Higher order Bol's inequality and its applications

Mingxiang Li *, Juncheng Wei

Abstract

Assuming that the conformal metric $g = e^{2u}|dx|^2$ on $\mathbb{R}^n$ is normal, our focus lies in investigating the following conjecture: if the Q-curvature of such a manifold is bounded from above by $(n-1)!$, then the volume is sharply bounded from below by the volume of the standard n-sphere. In specific instances, such as when u is radially symmetric or when the Q-curvature is represented by a polynomial, we provide a positive response to this conjecture, although the general case remains unresolved. Intriguingly, under the normal and radially symmetric assumptions, we establish that the volume is bounded from above by the volume of the standard n-sphere when the Q-curvature is bounded from below by $(n-1)!$, thereby addressing certain open problems raised by Hyder-Martinazzi (2021, JDE).

Keywords: Q-curvature, Bol's inequality, Volume comparison.

MSC2020: 53C18, 35A23

1 Introduction

The following conformally invariant equation plays an important role conformal geometry:

$$(-\Delta)^{\frac{n}{2}}u(x) = Q(x)e^{nu(x)} \quad \text{on } \mathbb{R}^n \tag{1.1}$$

where $n \geq 2$ is an even integer and $Q(x)$ is a smooth function. From geomtric point of view, for $n=2$, Q is the Gaussian curvature of the conformal metric $e^{2u}|dx|^2$ on $\mathbb{R}^2$. For $n \geq 4$, Q is the Q-curvature of the metric $e^{2u}|dx|^2$ on $\mathbb{R}^n$ which introduced by Branson [2]. We refer the interested readers to [5], [23], [6] for more details.

Supposing that $Qe^{nu} \in L^1(\mathbb{R}^n)$ for the equation (1.1), we say that the solution $u(x)$ is a normal solution to (1.1) if $u(x)$ satisfies the integral equation

$$u(x) = \frac{2}{(n-1)!|\mathbb{S}^n|}\int_{\mathbb{R}^n} \log\frac{|y|}{|x-y|}Q(y)e^{nu(y)}\mathrm{d}y + C \tag{1.2}$$

for some constant C. For brevity, denote the normalized integrated Q-curvature as

$$\alpha := \frac{2}{(n-1)!|\mathbb{S}^n|}\int_{\mathbb{R}^n} Qe^{nu}\mathrm{d}x.$$

More details about normal solutions can be found in Section 2 of [15]. Here, we assume that both $u(x)$ and $Q(x)$ are smooth functions on $\mathbb{R}^n$. Although similar results can also be obtained under weaker regularity assumptions, for the sake of brevity, we will focus on the smooth cases throughout this paper.

Our aim throughout this paper is to try to prove the following conjecture.

*M. Li, Nanjing University, limx@smail.nju.edu.cn

Conjecture 1. *Consider the normal solution $u(x)$ to (1.2) on $\mathbb{R}^n$ with $Q(x) \leq (n-1)!$ where $n \geq 4$ is an even integer. Then*

$$\int_{\mathbb{R}^n} e^{nu} \mathrm{d}x \geq |\mathbb{S}^n|$$

with equality holds if and only if $Q(x) \equiv (n-1)!$.

In [16], the Alexandrov-Bol's inequality (See Lemma 5.1) was used to rule out the "slow bubble" in the two-dimensional case. For brevity, Alexandrov-Bol's inequality on $\mathbb{R}^2$ shows that if the Guassian curvature has an upper bound, the volume has a sharp lower bound. More details can be found in [1]. Meanwhile, there are a lot of works devoted to study it including [21], [7], [9] and the references therein. For higher order cases, it is natural to ask whether similar Bol's inequality holds.

In the two-dimensional case, it is unnecessary to assume that the solution $u(x)$ is normal in order to derive the lower bound of the volume. For higher order cases $n \geq 4$, the situation becomes very different. In fact, even for the case $Q = 1$, the volume could be arbitraly small (See Theorem 1 in [3]). More details about the volume for non-normal solutions can be found in [24], [18], [10], [11] and the references therein. Therefore, we need to focus on the normal solutions to consider the higher order Bol's inequality.

While we haven't fully solved Conjecture 1, we establish a lower bound for the volume in Theorem 2.9. We intend to verify this conjecture in numerous specific scenarios in current paper.

Firstly, we aim to validate Conjecture 1 within the context of radially symmetric cases.

Theorem 1.1. *Supposing that $u(x)$ is a radially symmetric normal solution to (1.2) with $Q(x) \leq (n-1)!$ where even integer $n \geq 2$. Then there holds*

$$\int_{\mathbb{R}^n} e^{nu} \mathrm{d}x \geq |\mathbb{S}^n|$$

with equality holds if and only if $Q \equiv (n-1)!$.

Secondly, we want to deal with a sepical class of Q which comes from the blow up analysis of the asymptotic behavior of conformal metrics with null Q-curvature in [14] to validate Conjecture 1.

Theorem 1.2. *Consider $u(x)$ is a normal solution to (1.2) on $\mathbb{R}^4$ where $Q(x) \leq 6$ is a polynomial. Then there holds*

$$\int_{\mathbb{R}^4} e^{4u} \mathrm{d}x \geq |\mathbb{S}^4|$$

with equality holds if and only if $Q(x) \equiv 6$.

For $n \geq 6$, due to our technical constraints, we need introduce additional assumptions on the polynomial Q.

Theorem 1.3. *Consider the normal soultion $u(x)$ to (1.2) on $\mathbb{R}^n$ where even integer $n \geq 6$ with $Q(x) = (n-1)! + \varphi(x)$ where $\varphi(x) \leq 0$ is a polynomial. If $4 \leq \deg \varphi \leq n-2$, we additionally assume that*

$$x \cdot \nabla\varphi(x) \geq (\deg \varphi)\varphi(x) \tag{1.3}$$

up to a rotation or a translation of the coordinates. Then there holds

$$\int_{\mathbb{R}^n} e^{nu} \mathrm{d}x \geq |\mathbb{S}^n| \tag{1.4}$$

with equality holds if and only if $\varphi \equiv 0$.

Now, we are going to deal with the converse verison of Conjecture 1. We also leave it as a conjecture.

Conjecture 2. *Consider the normal solution $u(x)$ to (1.2) on $\mathbb{R}^n$ with $Q(x) \geq (n-1)!$ where $n \geq 2$ is an even integer. Then*

$$\int_{\mathbb{R}^n} e^{nu} \mathrm{d}x \leq |\mathbb{S}^n|$$

with equality holds if and only if $Q(x) \equiv (n-1)!$.

We give a positive answer to Conjecture 2 under the radial symmetric assumption.

Theorem 1.4. *Supposing that $u(x)$ is a radially symmetric normal solution to (1.2) with $Q(x) \geq (n-1)!$ and $Q(x) \leq C(|x|+1)^k$ for some $k \geq 0$ where even integer $n \geq 2$. Then there holds*

$$\int_{\mathbb{R}^n} e^{nu} \mathrm{d}x \leq |\mathbb{S}^n|$$

with equality holds if and only if $Q \equiv (n-1)!$.

Remark 1.5. In fact, for $n = 2$, Gui and Li give a proof for this result in Theorem 1.5 in [8]. However, their method doesn't work for higher order cases.

For the sake of convenience, we use the notation $B_R(p)$ to refer to an Euclidean ball in $\mathbb{R}^n$ centered at $p \in \mathbb{R}^n$ with a radius of R. For a function $\varphi(x)$, the positive part of $\varphi(x)$ is denoted as $\varphi(x)^+$ and the negative part of $\varphi(x)$ is denoted as $\varphi(x)^-$. Set $\fint_E \varphi(x)\mathrm{d}x = \frac{1}{|E|}\int_E \varphi(x)\mathrm{d}x$ for any measurable set E. For a constant C, C^+ denotes C if $C \geq 0$, otherwise, $C^+ = 0$. Here and thereafter, we denote by C a constant which may be different from line to line. For $s \in \mathbb{R}$, $[s]$ denotes the largest integer not greater than s.

The paper is organized as follows. In Section 2, we study the asymptotic behavior of the normal solutions to (1.1). We give a lower bound of the volume in Theorem 2.9 where Q-curvature is bounded from above. In Section 3, we give some Pohozaev identities with different restrictions which play an important role in proofs of our main theorems. In Section 4, we introduce an **s-cone** condition on Q-curvature and study the asymptotic behavior of the solutions. In Section 5, we deal with the polynomial Q-curvature to obtain Bol's inequality giving the proofs of Theorem 1.2 and Theorem 1.3. In Section 6, we deal with the radial symmetric solutions giving the proofs of Theorem 1.1 and Theorem 1.4. In the last Section 7, we give some applications of our higher order Bol's inequality and answer an open problem rasied by Hyder and Martinazzi in [12].

2 Asymptotic behavior

For reader's convenience, we repeat the following lemmas which have been established in [15].

Lemma 2.1. *(Lemma 2.3 in [15]) Consider the normal solution $u(x)$ to (1.2) with even integer $n \geq 2$. For $|x| \gg 1$, there holds*

$$u(x) = (-\alpha + o(1)) \log |x| + \frac{2}{(n-1)!|\mathbb{S}^n|} \int_{B_1(x)} \log \frac{1}{|x-y|} Q(y) e^{nu(y)} \mathrm{d}y \tag{2.1}$$

where $o(1) \to 0$ as $|x| \to \infty$.

