J. fixed point theory appl. *Online First*

DOI 10.1007/s11784-007-0033-6

Journal of Fixed Point Theory and Applications

Bound states for a coupled Schrödinger system

Thomas Bartsch, Zhi-Qiang Wang and Juncheng Wei

Dedicated to Albrecht Dold and Edward Fadell

Abstract. We consider the existence of bound states for the coupled elliptic system

$$\Delta u_1 - \lambda_1 u_1 + \mu_1 u_1^3 + \beta u_2^2 u_1 = 0 \quad \text{in } \mathbb{R}^n,$$
$$\Delta u_2 - \lambda_2 u_2 + \mu_2 u_2^3 + \beta u_1^2 u_2 = 0 \quad \text{in } \mathbb{R}^n,$$
$$u_1 > 0, \quad u_2 > 0, \quad u_1, u_2 \in \mathbb{H}^1(\mathbb{R}^n).$$

where $n \leq 3$. Using the fixed point index in cones we prove the existence of a five-dimensional continuum $\mathcal{C} \subset \mathbb{R}_+^5 \times \mathbb{H}^1(\mathbb{R}^n) \times \mathbb{H}^1(\mathbb{R}^n)$ of solutions $(\lambda_1, \lambda_2, \mu_1, \mu_2, \beta, u_1, u_2)$ bifurcating from the set of semipositive solutions (where $u_1 = 0$ or $u_2 = 0$) and investigate the parameter range covered by $\mathcal{C}$.

Mathematics Subject Classification (2000). Primary 35Q55; Secondary 35B32, 35J50, 35J55, 47J30, 58E05, 58E07.

Keywords. Bound states, coupled Schrödinger equations, global bifurcation.

1. Introduction

In this paper, we study solitary wave solutions of time-dependent coupled nonlinear Schrödinger equations given by

$$\begin{cases} -i\dfrac{\partial}{\partial t}\Phi_1 = \Delta\Phi_1 + \mu_1|\Phi_1|^2\Phi_1 + \beta|\Phi_2|^2\Phi_1 & \text{for } y \in \mathbb{R}^n,\ t > 0, \\ -i\dfrac{\partial}{\partial t}\Phi_2 = \Delta\Phi_2 + \mu_2|\Phi_2|^2\Phi_2 + \beta|\Phi_1|^2\Phi_2 & \text{for } y \in \mathbb{R}^n,\ t > 0, \\ \Phi_j = \Phi_j(y,t) \in \mathbb{C}, \quad j = 1,2, \\ \Phi_j(y,t) \to 0 \quad \text{as } |y| \to \infty,\ t > 0,\ j = 1,2 \end{cases} \tag{1.1}$$

where μ_1, μ_2 are positive constants, $n \leq 3$, and β is a coupling constant.

The system (1.1) has applications in many physical problems, especially in nonlinear optics. Physically, the solution Φ_j denotes the j-th component of the beam in Kerr-like photorefractive media ([1]). The positive constant μ_j is for self-focusing in the j-th component of the beam. The coupling constant β is the interaction between the first and the second component of the beam. The interaction is attractive $\beta > 0$, and repulsive if $\beta < 0$.

Problem (1.1) also arises in the Hartree–Fock theory for a double condensate, i.e. a binary mixture of Bose–Einstein condensates in two different hyperfine states $|1\rangle$ and $|2\rangle$ ([16]). Physically, Φ_1 and Φ_2 are the corresponding condensate amplitudes, μ_j and β are the intraspecies and interspecies scattering lengths. The sign of the scattering length β determines whether the interactions of states $|1\rangle$ and $|2\rangle$ are repulsive or attractive. When $\beta < 0$, the interactions are repulsive ([30]); in contrast, when $\beta > 0$, they are attractive. For atoms of the single state $|j\rangle$, when $\mu_j > 0$, the interactions of the single state $|j\rangle$ are attractive.

