition 3.2. We assume further that $v = w_1$ has an unbounded sequence of consecutive local maxima and minima near $t = \infty$. Then it follows that*

(3.23) $$\lim_{t\to\infty} \mathbf{w}(t) = P.$$

In particular, $\lim_{t\to\infty} v(t) = (-K_0)^{-1/(p+1)}$.

Proof. We first show that the limit of $v'(t)$ as $t \to \infty$ exists. Define $h(t) := \int_T^t v'(\xi)v''(\xi)d\xi$ for $t > T$. We easily see that the limit of $h(t)$ as $t \to \infty$ exists. Indeed, for any large t_1, t_2 with $T < t_1 < t_2$, we see from Lemma 3.6 that

$$|h(t_2) - h(t_1)| \le \Big(\int_{t_1}^{t_2} (v'(\xi))^2 d\xi\Big)^{1/2} \Big(\int_{t_1}^{t_2} |v''(\xi)|^2\Big)^{1/2} \to 0 \text{ as } t_1, t_2 \to \infty.$$

Thus, $\lim_{t\to\infty} h(t)$ exists and this implies $\lim_{t\to\infty} v'(t)$ exists. Lemma 3.6 implies that $\lim_{t\to\infty} v'(t) = 0$. Thus, we can obtain that $\lim_{t\to\infty} v''(t) = 0$, $\lim_{t\to\infty} v'''(t) = 0$ and $\lim_{t\to\infty} v^{(4)}(t) = 0$. It is easily seen from the equation (3.2) that

$$\lim_{t\to\infty} (v^{-p}(t) + K_0 v(t)) = 0.$$

This implies that

$$\lim_{t\to\infty} v(t) = (-K_0)^{-1/(p+1)}.$$

This completes the proof. $\square$

4. Properties of entire solutions: proof of Theorem 1.3

Let us define

$$\tilde{\nu}_j = \nu_j + \frac{4}{p+1}, \quad j = 1, 2, 3, 4. \tag{4.1}$$

A direct computation shows that $r^{\tilde{\nu}_j}$ are the four fundamental solutions to

$$\Delta^2 \psi = p U_0^{-(p+1)} \psi. \tag{4.2}$$

Using Proposition 3.1 and direct calculations, we have the following proposition:

Proposition 4.1. *(i) For any $N \ge 4$ and $p > 1$, we have*

$$\tilde{\nu}_2 > 0 > 2 - N > \tilde{\nu}_1. \tag{4.3}$$

(ii) For any $N = 4$ or $5 \le N \le 12$ and

$$p > p_c, \tag{4.4}$$

we have $\tilde{\nu}_3$, $\tilde{\nu}_4 \notin \mathbb{R}$ and $\Re(\tilde{\nu}_3) = \Re(\tilde{\nu}_4) = \frac{4-N}{2} \le 0$.

(iii) For any $N \ge 13$ or $5 \le N \le 12$ and $p < p_c$, we have

$$4 - N < \tilde{\nu}_3 < \frac{4-N}{2} < \tilde{\nu}_4 < 0, \quad \tilde{\nu}_3 + \tilde{\nu}_4 = 4 - N. \tag{4.5}$$

(iv) For any $5 \le N \le 12$ and $p = p_c$, we have $\tilde{\nu}_3 = \tilde{\nu}_4 = \frac{4-N}{2}$.

We need to consider two cases:

Case 1: $N = 4$ or $5 \le N \le 12$ and $p > p_c$.

Case 2: $N \ge 13$ or $5 \le N \le 12$ and $1 < p \le p_c$.

For the first case, we prove that $u(r) - U_0(r)$ must have infinitely many intersections. This amounts to the study of the following linearized equation

$$\Delta^2 \phi = p u^{-(p+1)} \phi, \quad \phi(r) \to 0 \text{ as } r \to +\infty. \tag{4.6}$$

First we have the following lemma.

Lemma 4.2. *(1) If $\phi(0) = 0$, then $\phi \equiv 0$,*

(2) If $\phi(0) = 1$, then $\Delta\phi(0) < 0$.

Proof. (1) Suppose $\phi(0) = 0$ and $\Delta\phi(0) \neq 0$. We may assume that $\Delta\phi(0) > 0$. Since $\phi(0) = \phi'(0) = 0$, we may assume that $\phi(r) > 0$ for $r \in (0, R)$ and $\phi(R) = 0$. (R can be $+\infty$.) Then in $(0, R)$, $(\Delta\phi)'(r) > 0$ and hence $\Delta\phi(r) > 0$ for $r \in (0, R)$. This implies that $\phi'(r) > 0$ and $\phi(r) > 0$ for $r \in (0, R]$. This contradicts with $\phi(R) = 0$.

