HOPF BIFURCATIONS FROM SPIKE SOLUTIONS FOR THE WEAK COUPLING GIERER-MEINHARDT SYSTEM

DANIEL GOMEZ, LINFENG MEI, AND JUNCHENG WEI

Abstract. The Hopf bifurcation for the classical Gierer-Meinhardt system in an one-dimensional interval is considered. The existence of time-periodic solution near the Hopf bifurcation parameter for a boundary spike is rigorously proved by the classical Crandall-Rabinowitz theory. The criteria for the stability of limit cycle is determined and it is numerically shown that the Hopf bifurcation is *subcritical*.

1. Introduction

In this paper we consider the following canonical one-dimensional Gierer-Meinhardt system ([6] [12])

$$\begin{cases} \tilde{A}_t = \epsilon^2 \tilde{A}_{xx} - \tilde{A} + \dfrac{\tilde{A}^2}{\tilde{H}}, & \tilde{A} > 0 \quad \text{for } 0 < x < 1,\ t > 0, \\ \tau \tilde{H}_t = D\tilde{H}_{xx} - \tilde{H} + \tilde{A}^2, & \tilde{H} > 0 \quad \text{for } 0 < x < 1,\ t > 0, \\ \tilde{A}_x = \tilde{H}_x = 0, & \text{for } x = 0, 1,\ t \geq 0, \end{cases} \tag{1.1}$$

where the unknowns $\tilde{A} = \tilde{A}(x,t)$ and $\tilde{H} = \tilde{H}(x,t)$ characterize the concentrations of the activator and inhibitor at a point $x \in (0,1)$ and at a time $t > 0$. Throughout this paper, we assume that

$\epsilon > 0$ is a small parameter independent of x and t,
$\tau > 0$ is a fixed constant independent of x, t and ϵ, and
$D > 0$ depends on ϵ but is independent of x and t.

We further assume that $D = D(\epsilon) \to \infty$ as $\epsilon \to 0$ and call this the weak coupling, or shadow limit, case.

Using the reduction techniques of [22], one can easily show that the stationary system of (1.1) has solutions with a single boundary spike at $x = 0$, as $\epsilon \to 0$ and $D = D(\epsilon) \to \infty$ at a suitable speed. (See also early work [17].) Since we consider a boundary single spike solution at $x = 0$, it is convenient to consider the even extension (with respect to the spatial variable x) of the system (1.1) on the interval $[-1, 1]$. In this case the spike solution becomes symmetric about $x = 0$.

The aim of this paper is to rigorously prove that, for $\epsilon > 0$ sufficiently small, there exists a Hopf bifurcation threshold for τ beyond which a time-periodic solution of (1.1) bifurcates from the single spike stationary solution. In addition, we prove that this Hopf bifurcation is *subcritical*, i.e. the bifurcating time-periodic solution is unstable. Previous studies into Hopf bifurcations for the one-dimensional Gierer-Meinhardt have used matched asymptotic expansions to derive leading order nonlocal eigenvalue problems (NLEPs) with purely imaginary eigenvalues for specific, numerically computed, values of τ [20, 21]. The numerical simulations in these studies suggest that the Hopf bifurcation is subcritical, though a rigorous proof has not yet been given. The

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aim of this paper is to give the first rigorous proof of the existence of time-periodic patterns and its sub-criticality.

To prove the existence, uniqueness, and stability of the Hopf bifurcation we use the classical Crandall-Rabinowitz bifurcation theory ([1]). More precisely we use a more concise formulation given in Theorem I.8.2 of [9]. The linear stability of the bifurcating periodic solutions is obtained using Corollary I.12.3 in [9]. Specifically, stability is determined by the sign of certain Floquet multipliers relative to a transversality condition. To apply these results we need to write (1.1) in the form of an evolution equation

$$U_t = \mathcal{F}_\epsilon(U) \equiv \mathcal{L}_\epsilon U + R(\tau, U), \tag{1.2}$$

where

$$U = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} \tilde{A} - A_\epsilon \\ \tilde{H} - H_\epsilon \end{bmatrix}$$

and

$$\mathcal{L}_\epsilon = \begin{bmatrix} \mathcal{L}_1 \\ \mathcal{L}_2 \end{bmatrix} = \begin{bmatrix} \epsilon^2 \frac{d^2}{dx^2} - 1 + \frac{2A_\epsilon}{H_\epsilon} & -\frac{A_\epsilon^2}{H_\epsilon^2} \\ \frac{2}{\tau} A_\epsilon & \frac{1}{\tau}(D\frac{d^2}{dx^2} - 1) \end{bmatrix}$$

denote the perturbation and linearization about the stationary single-spike solution $(A_\epsilon, H_\epsilon)^T$ respectively, and $R(\tau, U)$ indicates the remaining higher order term

$$R(\tau, U) = \begin{bmatrix} \frac{(A_\epsilon + U_1)^2}{H_\epsilon + U_2} - \frac{A_\epsilon^2}{H_\epsilon} - \frac{2A_\epsilon U_1}{H_\epsilon} + \frac{A_\epsilon^2 U_2}{H_\epsilon^2} \\ \frac{1}{\tau} U_1^2 \end{bmatrix}. \tag{1.3}$$

To motivate the remaining sections we outline briefly the key components of the Hopf bifurcation theorem derived in [9]. This theorem states that under suitable spectral conditions on the operator $\mathcal{L}_\epsilon$ at some critical parameter $\tau := \tau_\epsilon^h$, as well as additional regularity conditions on the nonlinear term, there exists a family of unique time-periodic solutions bifurcating from the stationary steady state. Central to the conditions is the study of the eigenvalue problem

$$\begin{cases} \epsilon^2 (\phi_\epsilon)_{xx} - \phi_\epsilon + 2\dfrac{A_\epsilon}{H_\epsilon}\phi_\epsilon - \dfrac{A_\epsilon^2}{H_\epsilon^2}\psi_\epsilon = \lambda_\epsilon \phi_\epsilon, \\ D(\epsilon)(\psi_\epsilon)_{xx} - \psi_\epsilon + 2A_\epsilon \phi_\epsilon = \tau \lambda_\epsilon \psi_\epsilon, \end{cases} \tag{1.4}$$

where λ_ϵ is some complex number,

$$\phi_\epsilon \in H_N^2([-1,1]), \qquad \psi_\epsilon \in H_N^2([-1,1]), \tag{1.5}$$

and (A_ϵ, H_ϵ) is the stationary solution of (1.1). Here

$$H_N^2([-1,1]) = \left\{ \phi \in H^2([-1,1]) : \phi_x(-1) = \phi_x(1) = 0 \right\}. \tag{1.6}$$

Closely related to $\mathcal{L}_\epsilon$ is its adjoint:

$$\mathcal{L}_\epsilon^* = \begin{bmatrix} \epsilon^2 \frac{d^2}{dx^2} - 1 + \frac{2A_\epsilon}{H_\epsilon} & \frac{2}{\tau} A_\epsilon \\ -\frac{A_\epsilon^2}{H_\epsilon^2} & \frac{1}{\tau}(D\frac{d^2}{dx^2} - 1) \end{bmatrix} \tag{1.7}$$

and the corresponding eigenvalue problem

$$\begin{cases} \epsilon^2 (\phi_\epsilon^*)_{xx} - \phi_\epsilon^* + 2\dfrac{A_\epsilon}{H_\epsilon}\phi_\epsilon^* + \dfrac{2}{\tau} A_\epsilon \psi_\epsilon^* = \lambda_\epsilon^* \phi_\epsilon^*, \\ D(\epsilon)(\psi_\epsilon^*)_{xx} - \psi_\epsilon^* - \tau \dfrac{A_\epsilon^2}{H_\epsilon^2}\phi_\epsilon^* = \tau \lambda_\epsilon^* \psi_\epsilon^*. \end{cases} \tag{1.8}$$

To make the definition of adjoint clear we establish the following definitions. For two functions $\phi_j \in L^2([-1,1]), j = 1, 2$, their inner product is defined by

$$\langle \phi_1, \phi_2 \rangle_{L^2([-1,1])} = \int_{-1}^{1} \phi_1(x)\overline{\phi_2}(x) dx,$$

where the overbar denotes the complex conjugate. Set $Z = L^2([-1,1]) \times L^2([-1,1])$. Then, for two function pairs $\Theta_j = (\phi_j, \psi_j) \in Z$ $(j = 1, 2)$, their inner product is defined by

$$\langle \Theta_1, \Theta_2 \rangle_Z = \langle \phi_1, \phi_2 \rangle_{L^2([-1,1])} + \langle \psi_1, \psi_2 \rangle_{L^2([-1,1])}. \tag{1.9}$$

With these definitions, the defining characteristic of the adjoint operator $\mathcal{L}^*$ is that

$$\langle \mathcal{L}_\epsilon \Theta_1, \Theta_2 \rangle_Z = \langle \Theta_1, \mathcal{L}_\epsilon^* \Theta_2 \rangle_Z, \tag{1.10}$$

for $\Theta_1, \Theta_2 \in Z$.

Additionally, we have the following relationships between the eigenvalues and eigenfunctions of $\mathcal{L}_\epsilon$ and $\mathcal{L}_\epsilon^*$. First, it is easy to see that $\lambda \in \mathbb{C}$ is an eigenvalue of $\mathcal{L}_\epsilon$ if and only if $\overline{\lambda}$ is an eigenvalue of $\mathcal{L}_\epsilon^*$. Furthermore, if $\lambda \in \mathbb{C}$ is a simple nonzero eigenvalue of $\mathcal{L}_\epsilon$ with a nontrivial eigenfunction Θ, and Θ^* is a nontrivial eigenfunction of $\mathcal{L}_\epsilon^*$ corresponding to $\overline{\lambda}$, i.e.

$$\mathcal{L}_\epsilon \Theta = \lambda \Theta, \qquad \mathcal{L}_\epsilon^* \Theta^* = \overline{\lambda} \Theta^*,$$

then

$$\lambda \langle \Theta, \overline{\Theta^*} \rangle = \langle \mathcal{L}_\epsilon \Theta, \overline{\Theta^*} \rangle = \langle \Theta, \mathcal{L}_\epsilon^* \overline{\Theta^*} \rangle = \lambda \langle \Theta, \overline{\Theta^*} \rangle,$$

and therefore

$$\langle \overline{\Theta}, \Theta^* \rangle = \langle \Theta, \overline{\Theta^*} \rangle = 0. \tag{1.11}$$

On the other hand if $(\lambda I - \mathcal{L}_\epsilon)^{-1}$ is compact for all $\lambda \in \rho(\mathcal{L}_\epsilon)$, we have that

$$\langle \Theta, \Theta^* \rangle \neq 0. \tag{1.12}$$

for the simple eigenvalue λ.

The main results of this paper can be summarized as follows: we first rigorously prove that there exists a unique $\tau = \tau_\epsilon^h$ at which a Hopf bifurcation appears. This Hopf bifurcation is transversal (and hence is simple and of codimension 1) (Lemmas 4.1 and 5.17). By applying Crandall-Rabinowitz theory we prove that near $\tau \sim \tau_\epsilon^h$ a time-periodic solution bifurcates (Theorem 6.1). To show stability/instability of this time periodic solution, we use Kielhöfer's formula and derive a stability criteria. By a numerical computation we then show that the Hopf bifurcation is *subcritical* (Theorem 7.1).

The study of localised patterns in the so-called Turing's diffusion-driven-instability reaction-diffusion systems has attracted lots of attention in the last couple of decades ([11]). The one-dimensional canonical model system such as the Gierer-Meinhardt system ([6] [12]) has been intensively studied in many papers. For the existence and stability of steady spiky patterns in a bounded interval or the whole space, we refer to [5], [3], [18], [8], [14], [24] and the book [25]. The dynamics of spiky patterns for one dimensional Gierer-Meinhardt system has been studied in [4] and [16]. For Hopf bifurcations out of spiky patterns for one-dimensional Gierer-Meinhadrt system, we refer to [20, 21].

The results in this paper can be easily extended to the whole $\mathbb{R}^1$. The existence of slowly varying amplitude Hopf bifurcation for the one-dimensional Gierer-Meinhardt system in $\mathbb{R}^1$ is studied in [19], by geometric singular perturbation technique and centre manifold analysis. It is unclear if the same technique works for bounded intervals. Furthermore, in [bottom of page 2218, [19]] it is assumed, without proof, that the Hopf bifurcation (eigenvalue) is simple and is of codimension 1. This is a key element in applying Crandall-Rabinowitz bifurcation theory. One of our main technical results in this paper is to give a rigorous proof of the transversality of the Hopf bifurcation. See Lemma 4.1 and also the formula (5.17). Our proof is more PDE-oriented. We believe that the techniques and computations presented in this paper can be used for the study of sub-criticality or super-criticality of Hopf bifurcations of spiky patterns in many other Turing systems.

The remainder of this paper is organized as follows. In Section 2 we summarize important properties of the stationary single spike solution $(A_\epsilon, H_\epsilon)^T$ for $0 < \epsilon \ll 1$. Then in Section 3 we discuss the spectral properties of the leading order NLEP obtained from (1.4) for $\epsilon \ll 1$. Sections 4 and 5 are dedicated to analyzing the spectral properties of the perturbed problem 1.4 for ϵ sufficiently small. This is followed by Sections 6 and 7 where we apply, set-up, and state the Hopf bifurcation theorem. Finally, in Section 8 we numerically compute an unknown quantity whose sign dictates the criticality of the Hopf bifurcation, while in Section 9 we perform some numerical simulations which illustrate the theoretical predictions.

2. Preliminaries

As remarked in the introduction, investigating the eigenvalue problem (1.4) is crucial to establishing the main results of this paper. It is therefore imperative that the properties of the stationary solution $(A, H)^T$, appearing as coefficients in (1.4), be well understood. Indeed the study of the stationary solutions to (1.1) has been the subject of numerous studies. Specifically the two-dimensional case for small $\epsilon > 0$ was studied in [22]. The one-dimensional case is similar and we review here the most pertinent characteristics for our analysis.