Lemma 2.2. *(Lemma 2.4 in [15]) Consider the normal solution $u(x)$ to (1.2) with even integer $n \geq 2$. For $|x| \gg 1$ and any $r_0 > 0$ fixed, there holds*

$$\fint_{B_{r_0}(x)} u(y) \mathrm{d}y = (-\alpha + o(1)) \log |x|. \tag{2.2}$$

Lemma 2.3. *(Lemma 2.10 in [15]) Consider the normal solution $u(x)$ to (1.2) with even integer $n \geq 2$. If Q^+ has compact support, there holds*

$$u(x) \leq -\alpha \log|x| + C, \quad |x| \gg 1.$$

Conversely, if Q^- has compact support, there holds

$$u(x) \geq -\alpha \log|x| - C, \quad |x| \gg 1.$$

Lemma 2.4. *(Lemma 2.5 in [15]) Consider the normal solution $u(x)$ to (1.2) with even integer $n \geq 2$. For $|x| \gg 1$ and any $0 < r_1 < 1$ fixed, there holds*

$$\fint_{B_{r_1|x|}(x)} u(y)\mathrm{d}y = (-\alpha + o(1))\log|x|. \tag{2.3}$$

Lemma 2.5. *(Lemma 2.6 in [15]) Consider the normal solution $u(x)$ to (1.2) with even integer $n \geq 2$. For $|x| \gg 1$ and any $r_2 > 0$ fixed, there holds*

$$\fint_{B_{|x|-r_2}(x)} u(y)\mathrm{d}y = (-\alpha + o(1))\log|x|. \tag{2.4}$$

By slightly modifying Lemma 2.8 in [15], we obtain the following property.

Lemma 2.6. *Consider the normal solution $u(x)$ to (1.2). For $R \gg 1$ and any $m > 0$ fixed, there holds*

$$\fint_{B_R(0)\backslash B_{R-1}(0)} e^{mu}\mathrm{d}x = R^{-m\alpha + o(1)}.$$

Proof. On one hand, with help of Jensen's inequality and Lemma 2.2, for any $r_3 > 0$ fixed, one has

$$\fint_{B_{r_3}(x)} e^{mu}\mathrm{d}y \geq \exp\left(\fint_{B_{r_3}(x)} mu\mathrm{d}y\right) = e^{(-m\alpha + o(1))\log|x|}. \tag{2.5}$$

Now, we are going to deal with the upper bound. For $|y| \gg 1$, there holds

$$\left|\int_{B_1(y)\backslash B_{1/4}(y)} \log\frac{1}{|y-z|}Q(z)e^{nu(z)}\mathrm{d}z\right| \leq C\int_{B_1(y)\backslash B_{1/4}(y)} |Q(z)|e^{nu}\mathrm{d}z \leq C.$$

Combing with Lemma 2.1, we obtain

$$u(y) = (-\alpha + o(1))\log|y| + \frac{2}{(n-1)!|\mathbb{S}^n|}\int_{B_{1/4}(y)} \log\frac{1}{|y-z|}Q(z)e^{nu(z)}\mathrm{d}z.$$

Then for $|x| \gg 1$ and $y \in B_{1/4}(x)$, one has

$$u(y) \leq (-\alpha + o(1))\log|x| + \frac{2}{(n-1)!|\mathbb{S}^n|}\int_{B_{1/2}(x)} \log\frac{1}{|y-z|}Q^+(z)e^{nu(z)}\mathrm{d}z \tag{2.6}$$

where we have used the fact $|y - z| \leq 1$.

Now, we claim that for $|x| \gg 1$ and $m > 0$, there holds

$$\int_{B_{1/4}(x)} e^{mu(y)}\mathrm{d}y \leq e^{(-m\alpha + o(1))\log|x|}. \tag{2.7}$$

If $Q^+(z)=0$ a.e. on $B_{1/2}(x)$, we immediately obatain (2.7) due to (2.6). Otherwise, Jensen's inequality yields that

$$\begin{aligned}
&\int_{B_{1/4}(x)} e^{mu}\mathrm{d}y\\
\leq&|x|^{-m\alpha+o(1)}\int_{B_{1/4}(x)}\exp\left(\frac{2m}{(n-1)!|\mathbb{S}^n|}\int_{B_{1/2}(x)}\log\frac{1}{|y-z|}Q^+(z)e^{nu(z)}\mathrm{d}z\right)\mathrm{d}y\\
\leq&|x|^{-m\alpha+o(1)}\int_{B_{1/4}(x)}\int_{B_{1/2}(x)}|y-z|^{-\frac{2m\|Q^+e^{nu}\|_{L^1(B_{1/2}(x))}}{(n-1)!|\mathbb{S}^n|}}\frac{Q^+(z)e^{nu(z)}}{\|Q^+e^{nu}\|_{L^1(B_{1/2}(x))}}\mathrm{d}z\mathrm{d}y.
\end{aligned}$$

Since $Qe^{nu}\in L^1(\mathbb{R}^n)$, there exists $R_2>0$ such that $|x|\geq R_2$, we have

$$\|Q^+e^{nu}\|_{L^1(B_{1/2}(x))}\leq\frac{(n-1)!|\mathbb{S}^n|}{4m}.$$

By applying Fuibini's theorem, we prove the claim (2.7).

For $r_3>0$ fixed, we could choose finite balls $1\leq j\leq C(r_3)$ such that $B_{r_3}(x)\subset\cup_j B_{1/4}(x_j)$ with $x_j\in B_{r_3}(x)$. Hence, using the estimate (2.7), for $|x|\gg1$, we have

$$⨍_{B_{r_3}(x)} e^{mu}\mathrm{d}y\leq C\sum_j ⨍_{B_{1/4}(x_j)} e^{mu}\mathrm{d}y\leq Ce^{(-m\alpha+o(1))\log|x|}=e^{(-m\alpha+o(1))\log|x|}$$

Combing with (2.5), one has

$$\log ⨍_{B_{r_3}(x)} e^{mu}\mathrm{d}y=(-m\alpha+o(1))\log|x|. \tag{2.8}$$

By a direct computation, we obtain the inequality $C^{-1}R^{n-1}\leq|B_R(0)\backslash B_{R-1}(0)|\leq CR^{n-1}$ for $R\gg1$, where C is independent of R. We can select an index $C^{-1}R^{n-1}\leq i_R\leq CR^{n-1}$ such that the balls $B_{1/4}(x_j)$ with $|x_j|=R-\frac{1}{2}$ and $1\leq j\leq i_R$ are pairwise disjoint, and the sum of the balls $B_4(x_j)$ cover the annulus $B_R(0)\backslash B_{R-1}(0)$. Applying the estimate (2.8), we obtain the following result

$$\begin{aligned}
⨍_{B_R(0)\backslash B_{R-1}(0)} e^{mu}\mathrm{d}y\leq&\frac{1}{C^{-1}R^{n-1}}\sum_{j=1}^{i_R}\int_{B_4(x_j)} e^{mu}\mathrm{d}y\\
\leq&CR^{1-n}\sum_{j=1}^{i_R}|x_j|^{-n\alpha+o(1)}\\
\leq&CR^{1-n}\cdot CR^{n-1}\cdot(R-\frac{1}{2})^{-m\alpha+o(1)}\\
=&R^{-m\alpha+o(1)}.
\end{aligned}$$

Similarly, there holds

$$⨍_{B_R(0)\backslash B_{R-1}(0)} e^{mu}\mathrm{d}y\geq\frac{1}{CR^{n-1}}\sum_{j=1}^{i_R}\int_{B_{1/4}(x_j)} e^{mu}\mathrm{d}y=R^{-m\alpha+o(1)}.$$

Finally, we get the desired result.

□

Lemma 2.7. *For $R \gg 1$ and $m > 0$ fixed, there holds*

$$\int_{B_{\frac{3R}{2}}(0)\backslash B_{\frac{R}{2}}(0)} e^{mu}\mathrm{d}x = R^{n-m\alpha+o(1)}.$$

Proof. On one hand, with help of Lemma 2.6, for any $\epsilon > 0$, there exists $R_1 > 0$ such that $R \geq R_1$,

$$R^{n-1-m\alpha-\epsilon} \leq \int_{B_{R+1}\backslash B_R(0)} e^{mu}\mathrm{d}x \leq R^{n-1-m\alpha+\epsilon}. \tag{2.9}$$

Using the above estimate (2.9), for $R \geq 2R_1 + 2$, there holds

$$\begin{aligned}\int_{B_{\frac{3R}{2}}(0)\backslash B_{\frac{R}{2}}(0)} e^{mu}\mathrm{d}x \leq & \sum_{i=[\frac{R}{2}]}^{[\frac{3R}{2}]} \int_{B_{i+1}(0)\backslash B_i(0)} e^{mu}\mathrm{d}x \\ \leq & \sum_{i=[\frac{R}{2}]}^{[\frac{3R}{2}]} i^{n-1-m\alpha+\epsilon} \\ \leq & CR \cdot R^{n-1-m\alpha+\epsilon} \\ \leq & CR^{n-m\alpha+\epsilon}\end{aligned}$$

On the other hand, for $R \geq 2R_1 + 2$, we have

$$\begin{aligned}\int_{B_{\frac{3R}{2}}(0)\backslash B_{\frac{R}{2}}(0)} e^{mu}\mathrm{d}x \geq & \sum_{i=[\frac{R}{2}]+1}^{[\frac{3R}{2}]-1} \int_{B_{i+1}(0)\backslash B_i(0)} e^{mu}\mathrm{d}x \\ \geq & \sum_{i=[\frac{R}{2}]+1}^{[\frac{3R}{2}]-1} i^{n-1-m\alpha_0-\epsilon} \\ \geq & CR \cdot R^{n-1-m\alpha+\epsilon} \\ = & CR^{n-m\alpha-\epsilon}.\end{aligned}$$

Thus we finish our proof. □

The following lemma has also been established in [15]. For readers' convenience, we give a brief proof here.

