CHAPTER 4

Weak Solutions, Part II

4.1. Guide

This chapter covers the well-known theory of De Giorgi-Nash-Moser. We present both the approach of De Giorgi and of Moser so students can make comparisons and can see that the ideas involved are essentially the same. The classical paper [**12**] is certainly very nice material for further reading. One may also wish to compare the results in [**12**] and [**7**].

4.2. Local Boundedness

In the following three sections we will discuss the De Giorgi-Nash-Moser theory for linear elliptic equations. In this section we will prove the local boundedness of solutions. In the next section we will prove Hölder continuity. Then in Section 4.4 we will discuss the Harnack inequality. For all results in these three sections there is no regularity assumption of coefficients.

The main theorem of this section is the following boundedness result.

THEOREM 4.1. *Suppose $a_{ij} \in L^\infty(B_1)$ and $c \in L^q(B_1)$ for some $q > n/2$ satisfy the following assumptions*

$$a_{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2 \quad \text{for any } x \in B_1,\ \xi \in \mathbb{R}^n \quad \text{and} \quad |a_{ij}|_{L^\infty} + \|c\|_{L^q} \leq \Lambda$$

for some positive constants λ and Λ. Suppose that $u \in H^1(B_1)$ is a subsolution in the following sense

$$(*) \qquad \int_{B_1} a_{ij} D_i u D_j \varphi + cu\varphi \leq \int_{B_1} f\varphi \quad \text{for any } \varphi \in H_0^1(B_1) \text{ and } \varphi \geq 0 \text{ in } B_1 .$$

If $f \in L^q(B_1)$, then $u^+ \in L^\infty_{\text{loc}}(B_1)$. Moreover, there holds for any $\theta \in (0,1)$ and any $p > 0$

$$\sup_{B_\theta} u^+ \leq C\left\{\frac{1}{(1-\theta)^{n/p}}\|u^+\|_{L^p(B_1)} + \|f\|_{L^q(B_1)}\right\}$$

where $C = C(n, \lambda, \Lambda, p, q)$ is a positive constant.

In the following we use two approaches to prove this theorem, the one by De Giorgi and the other by Moser.

PROOF. We first prove for $\theta = 1/2$ and $p = 2$.

METHOD 1. Approach by De Giorgi.

if k is sufficiently large. Obviously it is true for $\ell = 0$. Suppose it is true for $\ell - 1$. We write

$$[\varphi(k_{\ell-1}, r_{\ell-1})]^{1+\varepsilon} \leq \left\{ \frac{\varphi(k_0, r_0)}{\gamma^{\ell-1}} \right\}^{1+\varepsilon} = \frac{\varphi(k_0, r_0)^{\varepsilon}}{\gamma^{\ell\varepsilon-(1+\varepsilon)}} \cdot \frac{\varphi(k_0, r_0)}{\gamma^{\ell}} .$$

Then we obtain

$$\varphi(k_\ell, r_\ell) \leq \frac{C\gamma^{1+\varepsilon}}{1-\tau} \cdot \frac{k_0 + F + k}{k^{1+\varepsilon}} \cdot [\varphi(k_0, r_0)]^{\varepsilon} \cdot \frac{2^{\ell(1+\varepsilon)}}{\gamma^{\ell\varepsilon}} \cdot \frac{\varphi(k_0, r_0)}{\gamma^{\ell}} .$$

Choose γ first such that $\gamma^{\varepsilon} = 2^{1+\varepsilon}$. Note $\gamma > 1$. Next, we need

$$\frac{C\gamma^{1+\varepsilon}}{1-\tau} \cdot \left(\frac{\varphi(k_0, r_0)}{k} \right)^{\varepsilon} \cdot \frac{k_0 + F + k}{k} \leq 1 .$$

Therefore we choose

$$k = C_*\{k_0 + F + \varphi(k_0, r_0)\}$$

for C_* large. Let $\ell \to +\infty$ in (4.4). We conclude

$$\varphi(k_0 + k, \tau) = 0 .$$

Hence we have

$$\sup_{B_{1/2}} u^+ \leq (C_* + 1)\{k_0 + F + \varphi(k_0, r_0)\} .$$

Recall $k_0 = C\|u^+\|_{L^2(B_1)}$ and $\varphi(k_0, r_0) \leq \|u^+\|_{L^2(B_1)}$. This finishes the proof.

Next we give the second proof of Theorem 4.1.