(2) follows from the same arguments. $\square$

As a consequence of (1) of Lemma 4.2, we have

Lemma 4.3. *The solution to (4.6) is given by*

$$\phi(r) = C\Big(\frac{4}{p+1} u(r) - r u'(r)\Big) \tag{4.7}$$

for some $C \neq 0$.

The following theorem gives the asymptotic behavior of u for the first case.

Theorem 4.4. *For r sufficiently large,*

$$u(r) = (-K_0)^{-1/(p+1)} r^{4/(p+1)} + M_1 r^\alpha \cos(\beta \ln r) + M_2 r^\alpha \sin(\beta \ln r) + O(r^{\alpha-\delta}) \tag{4.8}$$

where $\delta = \frac{\sqrt{N_2 + 4\sqrt{N_3}}}{2(p+1)} > 0$, $\nu_4 = \alpha - \frac{4}{p+1} + i\beta$ and $M_1^2 + M_2^2 \neq 0$. (Note that it is known from Proposition 4.1 that $\alpha = (4-N)/2 \le 0$.)

Proof. Using the Emden-Fowler transformation (3.1) and letting $v(t) = (-K_0)^{-1/(p+1)} + h(t)$, we see that $h(t)$ satisfies

$$h^{(4)}(t) + K_3 h'''(t) + K_2 h''(t) + K_1 h'(t) + (1+p) K_0 h(t) + O(h^2) = 0, \quad t > 1. \tag{4.9}$$

Therefore in the leading order, we can write

$$h(t) = M_1 e^{(\alpha - \frac{4}{p+1})t} \cos\beta t + M_2 e^{(\alpha - \frac{4}{p+1})t} \sin\beta t + M_3 e^{\nu_1 t} + o(e^{\nu_1 t}) \tag{4.10}$$

(note that we have from Theorem 1.2 that $\lim_{t\to\infty} h(t) = 0$). This then implies that as $r \to +\infty$,

$$\varphi(r) = M_1 r^\alpha \cos(\beta \ln r) + M_2 r^\alpha \sin(\beta \ln r) + M_3 r^{\tilde{\nu}_1} + o(r^{\tilde{\nu}_1}) \tag{4.11}$$

where $\varphi(r) = r^{\frac{4}{p+1}} h(t) := u(r) - U_0(r)$.

We now show that $M_1^2 + M_2^2 \neq 0$.

Suppose now that $M_1 = M_2 = 0$. Then we have

$$\varphi \sim r^{2-N-\kappa} \ \text{ as } r \to +\infty \tag{4.12}$$

where $\kappa = -\tilde{\nu}_1 - (N-2) > 0$ by Proposition 4.1. Furthermore, $\varphi(r)$ has no zeroes for r large. We show that this is impossible. In fact, it is easy to see that φ must change sign in $(0, +\infty)$. Otherwise, we assume $\varphi(r) > 0$ for $r \geq 0$ (note that $u(r) > U_0(r)$ for r small). Then using the behavior of φ near ∞ and integrating the equation $\Delta^2 \varphi = U_0^{-p} - u^{-p}$ over $\mathbb{R}^N$, we see that

$$\int_0^\infty (U_0^{-p} - u^{-p}) r^{N-1} dr = 0$$

which contradicts with $\varphi = u - U_0 > 0$.

Suppose $\varphi(r)$ has exactly k zeroes in $(0, +\infty)$ (recalling that φ has no zeroes when r is large) and $\varphi(r) \sim r^{2-N-\kappa}$ as $r \to +\infty$, we easily see that $r^{N-1}\varphi'(r)$ has k zeroes. On the other hand, since the function $\eta(r) := r^{N-1}\varphi'(r)$ satisfies $\eta(0) = 0$ and $\eta(r) \to 0$ as $r \to +\infty$, we see that $\eta'(r)$ has $k+1$ zeroes. Thus $\Delta\varphi(r) = \frac{1}{r^{N-1}}\eta'(r)$ has at least $k+1$ zeroes. Similar idea implies that $r^{N-1}(\Delta\varphi)'(r)$ has at least k zeroes and $(r^{N-1}(\Delta\varphi)'(r))'$ has at least $k+1$ zeroes. Therefore, $\Delta^2\varphi = \frac{1}{r^{N-1}}(r^{N-1}(\Delta\varphi)'(r))'$ has at least $k+1$ zeroes. This contradicts our assumption that φ has k zeroes, since $\Delta^2\varphi = \frac{U_0^{-p} - u^{-p}}{u - U_0}\varphi$ and $\frac{U_0^{-p}(r) - u^{-p}(r)}{u(r) - U_0(r)} > 0$ for all $r > 0$. This proves our claim and completes the proof of the second part of Theorem 1.3. □

To prove the first part of Theorem 1.3, we need the following two theorems. (We need to consider Case 2.)