Lemma 2.8. *(Theorem 2.18 in [15]) Consider the normal solution $u(x)$ to (1.2) with even integer $n \geq 2$. Supposing that the volume is finite i.e. $e^{nu} \in L^1(\mathbb{R}^n)$, then there holds*

$$\int_{\mathbb{R}^n} Qe^{nu}\mathrm{d}x \geq \frac{(n-1)!|\mathbb{S}^n|}{2}.$$

Proof. With help of Lemma 2.4 and $|x| \gg 1$, Jensen's inequality implies that

$$\begin{aligned}&\int_{B_{|x|/2}(x)} e^{nu}\mathrm{d}y \\ \geq & |B_{|x|/2}(x)| \exp\left(\frac{1}{|B_{|x|/2}(x)|}\int_{B_{|x|/2}(x)} nu\mathrm{d}y\right) \\ = & C|x|^{n-n\alpha+o(1)}.\end{aligned}$$

Based on the assumption $e^{nu} \in L^1(\mathbb{R}^n)$, letting $|x| \to \infty$, there holds $\alpha \geq 1$ i.e.

$$\int_{\mathbb{R}^n} Qe^{nu}\mathrm{d}x \geq \frac{(n-1)!|\mathbb{S}^n|}{2}.$$

Finally, we finish our proof. □

With help of above lemma, we are able to give a lower bound of the volume for normal solutions with $Q \leq (n-1)!$.

Theorem 2.9. *Consider the normal solution $u(x)$ to (1.2) on $\mathbb{R}^n$ with $Q \leq (n-1)!$ where $n \geq 4$ is an even integer. Then there holds*

$$\int_{\mathbb{R}^n} e^{nu}\mathrm{d}x > \frac{|\mathbb{S}^n|}{2}. \tag{2.10}$$

Proof. If $\int_{\mathbb{R}^n} e^{nu}\mathrm{d}x = +\infty$, it is trivial that (2.10) holds.

Now, we suppose that $e^{nu} \in L^1(\mathbb{R}^n)$. On one hand, if $Q(x)$ is a constant, with help of Lemma 2.8, $Q(x)$ must be a positive constant. Based on the classification theorem for normal solutions in [17], [22], [19] or [25], one has

$$\int_{\mathbb{R}^n} Qe^{nu}\mathrm{d}x = (n-1)!|\mathbb{S}^n|.$$

Our assumption $Q \leq (n-1)!$ yields that

$$\int_{\mathbb{R}^n} e^{nu}\mathrm{d}x \geq |\mathbb{S}^n| > \frac{|\mathbb{S}^n|}{2}.$$

On the other hand, if $Q(x)$ is not a constant, using Lemma 2.8 again, there holds

$$\int_{\mathbb{R}^n} e^{nu}\mathrm{d}x > \frac{1}{(n-1)!}\int_{\mathbb{R}^n} Qe^{nu}\mathrm{d}x \geq \frac{|\mathbb{S}^n|}{2}.$$

Finally, we finish our proof. □

3 Pohozaev identity

The following Pohozaev-type inequality is based on the work of Xu (See Theorem 2.1 in [25]).

Lemma 3.1. *Suppose that $u(x)$ is a normal solution to (1.2) with $Q(x)$ doesn't change sign near infinity. Then there exists a sequence $R_i \to \infty$ such that*

$$\lim_{i\to\infty}\sup \frac{4}{n!|\mathbb{S}^n|}\int_{B_{R_i}(0)} x\cdot\nabla Qe^{nu}\mathrm{d}x \leq \alpha(\alpha-2).$$

Proof. By a direct computation, one has

$$\langle x, \nabla u\rangle = -\frac{2}{(n-1)!|\mathbb{S}^n|}\int_{\mathbb{R}^n}\frac{\langle x, x-y\rangle}{|x-y|^2}Q(y)e^{nu(y)}\mathrm{d}y \tag{3.1}$$

Multiplying by $Qe^{nu(x)}$ and integrating over the ball $B_R(0)$ for any $R>0$, we have

$$\int_{B_R(0)} Qe^{nu(x)}\left[-\frac{2}{(n-1)!|\mathbb{S}^n|}\int_{\mathbb{R}^n}\frac{\langle x, x-y\rangle}{|x-y|^2}Q(y)e^{nu(y)}\mathrm{d}y\right]\mathrm{d}x = \int_{B_R(0)} Qe^{nu(x)}\langle x, \nabla u(x)\rangle\mathrm{d}x. \tag{3.2}$$

With $x=\frac{1}{2}\left((x+y)+(x-y)\right)$, for the left-hand side of (3.2), one has the following identity

$$\begin{aligned}LHS=&\frac{1}{2}\int_{B_R(0)}Qe^{nu(x)}\left[-\frac{2}{(n-1)!|\mathbb{S}^n|}\int_{\mathbb{R}^n}Qe^{nu(y)}\mathrm{d}y\right]\mathrm{d}x\\&+\frac{1}{2}\int_{B_R(0)}Qe^{nu(x)}\left[-\frac{2}{(n-1)!|\mathbb{S}^n|}\int_{\mathbb{R}^n}\frac{\langle x+y,x-y\rangle}{|x-y|^2}Qe^{nu(y)}\mathrm{d}y\right]\mathrm{d}x.\end{aligned}$$

Now, we deal with the last term of above equation by changing variables x and y.

$$\begin{aligned}&\int_{B_R(0)}Q(x)e^{nu(x)}\left[\int_{\mathbb{R}^n}\frac{\langle x+y,x-y\rangle}{|x-y|^2}Q(y)e^{nu(y)}\mathrm{d}y\right]\mathrm{d}x\\=&\int_{B_R(0)}Q(x)e^{nu(x)}\left[\int_{\mathbb{R}^n\backslash B_R(0)}\frac{\langle x+y,x-y\rangle}{|x-y|^2}Q(y)e^{nu(y)}\mathrm{d}y\right]\mathrm{d}x\\=&\int_{B_{R/2}(0)}Q(x)e^{nu(x)}\left[\int_{\mathbb{R}^n\backslash B_R(0)}\frac{\langle x+y,x-y\rangle}{|x-y|^2}Q(y)e^{nu(y)}\mathrm{d}y\right]\mathrm{d}x\\&+\int_{B_R(0)\backslash B_{R/2}(0)}Q(x)e^{nu(x)}\left[\int_{\mathbb{R}^n\backslash B_{2R}(0)}\frac{\langle x+y,x-y\rangle}{|x-y|^2}Q(y)e^{nu(y)}\mathrm{d}y\right]\mathrm{d}x\\&+\int_{B_R(0)\backslash B_{R/2}(0)}Q(x)e^{nu(x)}\left[\int_{B_{2R(0)}\backslash B_R(0)}\frac{\langle x+y,x-y\rangle}{|x-y|^2}Q(y)e^{nu(y)}\mathrm{d}y\right]\mathrm{d}x\\=:&I_1(R)+I_2(R)+I_3(R).\end{aligned}$$

Notice that

$$|I_1|\leq 3\int_{B_{R/2}(0)}|Q(x)|e^{nu(x)}\mathrm{d}x\int_{\mathbb{R}^n\backslash B_R(0)}|Q(y)|e^{nu(y)}\mathrm{d}y$$

and

$$|I_2|\leq 3\int_{B_R(0)\backslash B_{R/2}(0)}|Q(x)|e^{nu(x)}\mathrm{d}x\int_{\mathbb{R}^n\backslash B_{2R}(0)}|Q(y)|e^{nu(y)}\mathrm{d}y.$$

Then both $|I_1|$ and $|I_2|$ tend to zero as $R\to\infty$ due to $Qe^{nu}\in L^1(\mathbb{R}^n)$. Now, we only need to deal with the term I_3. Since Q doesn't change sign near infinity, for $R\gg 1$,

$$I_3(R)=\int_{B_R(0)\backslash B_{R/2}(0)}Q(x)e^{nu(x)}\left[\int_{B_{2R(0)}\backslash B_R(0)}\frac{x^2-y^2}{|x-y|^2}Q(y)e^{nu(y)}\mathrm{d}y\right]\mathrm{d}x\leq 0.$$

As for the right-hand side of (3.2), by using divergence theorem, we have

$$\begin{aligned}RHS=&\frac{1}{n}\int_{B_R(0)}Q(x)\langle x,\nabla e^{nu(x)}\rangle\mathrm{d}x\\=&-\int_{B_R(0)}\left(Q(x)+\frac{1}{n}\langle x,\nabla Q(x)\rangle\right)e^{nu(x)}\mathrm{d}x\\&+\frac{1}{n}\int_{\partial B_R(0)}Q(x)e^{nu(x)}R\mathrm{d}\sigma.\end{aligned}$$