METHOD 2. Approach by Moser.

First we explain the idea. By choosing the test function appropriately, we will estimate the L^{p_1} norm of u in a smaller ball by the L^{p_2} norm of u for $p_1 > p_2$ in a larger ball, that is,

$$\|u\|_{L^{p_1}(B_{r_1})} \leq C\|u\|_{L^{p_2}(B_{r_2})}$$

for $p_1 > p_2$ and $r_1 < r_2$. This is a reversed Hölder inequality. As a sacrifice C behaves like $\frac{1}{r_2 - r_1}$. By iteration and careful choice of $\{r_i\}$ and $\{p_i\}$, we will obtain the result.

For some $k > 0$ and $m > 0$, set $\bar{u} = u^+ + k$ and

$$\bar{u}_m = \begin{cases} \bar{u} & \text{if } u < m \\ k + m & \text{if } u \geq m . \end{cases}$$

Then we have $D\bar{u}_m = 0$ in $\{u < 0\}$ and $\{u > m\}$ and $\bar{u}_m \leq \bar{u}$. Set the test function

$$\varphi = \eta^2(\bar{u}_m^{\beta}\bar{u} - k^{\beta+1}) \in H_0^1(B_1)$$

for some $\beta \geq 0$ and some nonnegative function $\eta \in C_0^1(B_1)$. Direct calculation yields

$$\begin{aligned} D\varphi &= \beta\eta^2\bar{u}_m^{\beta-1}D\bar{u}_m\bar{u} + \eta^2\bar{u}_m^{\beta}D\bar{u} + 2\eta D\eta(\bar{u}_m^{\beta}\bar{u} - k^{\beta+1}) \\ &= \eta^2\bar{u}_m^{\beta}(\beta D\bar{u}_m + D\bar{u}) + 2\eta D\eta(\bar{u}_m^{\beta}\bar{u} - k^{\beta+1}) . \end{aligned}$$

We should emphasize that later on we will begin the iteration with $\beta = 0$. Note $\varphi = 0$ and $D\varphi = 0$ in $\{u \le 0\}$. Hence if we substitute such φ in the equation we integrate in the set $\{u > 0\}$. Note also that $u^+ \le \bar{u}$ and $\bar{u}_m^\beta \bar{u} - k^{\beta+1} \le \bar{u}_m^\beta \bar{u}$ for $k > 0$. First we have by Hölder inequality

$$\begin{aligned}\int a_{ij} D_i u D_j \varphi &= \int a_{ij} D_i \bar{u}(\beta D_j \bar{u}_m + D_j \bar{u})\eta^2 \bar{u}_m^\beta \\ &\quad + 2\int a_{ij} D_i \bar{u} D_j \eta (\bar{u}_m^\beta \bar{u} - k^{\beta+1})\eta \\ &\ge \lambda\beta \int \eta^2 \bar{u}_m^\beta |D\bar{u}_m|^2 + \lambda \int \eta^2 \bar{u}_m^\beta |D\bar{u}|^2 - \Lambda \int |D\bar{u}||D\eta| \bar{u}_m^\beta \bar{u}\eta \\ &\ge \lambda\beta \int \eta^2 \bar{u}_m^\beta |D\bar{u}_m|^2 + \frac{\lambda}{2} \int \eta^2 \bar{u}_m^\beta |D\bar{u}|^2 - \frac{2\Lambda^2}{\lambda} \int |D\eta|^2 \bar{u}_m^\beta \bar{u}^2 .\end{aligned}$$

Hence we obtain by noting $\bar{u} \ge k$

$$\begin{aligned}\beta \int \eta^2 \bar{u}_m^\beta |D\bar{u}_m|^2 &+ \int \eta^2 \bar{u}_m^\beta |D\bar{u}|^2 \\ &\le C\left\{ \int |D\eta|^2 \bar{u}_m^\beta \bar{u}^2 + \int \left(|c|\eta^2 \bar{u}_m^\beta \bar{u}^2 + |f|\eta^2 \bar{u}_m^\beta \bar{u}\right\} \\ &\le C\left\{ \int |D\eta|^2 \bar{u}_m^\beta \bar{u}^2 + \int c_0 \eta^2 \bar{u}_m^\beta \bar{u}^2 \right\},\end{aligned}$$

where c_0 is defined as

$$c_0 = |c| + \frac{|f|}{k} .$$

Choose $k = \|f\|_{L^q}$ if f is not identically zero. Otherwise choose arbitrary $k > 0$ and eventually let $k \to 0+$. By assumption we have

$$\|c_0\|_{L^q} \le \Lambda + 1 .$$

Set $w = \bar{u}_m^{\frac{\beta}{2}} \bar{u}$. Note

$$|Dw|^2 \le (1+\beta)\{\beta \bar{u}_m^\beta |D\bar{u}_m|^2 + \bar{u}_m^\beta |D\bar{u}|^2\} .$$