We begin by supposing that

$$D(\epsilon) = \frac{1}{\beta^2(\epsilon)}, \tag{2.1}$$

so that $D = D(\epsilon) \to \infty$ is equivalent to $\beta = \beta(\epsilon) \to 0$. The stationary system for (1.1) is then

$$\begin{cases} \epsilon^2 A_{xx} - A + \dfrac{A^2}{H} = 0, & A > 0 \quad \text{in } [0,1], \\ \dfrac{1}{\beta^2} H_{xx} - H + A^2 = 0, & H > 0 \quad \text{in } [0,1], \\ A_x = H_x = 0, & \text{for } x = 0, 1. \end{cases} \tag{2.2}$$

As stated in the introduction, we consider the even extension of A and H to the interval $[-1,1]$. In this sense, (2.2) becomes

$$\begin{cases} \epsilon^2 A_{xx} - A + \dfrac{A^2}{H} = 0, & A > 0 \quad \text{in } (-1,1), \\ \dfrac{1}{\beta^2} H_{xx} - H + A^2 = 0, & H > 0 \quad \text{in } (-1,1), \\ A_x = H_x = 0, & \text{for } x = -1, 1. \end{cases} \tag{2.3}$$

The equation in H can be solved using a β-dependent Green's function whose properties we now review. Let $G_0(x,\xi)$ be the Green's function satisfying

$$\begin{cases} (G_0)_{xx}(x,\xi) - \dfrac{1}{2} + \delta(x-\xi) = 0 & \text{in } (-1,1), \\ (G_0)_x(x,\xi) = 0, & \text{for } x = -1, 1, \\ \displaystyle\int_{-1}^{1} G_0(x,\xi)dx = 0. \end{cases} \tag{2.4}$$

For a complex number $\beta \in \mathbb{C}$ such that $\frac{d^2}{dx^2} - \beta^2 I : H_N^2([-1,1]) \to L^2([-1,1])$ is invertible, we let $G_\beta(x,\xi)$ be the Green's function given by

$$\begin{cases} (G_\beta)_{xx} - \beta^2 G_\beta + \delta(x-\xi) = 0 & \text{in } [-1,1], \\ (G_\beta)_x(x,\xi) = 0, & \text{for } x = -1, 1, \end{cases} \tag{2.5}$$

We can relate G_β and G_0 as follows. From (2.5) we get

$$\int_{-1}^{1} G_\beta(x,\xi)dx = \beta^{-2}.$$

Set

$$G_\beta(x,\xi) = \frac{1}{2}\beta^{-2} + \bar{G}_\beta(x,\xi).$$

Then

$$\begin{cases} (\bar{G}_\beta)_{xx} - \beta^2 \bar{G}_\beta - \dfrac{1}{2} + \delta(x-\xi) = 0 & \text{in } [-1,1], \\ \displaystyle\int_{-1}^{1} \bar{G}_\beta(x,\xi)dx = 0, \\ (\bar{G}_\beta(x,\xi))_x = 0 & \text{for } x = -1, 1. \end{cases} \tag{2.6}$$

(2.4) and (2.6) imply that

$$\begin{aligned} \bar{G}_\beta(x,\xi) &= \left(\frac{d^2}{dx^2} - \beta^2 I\right)^{-1} \left(\frac{1}{2} - \delta(x-\xi)\right) \\ &= \left(\frac{d^2}{dx^2} - \beta^2 I\right)^{-1} \left[\left(\frac{d^2}{dx^2} - \beta^2 I\right) G_0(x,\xi) + \beta^2 G_0(x,\xi)\right] \\ &= G_0(x,\xi) + \beta^2 \left(\frac{d^2}{dx^2} - \beta^2 I\right)^{-1} G_0(x,\xi). \end{aligned}$$

Since $G_0(\cdot,\xi) \in L^2([-1,1])$, we have

$$\beta^2 \left(\frac{d^2}{dx^2} - \beta^2 I\right)^{-1} G_0(x,\xi) = O(1)$$

in the operator norm of $L^2([-1,1]) \to H^2([-1,1])$. Hence

$$G_\beta(x,\xi) = \frac{1}{2}\beta^{-2} + G_0(x,\xi) + \beta^2 \left(\frac{d^2}{dx^2} - \beta^2 I\right)^{-1} G_0 = \frac{1}{2}\beta^{-2} + G_0(x,\xi) + O(1) \tag{2.7}$$

in the operator norm of $L^2([-1,1]) \to H^2([-1,1])$.

We assume that for ϵ sufficiently small and $D = \frac{1}{\beta^2}$ sufficiently large such that

$$\beta(\epsilon) = O(\epsilon^\sigma) \qquad \text{for some constant } \sigma > 0. \tag{2.8}$$

From the argument found in [22], we have the following theorem.

Theorem 2.1. *Problem* (2.3) *has a solution with the following properties:*

(i) $A_\epsilon(-x) = A_\epsilon(x),\ x \in [-1,1]$, *and*

$$A_\epsilon(x) = \xi_\epsilon w\left(\frac{x}{\epsilon}\right) + O(\beta^2) \tag{2.9}$$

uniformly for $x \in [-1,1]$, *where*

$$\xi_\epsilon = \frac{2}{\epsilon \int_R w^2(y)dy}, \tag{2.10}$$

and w *is the unique solution of the problem*

$$\begin{cases} w_{yy} - w + w^2 = 0, & w > 0, \quad \text{in } R, \\ w(0) = \max_{y\in R} w(y), \\ w(y) \to 0, & \text{as } |y| \to \infty; \end{cases} \tag{2.11}$$

(ii) $H_\epsilon(-x) = H_\epsilon(x),\ x \in [-1,1]$

$$H_\epsilon(x) = \xi_\epsilon(1 + O(\beta^2)) \qquad \textit{uniformly for } x \in [-1,1]. \tag{2.12}$$

Remark 2.2. *The symmetry requirement of A_ϵ and H_ϵ implies that problem (2.2) has a boundary spike solution at $x = 0$ with corresponding properties.*

3. The nonlocal eigenvalue problems

In this section we study the following nonlocal eigenvalue problem (NLEP)

$$L\phi := \phi_{yy} - \phi + 2w\phi - \frac{2}{1+\tau\lambda_0}\frac{\int_{R_+} w\phi}{\int_{R_+} w^2} w^2 = \lambda_0\phi, \quad \phi \in H^2_N(R_+), \tag{3.1}$$

as well as the corresponding adjoint problem given by

$$L^*\phi^* := \phi^*_{yy} - \phi^* + 2w\phi^* - \frac{2}{1+\tau\lambda^*_0}\frac{\int_{R_+} w^2\phi^*}{\int_{R_+} w^2} w = \lambda^*_0\phi^*, \qquad \phi^* \in H^2_N(R_+). \tag{3.2}$$

As we will demonstrate in the next section, these two NLEPS serve as the limiting problems for both eigenvalue problems (1.4) and (1.8) respectively when $\epsilon > 0$ tends to zero.

It is easy to see that (3.1) can be extended to the entire real line

$$L\phi := \phi_{yy} - \phi + 2w\phi - \frac{2}{1+\tau\lambda_0}\frac{\int_R w\phi}{\int_R w^2} w^2 = \lambda_0\phi, \quad \phi \in H^2(R), \quad \phi(y) = \phi(-y). \tag{3.3}$$

We define the function

$$\psi \equiv \frac{2}{1+\tau\lambda_0}\frac{\int w\phi}{\int_R w^2}. \tag{3.4}$$

Similarly the adjoint problem (3.2) is equivalent to

$$L^*\phi^* := \phi^*_{yy} - \phi^* + 2w\phi^* - \frac{2}{1+\tau\lambda^*_0}\frac{\int_R w^2\phi^*}{\int_R w^2} w = \lambda^*_0\phi^*, \quad \phi \in H^2(R), \quad \phi^*(y) = \phi^*(-y). \tag{3.5}$$

For the remainder of this section we will establish several properties of the spectrum of (3.3).

We first recall the following well-known result:

Lemma 3.1. *The eigenvalue problem*

$$L_0\phi := \phi_{yy} - \phi + 2w\phi = \mu\phi, \qquad \phi \in H^2(R), \tag{3.6}$$

admits the set of eigenvalues

$$\mu_1 > 0, \quad \mu_2 = 0, \quad \mu_3 < 0, \cdots. \tag{3.7}$$

The eigenfunction ϕ_1 corresponding to μ_1 can be made positive and even; the space of eigenfunctions corresponding to the eigenvalue 0 is

$$K_0 := span\{w_y\}. \tag{3.8}$$

For the proof of this lemma we refer to Theorem 2.1 of [10] and Lemma C of [13]. In fact

$$w(y) = \frac{3}{2}\,\text{sech}^2\left(\frac{y}{2}\right). \tag{3.9}$$

Note that the nontrivial eigenfunctions corresponding to the eigenvalue 0 are odd functions.

A noteworthy identity for w is obtained as follows. Multiplying the equation for w by yw_y and integrating over R we obtain

$$-\frac{1}{2}\int_R w_y^2 + \frac{1}{2}\int_R w^2 - \frac{1}{3}\int_R w^3 = 0.$$

Multiplying the equation for w by w and integrating over R we obtain

$$-\int_R w_y^2 - \int_R w^2 + \int_R w^3 = 0.$$

Therefore we have the integral identities

$$\int_R w_y^2 = \frac{1}{5}\int_R w^2 = \frac{1}{6}\int_R w^3. \tag{3.10}$$

Integrating the equation for w over R we obtain

$$\int_R w = \int_R w^2. \tag{3.11}$$

Lemma 3.2. *There exists a unique $\tau = \tau_h > 0$ such that for $\tau < \tau_h$,* (3.1) *admits a positive eigenvalue, and for $\tau > \tau_h$, all nonzero eigenvalues of problem* (3.1) *satisfies $Re(\lambda_0) < 0$. At $\tau = \tau_h$,* (3.1) *has a pair of pure imaginary eigenvalues $\lambda_0(\tau_h) = \pm i\alpha_I$ with $\alpha_I \in (0,\infty)$ uniquely determined by τ_h. Moreover, the following transversality condition holds*

$$Re(\lambda_0'(\tau_h)) \neq 0. \tag{3.12}$$

Proof. The existence and uniqueness part of the lemma is essentially part of Theorem 2.2 and Lemma 2.4 of [22], which treats interior spike solutions in a two-dimensional space. The proof found there can be applied here almost without modification but for the sake of completeness we reproduce it here. The transversality condition (3.12) and its proof here are new.

Note we here only consider even functions. By Theorem 1.4 of [23], for $\tau = 0$ and by perturbation for τ small, all eigenvalues lie on the left half-plane. By [2], for τ large, there exist unstable eigenvalues. Therefore, for an intermediate value of $\tau = \tau_h$ an eigenvalue λ_0 must cross the imaginary axis into the positive real-part half-plane. We first show that this eigenvalue may not cross through the origin, and then we show the value of τ_h must be unique.

Suppose that there is a zero-eigenvalue crossing, $\lambda_0 = 0$, when $\tau = \tau_h$. Let

$$L_0\phi \equiv \phi_{yy} - \phi + 2w\phi,$$

so that at the zero-eigenvalue crossing the NLEP (3.3) becomes

$$L_0\phi - 2\frac{\int_R w\phi}{\int_R w^2}w^2 = 0,$$

and hence

$$L_0\left(\phi - 2\frac{\int_R w\phi}{\int_R w^2}w\right) = 0.$$

Thus

$$\phi - 2\frac{\int_R w\phi}{\int_R w^2}w \in K_0,$$

and since ϕ is even by Lemma 3.1 we must have

$$\phi - 2\frac{\int_R w\phi}{\int_R w^2}w = 0. \tag{3.13}$$

It follows from $\phi \not\equiv 0$ that

$$\int_R w\phi \neq 0.$$

But on the other hand, multiplying (3.13) by w and integrating over R, we arrive at the contradiction

$$\int_R w\phi = 2\int_R w\phi.$$

From the preceding argument we deduce that there must exist a $\tau_h \in (0, \infty)$ at which L has a pair of pure imaginary eigenvalues

$$\lambda_0(\tau_h) = \pm \alpha_I i,$$

where $i = \sqrt{-1}$ and $\alpha_I > 0$. Next we show that τ_h is unique. From

$$(L_0 - \lambda_0)\phi_0 = \frac{2}{1+\tau\lambda_0} \frac{\int_R w\phi_0}{\int_R w^2} w^2,$$

we obtain for $\lambda_0 = \alpha_I i$ that

$$\phi_0 = \frac{2}{1+\tau\lambda_0} \frac{\int_R w\phi_0}{\int_R w^2} (L_0 - \lambda_0)^{-1} w^2,$$

and hence $\alpha_I i$ is a simple eigenvalue in the sense that

$$\text{Ker}(L - \alpha_I i) = \text{Span}\{(L_0 - \alpha_I i)^{-1} w^2\}.$$

Thus we may assume that $\phi_0 = (L_0 - \alpha_I i)^{-1} w^2$ whence (3.3) becomes

$$\int_R w\phi_0 = \frac{1+\tau\alpha_I i}{2} \int_R w^2. \tag{3.14}$$

Let $\phi_0 = \phi_0^R + \phi_0^R i$. Then from (3.14) we obtain

$$\int_R w\phi_0^R = \frac{1}{2} \int_R w^2,$$

and

$$\int_R w\phi_0^I = \frac{\tau\alpha_I}{2} \int_R w^2.$$

But from

$$\phi_0 = (L_0 - \alpha_I i)^{-1} w^2 = (L_0 + \alpha_I i)(L_0^2 + \alpha_I^2)^{-1} w^2,$$

we have

$$\phi_0^R = L_0(L_0^2 + \alpha_I^2)^{-1} w^2, \qquad \phi_0^I = \alpha_I (L_0^2 + \alpha_I^2)^{-1} w^2.$$

It follows that

$$\int_R [w L_0 (L_0^2 + \alpha_I^2)^{-1} w^2] = \frac{1}{2} \int_R w^2, \tag{3.15}$$

$$\int_R [w (L_0^2 + \alpha_I^2)^{-1} w^2] = \frac{\tau}{2} \int_R w^2. \tag{3.16}$$

Let $h(\alpha_I) \equiv \int_R [w L_0 (L_0^2 + \alpha_I^2)^{-1} w^2]$. Then

$$h'(\alpha_I) = -2\alpha_I \int_R [w L_0 (L_0^2 + \alpha_I^2)^{-2} w^2].$$

By integration by parts, the last equation yields

$$h'(\alpha_I) = -2\alpha_I \int_R [w^2 (L_0^2 + \alpha_I^2)^{-2} w^2] < 0.$$

Since

$$h(0) = \int_R w(L_0^{-1} w^2) = \int_R w^2, \quad \text{and} \quad h(\alpha_I) \to 0 \qquad \text{as} \quad \alpha_I \to \infty,$$

there exists a unique $\alpha_I \in (0, \infty)$ that (3.15) holds. The unique value of $\tau = \tau_h \in (0, \infty)$ then comes from (3.16).