Since $Q(x)e^{nu(x)}\in L^1(\mathbb{R}^n)$, there exist a sequence $R_i\to\infty$ such that

$$\lim_{i\to\infty}R_i\int_{\partial B_{R_i}(0)}Qe^{nu}\mathrm{d}\sigma=0.$$

Otherwise, there exists $\epsilon_0 > 0$ such that for large $R_1 > 1$ and any $r \geq R_1$, there holds $|\int_{\partial B_r(0)} Qe^{nu}\mathrm{d}\sigma| \geq \frac{\epsilon_0}{r}$ and then

$$|\int_0^R \int_{\partial B_r(0)} Qe^{nu}\mathrm{d}\sigma\mathrm{d}r| \geq -C + \int_{R_1}^R \frac{\epsilon_0}{r}\mathrm{d}r \geq -C + \epsilon_0 \log R$$

which contradicts to $Qe^{nu} \in L^1(\mathbb{R}^n)$. Thus there holds

$$\begin{aligned}\frac{1}{n}\int_{B_{R_i}(0)} \langle x\cdot\nabla Q\rangle e^{nu}\mathrm{d}x =& -\int_{B_{R_i}(0)} Qe^{nu}\mathrm{d}x + \frac{1}{n}R_i\int_{\partial B_{R_i}(0)} Qe^{nu}\mathrm{d}\sigma + \frac{\alpha}{2}\int_{B_{R_i}(0)} Qe^{nu(x)}\mathrm{d}x \\ &+ I_1(R_i) + I_2(R_i) + I_3(R_i) \\ \leq& -\int_{B_{R_i}(0)} Qe^{nu}\mathrm{d}x + \frac{1}{n}R_i\int_{\partial B_{R_i}(0)} Qe^{nu}\mathrm{d}\sigma + \frac{\alpha}{2}\int_{B_{R_i}(0)} Qe^{nu(x)}\mathrm{d}x \\ &+ I_1(R_i) + I_2(R_i)\end{aligned}$$

which yields that

$$\lim_{i\to\infty}\sup \frac{4}{n!|\mathbb{S}^n|}\int_{B_{R_i}(0)} x\cdot\nabla Qe^{nu}\mathrm{d}x \leq \alpha(\alpha-2).$$

□

Corollary 3.2. *Consider the smooth conformally invariant equation*

$$-\Delta u = Ke^{2u} \quad \text{on } \mathbb{R}^2 \tag{3.3}$$

with $Ke^{2u} \in L^1(\mathbb{R}^2)$ and $e^{2u} \in L^1(\mathbb{R}^2)$. Suppose that $x\cdot\nabla K \geq 0$ and $K(x)$ is non-negative near infinity. Then

$$\int_{\mathbb{R}^2} Ke^{2u}\mathrm{d}x \geq 4\pi$$

with " $=$ " holds if and only if K is a positive constant.

Proof. With help of Theorem 2.2 in [15], the solution to (3.3) must be a normal solution. Making use of Lemma 3.1, one has $\alpha \geq 2$ i.e.

$$\int_{\mathbb{R}^2} Ke^{2u}\mathrm{d}x \geq 4\pi.$$

When the equality achieves, one has $x\cdot\nabla K = 0$ a.e. which shows that $K(x)$ is a non-negative constant since K is non-negative near infinity. If $K(x) \equiv 0$, since $u(x)$ is normal, we have $u \equiv C$. However, it contradicts to $e^{2u} \in L^1(\mathbb{R}^2)$. Hence, $K(x)$ must be a positive constant.

On the other hand, if $K(x)$ is a positive constant, with help of the classification theorem in [4], one has

$$\int_{\mathbb{R}^2} Ke^{2u}\mathrm{d}x = 4\pi.$$

Finally, we finish our proof. □

Remark 3.3. By utilizing this result, we are able to partially answer the question raised by Gui and Moradifam in Remark 5.1 of their paper [9].

With additional assumptions, we are able to obtain the Pohozaev identity.

Lemma 3.4. *(See Lemma 2.1 in [13]) Consider a normal solution $u(x)$ to (1.2). Supposing that*

$$|Q(x)|e^{nu} \leq C|x|^{-n}$$

near infinity, then there exists a sequence $R_i \to \infty$ such that

$$\lim_{i\to\infty}\frac{4}{n!|\mathbb{S}^n|}\int_{B_{R_i}(0)} x\cdot\nabla Qe^{nu}\mathrm{d}x = \alpha(\alpha-2).$$

Proof. The proof is essentially the same as Lemma 3.1, except for the treatment of the term $I_3(R)$. Firstly, a direct computation yields that

$$|I_3| \leq \int_{B_R(0)\backslash B_{R/2}(0)} |Q(x)|e^{nu(x)} \int_{B_{2R(0)}\backslash B_R(0)} \frac{|x+y|}{|x-y|}|Q(y)|e^{nu(y)}\mathrm{d}y\mathrm{d}x.$$

For each $x \in B_R(0)\backslash B_{R/2}(0)$ and $R \gg 1$, based on the assumption $|Q|e^{nu(x)} \leq C|x|^{-n}$ near infinity, a direct computation yields that

$$\begin{aligned}
&\int_{B_{2R(0)}\backslash B_R(0)} \frac{|x+y|}{|x-y|}|Q(y)|e^{nu(y)}\mathrm{d}y \\
\leq & CR^{1-n}\int_{B_{2R(0)}\backslash B_R(0)} \frac{1}{|x-y|}\mathrm{d}y \\
\leq & CR^{1-n}\int_{B_{3R}(0)} \frac{1}{|y|}\mathrm{d}y \leq C.
\end{aligned}$$

Thus we obtain that

$$|I_3| \leq C\int_{B_R(0)\backslash B_{R/2}(0)} |Q|e^{nu}\mathrm{d}x \to 0, \text{ as } R \to \infty.$$

Continuing along the same line of reasoning as presented in Lemma 3.1, we ultimately demonstrate the existence of a sequence $R_i \to \infty$ such that

$$\lim_{i\to\infty} \frac{4}{n!|\mathbb{S}^n|}\int_{B_{R_i}(0)} x\cdot\nabla Q e^{nu}\mathrm{d}x = \alpha(\alpha-2).$$

□

4 Polynomial cone condition

Definition 4.1. We say a function $\psi(x) \in L^\infty_{loc}(\mathbb{R}^n)$ satisfying **s-cone** condition if there exists $s \in \mathbb{R}$, $0 < r_0 < 1$ such that

$$|\psi(x)| \leq C(|x|+1)^s$$

and a sequence $\{x_i\} \subset \mathbb{R}^n$ with $|x_i| \geq 1$ and $|x_i| \to \infty$ as $i \to \infty$ such that for each i and $x \in B_{r_0|x_i|}(x_i)$ there holds

$$\frac{|\psi(x)|}{|x|^s} \geq c_1 > 0$$

where c_1 is a constant independent of i.

The definition mentioned here is derived from [13] where the first author focused on scenarios where $Q(x)$ is a polynomial, which is a common case that meets the **s-cone** condition.

Lemma 4.2. *Each non-constant polynomial $P(x)$ on $\mathbb{R}^n$ satisfies* **s-cone** *condition with $s = \deg P$.*

Proof. We can decompose the non-constant polynomial $P(x)$ as

$$P(x) = H_s(x) + P_{s-1}(x)$$

where $H_s(x)$ is a homogeneous funtion with degree equal to $s \geq 1$ and $P_{s-1}(x)$ is a polynomial with degree at most $s-1$. Immediately, one has

$$|P(x)| \leq C(|x|+1)^s.$$

We choose the polar coordinate such that $x = \xi(r, \theta)$ with $r \geq 0$ and $\theta \in \mathbb{S}^{n-1}$. Then one has

$$H_s(x) = r^s \varphi_s(\theta)$$

where $\varphi_s(\theta)$ is a non-zero smooth function defined on $\mathbb{S}^{n-1}$. There exist $c_1 > 0$ and a geodesic ball $\hat{B}_{s_1}(\theta_0) \subset \mathbb{S}^{n-1}$ such that

$$|\varphi_s(\theta)| \geq 2c_1 > 0$$

for any $\theta \in \hat{B}_{s_1}(\theta_0)$. Then

$$|P(x)| \geq 2c_1|x|^s - c_2|x|^{s-1}$$

where $c_2 > 0$ is a constant depending only on the cofficients of $P_{s-1}(x)$. We choose $R_1 = \max\{1, \frac{c_2}{c_1}\}$ and then one has

$$|P(x)| \geq c_1|x|^s$$

for $(r, \theta) \in [R_1, +\infty) \times \hat{B}_{s_1}(\theta_0)$. It is not hard to see that there exist $x_0 \in \mathbb{R}^n$ with $|x_0| = 1$ and $0 < s_0 < 1$ such that

$$\xi^{-1}(B_{s_0}(x_0)) \subset [0, +\infty) \times \hat{B}_{s_1}(\theta_0).$$

Meanwhile, one may check that for any $t > 0$

$$B_{s_0 t}(tx_0) = tB_{s_0}(x_0)$$

where $tB_{s_0}(x_0) := \{tx \in \mathbb{R}^n | x \in B_{s_0}(x_0)\}$. For any $x \in B_{s_0 t}(tx_0)$, there holds $|x| \geq (1 - s_0)t$. Then there exists $t_1 > 0$ such that $t \geq t_1$

$$\xi^{-1}(B_{s_0 t}(tx_0)) \subset [R_1, +\infty) \times \hat{B}_{s_1}(\theta_0).$$

In particular, for any $t \geq t_1$ and $x \in B_{ts_0}(tx_0)$ one has

$$|P(x)| \geq c_1|x|^s.$$

Thus $P(x)$ satisfies **s-cone** condition with $s = \deg P$.