Therefore we have

$$\int |Dw|^2 \eta^2 \le C\left\{ (1+\beta) \int w^2 |D\eta|^2 + (1+\beta) \int c_0 w^2 \eta^2 \right\},$$

or

$$\int |D(w\eta)|^2 \le C\left\{ (1+\beta) \int w^2 |D\eta|^2 + (1+\beta) \int c_0 w^2 \eta^2 \right\} .$$

Hölder inequality implies

$$\int c_0 w^2 \eta^2 \le \left(\int c_0^q \right)^{\frac{1}{q}} \left(\int (\eta w)^{\frac{2q}{q-1}} \right)^{1-\frac{1}{q}} \le (\Lambda + 1)\left(\int (\eta w)^{\frac{2q}{q-1}} \right)^{1-\frac{1}{q}}$$

By interpolation inequality and Sobolev inequality with $2^* = \frac{2n}{n-2} > \frac{2q}{q-1} > 2$ if $q > n/2$, we have

$$\begin{aligned}\|\eta w\|_{L^{\frac{2q}{q-1}}} &\le \varepsilon \|\eta w\|_{L^{2^*}} + C(n,q)\varepsilon^{-\frac{n}{2q-n}} \|\eta w\|_{L^2} \\ &\le \varepsilon \|D(\eta w)\|_{L^2} + C(n,q)\varepsilon^{-\frac{n}{2q-n}} \|\eta w\|_{L^2}\end{aligned}$$

for any small $\varepsilon > 0$. Therefore we obtain

$$\int |D(w\eta)|^2 \le C\left\{(1+\beta)\int w^2|D\eta|^2 + (1+\beta)^{\frac{2q}{2q-n}} \int w^2\eta^2\right\}$$

and in particular

$$\int |D(w\eta)|^2 \le C(1+\beta)^\alpha \int (|D\eta|^2 + \eta^2)w^2,$$

where α in a positive number depending only on n and q. Sobolev inequality then implies

$$\left(\int |\eta w|^{2\chi}\right)^{\frac{1}{\chi}} \le C(1+\beta)^\alpha \int (|D\eta|^2 + \eta^2)w^2$$

where $\chi = \frac{n}{n-2} > 1$ for $n > 2$ and $\chi > 2$ for $n = 2$. Choose the cut-off function as follows. For any $0 < r < R \le 1$ set $\eta \in C_0^1(B_R)$ with the property

$$\eta \equiv 1 \text{ in } B_r \quad \text{and} \quad |D\eta| \le \frac{2}{R-r}.$$

Then we obtain

$$\left(\int_{B_r} w^{2\chi}\right)^{\frac{1}{\chi}} \le C\frac{(1+\beta)^\alpha}{(R-r)^2} \int_{B_R} w^2.$$

Recalling the definition of w, we have

$$\left(\int_{B_r} \bar{u}^{2\chi}\bar{u}_m^{\beta\chi}\right)^{\frac{1}{\chi}} \le C\frac{(1+\beta)^\alpha}{(R-r)^2} \int_{B_R} \bar{u}^2\bar{u}_m^\beta.$$

Set $\gamma = \beta + 2 \ge 2$. Then we obtain

$$\left(\int_{B_r} \bar{u}_m^{\gamma\chi}\right)^{\frac{1}{\chi}} \le C\frac{(\gamma-1)^\alpha}{(R-r)^2} \int_{B_R} \bar{u}^\gamma$$

provided the integral in the right-hand side is bounded. By letting $m \to +\infty$ we conclude that

$$\|\bar{u}\|_{L^{\gamma\chi}(B_r)} \le \left(C\frac{(\gamma-1)^\alpha}{(R-r)^2}\right)^{\frac{1}{\gamma}} \|\bar{u}\|_{L^\gamma(B_R)}$$

provided $\|\bar{u}\|_{L^{\gamma}(B_R)} < +\infty$, where $C = C(n, q, \lambda, \Lambda)$ is a positive constant independent of γ. The above estimate suggests that we iterate, beginning with $\gamma = 2$, as $2, 2\chi, 2\chi^2, \ldots$. Now set for $i = 0, 1, 2, \ldots$,

$$\gamma_i = 2\chi^i \quad \text{and} \quad r_i = \frac{1}{2} + \frac{1}{2^{i+1}} .$$