Theorem 4.5. *We have $u(r) > U_0(r)$ for $r > 0$.*

Theorem 4.6. *The solution of (4.6) remains constant sign.*

The proofs of both theorems depend on the use of comparison principle for fourth order equations.

We prove Theorem 4.6 first.

Proof of Theorem 4.6: Assume Theorem 4.5 holds, i.e. $u(r) > U_0(r)$. Let $\phi(r)$ be a solution of (4.6). By Lemma 4.2, we may assume that $\phi(0) = 1$, $\Delta\phi(0) < 0$. Let $\psi(r) = r^{\tilde{\nu}_3}$. Then it is easy to see that

$$\Delta^2 \psi = p U_0^{-(p+1)} \psi. \tag{4.13}$$

By Proposition 4.1, we have $\tilde{\nu}_3 > 4 - N$. This implies that $\int_{B_r(0)} r^{-4}|\phi|\psi < +\infty$. So we can multiply (4.6) by ψ and (4.13) by ϕ and integrate over $B_r(0)$ to obtain

(4.14)
$$0 = \int_{B_r(0)} p(U_0^{-(p+1)} - u^{-(p+1)})\phi\psi + \int_{\partial B_r(0)} [(\Delta\phi)'\psi - \Delta\phi\psi'] + \int_{\partial B_r(0)} [\Delta\psi\phi' - (\Delta\psi)'\phi]$$
$$= I_1(r) + I_2(r) + I_3(r)$$

where $I_i(r)$ are defined at the last equality.

Let us assume that there exist $r_1, r_2 \in (0, +\infty]$ such that

$$\phi(r) > 0,\ r \in (0, r_1),\ \phi(r_1) = 0, \quad \Delta\phi(r) < 0, \quad r \in (0, r_2), \quad \Delta\phi(r_2) = 0. \tag{4.15}$$

We divide our proof into three cases:

Case 1: $r_1 = r_2$.

In this case, we take $r = r_1 = r_2$. Then we have $I_1(r) > 0$, $I_2(r) \geq 0$, $I_3(r) \geq 0$. The identity (4.14) gives a contradiction.

Case 2: $r_2 < r_1$.

In this case, we take $r = r_2$. Then it is easy to see that $I_1(r_2) \geq 0$, $I_2(r_2) = \int_{\partial B_{r_2}(0)} [(\Delta\phi)'\psi - \Delta\phi\psi'] \geq 0$. It remains to estimate $I_3(r_2)$.

To this end, we first show that $\Delta\phi > 0$ for $r \in (r_2, r_1)$. In fact, since $\Delta^2\phi = pu^{-(p+1)}\phi > 0$ in $(0, r_1)$, we see that $\Delta\phi$ must be positive for $r > r_2$ and near r_2. Suppose that there exists $r_3 \leq r_1$ such that $\Delta\phi(r_3) = 0$. Then we have $\Delta\phi > 0$, $\Delta(\Delta\phi) > 0$ in (r_2, r_3). This is impossible (since $\Delta\phi$ must attain its maximum in (r_2, r_3) where $\Delta(\Delta\phi) \leq 0$).

Now we consider the function $\Phi(r) = r^{N-1}(\Delta\psi\phi' - (\Delta\psi)'\phi)$. Its derivative is given by

$$\begin{aligned} \Phi'(r) &= (r^{N-1}\phi'(r))'\Delta\psi(r) - (r^{N-1}(\Delta\psi)'(r))'\phi(r) \\ &= r^{1-N}[\Delta\phi(r)\Delta\psi(r) - \phi(r)\Delta^2\psi(r)] < 0 \quad \text{for } r \in (r_2, r_1). \end{aligned}$$

(Here we have used the fact that $\Delta\psi < 0$.) So $\Phi(r_2) < \Phi(r_1) = r_1^{N-1}\Delta\psi(r_1)\phi'(r_1) \geq 0$. As a consequence, we have proved that $I_3(r_2) = r_2^{1-N}\int_{\partial B_{r_2}(0)} \Phi(r_2) \geq 0$. So again, we have $I_1(r_2) > 0$, $I_2(r_2) \geq 0$, $I_3(r_2) \geq 0$ and a contradiction to the identity (4.14).