□

Lemma 4.3. *Consider the normal solution $u(x)$ to (1.2) with $Q(x)$ satisfying* **s-cone** *condition. Then*

$$\alpha \geq 1 + \frac{s}{n}.$$

Proof. Due to $Q(x)$ satisfying s-cone conditon, there exits a sequence $\{x_i\}$ and $0 < s_0 < 1$ such that in each ball $x \in B_{s_0|x_i|}(x_i)$

$$|Q(x)| \geq C|x|^s$$

With help of Lemma 2.4 and Jensen's inequality, one has

$$\begin{aligned}
\int_{B_{s_0|x_i|}(x_i)} |Q|e^{nu}\mathrm{d}x \geq & C \int_{B_{s_0|x_i|}(x_i)} |x|^s e^{nu}\mathrm{d}x \\
\geq & C|x_i|^s \int_{B_{s_0|x_i|}(x_i)} e^{nu}\mathrm{d}x \\
\geq & C|x_i|^s |B_{s_0|x_i|}(x_i)| \exp\left(⨍_{B_{s_0|x_i|}(x_i)} nu\mathrm{d}x \right) \\
\geq & C|x_i|^{s+n-n\alpha+o(1)}.
\end{aligned}$$

Due to $Qe^{nu} \in L^1(\mathbb{R}^n)$, letting $i \to \infty$, we have

$$\alpha \geq 1 + \frac{s}{n}.$$

□

Lemma 4.4. *Consider the normal solution $u(x)$ to (1.2) with $Q(x)$ satisfying* **s-cone** *condition. Supposing that there exists $s_1 < s$ such that $Q^+ \leq C|x|^{s_1}$ or $Q^- \leq C|x|^{s_1}$ near infinity, then there holds*

$$|Q(x)|e^{nu} \leq C|x|^{-n}$$

near infinity.

Proof. By a direct computation, we have

$$\begin{aligned}
&\frac{(n-1)!|\mathbb{S}^n|}{2}(u(x)+\alpha\log|x|)\\
=&\int_{\mathbb{R}^n}\log\frac{|x|\cdot(|y|+1)}{|x-y|}Qe^{nu}\mathrm{d}y+\int_{\mathbb{R}^n}\log\frac{|y|}{|y|+1}Qe^{nu}\mathrm{d}y+C\\
=&\int_{\mathbb{R}^n}\log\frac{|x|\cdot(|y|+1)}{|x-y|}Q^+e^{nu}\mathrm{d}y-\int_{\mathbb{R}^n}\log\frac{|x|\cdot(|y|+1)}{|x-y|}Q^-e^{nu}\mathrm{d}y+C\\
=&:I_1-II_2+C
\end{aligned}$$

For $|x|\geq 1$, it is easy to check that

$$\frac{|x|\cdot(|y|+1)}{|x-y|}\geq 1$$

which shows that

$$\log\frac{|x|\cdot(|y|+1)}{|x-y|}\geq 0$$

and then $I_1\geq 0$, $II_2\geq 0$.

From now on, we suppose that $|x|\gg 1$.

If $Q^+(x)\leq C|x|^{s_1}$ near infinity, we split I_1 as follows

$$I_1=\int_{|x-y|\leq\frac{|x|}{2}}\log\frac{|x|\cdot(|y|+1)}{|x-y|}Q^+e^{nu}\mathrm{d}y+\int_{|x-y|\geq\frac{|x|}{2}}\log\frac{|x|\cdot(|y|+1)}{|x-y|}Q^+e^{nu}\mathrm{d}y=:I_{1,1}+I_{1,2}$$

Using the estimate (2.8) and Lemma 2.7, a direct computation and Hölder's inequality yield that

$$\begin{aligned}
I_{1,1}=&\int_{|x-y|\leq\frac{|x|}{2}}\log(|x|(|y|+1))Q^+e^{nu}\mathrm{d}y+\int_{|x-y|\leq\frac{|x|}{2}}\log\frac{1}{|x-y|}Q^+e^{nu}\mathrm{d}y\\
\leq&\int_{|x-y|\leq\frac{|x|}{2}}\log(|x|(|y|+1))Q^+e^{nu}\mathrm{d}y+\int_{|x-y|\leq 1}\log\frac{1}{|x-y|}Q^+e^{nu}\mathrm{d}y\\
\leq&C\log(4|x|^2)|x|^{s_1}\int_{|x-y|\leq\frac{|x|}{2}}e^{nu}\mathrm{d}y+C|x|^{s_1}\int_{|x-y|\leq 1}\log\frac{1}{|x-y|}e^{nu}\mathrm{d}y\\
\leq&C\log(4|x|^2)|x|^{s_1}\int_{|x-y|\leq\frac{|x|}{2}}e^{nu}\mathrm{d}y+C|x|^{s_1}(\int_{|x-y|\leq 1}(\log\frac{1}{|x-y|})^2\mathrm{d}y)^{\frac{1}{2}}(\int_{|x-y|\leq 1}e^{2nu}\mathrm{d}y)^{\frac{1}{2}}\\
\leq&C\log(4|x|^2)|x|^{s_1}\int_{\frac{|x|}{2}\leq|y|\leq\frac{3|x|}{2}}e^{nu}\mathrm{d}y+C|x|^{s_1}|x|^{-n\alpha+o(1)}\\
\leq&C\log(4|x|^2)|x|^{s_1}|x|^{n-n\alpha+o(1)}+C|x|^{s_1}|x|^{-n\alpha+o(1)}
\end{aligned}$$

With help of Lemma 4.3 and $s>s_1$, there holds

$$I_{1,1}\leq C\log(4|x|^2)|x|^{s_1-s+o(1)}+C|x|^{-n+s_1-s+o(1)}$$

which shows that

$$I_{1,1}\in L^\infty(\mathbb{R}^n). \tag{4.1}$$

With help of Lemma 2.6 and Lemma 4.3, choosing $\epsilon = \frac{s-s_1}{2}$, there exist an integer $R_1 > 0$ such that for any $i \geq R_1 + 1$

$$\int_{B_i(0)\backslash B_{i-1}(0)} e^{nu}\mathrm{d}x \leq i^{n-1-n\alpha+\epsilon} \leq i^{-1-s+\epsilon}.$$

As for the second term $I_{1,2}$, for small $\epsilon > 0$, Then there holds

$$\begin{aligned}
I_{1,2} \leq &\int_{|x-y|\geq \frac{|x|}{2}} \log 2(|y|+1)Q^+e^{nu}\mathrm{d}y\\
\leq &\int_{\mathbb{R}^n} \log(2|y|+2)Q^+e^{nu}\mathrm{d}y\\
\leq &C + C\lim_{k\to\infty}\sum_{i=R_1+1}^{k}\log(2i+2)i^{s_1}\int_{B_i(0)\backslash B_{i-1}(0)} e^{nu}\mathrm{d}y\\
\leq &C + C\lim_{k\to\infty}\sum_{i=R_1+1}^{k}\log(2i+2)i^{s_1}i^{-1-s+\epsilon} < +\infty.
\end{aligned}$$

Combing with (4.1), one has

$$I_1 \in L^\infty(\mathbb{R}^n).$$

Then there holds

$$u(x) \leq -\alpha\log|x| + C. \tag{4.2}$$

Since $Q(x)$ satisfies **s-cone** condition, Lemma 4.3 and (4.2) imply that

$$|Q(x)|e^{nu} \leq C|x|^s \cdot |x|^{-n\alpha} \leq C|x|^{-n}.$$

On the other hand, if $Q^- \leq C|x|^{s_1}$ near infinity, using similar argument, one has $II_2 \leq C$ which yields that

$$u(x) \geq -\alpha\log|x| - C.$$

Due to $Q(x)$ satisfying **s-cone** condition, there holds

$$\begin{aligned}
\int_{B_{s_0|x_i|}(x_i)} |Q|e^{nu}\mathrm{d}x \geq &C\int_{B_{s_0|x_i|}(x_i)} |x|^s e^{nu}\mathrm{d}x\\
\geq &C|x_i|^s\int_{B_{s_0|x_i|}(x_i)} e^{nu}\mathrm{d}x\\
\geq &C|x_i|^{s+n-n\alpha}
\end{aligned}$$

which yields that

$$\alpha > 1 + \frac{s}{n}.$$

Since $|Q(x)| \leq C|x|^s$ near infinity and $\alpha > 1+\frac{s}{n}$, similar argument yields that $I_1 \in L^\infty(\mathbb{R}^n)$. Finally, we obtain that

$$|u + \alpha\log|x|| \leq C.$$

Thus one has

$$|Q(x)|e^{nu}|x|^n \leq C|x|^{s+n-n\alpha} = o(1).$$

Finally, we finish our proof. □

As an application, we generalize Corollary 2.4 in [13].

Theorem 4.5. *Suppose that a smooth function $Q(x)$ satisfying* **s-cone** *condition with $s \geq n$ as well as*

$$x \cdot \nabla Q \leq 0.$$

There is no normal solution $u(x)$ to (1.2) on $\mathbb{R}^n$ with $Qe^{nu} \in L^1(\mathbb{R}^n)$ where even integer $n \geq 2$.