By $\gamma_i = \chi\gamma_{i-1}$ and $r_{i-1} - r_i = 1/2^{i+1}$, we have for $i = 1, 2, \ldots$,

$$\|\bar{u}\|_{L^{\gamma_i}(B_{r_i})} \le C(n, q, \lambda, \Lambda)^{\frac{i}{\chi^i}} \|\bar{u}\|_{L^{\gamma_{i-1}}(B_{r_{i-1}})}$$

provided $\|\bar{u}\|_{L^{\gamma_{i-1}}(B_{r_{i-1}})} < +\infty$. Hence by iteration we obtain

$$\|\bar{u}\|_{L^{\gamma_i}(B_{r_i})} \le C^{\sum \frac{i}{\chi^i}} \|\bar{u}\|_{L^2(B_1)} ;$$

in particular

$$\left(\int_{B_{1/2}} \bar{u}^{2\chi^i}\right)^{\frac{1}{2\chi^i}} \le C \left(\int_{B_1} \bar{u}^2\right)^{\frac{1}{2}} .$$

Letting $i \to +\infty$ we get

$$\sup_{B_{1/2}} \bar{u} \le C\|\bar{u}\|_{L^2(B_1)} \quad \text{or} \quad \sup_{B_{1/2}} u^+ \le C\{\|u^+\|_{L^2(B_1)} + k\} .$$

Recall the definition of k. This finishes the proof for $p = 2$.

REMARK 4.2. If the subsolution u is bounded, we may simply take the test function

$$\varphi = \eta^2(\bar{u}^{\beta+1} - k^{\beta+1}) \in H_0^1(B_1)$$

for some $\beta \ge 0$ and some nonnegative function $\eta \in C_0^1(B_1)$.

Next we discuss the general p case of Theorem 4.1. This is based on a dilation argument.

Take any $R \le 1$. Define

$$\tilde{u}(y) = u(Ry) \quad \text{for } y \in B_1 .$$

It is easy to see that $\tilde{u}$ satisfies the following equation

$$\int_{B_1} \tilde{a}_{ij} D_i\tilde{u} D_j\varphi + \tilde{c}\tilde{u}\varphi \le \int_{B_1} \tilde{f}\varphi \quad \text{for any } \varphi \in H_0^1(B_1) \text{ and } \phi \ge 0 \text{ in } B_1$$

where

$$\tilde{a}(y) = a(Ry), \quad \tilde{c}(y) = R^2 c(Ry) \quad \text{and} \quad \tilde{f}(y) = R^2 f(Ry)$$

for any $y \in B_1$. Direct calculation shows

$$|\tilde{a}_{ij}|_{L^\infty(B_1)} + \|\tilde{c}\|_{L^q(B_1)} = |a_{ij}|_{L^\infty(B_R)} + R^{2-\frac{n}{q}}\|c\|_{L^q(B_R)} \le \Lambda .$$

We may apply what we just proved to $\tilde{u}$ in B_1 and rewrite the result in terms of u. Hence we obtain for $p \ge 2$

$$\sup u^+ \le C\left\{\frac{1}{\cdots}\|u^+\|_{L^p(B_R)} + R^{2-\frac{n}{q}}\|f\|_{L^q(B_R)}\right\}$$

where $C = C(n, \lambda, \Lambda, p, q)$ is a positive constant.

The estimate in $B_{\theta R}$ can be obtained by applying the above result to $B_{(1-\theta)R}(y)$ for any $y \in B_{\theta R}$. Take $R = 1$. This is Theorem 4.1 for any $\theta \in (0, 1)$ and $p \geq 2$.