It is left to show that (3.12) holds. Setting $\lambda_0 = \lambda_R(\tau) + i\lambda_I(\tau)$ we have the system of equations

$$\begin{cases} \dfrac{1+\tau\lambda_R}{2}\displaystyle\int_R w^2 = \int_R w\frac{L_0-\lambda_R}{(L_0-\lambda_R)^2+\lambda_I^2}w^2, \\ \dfrac{\tau}{2}\displaystyle\int_R w^2 = \int_R w\frac{1}{(L_0-\lambda_R)^2+\lambda_I^2}w^2, \end{cases} \tag{3.17}$$

Suppose that $\frac{\partial(\lambda_R)}{\partial\tau}(\tau_h) = 0$ and differentiate the second equation of (3.17) with respect to τ and evaluate it at $\tau = \tau_h$ to obtain

$$\frac{1}{2}\int_R w^2 = -2\lambda_I(\tau_h)\frac{\partial(\lambda_I)}{\partial\tau}(\tau_h)\int_R w[L_0^2+\lambda_I^2(\tau_h)]^{-2}w^2, \tag{3.18}$$

where we have used $\lambda_R(\tau_h) = 0$. This implies that $\frac{\partial(\lambda_I)}{\partial\tau}(\tau_h) \neq 0$. If we now differentiate the first equation of (3.17) with respect to τ we obtain

$$0 = -\frac{\partial(\lambda_I^2)}{\partial\tau}(\tau_h)\int_R wL_0[L_0^2+\lambda_I^2(\tau_h)]^{-2}w^2. \tag{3.19}$$

However $\frac{\partial(\lambda_I^2)}{\partial\tau}(\tau_h) \neq 0$ and integrating by parts we see also that

$$\int_R wL_0[L_0^2+\lambda_I^2(\tau_h)]^{-2}w^2 = \int_R [w^2(L_0+\alpha_I^2)^{-2}w^2] > 0,$$

which yields a contradiction. Therefore $\frac{\partial(\lambda_R)}{\partial\tau}(\tau_h) \neq 0$.

□

The next lemma is a continuation of Lemma 3.2.

Lemma 3.3. *Let $\lambda_0 = \pm\alpha_I i$ be the unique imaginary eigenvalue pair described in Lemma 3.2 (at $\tau = \tau_h$). Then*

$$Re(\lambda_0'(\tau_h)) > 0. \tag{3.20}$$

Proof. Consider the eigenvalue problem

$$L_0\phi - \frac{2}{1+\tau\lambda_0}\frac{\int_R w\phi}{\int_R w^2}w^2 = \lambda_0\phi. \tag{3.21}$$

As in the proof the transversality condition of Lemma 3.2 we have

$$\phi = \frac{2}{1+\tau\lambda_0}\frac{\int_R w\phi}{\int_R w^2}(L_0-\lambda_0)^{-1}w^2,$$

so that multiplying by w and integrating gives

$$\frac{1+\tau\lambda_0}{2}\int_R w^2 = \int_R w(L_0-\lambda_0)^{-1}w^2. \tag{3.22}$$

Differentiating (3.22) with respect to τ we obtain

$$\frac{\lambda_0+\tau\lambda_0'}{2}\int_R w^2 = \lambda_0'\int_R w(L_0-\lambda_0)^{-2}w^2, \tag{3.23}$$

or equivalently

$$\lambda_0' = \lambda_0\frac{\int_R w^2}{2}\left(\int_R w(L_0-\lambda_0)^{-2}w^2 - \frac{\tau}{2}\int_R w^2\right)^{-1}. \tag{3.24}$$

Letting $\tau = \tau_h$ and using $Re(\lambda_0(\tau_h)) = 0$ we obtain

$$Re(\lambda_0'(\tau_h)) = -Im(\lambda_0(\tau_h))\frac{\int_R w^2}{2} Im\left[\left(\int_R w(L_0 - \lambda_0(\tau_h))^{-2}w^2 - \frac{\tau_h}{2}\int_R w^2\right)^{-1}\right]. \tag{3.25}$$

Denote

$$\int_R w(L_0 - \lambda_0(\tau_h))^{-2}w^2 = a + ib, \qquad c = \frac{\tau_h}{2}\int_R w^2, \qquad \text{with } a, b, c \in R.$$

Then we have

$$\begin{aligned} &Im\left[\left(\int_R w(L_0 - \lambda_0(\tau_h))^{-2}w^2 - \frac{\tau_h}{2}\int_R w^2\right)^{-1}\right] \\ &= Im\left[(a + bi - c)^{-1}\right] \\ &= \frac{-b}{(a-c)^2 + b^2}. \end{aligned} \tag{3.26}$$

On the other hand

$$\int_R w(L_0 - \lambda_0(\tau_h))^{-2}w^2 = \int_R w\frac{L_0^2 - \lambda_I(\tau_h)^2 + 2i\lambda_I(\tau_h)L_0}{(L_0^2 + \lambda_I(\tau_h)^2)^2}w^2, \tag{3.27}$$

and consequently by integration by parts we obtain

$$\begin{aligned} b &= 2\lambda_I(\tau_h)\int_R w\frac{L_0}{(L_0^2 + \lambda_I(\tau_h)^2)^2}w^2 \\ &= 2\lambda_I(\tau_h)\int_R (L_0 w)(L_0^2 + \lambda_I(\tau_h)^2)^{-2}w^2 \\ &= 2\lambda_I(\tau_h)\int_R w^2(L_0^2 + \lambda_I(\tau_h)^2)^{-2}w^2. \end{aligned}$$

Hence

$$Re(\lambda_0'(\tau_h)) = \frac{\lambda_I(\tau_h)^2\int_R w^2}{(a-c)^2 + b^2}\int_R w^2(L_0^2 + \lambda_I(\tau_h)^2)^{-2}w^2 > 0. \tag{3.28}$$

□

We conclude this section with an alternative representation of $\lambda_0'(\tau_h)$ and bounding the spectrum of (3.1). In (3.3) we write $\mu_0 = \tau\lambda_0$, ϕ as ϕ_0, and differentiate the equation with respect to τ

$$L_0\phi_0' - \frac{2}{1+\mu_0}\frac{\int_R w\phi_0'}{\int_R w^2}w^2 + \frac{2\mu_0'}{(1+\mu_0)^2}\frac{\int_R w\phi_0}{\int_R w^2}w^2 = \left(-\frac{\mu_0}{\tau^2} + \frac{\mu_0'}{\tau}\right)\phi_0 + \frac{\mu_0}{\tau}\phi_0'.$$

Multiplying by the conjugate of the adjoint eigenfunction $\overline{\phi_0^*}$ and integrating over R, we obtain

$$\begin{aligned} &\int_R [\overline{\phi_0^*}L_0\phi_0'] - \frac{2}{1+\mu_0}\frac{\int_R w\phi_0'}{\int_R w^2}\int_R w^2\overline{\phi_0^*} + \frac{2\mu_0'}{(1+\mu_0)^2}\frac{\int_R w\phi_0}{\int_R w^2}\int_R w^2\overline{\phi_0^*} \\ &= \left(-\frac{\mu_0}{\tau^2} + \frac{\mu_0'}{\tau}\right)\int_R \phi_0\overline{\phi_0^*} + \frac{\mu_0}{\tau}\int_R \overline{\phi_0^*}\phi_0'. \end{aligned} \tag{3.29}$$

Taking conjugate of (3.5) and recalling that $\lambda_0^* = \overline{\lambda_0}$ we obtain

$$L_0\overline{\phi_0^*} - \frac{2}{1+\mu_0}\frac{\int_R w^2\overline{\phi_0^*}}{\int_R w^2}w = \frac{\mu_0}{\tau}\overline{\phi_0^*}.$$

Multiplying by ϕ_0' and integrating over R, we obtain

$$\int_R [\phi_0' L_0 \overline{\phi_0^*}] - \frac{2}{1+\mu_0} \frac{\int_R w^2 \overline{\phi_0^*}}{\int_R w^2} \int_R w\phi_0' = \frac{\mu_0}{\tau} \int_R \overline{\phi_0^*}\phi_0'. \tag{3.30}$$

Note that by integration by parts,

$$\int_R [\overline{\phi_0^*} L_0 \phi_0'] = \int_R [\phi_0' L_0 \overline{\phi_0^*}].$$

We obtain from (3.29) and (3.30) that

$$\frac{2\mu_0'}{(1+\mu_0)^2} \frac{\int_R w\phi_0}{\int_R w^2} \int_R w^2 \overline{\phi_0^*} = \left(-\frac{\mu_0}{\tau^2} + \frac{\mu_0'}{\tau}\right) \int_R \phi_0 \overline{\phi_0^*}. \tag{3.31}$$

Therefore we have the formula

$$\mu_0'(\tau_h) = \frac{\lambda_0(\tau_h) \int_R \phi_0 \overline{\phi_0^*}}{\int_R \phi_0 \overline{\phi_0^*} - \frac{2\tau_h}{[1+\tau_h \lambda_0(\tau_h)]^2 \int_R w^2} \int_R w\phi_0 \int_R w^2 \overline{\phi_0^*}}. \tag{3.32}$$

Finally we have the following bound estimates for the spectrum of (3.1) which will play a key role in showing the unperturbed linear operator is sectorial.

Lemma 3.4. *Let λ_0 be an eigenvalue of* (3.1). *Then one of the following alternative cases happens:*

(i) $Im(\lambda_0) = 0$ and $\lambda_0 \le \mu_1$, where $\mu_1 > 0$ is the first eigenvalue of L_0, or
(ii) $Im(\lambda_0) \neq 0$ and $|\tau\lambda_0 - 1| \le \sqrt{2}$.

Proof. Multiplying (3.1) by w and integrating over R, we obtain

$$\int_R w^2 \phi = \left(\lambda_0 + \frac{2}{1+\tau\lambda_0} \frac{\int_R w^3}{\int_R w^2}\right) \int_R w\phi. \tag{3.33}$$

Using (3.10) we obtain It follows that

$$\int_R w^2 \phi = \left(\lambda_0 + \frac{12}{5(1+\tau\lambda_0)}\right) \int_R w\phi. \tag{3.34}$$

Taking the conjugate

$$\int_R w^2 \overline{\phi} = \left(\overline{\lambda_0} + \frac{12}{5(1+\tau\overline{\lambda_0})}\right) \int_R w\overline{\phi}. \tag{3.35}$$

Multiplying (3.1) by $\overline{\phi}$ and integrating over R, we obtain that

$$\int_R (|\phi_y|^2 + |\phi|^2 - 2w|\phi|^2) = -\lambda_0 \int_R |\phi|^2 - \frac{2}{1+\tau\lambda_0} \frac{\int_R w\phi}{\int_R w^2} \int_R w^2 \bar{\phi}. \tag{3.36}$$

Combining (3.35) and (3.36) we obtain

$$\int_R (|\phi_y|^2 + |\phi|^2 - 2w|\phi|^2) = -\lambda_0 \int_R |\phi|^2 - \left(\frac{2\bar{\lambda}_0}{1+\tau\lambda_0} + \frac{24}{5|1+\tau\lambda_0|^2}\right) \frac{|\int_R w\phi|^2}{\int_R w^2}. \tag{3.37}$$

Writing

$$\lambda_0 = \lambda_R + i\lambda_I, \qquad \phi = \phi_R + i\phi_I,$$

and considering the imaginary part of (3.37) we obtain

$$\lambda_I \int_R |\phi|^2 = \frac{2\lambda_I(1+2\tau\lambda_R)}{(1+\tau\lambda_R)^2 + \tau^2\lambda_I^2} \frac{|\int_R w\phi|^2}{\int_R w^2}. \tag{3.38}$$

We first consider the case that $\lambda_I \neq 0$. In this case we have

$$\int_R |\phi|^2 = \frac{2(1+2\tau\lambda_R)}{(1+\tau\lambda_R)^2+\tau^2\lambda_I^2}\frac{|\int_R w\phi|^2}{\int_R w^2}.$$

Using the Schwartz inequality

$$|\int_R w\phi|^2 \leq \int_R w^2 \int_R |\phi|^2,$$

we get

$$\frac{2(1+2\tau\lambda_R)}{(1+\tau\lambda_R)^2+\tau^2\lambda_I^2} \geq 1,$$

which is case (ii).