Proof. We argue by contradiction. Assume that such solution $u(x)$ exists. Since $x \cdot \nabla Q \leq 0$, we have $Q \leq C$. Based on Q satisfying **s-cone** condition with $s \geq n$, with help of Lemma 4.4 and Lemma 3.4, there exists a sequence $R_i \to \infty$ such that

$$\lim_{i \to \infty} \frac{4}{n!|\mathbb{S}^n|} \int_{B_{R_i}(0)} x \cdot \nabla Q e^{nu} \mathrm{d}x = \alpha(\alpha - 2).$$

Due to $x \cdot \nabla Q \leq 0$, we further have

$$\frac{4}{n!|\mathbb{S}^n|} \int_{\mathbb{R}^n} x \cdot \nabla Q e^{nu} \mathrm{d}x = \alpha(\alpha - 2)$$

which yields that

$$0 < \alpha < 2 \tag{4.3}$$

since Q is obviously not a constant. However, Lemma 4.3 yields that

$$\alpha \geq 1 + \frac{s}{n} \geq 2$$

which contradicts to (4.3).

□

In particular, suppose that $Q(x)$ is a non-constant polynomial splitting as

$$Q(x) = H_m(x) + P_{m-1}(x)$$

where H_m is a homogeneous function and $P_{m-1}(x)$ is a polynomial of degree at most $m-1$. If either $H_m(x) \geq 0$ or $H_m(x) \leq 0$, then we have either $Q^- \leq C|x|^{m-1}$ or $Q^+ \leq C|x|^{m-1}$ near infinity respectively.

5 Bol's inequality for polynomial Q-curvature

For readers' convenience, we establish the modified Ding's lemma (See Lemma 1.1 in [4]). It has also been established in Proposition 8.5 of [8] or Lemma 2.6 of [16].

Lemma 5.1. *Consider a smooth solution $u(x)$ to the following equation*

$$-\Delta u = fe^{2u} \quad \text{on } \mathbb{R}^2$$

with smooth function $f \leq 1$, then there holds

$$\int_{\mathbb{R}^2} e^{2u} \mathrm{d}x \geq 4\pi$$

when the equality achieves, one has $f \equiv 1$.

Proof of Theorem 1.2:

Proof. We only need to deal with the case $e^{4u}\in L^1(\mathbb{R}^4)$.

Firstly, if $Q(x)$ is a constant, Theorem 2.9 deduces that $Q(x)$ must be a positive constant. The classification theorem in [17] shows that

$$\int_{\mathbb{R}^4} Qe^{4u}\mathrm{d}x = 6|\mathbb{S}^4|$$

which yields that

$$\int_{\mathbb{R}^4} e^{4u}\mathrm{d}x \geq |\mathbb{S}^4|$$

with " $=$ " holds if and only if $Q\equiv 6$.

Secondly, if $Q(x)$ is a non-constant polynomial with $Q(x)\leq 6$, then the degree of $Q(x)$ must be an even integer. Lemma 4.2 yields that $Q(x)$ satisfying **s-cone** condition with $s=\deg Q$. If $\deg Q\geq 4$, Lemma 4.3 concludes that

$$\int_{\mathbb{R}^4} Qe^{4u}\mathrm{d}x \geq 6|\mathbb{S}^4|.$$

Since non-constant $Q(x)\leq 6$, we obtain that

$$\int_{\mathbb{R}^4} e^{4u}\mathrm{d}x > |\mathbb{S}^4|.$$

Finally, the remaining case is $\deg Q(x)=2$. Since $Q(x)\leq 6$, up to a rotation and a translation of the coordinates, we may suppose that

$$Q(x)=a+\sum_{i=1}^4 a_i x_i^2$$

where $a_i\leq 0$, $\sum_{i=1}^4 a_i^2\neq 0$ and $a\leq 6$. Using $e^{4u}\in L^1(\mathbb{R}^4)$ and $Qe^{4u}\in L^1(\mathbb{R}^4)$, one has

$$\sum_{i=1}^4 a_i x_i^2 e^{4u}\in L^1(\mathbb{R}^4)$$

which yields that $x\cdot\nabla Q e^{4u}\in L^1(\mathbb{R}^4)$. Applying Lemma 3.4 and Lemma 4.4, there holds

$$\alpha(\alpha-2)=\frac{1}{16\pi^2}\int_{\mathbb{R}^4} x\cdot\nabla Q e^{4u}\mathrm{d}x=\frac{1}{8\pi^2}\int_{\mathbb{R}^4}\sum_{i=1}^4 a_i x_i^2 e^{4u}\mathrm{d}x<0. \tag{5.1}$$

Combing (5.1) with Lemma 4.3, one has

$$\frac{3}{2}\leq\alpha<2. \tag{5.2}$$

Immediately, one has $a>0$. Using the identity (5.1) again, one has

$$\int_{\mathbb{R}^4} e^{4u}\mathrm{d}x=\frac{8\pi^2}{a}\alpha(3-\alpha).$$

Making using of the estimate (5.2) and $a\leq 6$, there holds

$$\int_{\mathbb{R}^4} e^{4u}\mathrm{d}x>\frac{8\pi^2}{3}=|\mathbb{S}^4|.$$

Finally, we finish our proof.

□

Proof of Theorem 1.3:

Proof. If $\int_{\mathbb{R}^n} e^{nu}\mathrm{d}x = +\infty$, it is trvial that the estimate (1.4) holds . We just need to consider the case $e^{nu} \in L^1(\mathbb{R}^n)$. Then one has

$$\int_{\mathbb{R}^n} |\varphi| e^{nu}\mathrm{d}x \leq (n-1)! \int_{\mathbb{R}^n} e^{nu}\mathrm{d}x + \int_{\mathbb{R}^n} |(n-1)! + \varphi(x)| e^{nu}\mathrm{d}x < +\infty.$$

If φ is a constant, with help of Theorem 2.9 and the the classification theorem in [22], [19] or [25], we must have $\varphi > -(n-1)!$ and

$$\int_{\mathbb{R}^n} e^{nu}\mathrm{d}x = \frac{(n-1)!}{(n-1)! + \varphi}|\mathbb{S}^n| \geq |\mathbb{S}^n|$$

with equality holds if and only if $\varphi \equiv 0$.

Now, we are going to deal with the non-constant case. Due to $\varphi \leq 0$, the degree of φ must be an even integer. If $\deg \varphi = 2$, following the same argument in the proof of Theorem 1.2, there holds

$$\int_{\mathbb{R}^n} e^{nu}\mathrm{d} > |\mathbb{S}^n|.$$

When $\deg(\varphi) \geq n$, with help of Lemma 4.3, one has $\alpha \geq 2$. Then there holds

$$\int_{\mathbb{R}^n} e^{nu}\mathrm{d}x > \int_{\mathbb{R}^n} \frac{(n-1)! + \varphi}{(n-1)!} e^{nu}\mathrm{d}x \geq |\mathbb{S}^n|.$$

If $4 \leq \deg \varphi < n$ and $\alpha \geq 2$, due to the same reason, the estimate (1.4) still holds. Thus the remaining case is $4 \leq \deg \varphi \leq n-2$ and $\alpha < 2$. With help of Lemma 4.3, there holds

$$1 + \frac{\deg \varphi}{n} \leq \alpha < 2. \tag{5.3}$$

Making use of Lemma 3.4, there exists a sequence $R_i \to \infty$ as $i \to \infty$ such that

$$\alpha(\alpha - 2) = \frac{4}{n!|\mathbb{S}^n|} \lim_{i\to\infty} \int_{B_{R_i}(0)} x \cdot \nabla\varphi e^{nu}\mathrm{d}x.$$

Using the assumption $x \cdot \nabla\varphi \geq \deg(\varphi)\varphi$, one has

$$\int_{B_{R_i}(0)} x \cdot \nabla\varphi e^{nu}\mathrm{d}x \geq \deg(\varphi) \int_{B_{R_i}(0)} \varphi e^{nu}\mathrm{d}x.$$

Then a direct computation yields that

$$\begin{aligned}\alpha(\alpha-2) \geq& \frac{4}{n!|\mathbb{S}^n|}\deg(\varphi)\int_{\mathbb{R}^n} \varphi e^{nu}\mathrm{d}x \\ =& \frac{2\deg(\varphi)}{n}\alpha - \frac{4\deg\varphi}{n|\mathbb{S}^n|}\int_{\mathbb{R}^n} e^{nu}\mathrm{d}x\end{aligned}$$

which shows that

$$\int_{\mathbb{R}^n} e^{nu}\mathrm{d}x \geq \frac{n|\mathbb{S}^n|}{4\deg\varphi}\alpha(2 + \frac{2\deg\varphi}{n} - \alpha).$$

Using (5.3), one has

$$\int_{\mathbb{R}^n} e^{nu}\mathrm{d}x > |\mathbb{S}^n|.$$

Finally, we finish our proof. □

6 Bol's inequality for radial solutions

Proof of Theorem 1.1:

Proof. We just need to deal with the case $e^{nu} \in L^1(\mathbb{R}^n)$. Set

$$v(x) := \frac{2}{(n-1)!|\mathbb{S}^n|} \int_{\mathbb{R}^n} \log \frac{|y|}{|x-y|} (n-1)! e^{nu(y)} \mathrm{d}y. \tag{6.1}$$