Now we prove the statement for $p \in (0, 2)$. We showed that for any $\theta \in (0, 1)$ and $0 < R \leq 1$ there holds

$$\|u^+\|_{L^\infty(B_{\theta R})} \leq C\left\{\frac{1}{[(1-\theta)R]^{\frac{n}{2}}}\|u^+\|_{L^2(B_R)} + R^{2-\frac{n}{q}}\|f\|_{L^q(B_R)}\right\}$$
$$\leq C\left\{\frac{1}{[(1-\theta)R]^{\frac{n}{2}}}\|u^+\|_{L^2(B_R)} + \|f\|_{L^q(B_1)}\right\}.$$

For $p \in (0, 2)$ we have

$$\int_{B_R}(u^+)^2 \leq \|u^+\|^{2-p}_{L^\infty(B_R)}\int_{B_R}(u^+)^p$$

and hence by Hölder inequality

$$\|u^+\|_{L^\infty(B_{\theta R})} \leq C\left\{\frac{1}{[(1-\theta)R]^{\frac{n}{2}}}\|u^+\|^{\frac{2-p}{2}}_{L^\infty(B_R)}\left(\int_{B_R}(u^+)^p dx\right)^{\frac{1}{2}} + \|f\|_{L^q(B_R)}\right\}$$
$$\leq \frac{1}{2}\|u^+\|_{L^\infty(B_R)} + C\left\{\frac{1}{[(1-\theta)R]^{\frac{n}{p}}}\left(\int_{B_R}(u^+)^p\right)^{\frac{1}{p}} + \|f\|_{L^q(B_R)}\right\}.$$

Set $f(t) = \|u^+\|_{L^\infty(B_t)}$ for $t \in (0, 1]$. Then for any $0 < r < R \leq 1$

$$f(r) \leq \frac{1}{2}f(R) + \frac{C}{(R-r)^{\frac{n}{p}}}\|u^+\|_{L^p(B_1)} + C\|f\|_{L^q(B_1)}.$$

We apply the following lemma to get for any $0 < r < R < 1$

$$f(r) \leq \frac{C}{(R-r)^{\frac{n}{p}}}\|u^+\|_{L^p(B_1)} + C\|f\|_{L^q(B_1)}.$$

Let $R \to 1-$. We obtain for any $\theta < 1$

$$\|u^+\|_{L^\infty(B_\theta)} \leq \frac{C}{(1-\theta)^{\frac{n}{p}}}\|u^+\|_{L^p(B_1)} + C\|f\|_{L^q(B_1)}.$$

□

We need the following simple lemma:

LEMMA 4.3. *Let $f(t) \geq 0$ be bounded in $[\tau_0, \tau_1]$ with $\tau_0 \geq 0$. Suppose for $\tau_0 \leq t < s \leq \tau_1$ we have*

$$f(t) \leq \theta f(s) + \frac{A}{(s-t)^\alpha} + B$$

for some $\theta \in [0, 1)$. Then for any $\tau_0 \leq t < s \leq \tau_1$ there holds

$$f(t) \leq c(\alpha, \theta)\left\{\frac{A}{(s-t)^\alpha} + B\right\}.$$

PROOF. Fix $\tau_0 \le t < s \le \tau_1$. For some $0 < \tau < 1$ we consider the sequence $\{t_i\}$ defined by

$$t_0 = t \quad \text{and} \quad t_{i+1} = t_i + (1-\tau)\tau^i(s-t).$$

Note $t_\infty = s$. By iteration

$$f(t) = f(t_0) \le \theta^k f(t_k) + \left[\frac{A}{(1-\tau)^\alpha}(s-t)^{-\alpha} + B\right]\sum_{i=0}^{k-1}\theta^i\tau^{-i\alpha}.$$

Choose $\tau < 1$ such that $\theta\tau^{-\alpha} < 1$, that is, $\theta < \tau^\alpha < 1$. As $k \to \infty$ we have

$$f(t) \le c(\alpha,\theta)\left\{\frac{A}{(1-\tau)^\alpha}(s-t)^{-\alpha} + B\right\}.$$

□

In the rest of this section we use Moser's iteration to prove a high integrability result, which is closely related to Theorem 4.1. For the next result we require $n \ge 3$.