Case 3: $r_1 < r_2$.

The proof is similar to Case 2. In this case, we take $r = r_1$. Then it is easy to see that $I_1(r_1) \geq 0$, $I_3(r_1) = \int_{\partial B_{r_1}(0)} [\Delta\psi\phi'] \geq 0$. It remains to estimate $I_2(r_1)$.

As before, we first show that $\phi(r) < 0$ for $r \in (r_1, r_2)$. In fact, since $\Delta\phi < 0$ in $(0, r_2)$, we see that ϕ must be negative for $r > r_1$ and near r_1. Suppose that there exists $r_3 \leq r_2$ such that $\phi(r_3) = 0$. Then we have $\Delta\phi < 0$, $\phi < 0$ in (r_1, r_3). This is impossible (since ϕ must attain its minimum in (r_3, r_2) where $\Delta\phi \geq 0$).

Now we consider the function $\Psi(r) = r^{N-1}((\Delta\phi)'\psi - \Delta\phi\psi')$. Its derivative is given by

$$\begin{aligned} \Psi'(r) &= (r^{N-1}(\Delta\phi)'(r))'\psi(r) - (r^{N-1}\psi'(r))'\Delta\phi(r) \\ &= r^{1-N}[\Delta^2\phi(r)\psi(r) - \Delta\phi(r)\Delta\psi(r)] < 0 \ \text{ for } r \in (r_1, r_2). \end{aligned}$$

So $\Psi(r_1) > \Psi(r_2) = r_2^{N-1}(\Delta\phi)'(r_2)\psi(r_2) \geq 0$. As a consequence, we have proved that $I_2(r_1) = r_1^{1-N}\int_{\partial B_{r_1}(0)} \Psi(r_1) \geq 0$. So again, we have $I_1(r_1) > 0$, $I_2(r_1) \geq 0$, $I_3(r_1) \geq 0$ and a contradiction to the identity (4.14). These contradictions imply that ϕ remains constant sign and this completes the proof. $\square$

Proof of Theorem 4.5: The proof is similar to that of Theorem 4.6. Let $\phi_0 = u(r) - U_0(r)$. Then it is easy to see that ϕ_0 satisfies

$$\Delta^2\phi_0 = U_0^{-p} - (U_0 + \phi_0)^{-p} \leq pU_0^{-(p+1)}\phi_0, \quad r > 0. \tag{4.16}$$

Now let $\psi_0 = r^{\tilde{\nu}_4}$. Then by Proposition 4.1, $\tilde{\nu}_4 \geq (4-N)/2$. Thus $\int_{B_R(0)} r^{-4}|\phi_0|\psi_0 \leq C\int_{B_R(0)} r^{-4}r^{(4-N)/2} < \infty$ since $N \geq 9$. Thus the integral $U_0^{-(p+1)}\phi_0\psi_0$ is integrable. Similar to (4.14), we have the following identity

$$\int_{\partial B_r(0)} [(\Delta\phi_0)'\psi_0 - \Delta\phi_0\psi_0'] + \int_{\partial B_r(0)} [\Delta\psi_0\phi_0' - (\Delta\psi_0)'\phi_0] \leq 0. \tag{4.17}$$

Now note that $\phi_0 > 0$, $\Delta\phi_0 < 0$ for r small. So we may assume (4.15). The case $r_1 = r_2$ is easy to exclude. We just need to show the case $r_2 < r_1$. To this end, we first show that $\Delta\phi_0 > 0$ for $r \in (r_2, r_1)$. In fact, since $\Delta^2\phi_0 = U_0^{-p} - (U_0 + \phi_0)^{-p} > 0$ in $(0, r_1)$, we see that $\Delta\phi_0$ must be positive for $r > r_2$ and near r_2. Suppose that there exists $r_3 \leq r_1$ such that $\Delta\phi_0(r_3) = 0$. Then we have $\Delta\phi_0 > 0$, $\Delta(\Delta\phi_0) > 0$ in (r_2, r_3). This is impossible (since $\Delta\phi_0$ must attain its maximum in (r_2, r_3) where $\Delta(\Delta\phi_0) \leq 0$). The rest of the proof is exactly the same as before. We omit the details. $\square$

Theorem 4.6 yields very important estimates on the asymptotic behavior of u.