Now assume that $\lambda_I = 0$. If $\tau\lambda_R + 1 = 0$, then

$$\lambda_0 = \lambda_R = -\frac{1}{\tau} < 0 < \mu_1.$$

If $\tau\lambda_R + 1 \neq 0$, we then use the Rayleigh's formula

$$\int_R |\phi_y|^2 + \int_R |\phi|^2 - 2\int_R w|\phi|^2 \geq -\mu_1 \int_R |\phi|^2,$$

and (3.37) to get that

$$\lambda_R \int_R |\phi|^2 + \left(\frac{2\lambda_R}{1+\tau\lambda_R} + \frac{6}{|1+\tau\lambda_R|^2}\right)\frac{|\int_R w\phi|^2}{\int_R w^2} \leq \mu_1 \int_R |\phi|^2.$$

If $\lambda_R \leq 0$, we are done. If $\lambda_R > 0$, we then have

$$\lambda_R \int_R |\phi|^2 \leq \mu_1 \int_R |\phi|^2.$$

Hence case (i) happens. □

4. Spectral analysis of (1.4)

We want to show that the operator $\mathcal{L}_\epsilon$ is an infinitesimal generator of the a strongly continuous and analytical semigroup. Since it suffices to show that $\mathcal{L}_\epsilon$ is a sectorial operator this naturally leads us to study the following eigenvalue problem

$$\begin{cases} (\phi_\epsilon)_{yy} - \phi_\epsilon + 2\dfrac{A_\epsilon}{H_\epsilon}\phi_\epsilon - \dfrac{A_\epsilon^2}{H_\epsilon^2}\psi_\epsilon = \lambda_\epsilon\phi_\epsilon, \\ \dfrac{1}{\beta^2}(\psi_\epsilon)_{xx} - \psi_\epsilon + 2A_\epsilon\phi_\epsilon = \tau\lambda_\epsilon\psi_\epsilon, \end{cases} \tag{4.1}$$

where $y = \epsilon^{-1}x$, $D = \beta^{-2}$, λ_ϵ is some complex number, and

$$\phi_\epsilon \in H_N^2([-\epsilon^{-1}, \epsilon^{-1}]), \qquad \psi \in H_N^2([-1,1]). \tag{4.2}$$

It is convenient to set $\hat{A}_\epsilon = \xi_\epsilon^{-1}A_\epsilon$ and $\hat{H}_\epsilon = \xi_\epsilon^{-1}H_\epsilon$ so that (4.1) becomes

$$\begin{cases} (\phi_\epsilon)_{yy} - \phi_\epsilon + 2\dfrac{\hat{A}_\epsilon}{\hat{H}_\epsilon}\phi_\epsilon - \dfrac{\hat{A}_\epsilon^2}{\hat{H}_\epsilon^2}\psi_\epsilon = \lambda_\epsilon\phi_\epsilon, \\ \dfrac{1}{\beta^2}(\psi_\epsilon)_{xx} - \psi_\epsilon + 2\xi_\epsilon\hat{A}_\epsilon\phi_\epsilon = \tau\lambda_\epsilon\psi_\epsilon. \end{cases} \tag{4.3}$$

The second equation in (4.3) is equivalent to

$$(\psi_\epsilon)_{xx} - \beta_{\lambda_\epsilon}^2\psi_\epsilon + 2\beta^2\xi_\epsilon\hat{A}_\epsilon\phi_\epsilon = 0. \tag{4.4}$$

where

$$\beta_{\lambda_\epsilon}^2 \equiv \beta^2(1+\tau\lambda_\epsilon). \tag{4.5}$$

We may assume that $\|\phi_\epsilon\|_{H^2([-\epsilon^{-1},\epsilon^{-1}])} = 1$.

Let χ be a smooth cut-off function which is equal to 1 in $[-\frac{1}{2},\frac{1}{2}]$ and equal to 0 in $R\setminus[-1,1]$. Let

$$\chi_\epsilon(y) = \chi(\epsilon y), \qquad y \in [-\epsilon^{-1}, \epsilon^1]. \tag{4.6}$$

Define the cut-off of ϕ_ϵ:

$$\phi_\epsilon^c(y) = \phi_\epsilon(y)\chi_\epsilon(y), \tag{4.7}$$

where $x = \epsilon y$. Then if $Re(1+\lambda_\epsilon) > c$, or $|Im(\lambda_\epsilon)| > c$, for a small constant $c > 0$, we have

$$\phi_\epsilon^c = \phi_\epsilon + e.s.t. \qquad \text{in } \ H^2([-\epsilon^{-1},\epsilon^{-1}]). \tag{4.8}$$

Then by the standard procedure, we extend ϕ_ϵ^c to a function defined on R such that

$$\begin{aligned} &\|\phi_\epsilon^c\|_{L^2(R)} \le C_0\|\phi_\epsilon\|_{L^2([-\epsilon^{-1},\epsilon^{-1}])}, \\ &\|(\phi_\epsilon^c)_y\|_{L^2(R)} \le C_0\|(\phi_\epsilon)_y\|_{L^2([-\epsilon^{-1},\epsilon^{-1}])}, \\ &\|(\phi_\epsilon^c)_{yy}\|_{L^2(R)} \le C_0\|(\phi_\epsilon)_{yy}\|_{L^2([-\epsilon^{-1},\epsilon^{-1}])}, \end{aligned} \tag{4.9}$$

for a constant $C_0 > 1$. Since $\|\phi_\epsilon\|_{H^2([-\epsilon^{-1},\epsilon^{-1}])} = 1$, we have $\|\phi_\epsilon^c\|_{H^2(R)} \le C_0$.

Using the Green's function introduced in Section 2 we write

$$\psi_\epsilon(x) = \int_{-1}^{1} 2\beta^2\xi_\epsilon G_{\beta_{\lambda_\epsilon}}(x,\xi)\hat{A}_\epsilon\left(\frac{\xi}{\epsilon}\right)\phi_\epsilon\left(\frac{\xi}{\epsilon}\right)d\xi. \tag{4.10}$$

At $x = 0$, we calculate

$$\begin{aligned} \psi_\epsilon(0) &= 2\beta^2\int_{-1}^{1} G_{\beta_{\lambda_\epsilon}}(0,\xi)\xi_\epsilon w\left(\frac{\xi}{\epsilon}\right)\phi_\epsilon^c\left(\frac{\xi}{\epsilon}\right)d\xi + o(1) \\ &= 2\beta^2\int_{-1}^{1}\left(\frac{(\beta_{\lambda_\epsilon})^{-2}}{2} + G_0(0,\xi) + O(1)\right)\xi_\epsilon w\left(\frac{\xi}{\epsilon}\right)\phi_\epsilon^c\left(\frac{\xi}{\epsilon}\right)d\xi + o(1) \\ &= 2\int_{-1}^{1}\left(\frac{1}{2(1+\tau\lambda_\epsilon)} + \beta^2 G_0(0,\xi) + O(\beta^2)\right)\xi_\epsilon w\left(\frac{\xi}{\epsilon}\right)\phi_\epsilon^c\left(\frac{\xi}{\epsilon}\right)d\xi + o(1) \\ &= \frac{1}{1+\tau\lambda_\epsilon}\xi_\epsilon\epsilon\int_R w(y)\phi_\epsilon^c(y)dy + O(\beta^2\xi_\epsilon\epsilon) + o(1) \\ &= \frac{1+o(1)}{1+\tau\lambda_\epsilon}\epsilon\xi_\epsilon\int_R w\phi_\epsilon^c \\ &= \frac{2[1+o(1)]}{(1+\tau\lambda_\epsilon)\int_R w^2}\int_R w\phi_\epsilon^c \qquad\qquad \text{as} \quad \epsilon \to 0. \end{aligned} \tag{4.11}$$

Substituting (4.11) into the first equation of (4.3) we arrive at

$$(\phi_\epsilon)_{yy} - \phi_\epsilon + 2w\phi_\epsilon - \frac{2[1+o(1)]}{1+\tau\lambda_\epsilon}\frac{\int_R w\phi_\epsilon^c}{\int_R w^2}w^2 = \lambda_\epsilon[1+o(1)]\phi_\epsilon \tag{4.12}$$

As in the proof of Theorem 1 in [2] one obtains

$$\lambda_\epsilon \to \lambda_0, \qquad \phi_\epsilon(y) \to \phi_0(y) \qquad \text{in } \ H^2_{loc}(R), \qquad \text{as } \ \epsilon \to 0, \tag{4.13}$$

where (λ_0, ϕ_0) is an eigenpair of the NLEP (3.1).

We can now prove the following spectral result for the eigenvalue problem (4.1).

Lemma 4.1. *If $\epsilon > 0$ is sufficiently small then there exists a unique value $\tau = \tau_\epsilon^h$ for which (4.1) has a pair of purely imaginary eigenvalues $\lambda_\pm^\epsilon = \pm i\alpha_I^\epsilon$ with $\alpha_I^\epsilon > 0$. Moreover this pair is unique in the sense that if $i\beta_I^\epsilon$ is an eigenvalue of (4.1), then $\beta_I^\epsilon = \alpha_I^\epsilon$ or $\beta_I^\epsilon = -\alpha_I^\epsilon$. Furthermore at this value of $\tau = \tau_\epsilon^h$ all other eigenvalues have negative real parts.*

Proof. For $\epsilon > 0$ sufficiently small, as in the proof of Lemma 3.2 all eigenvalues of (4.1) have negative real parts when $\tau > 0$ is small, whereas there exist eigenvalues with positive real part when $\tau > 0$ is sufficiently large. Furthermore, we can show that there are no zero eigenvalues for any $\tau > 0$. Thus, there exists a $\tau_\epsilon^h \in (0, \infty)$ such that (4.1) has a pair of pure imaginary eigenvalues.

The uniqueness comes from the fact that for $Re(\lambda_\epsilon) > -c$ we define $h_\epsilon(\lambda_I^\epsilon) := \int_R wRe(\phi_\epsilon^c)$ for the unperturbed problem (4.12) so that subject to a subsequence, $\alpha_I^\epsilon \to \alpha_I$ and $\phi_\epsilon \to \phi_0$ as $\epsilon \to 0$ we have

$$h'_\epsilon(\lambda_I^\epsilon) \to h'(\lambda_I) < 0 \qquad \text{as } \epsilon \to 0, \tag{4.14}$$

according to the calculation in the proof of Lemma 3.2 and the uniform continuity of $h'(\lambda_I)$ in λ_I.

□

The following two lemmas establish the semigroup framework.

Lemma 4.2. *Let $\lambda_\epsilon \in \mathbb{C}$ be an eigenvalue of problem* (4.1)*. Then for sufficiently small $\epsilon > 0$, one of the following cases happens:*

(i) $Im(\lambda_\epsilon) = 0$ and $\lambda_\epsilon \le 2\mu_1$, or
(ii) $Im(\lambda_\epsilon) \neq 0$ and $|\tau\lambda_\epsilon| \le 7$.

Proof. We may assume that the constant $C_0 > 1$ in (4.9) is arbitrarily close to 1. Multiplying (4.12) by $\overline{\phi_\epsilon^c}$ and integrating over R we get

$$-\int_R |(\phi_\epsilon^c)_y|^2 - \int_R |\phi_\epsilon^c|^2 + 2\int_R w|\phi_\epsilon^c|^2 - \frac{2[1+o(1)]}{1+\tau\lambda_\epsilon}\frac{\int_R w\phi_\epsilon^c}{\int_R w^2}\int_R w^2\overline{\phi_\epsilon^c} = \lambda_\epsilon[1+o(1)]\int_R |\phi_\epsilon^c|^2. \tag{4.15}$$

Multiplying (4.12) by w and integrating over R we get

$$[1+o(1)]\lambda_\epsilon \int_R w\phi_\epsilon^c = \int_R [w_{yy} - w + 2w^2]\phi_\epsilon^c - \frac{2[1+o(1)]}{1+\tau\lambda_\epsilon}\frac{\int_R w^3}{\int_R w^2}\int_R w\phi_\epsilon^c. \tag{4.16}$$

Using (3.10) we obtain

$$\int_R w^2\phi_\epsilon^c = [1+o(1)]\left(\lambda_\epsilon + \frac{12}{5(1+\tau\lambda_\epsilon)}\right)\int_R w\phi_\epsilon^c. \tag{4.17}$$

From (4.15) and (4.17) we obtain

$$[1+o(1)]\int_R (|(\phi_\epsilon^c)_y|^2 + |\phi_\epsilon^c|^2 - 2w|\phi_\epsilon^c|^2) = -\lambda_\epsilon\int_R |\phi_\epsilon^c|^2 - \left(\frac{2\overline{\lambda_\epsilon}}{1+\tau\lambda_\epsilon} + \frac{24}{5|1+\tau\lambda_\epsilon|^2}\right)\frac{|\int_R w\phi_\epsilon^c|^2}{\int_R w^2}. \tag{4.18}$$

Considering the imaginary part of (4.18) we get

$$[1+o(1)]\lambda_I^\epsilon \int_R |\phi_\epsilon^c|^2 = \frac{2\lambda_I^\epsilon(1+2\tau\lambda_R^\epsilon)}{(1+\tau\lambda_R^\epsilon)^2 + \tau^2(\lambda_I^\epsilon)^2}\frac{|\int_R w\phi_\epsilon^c|^2}{\int_R w^2}. \tag{4.19}$$

If $\lambda_I^\epsilon \neq 0$, we have

$$\frac{2(1+2\tau\lambda_R^\epsilon)}{(1+\tau\lambda_R^\epsilon)^2 + \tau^2(\lambda_I^\epsilon)^2} \ge \frac{1}{2} \qquad \text{for sufficiently small } \epsilon > 0,$$

or equivalently, for small $\epsilon > 0$,

$$(\tau\lambda_R^\epsilon - 3)^2 + (\tau\lambda_I^\epsilon)^2 \le 13, \tag{4.20}$$

From here we obtain the coarse bounds

$$3 - \sqrt{13} \le \tau\lambda_R^\epsilon \le 3 + \sqrt{13}, \qquad -\sqrt{13} \le \tau\lambda_I^\epsilon \le \sqrt{13},$$

and hence

$$|\tau\lambda_\epsilon| \leq 3 + \sqrt{13} \leq 7. \tag{4.21}$$

If $\lambda_I^\epsilon = 0$, then $\lambda_\epsilon = \lambda_R^\epsilon$, and (4.18) becomes

$$[1+o(1)]\int_R (|(\phi_\epsilon^c)_y|^2 + |\phi_\epsilon^c|^2 - 2w|\phi_\epsilon^c|^2) = -\lambda_R^\epsilon \int_R |\phi_\epsilon^c|^2 - \left(\frac{2\lambda_R^\epsilon}{1+\tau\lambda_R^\epsilon} + \frac{24}{5|1+\tau\lambda_R^\epsilon|^2}\right)\frac{|\int_R w\phi_\epsilon^c|^2}{\int_R w^2}.$$

Using the inequality

$$\int_R (|(\phi_\epsilon^c)_y|^2 + |\phi_\epsilon^c|^2 - 2w|\phi_\epsilon^c|^2) \geq -\mu_1 \int_R |\phi_\epsilon^c|^2,$$

we obtain that for $\epsilon > 0$ sufficiently small

$$-2\mu_1 \int_R |\phi_\epsilon^c|^2 \leq -\lambda_R^\epsilon \int_R |\phi_\epsilon^c|^2 - \left(\frac{2\lambda_R^\epsilon}{1+\tau\lambda_R^\epsilon} + \frac{24}{5|1+\tau\lambda_R^\epsilon|^2}\right)\frac{|\int_R w\phi_\epsilon^c|^2}{\int_R w^2}. \tag{4.22}$$

Then $\lambda_R^\epsilon \leq 0$, or $\lambda_R^\epsilon > 0$. In the case $\lambda_R^\epsilon > 0$, we obtain from (4.22) that

$$\lambda_R^\epsilon \int_R |\phi_\epsilon^c|^2 \leq 2\mu_1 \int_R |\phi_\epsilon^c|^2,$$

and hence

$$\lambda_R^\epsilon \leq 2\mu_1. \tag{4.23}$$

This finishes the proof of the lemma. □

In view of Lemma 4.2, there exist constants $\epsilon_0 > 0$, $a > 0$ and $\theta \in (\frac{\pi}{2}, \pi)$ such that the sector

$$S_{a,\theta} := \{\lambda \in \mathbb{C} : |\arg(\lambda - a)| < \theta\} \cup \{a\} \tag{4.24}$$

is contained in the resolvent set of $\mathcal{L}_\epsilon$ for all $\epsilon \in (0, \epsilon_0]$.