THEOREM 4.4. *Suppose $a_{ij} \in L^\infty(B_1)$ and $c \in L^{n/2}(B_1)$ satisfy the following assumption:*

$$\lambda|\xi|^2 \le a_{ij}(x)\xi_i\xi_j \le \Lambda|\xi|^2 \quad \textit{for any } x \in B_1,\ \xi \in \mathbb{R}^n$$

for some positive constants λ and Λ. Suppose that $u \in H^1(B_1)$ is a subsolution in the following sense:

$$\int_{B_1} a_{ij}D_iuD_j\varphi + cu\varphi \le \int_{B_1} f\varphi \quad \textit{for any } \varphi \in H_0^1(B_1) \textit{ and } \varphi \ge 0 \textit{ in } B_1.$$

If $f \in L^q(B_1)$ for some $q \in [\frac{2n}{n+2}, \frac{n}{2})$, then $u^+ \in L^{q^*}_{\mathrm{loc}}(B_1)$ for $\frac{1}{q^*} = \frac{1}{q} - \frac{2}{n}$. Moreover, there holds

$$\|u^+\|_{L^{q^*}(B_{1/2})} \le C\left\{\|u^+\|_{L^2(B_1)} + \|f\|_{L^q(B_1)}\right\}$$

where $C = C(n,\lambda,\Lambda,q,\varepsilon(K))$ is a positive constant with

$$\varepsilon(K) = \left(\int_{\{|c|>K\}} |c|^{\frac{n}{2}}\right)^{\frac{2}{n}}.$$

PROOF. For $m > 0$, set $\bar{u} = u^+$ and

$$\bar{u}_m = \begin{cases} \bar{u} & \text{if } u < m \\ m & \text{if } u \ge m. \end{cases}$$

Then set the test function

for some $\beta \geq 0$ and some nonnegative function $\eta \in C_0^1(B_1)$. By similar calculations as in the proof of Theorem 4.1 we conclude

$$\left(\int \eta^{2\chi}\bar{u}_m^{\beta\chi}\bar{u}^{2\chi}\right)^{\frac{1}{\chi}} \leq C(1+\beta)\left\{\int |D\eta|^2\bar{u}_m^{\beta}\bar{u}^2 + \int |c|\eta^2\bar{u}_m^{\beta}\bar{u}^2 + \int |f|\eta^2\bar{u}_m^{\beta}\bar{u}\right\}$$

where $\chi = \frac{n}{n-2} > 1$. Hölder inequality implies for any $K > 0$

$$\begin{aligned}\int |c|\eta^2\bar{u}_m^{\beta}\bar{u}^2 &\leq K \int_{\{|c|\leq K\}} \eta^2\bar{u}_m^{\beta}\bar{u}^2 + \int_{\{|c|>K\}} |c|\eta^2\bar{u}_m^{\beta}\bar{u}^2 \\ &\leq K\int \eta^2\bar{u}_m^{\beta}\bar{u}^2 + \left(\int_{\{|c|>K\}} |c|^{\frac{n}{2}}\right)^{\frac{2}{n}} \left(\int (\eta^2\bar{u}_m^{\beta}\bar{u}^2)^{\frac{n}{n-2}}\right)^{\frac{n-2}{n}} \\ &\leq K\int \eta^2\bar{u}_m^{\beta}\bar{u}^2 + \varepsilon(K)\left(\int \eta^{2\chi}\bar{u}_m^{\beta\chi}\bar{u}^{2\chi}\right)^{\frac{1}{\chi}}.\end{aligned}$$

Note $\varepsilon(K) \to 0$ as $K \to +\infty$ since $c \in L^{n/2}(B_1)$. Hence for bounded β we obtain by choosing large $K = K(\beta)$

$$\left(\int \eta^{2\chi}\bar{u}_m^{\beta\chi}\bar{u}^{2\chi}\right)^{\frac{1}{\chi}} \leq C(1+\beta)\left\{\int (|D\eta|^2+\eta^2)\bar{u}_m^{\beta}\bar{u}^2 + \int |f|\eta^2\bar{u}_m^{\beta}\bar{u}\right\}.$$

Observe

$$\bar{u}_m^{\beta}\bar{u} \leq \bar{u}_m^{\beta-\frac{\beta}{\beta+2}}\bar{u}^{1+\frac{\beta}{\beta+2}} = (\bar{u}_m^{\beta}\bar{u}^2)^{\frac{\beta+1}{\beta+2}}.$$