To obtain solitary wave solutions of the system (1.1), we set $\Phi_j(y,t) = e^{i\lambda_j t}u_j(y)$, $j = 1, 2$, and we may transform the system (1.1) to a coupled elliptic system given by

$$\begin{cases} \Delta u_1 - \lambda_1 u_1 + \mu_1 u_1^3 + \beta u_2^2 u_1 = 0 \quad \text{in } \mathbb{R}^n, \\ \Delta u_2 - \lambda_2 u_2 + \mu_2 u_2^3 + \beta u_1^2 u_2 = 0 \quad \text{in } \mathbb{R}^n. \end{cases} \tag{1.2}$$

The purpose of this paper is to study the existence of *bound states*, i.e., solutions (u_1, u_2) satisfying (1.2) and the following conditions:

$$u_1, u_2 > 0 \quad \text{in } \mathbb{R}^n, \qquad u_1(y), u_2(y) \to 0 \quad \text{as } |y| \to +\infty. \tag{1.3}$$

We consider $b = (\lambda_1, \lambda_2, \mu_1, \mu_2, \beta) \in \mathbb{R}^5_+$ as parameter in (1.2) and want to investigate the parameter range for which solutions of (1.2), (1.3) exist.

When the spatial dimension is one, i.e. $n = 1$, system (1.2) can become integrable for some special parameters, and there are many analytical and numerical results on solitary wave solutions of coupled nonlinear Schrödinger equations ([10], [18], [19], [20]).

Note that for $b = (\lambda_1, \lambda_2, \mu_1, \mu_2, \beta) \in \mathbb{R}^5_+$ equation (1.2) admits three trivial solutions

$$U_0 = (0,0), \qquad U_{b,1} = (w_{\lambda_1,\mu_1}, 0), \qquad U_{b,2} = (0, w_{\lambda_2,\mu_2}) \tag{1.4}$$

where w_{λ_j,μ_j} is the unique solution of the elliptic problem

$$\begin{cases} \Delta w - \lambda_j w + \mu_j w^3 = 0, \quad w > 0 \quad \text{in } \mathbb{R}^n, \\ w(0) = \max\limits_{y\in\mathbb{R}^n} w(y), \quad w \in \mathbb{H}^1(\mathbb{R}^n). \end{cases} \tag{1.5}$$

Setting $w = w_{1,1}$ we have $w_{\lambda_j,\mu_j}(y) = (\lambda_j/\mu_j)^{1/2} w(\lambda_j^{1/2} y)$.

A natural question is whether there are bound state solutions other than $U_{b,1}, U_{b,2}$. We shall investigate this question and give positive answers for certain parameter ranges.

A more general N coupled Schrödinger equation is studied in [21]. It was proved that, for $\beta < 0$, ground state solutions (i.e. bound states with minimal "energy") do not exist; while for $\beta > 0$, there exists a $\beta_0 \in (0, \sqrt{\mu_1\mu_2}]$ such that for $\beta \le \beta_0$ a ground state solution exists. We also remark that problem (1.2) in a bounded domain was studied in [21].

In this paper, we concentrate on the case of $\beta > 0$. First, since the problem has radial symmetry, by the symmetric criticality principle we can restrict the problem to the space of all $\vec{u} = (u_1, u_2) \in X := \mathbb{H}^1_r(\mathbb{R}^n) \times \mathbb{H}^1_r(\mathbb{R}^n)$. Here $\mathbb{H}^1_r(\mathbb{R}^n)$ consists of all radially symmetric functions in $\mathbb{H}^1(\mathbb{R}^n)$. In fact, due to the work of Busca and Sirakov [8], solutions of (1.2), (1.3) must be radially symmetric and strictly decreasing (this was shown by the moving plane method). Since there exist solutions having negative or sign-changing components we confine the problem to the nonnegative cone