Lemma 4.3. *The operator $\mathcal{L}_\epsilon$ is a sectorial operator and hence generates a strongly continuous and analytic semigroup on the space $L^2([-1,1]) \times L^2([-1,1])$. Moreover, for $\lambda \in S_{a,\theta}$ with $a \gg 1$, the operator $\mathcal{R}(\lambda, a) = (\lambda - \mathcal{L}_\epsilon)^{-1}$ is compact as an operator mapping $L^2([-1,1]) \times L^2([-1,1])$ into itself and there exists a constant $M > 0$ such that*

$$\|\mathcal{R}(\lambda, a)\| \leq \frac{M}{|\lambda - a|}, \qquad \text{for } \lambda \in S_{a,\theta}. \tag{4.25}$$

Proof. For any $\lambda \in S_{a,\theta}$ we consider the resolvent equation

$$(\mathcal{L}_\epsilon - \lambda)\begin{bmatrix}\phi \\ \psi\end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \tag{4.26}$$

namely,

$$\begin{cases} (\phi_\epsilon)_{yy} - \phi_\epsilon + 2\dfrac{\hat{A}_\epsilon}{\hat{H}_\epsilon}\phi_\epsilon - \dfrac{\hat{A}_\epsilon^2}{\hat{H}_\epsilon^2}\psi_\epsilon = \lambda\phi_\epsilon + f_1, \\ \dfrac{1}{\beta^2}(\psi_\epsilon)_{xx} - \psi_\epsilon + 2\xi_\epsilon \hat{A}_\epsilon \phi_\epsilon = \tau\lambda\psi_\epsilon + \tau f_2. \end{cases} \tag{4.27}$$

From the second equation of (4.27) we get

$$\psi_\epsilon(x) = \int_{-1}^1 G_{\beta_\lambda}(x,\xi)\left[2\beta^2\xi_\epsilon \hat{A}_\epsilon\left(\frac{\xi}{\epsilon}\right)\phi_\epsilon\left(\frac{\xi}{\epsilon}\right) - \tau\beta^2 f_2\left(\frac{\xi}{\epsilon}\right)\right]d\xi. \tag{4.28}$$

As before we calculate at $x = 0$ to get that

$$\psi_\epsilon(0) = [1+o(1)]\left(\frac{2}{1+\tau\lambda}\frac{\int_R w\phi_\epsilon^c}{\int_R w^2} - \frac{2\tau}{1+\tau\lambda}\int_R f_2^c\right). \tag{4.29}$$

We assume $a \gg 1$ and θ be fixed. Then from the first equation in (4.27) we get

$$\phi_\epsilon = \left[\epsilon^2 \frac{d^2}{dx^2} - (1+\lambda) + 2\frac{A_\epsilon}{H_\epsilon}\right]^{-1} \left(\frac{A_\epsilon^2}{H_\epsilon^2}\psi_\epsilon + f_1\right) \tag{4.30}$$

Since for ϵ small,

$$\max_{[-1,1]} \frac{A_\epsilon}{H_\epsilon} \le 2w(0) = 2\max_R w,$$

there exists, by the resolvent estimate, a constant $M > 0$, such that

$$\|\phi_\epsilon\|_{L^2([-1,1])} \le \frac{M}{|\lambda + 1 - 4w(0)|}(w^2(0)\|\psi_\epsilon\|_{L^2([-1,1])} + \|f_1\|_{L^2([-1,1])}).$$

While

$$\begin{aligned}\|\psi_\epsilon\|_{L^2([-1,1])} &\le \frac{4}{\|w\|_{L^2(R)}|1+\tau\lambda|}\|\phi_\epsilon^c\|_{L^2(R)} + \frac{4\tau}{|1+\tau\lambda|}\|f_2^c\|_{L^2(R)} \\ &\le \frac{C}{|1+\tau\lambda|}(\|\phi_\epsilon\|_{L^2([-1,1])} + \|f_2\|_{L^2([-1,1])}).\end{aligned}$$

Let $a > 0$ be sufficiently large, then if $\lambda \in S_{a,\theta}$, we have

$$\frac{Mw^2(0)C}{|1+\tau\lambda||\lambda + 1 - 4w(0)|} < \frac{1}{2},$$

and hence

$$\|\phi_\epsilon\|_{L^2([-1,1])} \le \frac{CM}{|\lambda - a|}\left(\|f_1\|_{L^2([0,1])}) + \|f_2\|_{L^2([-1,1])}\right). \tag{4.31}$$

From (4.29) we then have

$$\|\psi_\epsilon\|_{L^2([-1,1])} \le \frac{CM}{|\lambda - a|}\left(\|f_1\|_{L^2([-1,1])}) + \|f_2\|_{L^2([-1,1])}\right), \tag{4.32}$$

and therefore

$$\|\mathcal{R}(\lambda, a)\| \le \frac{CM}{|\lambda - a|}, \qquad \text{for } \lambda \in S_{a,\epsilon}. \tag{4.33}$$

The compactness of $(\lambda - \mathcal{L}_\epsilon)^{-1}$ is obvious. This finishes the proof of the lemma. □

The semigroup generated by $\mathcal{L}_\epsilon$ is defined by the formula

$$T(t) = e^{\mathcal{L}_\epsilon t} = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}\mathcal{R}(\lambda, a)d\lambda, \tag{4.34}$$

where Γ is a smooth curve in $S_{a,\theta}$ that connects $\infty e^{-\theta i}$ and $\infty e^{\theta i}$.

5. The transversality condition for the perturbed system

We begin from the eigenvalue problem

$$\begin{cases} (\phi_\epsilon)_{yy} - \phi_\epsilon + 2\dfrac{\hat{A}_\epsilon}{\hat{H}_\epsilon}\phi_\epsilon - \dfrac{\hat{A}_\epsilon^2}{\hat{H}_\epsilon^2}\psi_\epsilon = \lambda_\epsilon\phi_\epsilon, \\ \dfrac{1}{\beta^2}(\psi_\epsilon)_{xx} - \psi_\epsilon + 2\xi_\epsilon\hat{A}_\epsilon\phi_\epsilon = \tau\lambda_\epsilon\psi_\epsilon. \end{cases} \tag{5.1}$$

We let $\mu_\epsilon = \tau\lambda_\epsilon$. Then (5.1) is equivalent to the following eigenvalue problem

$$\begin{cases} \tau\{(\phi_\epsilon)_{yy} - \phi_\epsilon + 2\dfrac{\hat{A}_\epsilon}{\hat{H}_\epsilon}\phi_\epsilon - \dfrac{\hat{A}_\epsilon^2}{\hat{H}_\epsilon^2}\psi_\epsilon\} = \mu_\epsilon\phi_\epsilon, \\ \dfrac{1}{\beta^2}(\psi_\epsilon)_{xx} - \psi_\epsilon + 2\xi_\epsilon\hat{A}_\epsilon\phi_\epsilon = \mu_\epsilon\psi_\epsilon. \end{cases} \tag{5.2}$$

Namely,

$$\mathscr{L}\begin{bmatrix}\phi_\epsilon\\ \psi_\epsilon\end{bmatrix}=\mu_\epsilon\begin{bmatrix}\phi_\epsilon\\ \psi_\epsilon\end{bmatrix}, \tag{5.3}$$

with $\mathscr{L}=\tau\mathcal{L}$. We note that $\mathscr{L}^*=\tau\mathcal{L}^*$.

Let τ_ϵ be the parameter value from Lemma 4.1, so that $Re(\lambda_\epsilon(\tau_\epsilon^h))=0$. Then, via the relationship

$$\mu_\epsilon'(\tau)=\tau\lambda_\epsilon'(\tau)+\lambda_\epsilon(\tau), \tag{5.4}$$

we obtain that $Re(\mu_\epsilon'(\tau_\epsilon^h))=\tau_\epsilon^h Re(\lambda_\epsilon'(\tau_\epsilon^h))$. We now show that $\mu_\epsilon'(\tau_\epsilon^h)>0$ for $\epsilon>0$ sufficiently small.

Let $\Theta_\epsilon=(\phi_\epsilon,\psi_\epsilon)^T$ be a nontrivial eigenfunction of $\mathscr{L}$ corresponding to μ_ϵ and $\Theta_\epsilon^*=(\phi_\epsilon^*,\psi_\epsilon^*)^T$ be a nontrivial eigenfunction of $\mathscr{L}^*$ corresponding to μ_ϵ^*. We have by definition

$$\langle\Theta_\epsilon,\overline{\Theta_\epsilon^*}\rangle=\langle\overline{\Theta_\epsilon},\Theta_\epsilon^*\rangle=0. \tag{5.5}$$

Since λ_0 is a simple eigenvalue, μ_ϵ is simple. Moreover we also have

$$\langle\Theta_\epsilon,\Theta_\epsilon^*\rangle=\langle\overline{\Theta_\epsilon},\overline{\Theta_\epsilon^*}\rangle\neq 0. \tag{5.6}$$

Write

$$\Theta_\epsilon=\begin{bmatrix}\phi_\epsilon\\ \psi_\epsilon\end{bmatrix},\qquad \Theta_\epsilon^*=\begin{bmatrix}\phi_\epsilon^*\\ \psi_\epsilon^*\end{bmatrix}. \tag{5.7}$$

Using the Green's function introduced in Section 2 we write

$$\psi_\epsilon(x)=\int_{-1}^1 2\beta^2\xi_\epsilon G_{\beta_{\lambda_\epsilon}}(x,\xi)\hat{A}_\epsilon\left(\frac{\xi}{\epsilon}\right)\phi_\epsilon\left(\frac{\xi}{\epsilon}\right)d\xi. \tag{5.8}$$

By (4.11) we have

$$\psi_\epsilon(0)=\frac{(1+o(1))}{(1+\tau\lambda_\epsilon)}\epsilon\xi_\epsilon\int_R w\phi_0 \tag{5.9}$$

Similar to the calculation of (4.11), we write

$$\psi_\epsilon^*(x)=-\int_{-1}^1\tau\beta^2 G_{\beta_{\lambda_\epsilon^*}}(x,\xi)\frac{\hat{A}_\epsilon^2}{\hat{H}_\epsilon^2}\left(\frac{\xi}{\epsilon}\right)\phi_\epsilon^*\left(\frac{\xi}{\epsilon}\right)d\xi,$$

and calculate

$$\begin{aligned}
\xi_\epsilon\psi_\epsilon^*(0)&=-\beta^2\tau\xi_\epsilon\int_{-1}^1 G_{\beta_{\lambda_\epsilon^*}}(0,\xi)w^2\left(\frac{\xi}{\epsilon}\right)(\phi_\epsilon^*)^c\left(\frac{\xi}{\epsilon}\right)d\xi+o(1)\\
&=-\beta^2\tau\xi_\epsilon\int_{-1}^1\left(\frac{(\beta_{\lambda_\epsilon^*})^{-2}}{2}+G_0(0,\xi)+O(1)\right)w^2\left(\frac{\xi}{\epsilon}\right)(\phi_\epsilon^*)^c\left(\frac{\xi}{\epsilon}\right)d\xi+o(1)\\
&=-\tau\xi_\epsilon\int_{-1}^1\left(\frac{1}{2(1+\tau\lambda_\epsilon^*)}+\beta^2G_0(0,\xi)+O(\beta^2)\right)w^2\left(\frac{\xi}{\epsilon}\right)(\phi_\epsilon^*)^c\left(\frac{\xi}{\epsilon}\right)d\xi+o(1)\\
&=-\frac{\tau\epsilon\xi_\epsilon}{2(1+\tau\lambda_\epsilon^*)}\int_R w(y)^2(\phi_\epsilon^*)^c(y)dy+O(\beta^2)\\
&=-\frac{\tau(1+o(1))}{(1+\tau\lambda_\epsilon^*)\int_R w^2}\int_R w^2(\phi_\epsilon^*)^c\\
&=-\frac{\tau(1+o(1))}{(1+\tau\lambda_\epsilon^*)\int_R w^2}\int_R w^2\phi_0^* \qquad \text{as}\quad \epsilon\to 0.
\end{aligned} \tag{5.10}$$

Differentiating (5.3) with respect to τ we find that

$$\frac{\partial\mathscr{L}}{\partial\tau}\Theta_\epsilon+\mathscr{L}\frac{\partial\Theta_\epsilon}{\partial\tau}=\frac{\partial\mu_\epsilon}{\partial\tau}\Theta_\epsilon+\mu_\epsilon\frac{\partial\Theta_\epsilon}{\partial\tau}. \tag{5.11}$$

Taking the inner product with Θ_ϵ^* gives

$$\langle \frac{\partial \mathscr{L}}{\partial \tau}\Theta_\epsilon, \Theta_\epsilon^*\rangle + \langle \mathscr{L}\frac{\partial \Theta_\epsilon}{\partial \tau}, \Theta_\epsilon^*\rangle = \langle \frac{\partial \mu_\epsilon}{\partial \tau}\Theta_\epsilon, \Theta_\epsilon^*\rangle + \langle \mu_\epsilon \frac{\partial \Theta_\epsilon}{\partial \tau}, \Theta_\epsilon^*\rangle, \tag{5.12}$$

and then using

$$\langle \mathscr{L}\frac{\partial \Theta_\epsilon}{\partial \tau}, \Theta_\epsilon^*\rangle = \mu_\epsilon \langle \frac{\partial \Theta_\epsilon}{\partial \tau}, \Theta_\epsilon^*\rangle,$$

we obtain

$$\mu_\epsilon'(\tau_\epsilon^h) = \frac{\partial \mu_\epsilon}{\partial \tau}(\tau_\epsilon^h) = \frac{\langle \frac{\partial \mathscr{L}}{\partial \tau}\Theta_\epsilon, \Theta_\epsilon^*\rangle}{\langle \Theta_\epsilon, \Theta_\epsilon^*\rangle} = \frac{\mu_\epsilon}{\tau_\epsilon^h}\frac{\int_{-1}^1 \phi_\epsilon \overline{\phi_\epsilon^*}}{\langle \Theta_\epsilon, \Theta_\epsilon^*\rangle}. \tag{5.13}$$