Corollary 4.7. *(1) Under the assumptions of Case 2. Then the set of solutions $\{u_a(r)\}$ to (1.1) is well ordered. That is if $a > b$ then $u_a(r) > u_b(r)$ for all $r > 0$.*

(2) If $N \geq 13$ or $9 \leq N \leq 12$ and $p_m < p < p_c$, then we have the following asymptotic expansion for u:

$$u(r) = (-K_0)^{-1/(p+1)} r^{4/(p+1)} + M_1 r^{\tilde{\nu}_4} + O(r^{\max(2\tilde{\nu}_4, \tilde{\nu}_3)}) \tag{4.18}$$

where $M_1 \neq 0$. If $9 \leq N \leq 12$ and $p = p_c$, then we have the following asymptotic expansion for u:

$$u(r) = (-K_0)^{-1/(p+1)} r^{4/(p+1)} + (M_1 + M_2 \log r) r^{\frac{4-N}{2}} + O(r^{4-N}). \tag{4.19}$$

Proof of Corollary 4.7: For (1), we note that $\phi = \frac{\partial u_a}{\partial a}$ satisfies (4.6) with $\phi(0) = 1$. By Theorem 4.6, $\phi > 0$. Thus $u_a(r) > u_b(r)$ for $a > b$.

For (2), we have

$$u(r) = (-K_0)^{-1/(p+1)} r^{4/(p+1)} + M_1 r^{\tilde{\nu}_3} + M_2 r^{\tilde{\nu}_4} + O(r^{\max(2\tilde{\nu}_4, \tilde{\nu}_3)}). \tag{4.20}$$

If $M_2 = 0$, then

$$u(r) = (-K_0)^{-1/(p+1)} r^{4/(p+1)} + O(r^{\tilde{\nu}_3}) \tag{4.21}$$

which implies that $\phi = O(r^{\tilde{\nu}_3})$. Now as in the proof of Theorem 4.6, we have

$$\int_0^\infty p(U_0^{-(p+1)} - u^{-(p+1)}) \phi r^{\tilde{\nu}_3} r^{N-1} dr = 0 \tag{4.22}$$

where the integral is finite because $\tilde{\nu}_3 > 4 - N$ and $2\tilde{\nu}_3 < 4 - N$. This is impossible since $\phi > 0$ and $u(r) > U_0(r)$. So $M_2 \neq 0$.

When $p = p_c$, (4.19) follows from the fact that $\tilde{\nu}_3 = \tilde{\nu}_4 = \frac{4-N}{2}$. $\square$

5. Structure of radial solutions of (1.13): proof of Theorem 1.4

In this section we study the structure of radial solutions of (1.13) and prove Theorem 1.4.

Note that (1.13) is reduced to

$$\begin{cases} u^{(4)}(r) + \frac{2(N-1)}{r} u'''(r) + \frac{(N-1)(N-3)}{r^2} u''(r) - \frac{(N-1)(N-3)}{r^3} u'(r) == \frac{\lambda}{(1-u(r))^{-p}} & \text{for } r \in (0,1) \\ 0 \leq u(r) < 1 \\ u(1) = 0, \quad u''(1) + (N-1)u'(1) = 0, \quad u'(0) = u'''(0) = 0 \end{cases} \tag{5.1}$$

where $u = u(r)$ for $r = |x|$. We apply the phase plane analysis as in [12].

Next we introduce the initial value problem

$$\begin{cases} u^{(4)}(r) + \frac{2(N-1)}{r} u'''(r) + \frac{(N-1)(N-3)}{r^2} u''(r) - \frac{(N-1)(N-3)}{r^3} u'(r) == \frac{\lambda}{(1-u(r))^{-p}} & \text{for } r \in (0,1) \\ u(0) = A \in (0,1), \quad u'(0) = u'''(0) = 0. \end{cases} \tag{5.2}$$

Make the changes:

$$v(y) = \frac{1 - u(r)}{1 - A}, \quad y = \lambda^{1/4}(1 - A)^{-(p+1)/4} r.$$

Then (5.2) is reduced to (for $y \in (0, \lambda^{1/4}(1 - A)^{-(p+1)/4})$)

$$(5.3) \quad \begin{cases} v^{(4)}(y) + \frac{2(N-1)}{y} v'''(y) + \frac{(N-1)(N-3)}{y^2} v''(y) - \frac{(N-1)(N-3)}{y^3} v'(y) == -v^{-p}(y) \\ 0 < v \le \frac{1}{1-A} \\ v(0) = 1, \quad v'(0) = v'''(0) = 0. \end{cases}$$