$$\mathbb{P} := \{(u_1, u_2) \in X : u_1 \ge 0,\ u_2 \ge 0\}.$$

Using the fixed point index we prove a global bifurcation theorem for positive solutions. We also apply Morse theory to the associated energy functional constrained to $\mathbb{P}$ to obtain further information on the set of all positive bound states. In order to do this we compute the critical groups at the semipositive solutions as well as the critical groups at infinity. This idea was used initially in [31] for semilinear elliptic problems. Since there exist solutions having negative or sign-changing components we make use of invariant sets of the negative gradient flow and confine the problem to the positive cone. To carry out this idea, critical groups with respect to the cone have to be computed. We remark that the notion of invariant sets has been used recently to treat sign-changing solutions in [5, 12, 24].

In order to state our main result we denote the set of *trivial*, i.e. semipositive, solutions by

$$\mathcal{T}_j := \{(b, U_{b,j}) : b = (\lambda_1, \lambda_2, \mu_1, \mu_2, \beta) \in \mathbb{R}^5_+\}, \qquad j = 1, 2,$$

and the set of nontrivial positive solutions by

$$\mathcal{S} := \{(b, \vec{u}) \in \mathbb{R}^5_+ \times \mathbb{P} : (b, \vec{u}) \text{ solves (1.2)},\ u_1, u_2 > 0\}.$$

We also need the function $\beta : \mathbb{R}_+ \to \mathbb{R}_+$ defined by

$$\beta(s) := \inf_{\phi \in \mathbb{H}^1_r(\mathbb{R}^n)} \frac{\int (|\nabla \phi|^2 + s\phi^2)}{\int w^2 \phi^2} \tag{1.6}$$

where $w \in \mathbb{H}^1_r(\mathbb{R}^n)$ is the ground state of (1.5) for $\lambda_j = \mu_j = 1$. It follows that $\beta(1) = \int (|\nabla w|^2 + w^2) / \int w^4$.

Theorem 1.1. *There exist connected sets $\mathcal{S}_1, \mathcal{S}_2 \subset \mathcal{S}$ with*

$$\overline{\mathcal{S}}_1 \cap \mathcal{T}_1 = \{(b, U_{b,1}) : b = (\lambda_1, \lambda_2, \mu_1, \mu_2, \mu_1 \beta(\lambda_2/\lambda_1))\},$$
$$\overline{\mathcal{S}}_2 \cap \mathcal{T}_2 = \{(b, U_{b,2}) : b = (\lambda_1, \lambda_2, \mu_1, \mu_2, \mu_2 \beta(\lambda_1/\lambda_2))\}.$$

These sets have topological dimension at least 5 *at every point. Moreover, $\mathcal{S}_1 \cup \mathcal{S}_2$ covers the set of all $b = (\lambda_1, \lambda_2, \mu_1, \mu_2, \beta) \in \mathbb{R}^5_+$ satisfying*

$$\beta < \min\{\mu_1 \beta(\lambda_2/\lambda_1), \mu_2 \beta(\lambda_1/\lambda_2)\}$$

or

$$\beta > \max\{\mu_1\beta(\lambda_2/\lambda_1), \mu_2\beta(\lambda_1/\lambda_2)\}.$$

Remark 1.2. Similarly we can consider the bounded ball case:

$$\begin{cases} \Delta u_1 - \lambda_1 u_1 + \mu_1 u_1^3 + \beta u_2^2 u_1 = 0 \quad \text{in } B_R(0), \\ \Delta u_2 - \lambda_2 u_2 + \mu_2 u_2^3 + \beta u_1^2 u_2 = 0 \quad \text{in } B_R(0), \\ u_1, u_2 > 0 \quad \text{in } B_R(0), \quad u_1 = u_2 = 0 \quad \text{on } \partial B_R(0). \end{cases} \tag{1.7}$$

The proof of Theorem 1.1 can be extended without any difficulty to (1.7). We omit the details.