We compute

$$\begin{aligned}\int_{-1}^1 \phi_\epsilon \overline{\phi_\epsilon^*}\,dx &= \epsilon \int_{-\epsilon^{-1}}^{\epsilon^{-1}} \phi_\epsilon \overline{\phi_\epsilon^*}(y)\,dy \\ &= \epsilon[1+o(1)]\int_R \phi_0 \overline{\phi_0^*}\,dy,\end{aligned} \tag{5.14}$$

and

$$\begin{aligned}\int_{-1}^1 \psi_\epsilon \overline{\psi_\epsilon^*}\,dx &= \frac{1}{\xi_\epsilon}\int_{-1}^1 \psi_\epsilon(x)\overline{\xi_\epsilon \psi_\epsilon^*}(x)\,dx \\ &= -\epsilon[1+o(1)]\frac{2\tau_\epsilon^h}{[1+\mu_\epsilon(\tau_\epsilon^h)]^2\int_R w^2}\int_R w^2\overline{\phi_0^*}\int_R w\phi_0,\end{aligned} \tag{5.15}$$

so that in view of (3.32), we obtain

$$\mu_\epsilon'(\tau_\epsilon^h) = \frac{[1+o(1)]\lambda_0(\tau_h)\int_R \phi_0\overline{\phi_0^*}}{\int_R \phi_0\overline{\phi_0^*} - \frac{2\tau_h}{(1+\tau_h\lambda_0(\tau_h))^2\int_R w^2}\int_R w\phi_0 \int_R w^2\overline{\phi_0^*}} = [1+o(1)]\mu_0'(\tau_h). \tag{5.16}$$

As a consequence of Lemma 3.3 we therefore have

$$Re(\lambda_\epsilon'(\tau_\epsilon^h)) = \frac{1}{\tau_\epsilon^h}Re(\mu_\epsilon') = [1+o(1)]Re(\lambda_0'(\tau_h)) > 0, \qquad \text{for sufficiently small } \epsilon > 0. \tag{5.17}$$

6. Hopf bifurcation: existence, uniqueness and symmetry

We have now established all the assumptions of the Hopf bifurcation theorem of [9]. Indeed, the relevant spectral and semigroup assumptions on the linearization $D_U\mathcal{F}_\epsilon = \mathcal{L}_\epsilon$ at $\tau = \tau_\epsilon^h$ were established in Sections 4 and 5. Furthermore, with $X = H_N^2([0,1]) \times H_N^2([0,1])$ and $Z = L^2([0,1]) \times L^2([0,1]$, the map $\mathcal{F}_\epsilon : X \to Z$ satisfies the required regularity assumptions. We introduce the spaces

$$\begin{aligned} C_{2\pi\rho}^\gamma(\mathbb{R}, X) := \Big\{ & U : \mathbb{R} \to X \,\Big|\, U(t+2\pi\rho) = U(t) \quad t \in \mathbb{R}, \\ & ||U||_{X,\gamma} := \max_{t\in\mathbb{R}} ||U(t)||_X + \sup_{s\neq t}\frac{||U(t)-U(s)||_X}{|t-s|^\gamma} < \infty \Big\},\end{aligned} \tag{6.1}$$

and

$$\begin{aligned} C_{2\pi\rho}^{1+\gamma}(\mathbb{R}, Z) := \Big\{ & U : \mathbb{R} \to Z \,\Big|\, U \in C_{2\pi\rho}^\gamma(\mathbb{R}, Z),\ \tfrac{dU}{dt} \in C_{2\pi\rho}^\gamma(\mathbb{R}, Z), \\ & ||U||_{Z,1+\gamma} := ||U||_{Z,\gamma} + ||\tfrac{dU}{dt}||_{Z,\gamma} < \infty \Big\},\end{aligned} \tag{6.2}$$

where $\gamma \in (0,1]$ is the Hölder exponent. The relevant space for solutions to (1.2) is $Y \equiv C^{\gamma}_{2\pi\rho}(\mathbb{R}, X) \cap C^{1+\gamma}_{2\pi\rho}(\mathbb{R}, Z)$ with the norm

$$||U||_Y \equiv ||U||_{X,\gamma} + ||\tfrac{dU}{dt}||_{Z,\gamma}. \tag{6.3}$$

The Hopf bifurcation theorem thus applies and yields the following result.

Theorem 6.1. *There exists an $\epsilon_0 > 0$ such that for every $0 < \epsilon \leq \epsilon_0$ there are numbers $\delta_\epsilon, \eta_\epsilon > 0$ and continuously differentiable functions $\rho_\epsilon(s)$, $\tau_\epsilon(s)$, and $(\tilde{A}_\epsilon(s), \tilde{H}_\epsilon(s)) \in Y$ defined in $-\eta_\epsilon < s < \eta_\epsilon$ such that $(\tilde{A}_\epsilon(s), \tilde{H}_\epsilon(s))$ is a $2\pi\rho_\epsilon(s)$-periodic solution to (1.1) and*

$$\tau_\epsilon(0) = \tau^h_\epsilon, \quad \rho_\epsilon(0) = 1/\alpha^\epsilon_I, \quad \tilde{A}_\epsilon(0) = A_\epsilon, \quad \tilde{H}_\epsilon(0) = H_\epsilon.$$

In addition the solutions are nontrivial in that $(\tilde{A}_\epsilon(s), \tilde{H}_\epsilon(s)) \neq (A_\epsilon, H_\epsilon)$ for $0 < |s| < \eta_\epsilon$. Furthermore we have uniqueness in the sense that if $(\tau_{\epsilon_1}, \tilde{A}_{\epsilon,1}, \tilde{H}_{\epsilon,1})$ is a $2\pi\rho_{\epsilon,1}$-periodic solution of (1.1) with $|\rho_{\epsilon,1} - 1/\alpha^\epsilon_I| < \delta_\epsilon$, $|\tau_{\epsilon,1} - \tau^h_\epsilon| < \delta_\epsilon$, and $||(\tilde{A}_{\epsilon,1}, \tilde{H}_{\epsilon,1}) - (A_\epsilon, H_\epsilon)||_Y < \delta_\epsilon$, then there exist numbers $s \in [0, \eta_\epsilon)$ and $\theta \in [0, 2\pi\rho_{\epsilon,1})$ so that $\tau_{\epsilon,1} = \tau_\epsilon(s)$ and the solution $(\tilde{A}_{\epsilon,1}, \tilde{H}_{\epsilon,1})$ is obtained from a θ-phase shift of $(\tilde{A}_\epsilon(s), \tilde{H}_\epsilon(s))$, i.e.

$$(\tilde{A}_{\epsilon,1}, \tilde{H}_{\epsilon,1})(t) = [S_\theta(\tilde{A}_\epsilon(s), \tilde{H}_\epsilon(s))](t) \equiv (\tilde{A}_\epsilon(s), \tilde{H}_\epsilon(s))(t+\theta) \qquad \text{for all} \quad t \in \mathbb{R}.$$

Finally, the bifurcating solutions have the following symmetry property

$$(\tilde{A}_\epsilon(-s), \tilde{H}_\epsilon(-s)) = S_{\pi\rho_\epsilon(s)}(\tilde{A}_\epsilon(s), \tilde{H}_\epsilon(s)), \quad \tau_\epsilon(-s) = \tau_\epsilon(s), \quad \rho_\epsilon(-s) = \rho_\epsilon(s) \qquad \text{for all} \quad -\eta_\epsilon < s < \eta_\epsilon.$$

7. Linearized stability of the Hopf bifurcation

In this section we investigate the linearized stability of the periodic solutions obtained in Theorem 6.1 from the previous section. This stability analysis is carried out in the context of a generalization of Floquet Theory from ODEs to semilinear parabolic PDEs and we refer here to Section I.12 of [9]. We briefly summarize the main aspects of this theory so that our stability result may be accurately stated.

Suppose $A(t)$ is a time-dependent linear operator which is p-periodic in t and consider the problem

$$\frac{dw}{dt} - A(t)w = 0. \tag{7.1}$$

The Floquet multipliers of (7.1) are the eigenvalues of $U(p)$, where $w(t) = U(t)x$ is the solution of (7.1) satisfying $w(0) = x$. We say that κ is a Floquet exponent of (7.1) if and only if $e^{-p\kappa}$ is a Floquet multiplier, or equivalently if κ is an eigenvalue of $\frac{d}{dt} - A(t)$ in the space of p-periodic functions.

The concepts of Floquet Theory arise in the study of periodic solutions as follows. If u is a p-periodic solution of the nonlinear problem

$$\frac{du}{dt} = g(u), \tag{7.2}$$

then the linearization about this periodic solution results in the variational equation

$$\tfrac{dv}{dt} - g_u(u(t))v = 0, \tag{7.3}$$

from which the Floquet multipliers and exponents are defined as for (7.1) with $A(t) = g_u(u(t))$. If $\dot{u} = \frac{du}{dt} \not\equiv 0$, formally differentiating (7.2) shows that

$$\frac{d\dot{u}}{dt} = g_u(u(t))\dot{u},$$

so that 0 is always a Floquet exponent and 1 is a Floquet multiplier for u. It has been shown that the stability properties of a periodic solution to (7.2) are determined by the moduli of its Floquet multipliers (see Section 8. 2 of [7]). Specifically, if the Floquet exponent $\kappa = 0$ is simple and all remaining Floquet exponents have positive real parts, then the p-periodic solution u is linearly stable.

The Floquet exponent for the $2\pi\rho_\epsilon(s)$-periodic solutions $U_\epsilon(s) = (\tilde{A}_\epsilon(s), \tilde{H}_\epsilon(s))$ from Theorem 6.1 are therefore numbers κ such that the problem

$$\frac{1}{\rho_\epsilon(s)}\frac{dw}{dt} - \big(\mathcal{L}_\epsilon + R_U(\tau_\epsilon(s), U_\epsilon(s)(\rho_\epsilon(s)t))\big)w = \kappa w, \qquad w(0) = w(2\pi) \tag{7.4}$$

has a nontrivial solution. At $s = 0$, (7.4) becomes

$$\alpha_I^\epsilon \frac{dw}{dt} - \mathcal{L}_\epsilon w = \kappa w, \qquad w(0) = w(2\pi). \tag{7.5}$$

The set of values of κ for which (7.5) has a nontrivial solution is $\{\alpha_I^\epsilon ni - \sigma(\mathcal{L}_\epsilon) : n = \pm1, \pm2, , \ldots\}$, so the corresponding multipliers are $e^{2\pi\sigma(\mathcal{L}_\epsilon)/\alpha_I^\epsilon}$. Thus, 1 is clearly a Floquet multiplier with multiplicity two corresponding to the double eigenvalue $\kappa = 0$ inherited from $\pm i\alpha_I \in \sigma(\mathcal{L}_\epsilon)$. On the other hand, Lemma 4.1 implies that the remaining eigenvalues of $\mathcal{L}_\epsilon$ at $s = 0$ have negative real part and therefore the remaining Floquet exponents have positive real parts.

Since a zero Floquet exponent persists for all values of $-\eta_\epsilon < s < \eta_\epsilon$, it is a second, nontrivial, Floquet exponent, $\kappa_\epsilon(s)$, with $\kappa_\epsilon(0) = 0$ which determines the linear stability of the periodic solution. Specifically, if $Re(\kappa_\epsilon(s)) > 0$ then the periodic solution is linearly stable in the sense of [7], and is otherwise unstable. With $\cdot$ denoting a derivative with respect to s, Theorem I.12.2 of [9] implies that $\dot{\kappa}_\epsilon(0) = 0$ and $\dot{\tau}_\epsilon(0) = 0$. Moreover, formula (I.12.34) of [9] relates the second derivatives according to

$$\ddot{\kappa}_\epsilon(0) = 2\ddot{\tau}_\epsilon(0)Re(\lambda'_\epsilon(\tau_\epsilon^h)).$$

From Section 5 we know $Re(\lambda'_\epsilon(\tau_\epsilon^h)) > 0$ and therefore the first part of Corollary I.12.3, or the *Principle of Exchange of Stability*, of [9] applies.

Theorem 7.1. *Let the hypotheses of Theorem 6.1 be satisfied. Then*

$$sgn(\tau_\epsilon(s) - \tau_\epsilon^h) = sgn(\kappa_\epsilon(s)) \quad \textit{for} \quad -\eta_\epsilon < s < \eta_\epsilon.$$

Hence, the bifurcating periodic solutions of Theorem 6.1 are linearly stable (resp. unstable) if the bifurcation is supercritical (resp. subcritical).