Setting $\theta = \lambda^{1/4}(1 - A)^{-(p+1)/4}$, we see that the solution $v(y)$ of (5.3) depends on θ, we denote it by v_θ. Moreover, $v_\theta(\theta) = \frac{1}{1-A}$, $(\Delta_y v_\theta)(\theta) = 0$. We claim that $v_\theta(y) \to u_{\gamma^*}(y)$ for all $y \in (0, \infty)$ as $\theta \to \infty$. This can be seen from Theorem 1.1. Note that for each θ, there is a unique γ_θ such that $(v_\theta)''(0) = \gamma_\theta$. We easily see that $\gamma_\theta \to \gamma^*$ as $\theta \to \infty$, where γ^* is defined in Theorem 1.1. The standard ODE theory implies that our claim holds.

We apply the Emden-Fowler transformation:

$$z_\tau(t) = y^{-\frac{4}{p+1}} v_\theta(y), \quad t = \ln y$$

where $\tau = \ln \theta$. Then (5.3) changes to

(5.4)

$$\begin{cases} z_\tau^{(4)}(t) + K_3 z_\tau'''(t) + K_2 z_\tau''(t) + K_1 z_\tau'(t) + K_0 z_\tau(t) = -z_\tau^{-p} \ \text{ for } t \in (-\infty, \tau) \\ 0 < z_\tau(t) < \frac{1}{1-A} e^{-\frac{4}{p+1}t} \\ \lim_{t\to-\infty} e^{\frac{4}{p+1}t} z_\tau(t) = 1, \ \lim_{t\to-\infty} e^{\frac{4}{p+1}t} z_\tau'(t) = -\frac{4}{p+1}, \ \lim_{t\to-\infty} e^{\frac{4}{p+1}t} z_\tau''(t) = \frac{16}{(p+1)^2}. \end{cases}$$

Through the above transformation, the boundary conditions: $u(1) = \Delta u(1) = 0$ correspond to

$$z_\tau(\tau) = \lambda^{-1/(p+1)}, \quad (z_\tau)''(\tau) + \Big(N - 1 + \frac{7-p}{p+1}\Big) z_\tau'(\tau) + \frac{4}{p+1}\Big(N - 1 + \frac{3-p}{p+1}\Big) z_\tau(\tau) = 0.$$

In other words for any $\tau \in \mathbb{R}$, (λ_τ, u_τ) defined by

$$(5.5) \quad \begin{cases} u_\tau(r) = 1 - \frac{z_\tau(\tau + \ln r)}{z_\tau(\tau)} r^{\frac{4}{p+1}}, \\ \lambda_\tau = \frac{1}{z_\tau^{p+1}(\tau)}, \\ A_\tau = 1 - \frac{1}{e^{\frac{4}{p+1}\tau} z_\tau(\tau)}, \\ (z_\tau)''(\tau) + \Big(N - 1 + \frac{7-p}{p+1}\Big) z_\tau'(\tau) + \frac{4}{p+1}\Big(N - 1 + \frac{3-p}{p+1}\Big) z_\tau(\tau) = 0 \end{cases}$$

satisfies (5.1), and conversely, every solution of (5.1) is written in the form of (5.5). Hence $\mathcal{C}_r$ is homeomorphic to $\mathbb{R}$. Since $v_\theta(y) \to V(y)$ as $\theta \to \infty$, where $V(y)$ is the solution of the equation

$$\begin{cases} V^{(4)}(y) + \frac{2(N-1)}{y} V'''(y) + \frac{(N-1)(N-3)}{y^2} V''(y) - \frac{(N-1)(N-3)}{y^3} V'(y) = -V^{-p}(y) \ \ y \in (0, \infty) \\ V(0) = 1, \quad V'(0) = V'''(0) = 0. \end{cases}$$