2. Proof of Theorem 1.1

Theorem 1.1 is proved through several lemmas. The energy functional associated with (1.2) is

$$\begin{aligned} E_b(u_1, u_2) = &\frac{1}{2}\int_{\mathbb{R}^n} (|\nabla u_1|^2 + |\nabla u_2|^2) + \frac{1}{2}\int_{\mathbb{R}^n} (\lambda_1 u_1^2 + \lambda_2 u_2^2) \\ &- \frac{1}{4}\int_{\mathbb{R}^n} \mu_1 u_1^4 - \frac{1}{4}\int_{\mathbb{R}^n} \mu_2 u_2^4 - \frac{\beta}{2}\int_{\mathbb{R}^n} u_1^2 u_2^2 \end{aligned} \tag{2.1}$$

for $b = (\lambda_1, \lambda_2, \mu_1, \mu_2, \beta) \in \mathbb{R}_+^5$ and $\vec{u} = (u_1, u_2) \in X = \mathbb{H}_r^1(\mathbb{R}^n) \times \mathbb{H}_r^1(\mathbb{R}^n)$. For each such b, X is a Hilbert space with inner product

$$\langle \vec{u}, \vec{v} \rangle_b = \langle \nabla u_1, \nabla v_1 \rangle_{L^2} + \lambda_1 \langle u_1, v_1 \rangle_{L^2} + \langle \nabla u_2, \nabla v_2 \rangle_{L^2} + \lambda_2 \langle u_2, v_2 \rangle_{L^2}.$$

Clearly, the associated norms

$$\|\vec{u}\|_b = (|\nabla u_1|_2^2 + \lambda_1 |u_1|_2^2)^{1/2} + (|\nabla u_2|_2^2 + \lambda_2 |u_2|_2^2)^{1/2}$$

are equivalent.

Assume that $n \le 3$. As a consequence of the Sobolev embedding theorem it follows easily that $E : X \to \mathbb{R}$ is well-defined and is a $\mathcal{C}^2$-functional. The gradient of E with respect to $\langle\,\cdot\,,\,\cdot\,\rangle_b$ can be computed as

$$\nabla_b E_b(\vec{u}) = \vec{u} - (-\Delta + \Lambda)^{-1} f_b(\vec{u}) =: \vec{u} - \mathbb{A}_b(\vec{u}) \tag{2.2}$$

where $\Lambda = \operatorname{diag}(\lambda_1, \lambda_2)$ and $f_b(\vec{u}) = (\mu_1 u_1^3 + \beta u_2^2 u_1, \mu_2 u_2^3 + \beta u_1^2 u_2)$. By the compact embedding from $\mathbb{H}_r^1(\mathbb{R}^n)$ into $L^q(\mathbb{R}^n)$ for $2 < q < 2^*$ the map

$$\mathbb{A} : \mathbb{R}_+^5 \times X \to X, \quad \mathbb{A}(b, \vec{u}) = \mathbb{A}_b(\vec{u}) = (-\Delta + \Lambda)^{-1} f_b(\vec{u}),$$

is completely continuous. It is now standard to show that E satisfies the Palais–Smale condition, so Morse type methods can be applied. In order to obtain connected sets of solutions we shall apply fixed point index theory to the equation

$$\mathbb{A}(b, \vec{u}) = \vec{u}, \qquad b \in \mathbb{R}_+^5,\ \vec{u} \in \mathbb{P}. \tag{2.3}$$

Clearly $\mathbb{A}_b(\mathbb{P}) \subset \mathbb{P}$. We begin with

Lemma 2.1. *If $B \subset \mathbb{R}_+^5$ is compact then there exists a uniform bound $R > 0$ such that $\mathcal{S} \cap (B \times \mathbb{P}) \subset B \times B_R(0)$.*

Proof. This can be proved following the nonexistence result for supersolutions in [17] (see also [14]). □

Lemma 2.2. $\overline{\mathcal{S}} \cap \mathcal{T}_1 \subset \{(b, U_{b,1}) : b = (\lambda_1, \lambda_2, \mu_1, \mu_2, \mu_1 \beta(\lambda_2/\lambda_1))\}$ *and* $\overline{\mathcal{S}} \cap \mathcal{T}_2 \subset \{(b, U_{b,2}) : b = (\lambda_1, \lambda_2, \mu_1, \mu_2, \mu_2 \beta(\lambda_1/\lambda_2))\}$.