Firstly, we claim that $v(x)$ is also radial symmetric. For any rotation T, we have $u(Ty) = u(y)$ due to $u(x)$ is radial symmetric. In fact, by rotating the coordinates, there holds

$$\begin{aligned} v(Tx) =& \frac{2}{(n-1)!|\mathbb{S}^n|} \int_{\mathbb{R}^n} \log \frac{|y|}{|Tx-y|} (n-1)! e^{nu(y)} \mathrm{d}y \\ =& \frac{2}{(n-1)!|\mathbb{S}^n|} \int_{\mathbb{R}^n} \log \frac{|Ty|}{|Tx-Ty|} (n-1)! e^{nu(Ty)} \mathrm{d}y \\ =& \frac{2}{(n-1)!|\mathbb{S}^n|} \int_{\mathbb{R}^n} \log \frac{|y|}{|x-y|} (n-1)! e^{nu(y)} \mathrm{d}y \\ =& v(x). \end{aligned}$$

Setting $h := u - v$, for $n = 2$, there holds

$$\Delta h = (1-Q)e^{nu} \geq 0. \tag{6.2}$$

For $n \geq 4$, a direct computation yields that

$$\Delta h = \frac{2(n-2)}{(n-1)!|\mathbb{S}^n|} \int_{\mathbb{R}^n} \frac{1}{|x-y|^2} \left((n-1)! - Q\right) e^{nu} \mathrm{d}y \geq 0 \tag{6.3}$$

and there holds

$$(-\Delta)^{\frac{n}{2}} v = (n-1)! e^{nh} e^{nv}.$$

Combing with (6.1), we find that v is a normal solution with Q-curvature equal to $(n-1)!e^{nh}$. For brevity, we denote the normalized integrated Q-curvature as

$$\alpha_0 := \frac{2}{(n-1)!|\mathbb{S}^n|} \int_{\mathbb{R}^n} (n-1)! e^{nh} e^{nv} \mathrm{d}x = \frac{2}{|\mathbb{S}^n|} \int_{\mathbb{R}^n} e^{nu} \mathrm{d}x.$$

By using Lemma 3.1, there exists a sequence $R_i \to \infty$ such that

$$\lim_{i \to \infty} \frac{4}{n!|\mathbb{S}^n|} \int_{B_{R_i}(0)} x \cdot \nabla((n-1)! e^{nh}) e^{nv} \mathrm{d}x \leq \alpha_0(\alpha_0 - 2). \tag{6.4}$$

Using divergence theorem and the condition that $u(x)$ is radially symmetric, there holds

$$\begin{aligned} & \frac{4}{n!|\mathbb{S}^n|} \int_{B_{R_i}(0)} x \cdot \nabla((n-1)! e^{nh}) e^{nv} \mathrm{d}x \\ =& \frac{4}{|\mathbb{S}^n|} \int_{B_{R_i}(0)} x \cdot \nabla h e^{nu} \mathrm{d}x \\ =& \frac{4}{|\mathbb{S}^n|} \int_0^{R_i} \int_{\mathbb{S}^{n-1}(r)} r \frac{\partial h}{\partial r} e^{nu} \mathrm{d}\sigma \mathrm{d}r \\ =& \frac{4}{|\mathbb{S}^n|} \int_0^{R_i} r e^{nu(r)} \int_{\mathbb{S}^{n-1}(r)} \frac{\partial h}{\partial r} \mathrm{d}\sigma \mathrm{d}r \end{aligned}$$

$$
\begin{aligned}
=&\frac{4}{|\mathbb{S}^n|}\int_0^{R_i} re^{nu(r)}\int_{B_r(0)}\Delta h\mathrm{d}x\mathrm{d}r\\
\geq&0
\end{aligned}
$$

where the last term comes from (6.2) and (6.3). Finally, the estimate (6.4) yields that

$$\alpha_0 \geq 2$$

i.e.

$$\int_{\mathbb{R}^n} e^{nu}\mathrm{d}x \geq |\mathbb{S}^n|$$

when the equality achieves, one has $\Delta h \equiv 0$ which yields that $Q \equiv (n-1)!$.

On the other hand, if $Q \equiv (n-1)!$, using Lemma 3.4 and Lemma 4.4, we also obtian that $\alpha_0 = 2$.

□

Proof of Theorem 1.4:

Proof. Due to the assumptions $Qe^{nu} \in L^1(\mathbb{R}^n)$ and $Q \geq (n-1)!$, one has $e^{nu} \in L^1(\mathbb{R}^n)$. Making use of Lemma 2.3, for $|x| \gg 1$, there holds

$$
\begin{aligned}
\int_{B_{\frac{|x|}{2}}(x)} e^{nu}\mathrm{d}y \geq& C\int_{B_{\frac{|x|}{2}}(x)} |y|^{-n\alpha}\mathrm{d}y\\
\geq& C|x|^{n-n\alpha}
\end{aligned}
$$

which yields that

$$\alpha > 1. \tag{6.5}$$

Using the assumption $(n-1)! \leq Q(x) \leq C(|x|+1)^k$, we claim that

$$u(x) = (-\alpha + o(1))\log|x|. \tag{6.6}$$

For $|x| \gg 1$, using $Qe^{nu} \in L^1(\mathbb{R}^n)$, there holds

$$|\int_{|x|^{-2(k+1)}\leq|x-y|\leq 1}\log\frac{1}{|x-y|}Qe^{nu}\mathrm{d}y| \leq 2(k+1)\left(\int_{|x|^{-2(k+1)}\leq|x-y|\leq 1}|Q|e^{nu}\mathrm{d}y\right)\log|x| = o(1)\log|x|$$

With help of Lemma 2.1, one has

$$u(x) = (-\alpha + o(1))\log|x| + \frac{2}{(n-1)!|\mathbb{S}^n|}\int_{|x-y|\leq|x|^{-2(k+1)}}\log\frac{1}{|x-y|}Qe^{nu}\mathrm{d}y. \tag{6.7}$$

Using the assumption $Q \leq C(|x|+1)^k$ and the estimate (2.7), for $|x| \gg 1$, Holder's inequality yields that

$$
\begin{aligned}
&\int_{|x-y|\leq|x|^{-2(k+1)}}\log\frac{1}{|x-y|}Qe^{nu}\mathrm{d}y\\
\leq& C(|x|+1)^k\left(\int_{|x-y|\leq|x|^{-2(k+1)}}(\log\frac{1}{|x-y|})^2\mathrm{d}y\right)^{\frac{1}{2}}\left(\int_{|x-y|\leq|x|^{-2(k+1)}}e^{2nu}\mathrm{d}y\right)^{\frac{1}{2}}\\
\leq& C|x|^k\left(\int_{|y|\leq|x|^{-2(k+1)}}(\log\frac{1}{|y|})^2\mathrm{d}y\right)^{\frac{1}{2}}\left(\int_{B_{1/4}(x)}e^{2nu}\mathrm{d}y\right)^{\frac{1}{2}}
\end{aligned}
$$

$$
\begin{aligned}
&\leq C|x|^k(\int_{(2k+2)\log|x|}^{\infty} e^{-nt}t^2\mathrm{d}t)^{\frac{1}{2}}|x|^{-\alpha+o(1)}\\
&\leq C|x|^k|x|^{-n(2k+2)}(\log|x|)^2|x|^{-\alpha+o(1)}.
\end{aligned}
$$

Using above estimate and the (6.5) as well as $Q>0$, for $|x|\gg 1$, one has

$$
0\leq \int_{|x-y|\leq |x|^{-2(k+1)}} \log\frac{1}{|x-y|}Qe^{nu}\mathrm{d}y\leq C.
$$

Due to (6.7), we prove our claim (6.6). Moreover, using (6.6) and (6.5), one has

$$
e^{nu}\leq C|x|^{-n} \tag{6.8}
$$

near infinity. Set

$$
v(x):=\frac{2}{(n-1)!|\mathbb{S}^n|}\int_{\mathbb{R}^n}\log\frac{|y|}{|x-y|}(n-1)!e^{nu(y)}\mathrm{d}y.
$$

Firstly, we know that $v(x)$ is also radial symmetric following the same argument before. Samely, setting $h:=u-v$, for $n=2$, one has

$$
\Delta h=(1-Q)e^{2u}\leq 0 \tag{6.9}
$$

For $n\geq 4$, one has

$$
\Delta h=\frac{2(n-2)}{(n-1)!|\mathbb{S}^n|}\int_{\mathbb{R}^n}\frac{1}{|x-y|^2}\left((n-1)!-Q\right)e^{nu}\mathrm{d}y\leq 0 \tag{6.10}
$$

and there holds

$$
(-\Delta)^{\frac{n}{2}}v=(n-1)!e^{nh}e^{nv}.
$$

For brevity, we denote

$$
\alpha_0:=\frac{2}{(n-1)!|\mathbb{S}^n|}\int_{\mathbb{R}^n}(n-1)!e^{nh}e^{nv}\mathrm{d}x=\frac{2}{|\mathbb{S}^n|}\int_{\mathbb{R}^n}e^{nu}\mathrm{d}x.
$$

By using Lemma 3.4 and the estimate (6.8), there exists a sequence $R_i\to\infty$ such that

$$
\lim_{i\to\infty}\frac{4}{n!|\mathbb{S}^n|}\int_{B_{R_i}(0)}x\cdot\nabla((n-1)!e^{nh})e^{nv}\mathrm{d}x=\alpha_0(\alpha_0-2). \tag{6.11}
$$