Moreover, we easily see that $z_\tau(t) \to Z(t)$ for all $t \in (-\infty, \infty)$ as $\tau \to \infty$, where $Z(t)$ is the unique solution of the problem

$$\begin{cases} Z^{(4)}(t) + K_3 Z'''(t) + K_2 Z''(t) + K_1 Z'(t) + K_0 Z(t) = -Z^{-p}(t) \text{ for } t \in \mathbb{R} \\ \lim_{t\to-\infty} e^{\frac{4}{p+1}t} Z(t) = 1, \ \lim_{t\to-\infty} e^{\frac{4}{p+1}t} Z'(t) = -\frac{4}{p+1}, \ \lim_{t\to-\infty} e^{\frac{4}{p+1}t} Z''(t) = \frac{16}{(p+1)^2}. \end{cases}$$

The singular point $\mathbf{w} = P$ corresponds to $(\lambda, u) = (\lambda_*, 1 - |x|^{\frac{4}{p+1}})$ since $z_\tau(\tau) \to (-K_0)^{-1/(p+1)}$ as $\tau \to \infty$, where $\lambda_* = -K_0$.

To prove that $\mathcal{C}_r$ bends infinitely many times with respect to λ around λ_*, we only need to show that P is a spiral attractor. Since $\mathbf{w}$ is on the stable manifold of the singular point P, we see that all trajectories of system (3.3) are eventually tangential to the space

$$S := \{s_1\mathbf{x}_1 + s_2\mathbf{x}_2 + b\mathbf{y} : \ \ s_1, s_2, b \in \mathbb{R}\}.$$

Where $\mathbf{x}_1 \pm i\mathbf{x}_2$ denotes eigenvectors of the matrix M_P defined in Section 3 corresponding to the complex eigenvalues ν_3, ν_4. $\mathbf{y}$ denotes the eigenvector of the matrix M_P corresponding to the real negative eigenvalue ν_1. But by Theorem 4.4, we have

(5.6) $v(t) = (K_0)^{-1/(p+1)} + M_1 e^{(\alpha - \frac{4}{p+1})t} \cos\beta t + M_2 e^{(\alpha-\frac{4}{p+1})t} \sin\beta t + M_3 e^{\nu_1 t} + o(e^{\nu_1 t})$

where $M_1^2 + M_2^2 \neq 0$. Thus P is a spiral attractor. This shows that $\mathcal{C}_r$ must bend infinitely many times with respect to λ around λ_*.

Next we show that the secondary bifurcation point of $\mathcal{C}_r$ does not occur, which is the content of the following lemma.

Lemma 5.1. *For any $\kappa \in (0,1)$, there is at most one $\tilde\lambda := \tilde\lambda(\kappa) \in (0, \lambda_c]$ with $(\tilde\lambda, u_{\tilde\lambda}) \in \mathcal{C}_r$ and $u_{\tilde\lambda}(0) = \kappa$.*

Proof. Suppose there are $\lambda_1, \lambda_2 \in (0, \lambda_c]$ with $\lambda_1 \neq \lambda_2$, say $\lambda_1 > \lambda_2$ and $(\lambda_1, u_{\lambda_1})$, $(\lambda_2, u_{\lambda_2}) \in \mathcal{C}_r$ such that $u_{\lambda_1}(0) = u_{\lambda_2}(0) = \kappa$. If we set $u_1 \equiv u_{\lambda_1}$, $u_2 \equiv u_{\lambda_2}$ and $z_j = 1 - u_j(r)$ for $j = 1, 2$, then

(5.7)

$$-\Delta^2 z_j = \lambda_j z_j^{-p}, \ \ z_j(0) = 1-\kappa, \ \ z_j'(0) = z_j'''(0) = 0, \ \ z_j(1) = 1, \ \ (\Delta z_j)(1) = 0.$$

Let $\tilde z_j(y) = \frac{z_j((1-\kappa)^{(p+1)/4}\lambda_j^{-1/4} y)}{1-\kappa}$. We see that $\tilde z_j$ $(j = 1, 2)$ satisfies

(5.8)

$$\Delta_y^2 v_j = -v_j^{-p}, \ \ v_j(0) = 1, \ \ v_j'(0) = v_j'''(0) = 0, \ \ v_j(\tau_j) = \frac{1}{1-\kappa}, \ \ (\Delta_y v_j)(\tau_j) = 0$$

where $\tau_j = \lambda_j^{1/4}(1-\kappa)^{-(p+1)/4}$. Since $\lambda_1 > \lambda_2$, we see that $\tau_1 > \tau_2$. Suppose $(v_1)_{yy}(0) > (v_2)_{yy}(0)$, by the comparison principle (see Lemma 2.1), we see that