Proof. Linearizing the equation (1.2) at $U_{b,1} = (w_{\lambda_1,\mu_1}, 0)$ gives two linear equations:

$$\Delta\phi_1 - \lambda_1\phi_1 + 3\mu_1 w^2_{\lambda_1,\mu_1}\phi_1 = 0, \qquad \phi_1 \in \mathbb{H}^1_r,$$
$$\Delta\phi_2 - \lambda_2\phi_2 + \beta w^2_{\lambda_1,\mu_1}\phi_2 = 0, \qquad \phi_2 \in \mathbb{H}^1_r.$$

The change of variables $z = \sqrt{\lambda_1}y$ yields

$$\Delta\phi_1 - \phi_1 + 3w^2\phi_1 = 0, \qquad \phi_1 \in \mathbb{H}^1_r, \tag{2.4}$$
$$\Delta\phi_2 - \frac{\lambda_2}{\lambda_1}\phi_2 + \frac{\beta}{\mu_1}w^2\phi_2 = 0, \qquad \phi_2 \in \mathbb{H}^1_r. \tag{2.5}$$

It is known that $\phi_1 = 0$ is the only solution of (2.4); see Appendix C of [27].

For fixed $(\lambda_1, \lambda_2, \mu_1)$, problem (2.5) can be considered as an eigenvalue problem in terms of β. The only eigenvalue having a positive eigenfunction is the first one given by

$$\beta_1 = \mu_1 \inf_{\phi\in\mathbb{H}^1_r(\mathbb{R}^n)} \frac{\int(|\nabla\phi|^2 + \frac{\lambda_2}{\lambda_1}\phi^2)}{\int w^2\phi^2} = \mu_1\beta(\lambda_2/\lambda_1). \tag{2.6}$$

The eigenfunctions associated to higher eigenvalues change sign. The implicit function theorem now implies that $(b, U_{b,1})$ with $b = (\lambda_1, \lambda_2, \mu_1, \mu_2, \beta) \in \mathbb{R}^5_+$ is a possible bifurcation point for solutions of (2.3) only if $\beta = \beta_1$.

The second inclusion is proved analogously. □

Remark 2.3. In the one-dimensional case, the eigenvalue problem (2.5) can be computed explicitly by using hypergeometric functions; see [32].

In order to prove equality in Lemma 2.2 we prove that the local fixed point index $\mathrm{ind}_{\mathbb{P}}(\mathbb{A}_b, U_{b,1})$, $b = (\lambda_1, \lambda_2, \mu_1, \mu_2, \beta) \in \mathbb{R}^5_+$, changes when β passes $\mu_1\beta(\lambda_2/\lambda_1)$. An analogous result holds with $U_{b,2}$ instead of $U_{b,1}$. Recall that this index is defined as follows. Since $\mathbb{P} \subset X$ is closed and convex there exists a retraction $r : X \to \mathbb{P}$. The fixed point problem (2.3) is equivalent to the equation

$$\vec{u} - \mathbb{A}_b(r(\vec{u})) = 0, \qquad b \in \mathbb{R}^5_+,\ u \in X, \tag{2.7}$$