To conclude the stability question it remains therefore to determine the sign of $\ddot{\tau}_\epsilon(0)$. For this we use the formula (see equation (I.9.11) of [9])

$$\ddot{\tau}_\epsilon(0) = \frac{1}{Re(\lambda'_\epsilon(\tau_\epsilon^h))}Re(K(\epsilon)), \tag{7.6}$$

where

$$\begin{aligned} K(\epsilon) &= -\langle D^3_{UUU}R(\tau_\epsilon^h, 0)[\Theta_\epsilon, \Theta_\epsilon, \overline{\Theta_\epsilon}], \Theta_\epsilon^*\rangle \\ &\quad + \langle D^2_{UU}R(\tau_\epsilon^h, 0)[\overline{\Theta_\epsilon}, (\mathcal{L}_\epsilon - 2\alpha_I^\epsilon i)^{-1}D^2_{UU}R(\tau_\epsilon^h, 0)[\Theta_\epsilon, \Theta_\epsilon]], \Theta_\epsilon^*\rangle \\ &\quad + 2\langle D^2_{UU}R(\tau_\epsilon^h, 0)[\Theta_\epsilon, \mathcal{L}_\epsilon^{-1}D^2_{UU}R(\tau_\epsilon^h, 0)[\Theta_\epsilon, \overline{\Theta}_\epsilon]], \Theta_\epsilon^*\rangle \\ &= K_1(\epsilon) + K_2(\epsilon) + K_3(\epsilon), \end{aligned} \tag{7.7}$$

where $\Theta_\epsilon = (\phi_\epsilon, \psi_\epsilon)$ is a nontrivial eigenfunction of $\mathcal{L}_\epsilon$ corresponding to the eigenvalue $\alpha_I i$, $\Theta_\epsilon^* = (\phi_\epsilon^*, \psi_\epsilon^*)$ is a nontrivial eigenfunction of $\mathcal{L}_\epsilon^*$ corresponding to the eigenvalue $-\alpha_I i$, and moreover

$$\langle \Theta_\epsilon, \Theta_\epsilon^* \rangle = 1. \tag{7.8}$$

As calculated before

$$\begin{aligned}\langle\Theta_\epsilon,\Theta_\epsilon^*\rangle &= \int_{-1}^1 \phi_\epsilon\overline{\phi_\epsilon^*}\,dx+\int_{-1}^1\psi_\epsilon\overline{\psi_\epsilon^*}\,dx \\ &= \epsilon[1+o(1)]\left[\int_R \phi_0\overline{\phi_0^*}-\frac{2\tau_h}{(1+\tau_h\lambda_0(\tau_h))^2\int_R w^2}\int_R w\phi_0\int_R w^2\overline{\phi_0^*}\right].\end{aligned} \tag{7.9}$$

Therefore, we have

$$\int_R \phi_0\overline{\phi_0^*}-\frac{2\tau_h}{(1+\tau_h\lambda_0(\tau_h))^2\int_R w^2}\int_R w\phi_0\int_R w^2\overline{\phi_0^*}=\frac{1+o(1)}{\epsilon}. \tag{7.10}$$

Recall that

$$R(\tau,U)=\begin{bmatrix}R_1(\tau,U)\\ R_2(\tau,U)\end{bmatrix} \tag{7.11}$$

with

$$R_1(\tau,U)=\frac{(A_\epsilon+U_1)^2}{H_\epsilon+U_2}-\frac{A_\epsilon^2}{H_\epsilon}-\frac{2A_\epsilon U_1}{H_\epsilon}+\frac{A_\epsilon^2U_2}{H_\epsilon^2},$$

and

$$R_2(\tau,U)=\frac{1}{\tau}\left((A_\epsilon+U_1)^2-A_\epsilon^2-2A_\epsilon U_1\right)=\frac{2}{\tau}U_1^2.$$

For functions

$$g=\begin{bmatrix}g_1\\ g_2\end{bmatrix},\quad h=\begin{bmatrix}h_1\\ h_2\end{bmatrix},\quad l=\begin{bmatrix}l_1\\ l_2\end{bmatrix}\qquad\in Z,$$

We calculate

$$D_{UU}^2R_1(\tau,0)[g,h]=\frac{2}{H_\epsilon}g_1h_1-\frac{2A_\epsilon}{H_\epsilon^2}[g_1h_2+g_2h_1]+\frac{2A_\epsilon^2}{H_\epsilon^3}g_2h_2,$$

$$D_{UUU}^3R_1(\tau,0)[g,h,l]=-\frac{2}{H_\epsilon^2}[g_1h_2l_1+g_2h_1l_1+g_1h_1l_2]+\frac{4A_\epsilon}{H_\epsilon^3}[g_2h_2l_1+g_1h_2l_2+g_2h_1l_2]-\frac{6A_\epsilon^2}{H_\epsilon^4}g_2h_2l_2,$$

$$D_{UU}^2R_2(\tau,0)[g,h]=\frac{4}{\tau}g_1h_1,$$

$$D_{UUU}^3R_2(\tau,0)[g,h,l]=0.$$

Therefore,

$$\begin{aligned}K_1(\epsilon)&=-\langle D_{UUU}^3R(\tau_\epsilon^h,0)[\Theta_\epsilon,\Theta_\epsilon,\overline{\Theta_\epsilon}],\Theta_\epsilon^*\rangle\\ &=\int_{-1}^1\left[\frac{2}{H_\epsilon^2}(2|\phi_\epsilon|^2\psi_\epsilon+\phi_\epsilon^2\overline{\psi_\epsilon})-\frac{4A_\epsilon}{H_\epsilon^3}(\psi_\epsilon^2\overline{\phi_\epsilon}+2\phi_\epsilon|\psi_\epsilon|^2)+\frac{6A_\epsilon^2}{H_\epsilon^4}\psi_\epsilon|\psi_\epsilon|^2\right]\overline{\phi_\epsilon^*}\,dx,\end{aligned} \tag{7.12}$$

$$\begin{aligned}\xi_\epsilon K_2(\epsilon)&=\langle D_{UU}^2R(\tau_\epsilon^h,0)[\overline{\Theta_\epsilon},\xi_\epsilon(\mathcal{L}_\epsilon-2\alpha_I^\epsilon i)^{-1}D_{UU}^2R(\tau_\epsilon^h,0)[\Theta_\epsilon,\Theta_\epsilon]],\Theta_\epsilon^*\rangle\\ &=\int_{-1}^1\left[\frac{2}{H_\epsilon}\overline{\phi_\epsilon}z_1^\epsilon-\frac{2A_\epsilon}{H_\epsilon^2}(\overline{\phi_\epsilon}z_2^\epsilon+\overline{\psi_\epsilon}z_1^\epsilon)+\frac{2A_\epsilon^2}{H_\epsilon^3}\overline{\psi_\epsilon}z_2^\epsilon\right]\overline{\phi_\epsilon^*}\,dx+\frac{4}{\tau_\epsilon^h}\int_{-1}^1 z_1^\epsilon\overline{\phi_\epsilon\psi_\epsilon^*}\,dx,\end{aligned} \tag{7.13}$$

$$\begin{aligned}\xi_\epsilon K_3(\epsilon)&=2\langle D_{UU}^2R(\tau_\epsilon^h,0)[\Theta_\epsilon,\xi_\epsilon\mathcal{L}_\epsilon^{-1}D_{UU}^2R(\tau_\epsilon^h,0)[\Theta_\epsilon,\overline{\Theta}_\epsilon]],\Theta_\epsilon^*\rangle\\ &=2\int_{-1}^1\left[\frac{2}{H_\epsilon}\phi_\epsilon h_1^\epsilon-\frac{2A_\epsilon}{H_\epsilon^2}(\phi_\epsilon h_2^\epsilon+\psi_\epsilon h_1^\epsilon)+\frac{2A_\epsilon^2}{H_\epsilon^3}\psi_\epsilon h_2^\epsilon\right]\overline{\phi_\epsilon^*}\,dx+\frac{8}{\tau_\epsilon^h}\int_{-1}^1 h_1^\epsilon\phi_\epsilon\overline{\psi_\epsilon^*}\,dx.\end{aligned} \tag{7.14}$$

Here

$$\begin{bmatrix}z_1^\epsilon\\ z_2^\epsilon\end{bmatrix}=\xi_\epsilon(\mathcal{L}_\epsilon-2\alpha_I^\epsilon i)^{-1}D_{UU}^2(\tau_\epsilon^h,0)[\Theta_\epsilon,\Theta_\epsilon]=\xi_\epsilon(\mathcal{L}_\epsilon-2\alpha_I^\epsilon i)^{-1}\begin{bmatrix}\frac{2}{H_\epsilon}\phi_\epsilon^2-\frac{4A_\epsilon}{H_\epsilon^2}\phi_\epsilon\psi_\epsilon+\frac{2A_\epsilon^2}{H_\epsilon^3}\psi_\epsilon^2\\ \frac{4}{\tau_\epsilon^h}\phi_\epsilon^2\end{bmatrix}.$$

Namely,

(7.15)

$$\begin{cases} \epsilon^2 (z_1^\epsilon)'' - (1+2\alpha_I^\epsilon i) z_1^\epsilon + \dfrac{2A_\epsilon}{H_\epsilon} z_1^\epsilon - \dfrac{A_\epsilon^2}{H_\epsilon^2} z_2^\epsilon = \dfrac{2\xi_\epsilon}{H_\epsilon}\phi_\epsilon^2 - \dfrac{4\xi_\epsilon A_\epsilon}{H_\epsilon^2}\phi_\epsilon\psi_\epsilon + \dfrac{2\xi_\epsilon A_\epsilon^2}{H_\epsilon^3}\psi_\epsilon^2 & \text{in } [-1,1], \\ (z_2^\epsilon)'' - \beta^2(1+2\tau_\epsilon^h \alpha_I^\epsilon i) z_2^\epsilon + 2\beta^2 A_\epsilon z_1^\epsilon = 4\beta^2 \xi_\epsilon \phi_\epsilon^2 & \text{in } [-1,1], \\ (z_1^\epsilon)' = (z_2^\epsilon)' = 0 & \text{for } x = -1, 1. \end{cases}$$

By the discussions in previous sections, we can derive a limit equation of (7.15)

$$\begin{cases} z_1'' - (1+2\alpha_I i) z_1 + 2wz_1 - \dfrac{2}{1+2\tau_h\alpha_I i}\dfrac{\int_R (wz_1 - 2\phi_0^2)}{\int_R w^2} w^2 = 2\phi_0^2 - 4w\phi_0\psi_0 + 2w^2\psi_0^2 & \text{in } R, \\ z_2 = \dfrac{2}{1+2\tau_h\alpha_I i}\dfrac{\int_R (wz_1 - 2\phi_0^2)}{\int_R w^2} & \text{in } R. \end{cases}$$

While

$$\begin{bmatrix} h_1^\epsilon \\ h_2^\epsilon \end{bmatrix} = \xi_\epsilon (\mathcal{L}_\epsilon)^{-1} D_{UU}^2(\tau_\epsilon^h, 0)[\Theta_\epsilon, \Theta_\epsilon] = \xi_\epsilon (\mathcal{L}_\epsilon)^{-1} \begin{bmatrix} \frac{2}{H_\epsilon}|\phi_\epsilon|^2 - \frac{2A_\epsilon}{H_\epsilon^2}(\phi_\epsilon\overline{\psi_\epsilon} + \psi_\epsilon\overline{\phi_\epsilon}) + \frac{2A_\epsilon^2}{H_\epsilon^3}|\psi_\epsilon|^2 \\ \frac{4}{\tau_\epsilon^h}|\phi_\epsilon|^2 \end{bmatrix}.$$

Namely,

(7.16)

$$\begin{cases} \epsilon^2 (h_1^\epsilon)'' - h_1^\epsilon + \dfrac{2A_\epsilon}{H_\epsilon} h_1^\epsilon - \dfrac{A_\epsilon^2}{H_\epsilon^2} h_2^\epsilon = \dfrac{2\xi_\epsilon}{H_\epsilon}|\phi_\epsilon|^2 - \dfrac{2\xi_\epsilon A_\epsilon}{H_\epsilon^2}(\phi_\epsilon\overline{\psi_\epsilon} + \psi_\epsilon\overline{\phi_\epsilon}) + \dfrac{2\xi_\epsilon A_\epsilon^2}{H_\epsilon^3}|\psi_\epsilon|^2 & \text{in } [-1,1], \\ (h_2^\epsilon)'' - \beta^2 h_2^\epsilon + 2\beta^2 A_\epsilon h_1^\epsilon = 4\beta^2\xi_\epsilon|\phi_\epsilon|^2 & \text{in } [-1,1], \\ (h_1^\epsilon)' = (h_2^\epsilon)' = 0 & \text{for } x = -1, 1. \end{cases}$$

Accordingly, the limit equation of (7.16) is

$$\begin{cases} h_1'' - h_1 + 2wh_1 - 2\dfrac{\int_R (wh_1 - 2|\phi_0|^2)}{\int_R w^2} w^2 = 2|\phi_0|^2 - 2w(\phi_0\overline{\psi_0} + \psi_0\overline{\phi_0}) + 2w^2|\psi_0|^2 & \text{in } R, \\ h_2 = 2\dfrac{\int_R (wh_1 - 2|\phi_0|^2)}{\int_R w^2} & \text{in } R. \end{cases}$$

Therefore we have, as $\epsilon \to 0$, that

(7.17)

$$\epsilon^{-1}\xi_\epsilon^2 K_1(\epsilon) = [1+o(1)] \int_R \left[2(2|\phi_0|^2\psi_0 + \phi_0^2\overline{\psi_0}) - 4w(\psi_0^2\overline{\phi_0} + 2\phi_0|\psi_0|^2) + 6w^2\psi_0|\psi_0|^2\right] \overline{\phi_0^*}\, dy.$$

Using the estimate (5.10) we obtain, as $\epsilon \to 0$, that

$$\begin{aligned} \epsilon^{-1}\xi_\epsilon^2 K_2(\epsilon) =& [1+o(1)] \int_R \left[2\overline{\phi_0} z_1 - 2w(\overline{\phi_0} z_2 + \overline{\psi_0} z_1) + 2w^2 \overline{\psi_0} z_2\right] \overline{\phi_0^*}\, dy \\ & - [1+o(1)] \frac{4}{(1+\tau_h\alpha_I i)\int_R w^2} \int_R z_1\overline{\phi_0} \int_R w^2 \overline{\phi_0^*}\, dy. \end{aligned} \tag{7.18}$$

Similarly, as $\epsilon \to 0$,

$$\begin{aligned} \epsilon^{-1}\xi_\epsilon^2 K_3(\epsilon) =& 2[1+o(1)] \int_R \left[2\phi_0 h_1 - 2w(\phi_0 h_2 + \psi_0 h_1) + 2w^2\psi_0 h_2\right] \overline{\phi_0^*}\, dy \\ & - [1+o(1)] \frac{8}{(1+\tau_h\alpha_I i)\int_R w^2} \int_R h_1\phi_0 \int_R w^2 \overline{\phi_0^*}\, dy. \end{aligned} \tag{7.19}$$

Here

$$\psi_0 \equiv \frac{2}{1+\tau_h\alpha_I i}\frac{\int_R w\phi_0}{\int_R w^2}. \tag{7.20}$$

Remark 7.2. *Thus the criticality of the Hopf bifurcation is the same as for the corresponding limiting $\epsilon \to 0$ problem.*

8. Numerical Computation of $Re(K(\epsilon))$

It remains to compute the sign of $Re(K(\epsilon))$ as given by (7.7) and using the limiting behaviour as $\epsilon \to 0$ of K_1, K_2, and K_3 found in equations (7.17), (7.18), and (7.19) respectively. This requires us to first calculate the Höpf bifurcation time constant τ_h and purely imaginary eigenvalue λ_0 as well as its corresponding eigenfunction ϕ_0 and adjoint eigenfunction ϕ_0^*. Following this we must evaluate the auxiliary functions z_k and h_k for $k = 1,2$ satisfying the limiting equations of (7.15) and (7.16) respectively.