Therefore by Hölder inequality again we have for $\eta \leq 1$

$$\begin{aligned}\int |f|\eta^2\bar{u}_m^{\beta}\bar{u} &\leq \left(\int |f|^q\right)^{\frac{1}{q}}\left(\int (\eta^2\bar{u}_m^{\beta}\bar{u}^2)^{\chi}\right)^{\frac{\beta+1}{(\beta+2)\chi}} |\operatorname{supp}\eta|^{1-\frac{1}{q}-\frac{\beta+1}{(\beta+2)\chi}} \\ &\leq \varepsilon\left(\int \eta^{2\chi}\bar{u}^{\chi}\bar{u}_m^{\beta\chi}\right)^{\frac{1}{\chi}} + C(\varepsilon,\beta)\left(\int |f|^q\right)^{\frac{\beta+2}{q}},\end{aligned}$$

provided

$$1 - \frac{1}{q} - \frac{\beta+1}{(\beta+2)\chi} \geq 0$$

which is equivalent to

$$\beta + 2 \leq \frac{q(n-2)}{n-2q}.$$

Hence β is required to be bounded, depending only on n and q. Then we obtain

$$\left(\int \eta^{2\chi}\bar{u}_m^{\beta\chi}\bar{u}^{2\chi}\right)^{\frac{1}{\chi}} \leq C\left\{\int (|D\eta|^2+\eta^2)\bar{u}_m^{\beta}\bar{u}^2 + \|f\|_{L^q}^{\beta+2}\right\}.$$

By setting $\gamma = \beta + 2$, we have by definition of q^*

$$2 \leq \gamma \leq \frac{q(n-2)}{n-2q} = \frac{q^*}{\chi}. \tag{4.5}$$

We conclude, as before, for any such γ in (4.5) and any $0 < r < R \le 1$

$$\|\bar{u}\|_{L^{\chi\gamma}(B_r)} \le C\left\{\frac{1}{(R-r)^{\frac{2}{\gamma}}}\|\bar{u}\|_{L^{\gamma}(B_R)} + \|f\|_{L^q(B_1)}\right\} \tag{4.6}$$

provided $\|\bar{u}\|_{L^{\gamma}(B_R)} < +\infty$. Again this suggests the iteration $2, 2\chi, 2\chi^2, \dots$.

For given $q \in [\frac{2n}{n+2}, \frac{n}{2})$, there exists a positive integer k such that

$$2\chi^{k-1} \le \frac{q(n-2)}{n-2q} < 2\chi^k .$$

Hence for such k we get by finitely many iterations of (4.6)

$$\|\bar{u}\|_{L^{2\chi^k}(B_{3/4})} \le C\left\{\|\bar{u}\|_{L^2(B_1)} + \|f\|_{L^q(B_1)}\right\}$$

in particular

$$\|\bar{u}\|_{L^{\frac{q^*}{\chi}}(B_{3/4})} \le C\left\{\|\bar{u}\|_{L^2(B_1)} + \|f\|_{L^q(B_1)}\right\}.$$

While with $\gamma = \frac{q^*}{\chi}$ in (4.6) we obtain

$$\|\bar{u}\|_{L^{q^*}(B_{1/2})} \le C\left\{\|\bar{u}\|_{L^{\frac{q^*}{\chi}}(B_{3/4})} + \|f\|_{L^q(B_1)}\right\}.$$

This finishes the proof. □

4.3. Hölder Continuity

We first discuss homogeneous equations with no lower-order terms. Consider

$$Lu \equiv -D_i(a_{ij}(x)D_ju) \quad \text{in } B_1(0) \subset \mathbb{R}^n$$

where $a_{ij} \in L^\infty(B_1)$ satisfies

$$\lambda|\xi|^2 \le a_{ij}(x)\xi_i\xi_j \le \Lambda|\xi|^2 \quad \text{for all } x \in B_1(0) \text{ and } \xi \in \mathbb{R}^n$$

for some positive constants λ and Λ.

DEFINITION 4.5. The function $u \in H^1_{\mathrm{loc}}(B_1)$ is called a *subsolution* (*supersolution*) of the equation

$$Lu = 0$$

if

$$\int_{B_1} a_{ij}D_iuD_j\varphi \le 0 (\ge 0)$$

for all $\varphi \in H^1_0(B_1)$ and $\varphi \ge 0$.

LEMMA 4.6. *Let $\Phi \in C^{0,1}_{\mathrm{loc}}(\mathbb{R})$ be convex. Then*

(i) *if u is a subsolution and $\Phi' \ge 0$, then $v = \Phi(u)$ is also a subsolution provided $v \in H^1_{\mathrm{loc}}(B_1)$.*

(ii) *if u is a supersolution and $\Phi' \le 0$, then $v = \Phi(u)$ is a subsolution provided*