Using divergence theorem, there holds

$$
\begin{aligned}
&\frac{4}{n!|\mathbb{S}^n|}\int_{B_{R_i}(0)}x\cdot\nabla((n-1)!e^{nh})e^{nv}\mathrm{d}x\\
=&\frac{4}{|\mathbb{S}^n|}\int_{B_{R_i}(0)}x\cdot\nabla h e^{nu}\mathrm{d}x\\
=&\frac{4}{|\mathbb{S}^n|}\int_0^{R_i}\int_{\mathbb{S}^{n-1}(r)}r\frac{\partial h}{\partial r}e^{nu}\mathrm{d}\sigma\mathrm{d}r\\
=&\frac{4}{|\mathbb{S}^n|}\int_0^{R_i}re^{nu(r)}\int_{\mathbb{S}^{n-1}(r)}\frac{\partial h}{\partial r}\mathrm{d}\sigma\mathrm{d}r\\
=&\frac{4}{|\mathbb{S}^n|}\int_0^{R_i}re^{nu(r)}\int_{B_r(0)}\Delta h\mathrm{d}x\mathrm{d}r\\
\leq&0.
\end{aligned}
$$

Finally, the estimate (6.11) yields that

$$\alpha_0 \le 2$$

i.e.

$$\int_{\mathbb{R}^n} e^{nu} \mathrm{d}x \le |\mathbb{S}^n|$$

when the equality achieves, one has $\Delta h \equiv 0$ which yields that $Q \equiv (n-1)!$. On the other hand, if $Q \equiv (n-1)!$, due to the same reason as before, one has

$$\int_{\mathbb{R}^n} e^{nu} \mathrm{d}x = |\mathbb{S}^n|.$$

□

7 Applications

We are going to show some applications of these higher order Bol's inequality. In [20], Poliakovsky and Tarantello consider the equation

$$-\Delta u = (1 + |x|^{2p}) e^{2u} \tag{7.1}$$

where $p > 0$ and $x \in \mathbb{R}^2$ with $\beta = \frac{1}{2\pi} \int_{\mathbb{R}^2} (1 + |x|^{2p}) e^{2u} \mathrm{d}x < +\infty$ to study selfgravitating strings. They make use of very techinial methods to show that the existence of radial solution if and only if

$$\max\{2, 2p\} < \beta < 2 + 2p. \tag{7.2}$$

Here, we will use Theorem 1.4 to show for each radial solution, the estimate (7.2) holds. Firstly, due to $(1 + |x|^{2p}) e^{2u} \in L^1(\mathbb{R}^2)$, Theorem 2.2 in [15] shows that the solution $u(x)$ is normal. Making use of the following theorem, we can easily show that (7.2) is necessary for the existence of radial solutions.

For higher order cases, we will consider the normal solutions

$$u(x) = \frac{2}{(n-1)!|\mathbb{S}^n|} \int_{\mathbb{R}^n} \log \frac{|y|}{|x-y|} (1 + |x|^{np}) e^{nu} \mathrm{d}x + C \tag{7.3}$$

with $(1 + |x|^{np}) e^{nu} \in L^1(\mathbb{R}^n)$. Set the same notation as before

$$\beta := \frac{2}{(n-1)!|\mathbb{S}^n|} \int_{\mathbb{R}^n} (1 + |x|^{np}) e^{nu} \mathrm{d}x.$$

Theorem 7.1. *For $p > 0$, consider the normal solution to (7.3) with even integer $n \ge 2$. For each radial solution to (7.3), there holds*

$$\max\{2, 2p\} < \beta < 2 + 2p.$$

Proof. Making use of Lemma 3.4 and Lemma 4.4, the following Pohozaev identity holds

$$\beta(\beta - 2) = \frac{4p}{(n-1)!|\mathbb{S}^n|} \int_{\mathbb{R}^n} |x|^{np} e^{nu} \mathrm{d}x \tag{7.4}$$

which yields that $\beta > 2$. It is obvious to see that

$$\frac{4p}{(n-1)!|\mathbb{S}^n|} \int_{\mathbb{R}^n} |x|^{np} e^{nu} \mathrm{d}x < \frac{4p}{(n-1)!|\mathbb{S}^n|} \int_{\mathbb{R}^n} (1 + |x|^{np}) e^{nu} \mathrm{d}x = 2p\beta.$$

Thus the identity (7.4) yields that

$$2 < \beta < 2 + 2p. \tag{7.5}$$

On the other hand, the identity (7.4) is equivalent to

$$\int_{\mathbb{R}^n} e^{nu}\mathrm{d}x = \frac{(n-1)!|\mathbb{S}^n|}{4p}\beta(2+2p-\beta). \tag{7.6}$$

For each radial solution to (7.3), sicne $1+|x|^{np}\geq 1$, Theorem 1.4 yields that

$$\int_{\mathbb{R}^n} e^{nu}\mathrm{d}x < (n-1)!|\mathbb{S}^n|. \tag{7.7}$$

Combing (7.6) with (7.7), one has

$$(\beta-2)(\beta-2p) > 0.$$

With help of (7.5), we finally have

$$\max\{2, 2p\} < \beta < 2+2p.$$

□

With help of above theorem, we answer an open problem in [12]. Combing with Theorem 1.5 in [12], we obtain the following result which generalize the reuslt of Poliakovsky and Tarantello in [20] for $n=2$ to higher order cases.

Corollary 7.2. *For $n=4$, the integeral equation (7.3) has radial normal solutions if and only if*

$$\max\{2, 2p\} < \beta < 2+2p.$$

Acknowledgements: The first author would like to thank Professor Xingwang Xu for his helpful discussion and constant encouragement. Besides, the author also wants to thank Professor Yuxin Ge for his useful discussion and hospitality in University Paul Sabatier in Toulouse.

References

[1] C. Bandle, On a differential inequality and its applications to geometry. Math. Z. 147 (1976), no. 3, 253261.

[2] T. P. Branson, Differential operators canonically associated to a conformal structure. Math. Scand. 57:2 (1985), 293345.

[3] S.-Y. A. Chang and W. Chen, A note on a class of higher order conformally covariant equations. Discrete Contin. Dynam. Systems 7 (2001), no. 2, 275281.

[4] W. Chen and C. Li, Classification of solutions of some nonlinear elliptic equations. Duke Math. J. 63,615-622(1991).

[5] S.-Y. A. Chang and P. C. Yang, Extremal metrics of zeta function determinants on 4-manifolds. Ann. of Math. (2) 142 (1995), no. 1, 171212.

[6] Z. Djadli and A. Malchiodi, Existence of conformal metrics with constant Q-curvature. Ann. of Math. (2) 168 (2008), no. 3, 813858.

[7] C. Gui, F. Hang and A. Moradifam, The sphere covering inequality and its dual. Comm. Pure Appl. Math. 73 (2020), no. 12, 26852707.

[8] C. Gui and Q. Li, Some geometric inequalities related to Liouville equation. arXiv:2208.03612.

[9] C. Gui and A. Moradifam, The sphere covering inequality and its applications, Invent. math. (2018) 214:11691204.

[10] X. Huang and D. Ye, Conformal metrics in $\mathbb{R}^{2m}$ with constant Q-curvature and arbitrary volume. Calc. Var. Partial Differential Equations 54 (2015), no. 4, 33733384.

[11] A. Hyder, Conformally Euclidean metrics on $\mathbb{R}^n$ with arbitrary total Q-curvature. Anal. PDE 10 (2017), no. 3, 635652.

[12] A. Hyder and L. Martinazzi, Normal conformal metrics on $\mathbb{R}^4$ with Q-curvature having power-like growth. J. Differential Equations 301 (2021), 3772.

[13] M. Li, A Liouville-type theorem in conformally invariant equations, arXiv:2306.15754.

[14] M. Li, Asymptotic behavior of conformal metrics with null Q-curvature, preprint.

[15] M. Li, The total Q-curvature, volume entropy and polynomial growth polyharmonic functions, arXiv:2306.15623.

[16] M. Li and X. Xu, Asymptotic behavior of conformal metrics on torus, preprint.

[17] C. Lin, A classification of solutions of a conformally invariant fourth order equation in R^n. Comment. Math. Helv. 73 (1998), no. 2, 206231.

[18] L. Martinazzi, Conformal metrics on $\mathbb{R}^{2m}$ with constant Q-curvature and large volume. Ann. Inst. Henri Poincar, Anal. Non Linaire 30, No. 6, 969-982 (2013).

[19] L. Martinazzi, Classification of solutions to the higher order Liouville's equation on $\mathbb{R}^{2m}$. Math. Z. 263 (2009), no. 2, 307329.

[20] A. Poliakovsky and G. Tarantello, On a planar Liouville-type problem in the study of selfgravitating strings, J. Differ. Equations 252, No. 5, 3668-3693 (2012).

[21] T. Suzuki, Global analysis for a two-dimensional elliptic eigenvalue problem with the exponential nonlinearity. Ann. Inst. H. Poincar C Anal. Non Linaire 9 (1992), no. 4, 367397.

[22] J. Wei and X. Xu, Classification of solutions of higher order conformally invariant equations. Math. Ann. 313 (1999), no. 2, 207228.

[23] J. Wei and X. Xu, On conformal deformations of metrics on S^n, J. Funct. Anal. 157, No. 1, 292-325 (1998).

[24] J. Wei and D. Ye, Nonradial solutions for a conformally invariant fourth order equation in $\mathbb{R}^4$. Calc. Var. Partial Differential Equations 32 (2008), no. 3, 373386.

[25] X. Xu, Uniqueness and non-existence theorems for conformally invariant equations. J. Funct. Anal. 222 (2005), no.1, 128.