To calculate the Höpf bifurcation threshold τ_h and eigenvalue $\lambda_0 = i\alpha_I$ we first rewrite the NLEP (3.1) as

$$1 + \tau_h\lambda_0 - 2\frac{\int_{-\infty}^{\infty} w(L_0 - \lambda_0)^{-1}w^2}{\int_{-\infty}^{\infty} w^2} = 0. \tag{8.1}$$

The term $(L_0 - \lambda_0)^{-1}w^2$ appearing in the numerator is calculated by solving the boundary value problem $(L_0 - \lambda_0)\zeta = w^2$ with boundary conditions $\zeta'(0) = 0$ and $\zeta(y) \to 0$ as $y \to \infty$. Numerically this is solved on the truncated domain $0 < y < L$ for which the exponential decay of the solution can be leveraged to replace the decay at infinity with $\zeta(L) = 0$ provided L is sufficiently large. For this and subsequent truncated domain computations we will use a value of $L = 500$. Additionally we use the solve_bvp routine from the scipy.integrate library. Having computed the relevant boundary value problem it is then straightforward to solve (8.1) for τ_h and $\lambda_0 = i\alpha_I$ using a zero-finding routine. Specifically, by equating real and imaginary parts, we first solve

$$1 - 2Re\left\{\frac{\int_{-\infty}^{\infty} w(L_0 - i\alpha_I)^{-1}w^2}{\int_{-\infty}^{\infty} w^2 dy}\right\} = 0,$$

for α_I and then obtain τ_h from

$$\tau_h = \frac{2}{\alpha_I} Im\left\{\frac{\int_{-\infty}^{\infty} w(L_0 - i\alpha_I)^{-1}w^2}{\int_{-\infty}^{\infty} w^2 dy}\right\}.$$

Using the brentq routine from the scipy library we compute

$$\tau_h = 0.77107, \qquad \lambda_0 = i\alpha_I = 1.2376i, \tag{8.2}$$

for which the left hand side of (8.1) evaluates to an $O(10^{-13})$ value. We remark that these values are in agreement with those found in Table 1 of [20].

The corresponding eigenfunction ϕ_0 can then be found by solving the boundary value problem

$$(L_0 - \lambda_0)\phi_0 = w^2, \quad 0 < y < \infty, \qquad \phi_0'(0) = 0, \qquad \phi_0(y) \to 0 \quad \text{as } y \to +\infty$$

Numerical integration then gives $\psi_0 \approx 1$ which can be verified explicitly from the definition of ψ_0. The adjoint eigenfunction ϕ_0^* is found similarly. We first solve the problem

$$(L_0 - \bar{\lambda}_0)q_0^* = w, \quad 0 < y < \infty, \qquad \phi_0'(0) = 0, \qquad \phi_0(y) \to 0 \quad \text{as } y \to +\infty,$$

and then let $\phi_0^* = \bar{\beta}q_0^*$ where the constant β is chosen so that ϕ_0 and ϕ_0^* adhere to the normalization (7.10) which yields

$$\beta = \frac{1}{\epsilon}\left[\int_{-\infty}^{\infty} \phi_0\bar{q}_0^* - \frac{2\tau_h}{(1 + i\tau_h\alpha_I)^2}\frac{\int_{-\infty}^{\infty} w\phi_0 \int_{-\infty}^{\infty} w^2\bar{q}_0^*}{\int_{-\infty}^{\infty} w^2}\right]^{-1}.$$

To calculate z_1 and z_2 we first rewrite the z_1 limit equation of (7.15) as

$$(L_0 - 2\lambda_0)z_1 = f_1 + \frac{2}{1 + 2\tau_h\lambda_0}\frac{\int_{-\infty}^{\infty} wz_1}{\int_{-\infty}^{\infty} w^2}w^2, \qquad z_1'(0) = 0, \qquad z_1(y) \to 0 \quad \text{as } y \to +\infty,$$

where

$$f_1 := 2\phi_0^2 - 4w\phi_0\psi_0 + 2w^2\psi_0^2 - \frac{4}{1+2\tau_h\lambda_0}\frac{\int_{-\infty}^{\infty}\phi_0^2}{\int_{-\infty}^{\infty}w^2}w^2.$$

Let ξ_1 and ξ_2 be the solutions to

$$(L_0 - 2\lambda_0)\xi_1 = f_1, \quad 0 < y < \infty, \qquad \xi_1'(0) = 0, \qquad \xi_1(y) \to 0 \quad \text{as } y \to +\infty, \tag{8.3}$$

and

$$(L_0 - 2\lambda_0)\xi_2 = w^2, \quad 0 < y < \infty, \qquad \xi_2'(0) = 0, \qquad \xi_2(y) \to 0 \quad \text{as } y \to +\infty, \tag{8.4}$$

respectively. Then

$$z_1 = \xi_1 + \frac{2}{1+2\tau_h\lambda_0}\frac{\int_{-\infty}^{\infty}wz_1}{\int_{-\infty}^{\infty}w^2}\xi_2,$$

so that multiplying by w and integrating allows us to solve for $\int wz_1$ from which we deduce

$$z_1 = \xi_1 + \frac{\frac{2}{1+2\tau_h\lambda_0}\frac{\int_{-\infty}^{\infty}w\xi_1}{\int_{-\infty}^{\infty}w^2}}{1 - \frac{2}{1+2\tau_h\lambda_0}\frac{\int_{-\infty}^{\infty}w\xi_2}{\int_{-\infty}^{\infty}w^2}}\xi_2. \tag{8.5}$$

Therefore z_1 can be computed by solving the two corresponding boundary value problems numerically. It is then straightforward to numerically calculate $z_2 \approx -1.402 - 1.373i$. The function h_1 can be found similarly by writing the limit equation of (7.16) as

$$L_0 h_1 = f_2 + \frac{2\int_{-\infty}^{\infty}wh_1}{\int_{-\infty}^{\infty}w^2}w^2, \qquad h_1'(0) = 0, \qquad h_1(y) \to 0 \quad \text{as } y \to +\infty,$$

where

$$f_2 := 2|\phi_0|^2 - 2(\phi_0\bar{\psi}_0 + \bar{\phi}_0\psi_0)w + 2w^2|\psi_0|^2 - \frac{4\int_{-\infty}^{\infty}|\phi_0|^2}{\int_{-\infty}^{\infty}w^2}w^2,$$

We then let η_1 and η_2 be the solutions to

$$L_0\eta_1 = f_2, \quad 0 < y < \infty, \qquad \eta_1'(0) = 0, \qquad \eta_1(y) \to 0 \quad \text{as } y \to +\infty, \tag{8.6}$$

and

$$L_0\eta_2 = w^2, \quad 0 < y < \infty, \qquad \eta_2'(0) = 0, \qquad \eta_2(y) \to 0 \quad \text{as } y \to +\infty, \tag{8.7}$$

respectively. Solving these two boundary value problems we obtain h_1 in the form

$$h_1 = \eta_1 + \frac{\frac{2\int_{-\infty}^{\infty}w\eta_1}{\int_{-\infty}^{\infty}w^2}}{1 - \frac{2\int_{-\infty}^{\infty}w\eta_2}{\int_{-\infty}^{\infty}w^2}}\eta_2, \tag{8.8}$$

and obtain $h_2 \approx -0.14669$. Using (7.17), (7.18), and (7.19) we thus calculate

$$\xi_\epsilon^2 K_1 = -1.2732 - 2.5039i + o(1),$$
$$\xi_\epsilon^2 K_2 = -1.3820 - 0.39262i + o(1),$$
$$\xi_\epsilon^2 K_3 = 2.6454 + 7.0406i + o(1),$$

and therefore

$$\xi_\epsilon^2 K(\varepsilon) = -0.0098061 + 4.1441i + o(1), \tag{8.9}$$

where the ϵ^{-1} term from the normalization of ϕ_0^* has cancelled out the ϵ^{-1} in front of the expressions (7.17), (7.18), and (7.19). The negative sign of $Re(K(\varepsilon))$ indicates that the Höpf bifurcation is subcritical and the bifurcating periodic solutions are therefore linearly unstable.

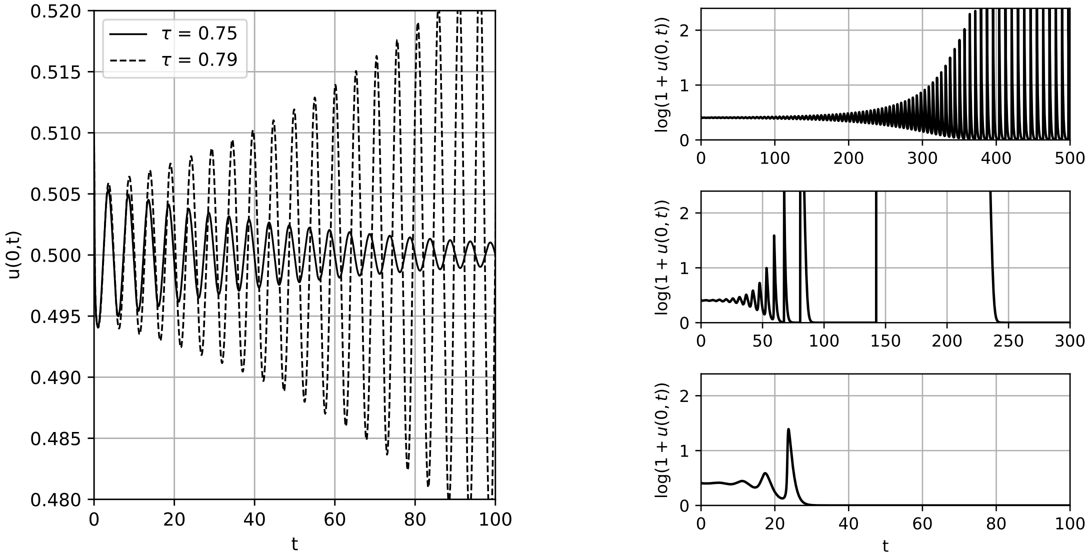

FIGURE 1. Numerical simulations performed for $D = 5000$ and $\epsilon = 0.02$. (A) The onset of oscillatory instabilities as τ exceeds the Höpf bifurcation threshold. (B) Long-time behaviour illustrating the instability and annihilation of a time-periodic solution for values of $\tau = 0.79$ (top), 0.9 (middle), and 1.2 (bottom).

9. Numerical Verification

In this section we illustrate the theoretical results of the previous sections by numerically computing solutions of the time-dependent system (1.1) for a variety of τ values and fixed values of $D = 5000$ and $\epsilon = 0.02$. For convenience we introduce the scaling

$$\tilde{A}(x,t) = \epsilon^{-1}u(x,t), \qquad \tilde{H}(x,t) = \epsilon^{-1}v(x,t),$$

so that the nontrivial equilibrium from Theorem 2.1 becomes $O(1)$. Furthermore the system (1.1) becomes

$$\begin{cases} u_t = \epsilon^2 u_{xx} - u + \dfrac{u^2}{v}, & u > 0 \quad \text{for } 0 < x < 1,\ t > 0, \\ \tau v_t = D v_{xx} - v + \epsilon^{-1}u^2, & v > 0 \quad \text{for } 0 < x < 1,\ t > 0, \\ u_x = v_x = 0, & \quad\quad\quad \text{for } x = 0, 1,\ t \geq 0. \end{cases} \tag{9.1}$$

With the (scaled) equilibrium from Theorem 2.1 as the initial condition we can illustrate the theoretical results given above by solving (9.1) numerically for values of τ below and above the predicted Höpf bifurcation threshold.

The numerical solutions are calculated by discretizing the interval $0 \leq x \leq 1$ into 1000 equidistant points and using a second-order semi-implicit backwards difference (2-SBDF) implicit-explicit (IMEX) time stepping scheme (see [15] for details) with a time-step size of 0.0001. Since IMEX schemes use explicit (resp. implicit) methods for the nonlinear (resp. diffusive) terms they are well suited for reaction diffusion systems where they can avoid the nonlinear solvers used in fully implicit schemes and the small time-steps required in fully explicit schemes. In Figure 1 we collect results of the numerical simulations for different values of τ. In 1a we observe the onset of an oscillatory instability of the value of $\tau = 0.79$ exceeding the Höpf bifurcation threshold $\tau_h = 0.77107$. On the other hand, when $\tau = 0.75 < \tau_h$ we observe the solution settles to the original equilibrium. The long-time behaviour is shown in Figure 1b, where we have chosen to plot $\log(1 + u(0, t))$ to better demonstrate the solution's variability. While the uppermost subplot ($\tau = 0.79$) appears to exhibit a stable limit-cycle solution, these oscillations are instead large-amplitude instabilities cased by the instability of the trivial equilibrium for $\tau < 1$ (see [20] for details). Indeed the middle subplot ($\tau = 0.9$) shows how the oscillations eventually subside and then lead to a substantial jump from the unstable zero-solution. Meanwhile the bottom subplot ($\tau = 1.2$) shows how the initial oscillatory instabilities subside and the solutions settles to the trivial equilibrium solution. Together, the numerical results shown in Figure 1 illustrate the theoretical prediction that the Höpf bifurcation is subcritical.

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Department of Mathematics, The University of British Columbia, Vancouver, BC Canada V6T 1Z2

E-mail address: `dagubc@math.ubc.ca`

Department of Mathematics, Henan Normal University, Xinxiang 453007, P.R. China

E-mail address: `mlf@htu.edu.cn`

Department of Mathematics, The University of British Columbia, Vancouver, BC Canada V6T 1Z2

E-mail address: `jcwei@math.ubc.ca`