$v_1(y) > v_2(y)$ for $y \in (0, \tau_2]$. This contradicts the fact that $v_1(\tau_2) < v_2(\tau_2) = \frac{1}{1-\kappa}$. Suppose $(v_1)_{yy}(0) < (v_2)_{yy}(0)$, by the comparison principle again, we see that $(\Delta v_1)(\tau_2) < (\Delta v_2)(\tau_2) = 0$, but this contradicts the fact that $(\Delta v_1)(\tau_2) > (\Delta v_1)(\tau_1) = 0$. Thus, $(v_1)_{yy}(0) = (v_2)_{yy}(0)$ and thus, $v_1 \equiv v_2$, $\tau_1 = \tau_2$. Therefore, $\lambda_1 = \lambda_2$. This is a contradiction and completes the proof. □

6. Structure of radial solutions of (1.13): proof of Theorem 1.5

In this section we prove Theorem 1.5. We also apply the phase analysis to consider the problem (5.1).

Consider the initial value problem

(6.1)
$$\begin{cases} u^{(4)}(r) + \frac{2(N-1)}{r}u'''(r) + \frac{(N-1)(N-3)}{r^2}u''(r) - \frac{(N-1)(N-3)}{r^3}u'(r) = \frac{\lambda}{(1-u(r))^{-p}} & \text{for } r \in (0,1) \\ u(0) = A \in (0,1), \quad u'(0) = u'''(0) = 0. \end{cases}$$

By making the changes:

$$v(y) = \frac{1-u(r)}{1-A}, \quad y = \lambda^{1/4}(1-A)^{-(p+1)/4}r,$$

we see that (6.1) is reduced to (for $y \in (0, \lambda^{1/4}(1-A)^{-(p+1)/4}))$

$$(6.2) \quad \begin{cases} v^{(4)}(y) + \frac{2(N-1)}{y}v'''(y) + \frac{(N-1)(N-3)}{y^2}v''(y) - \frac{(N-1)(N-3)}{y^3}v'(y) == -v^{-p}(y) \\ 0 < v \leq \frac{1}{1-A} \\ v(0) = 1, \quad v'(0) = v'''(0) = 0. \end{cases}$$

Setting $\theta = \lambda^{1/4}(1-A)^{-(p+1)/4}$ and using arguments similar to those in the proof of Theorem 1.4, we see that the solution $v(y)$ of (6.2) depends on θ, we denote it by v_θ. Moreover, $v_\theta(\theta) = \frac{1}{1-A}$, $(\Delta_y v_\theta)(\theta) = 0$. By arguments similar to those in the proof of Theorem 1.4, we see that $v_\theta(y) \to u_{\gamma^*}(y)$ for all $y \in (0,\infty)$ as $\theta \to \infty$. By using the expressions as in (5.4) and (5.5), we see that $\mathcal{C}_r$ is homeomorphic to $\mathbb{R}$ and

$$\lambda_\tau \to \lambda_*(= -K_0), \quad A_\tau \to 1, \quad u_\tau(r) \to 1 - r^{\frac{4}{p+1}} \quad \text{as } \tau \to \infty.$$

Moreover, for any $\varphi \in H_0^2(B)$, we see that

$$\begin{aligned} &\int_B [(\Delta\varphi)^2 - p(-K_0)\varphi^2]dx \\ &\geq \int_B \left[(\Delta\varphi)^2 - \left(\frac{N(N-4)}{4}\right)^2 \frac{\varphi^2}{r^4}\right]dx \\ &> 0. \end{aligned}$$

The latter inequality is the well-known Hardy's inequality. Note that for $N \geq 13$ or $5 \leq N \leq 12$ and $1 < p < p_c$, we see that

$$p(-K_0) \leq \Big(\frac{N(N-4)}{4}\Big)^2.$$

This implies that $(-(K_0), 1 - r^{\frac{4}{p+1}})$ is a stable solution of (1.12).

On the other hand, it is known from [22] that for any $\lambda \in (0, \lambda_c)$, (1.13) has a minimal solution $\underline{u}_\lambda$ satisfying $\underline{u}_\lambda > 0$ in B. Arguments similar to those in the proof of Lemma 5.1 imply that the secondary bifurcation point of $\mathcal{C}_r$ does not occur. Thus, $\lambda_c = \lambda_* = -K_0$. This completes the proof. □

Acknowledgments

The research of the first author is supported by a grant of NSFC (10571022). The research of the second author is partially supported by an Earmarked Grant from RGC of Hong Kong. We thank Professor Ghoussoub for helpful conversations.

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