with a completely continuous map $\mathbb{R}^5_+ \times X \to X$, $(b, \vec{u}) \mapsto \mathbb{A}_b(r(\vec{u}))$. With this notation, $\mathrm{ind}_{\mathbb{P}}(\mathbb{A}_b, U_{b,1}) = \deg(\mathrm{id} - \mathbb{A}_b \circ r, N_\varepsilon(U_{b,1}), 0)$, where $\varepsilon > 0$ is small, N_ε denotes the ε-neighborhood in X, and deg denotes the Leray–Schauder degree. This is defined when $U_{b,1}$ is an isolated fixed point of $\mathbb{A}_b$ in $\mathbb{P}$, hence when $b = (\lambda_1, \lambda_2, \mu_1, \mu_2, \beta) \in \mathbb{R}^5_+$ with $\beta \neq \mu_1\beta(\lambda_2/\lambda_1)$. Its value can only change when β passes the critical value $\mu_1\beta(\lambda_2/\lambda_1)$.

Lemma 2.4. *For $b = (\lambda_1, \lambda_2, \mu_1, \mu_2, \beta) \in \mathbb{R}^5_+$ we have*

$$\mathrm{ind}_{\mathbb{P}}(\mathbb{A}_b, U_{b,1}) = \begin{cases} -1 & \text{if } \beta < \mu_1\beta(\lambda_2/\lambda_1), \\ 0 & \text{if } \beta > \mu_1\beta(\lambda_2/\lambda_1) \end{cases}$$

and

$$\mathrm{ind}_{\mathbb{P}}(\mathbb{A}_b, U_{b,2}) = \begin{cases} -1 & \text{if } \beta < \mu_2\beta(\lambda_1/\lambda_2), \\ 0 & \text{if } \beta > \mu_2\beta(\lambda_1/\lambda_2). \end{cases}$$

Proof. It is sufficient to deal with the case where $U_{b,1}$ is a nondegenerate fixed point of $\mathbb{A}_b$. We set $\pi : X \to \mathbb{H}^1_r$, $\pi(u_1, u_2) = u_2$, and $L\phi := \pi \circ D\mathbb{A}_b(U_{b,1})[0, \phi]$ for $\phi \in \mathbb{H}^1_r$, and let $r(L)$ denote the spectral radius of L. Finally, $\overline{\mathrm{lin}}\,\mathbb{P} = X$, so [13, Theorem 1] applies and yields

$$\mathrm{ind}_{\mathbb{P}}(\mathbb{A}_b, U_{b,1}) = \begin{cases} \mathrm{ind}_X(D\mathbb{A}_b(U_{b,1}), 0) & \text{if } r(L) < 1, \\ 0 & \text{if } r(L) > 1. \end{cases}$$

Now $L\phi = (-\Delta + \lambda_2)^{-1}(\beta w^2_{\lambda_1,\mu_1}\phi_2)$, so a simple calculation shows that $r(L) < 1$ if and only if $\beta < \mu_1\beta(\lambda_2/\lambda_1)$. Moreover, in this case $D\mathbb{A}_b(U_{b,1})$ has precisely one eigenvalue larger than 1, hence $\mathrm{ind}_X(D\mathbb{A}_b(U_{b,1}), 0) = -1$. This follows from the fact that

$$D\mathbb{A}_b(U_{b,1})[\phi_1, \phi_2] = ((-\Delta + \lambda_1)^{-1}(\mu_1 w^2_{\lambda_1,\mu_1}\phi_1), (-\Delta + \lambda_2)^{-1}(\mu_2 w^2_{\lambda_1,\mu_1}\phi_2))$$

splits, the second component being $L\phi_2$ has no eigenvalue larger than 1, and the first component has precisely one eigenvalue larger than 1 because w_{λ_1,μ_1} is a mountain pass critical point. This proves the statement about $\mathrm{ind}_{\mathbb{P}}(\mathbb{A}_b, U_{b,1})$, and the other one follows analogously. □

Proof of Theorem 1.1. We first prove bifurcation from $\mathcal{T}_1$. We set

$$\mathcal{B}_1^- := \{b = (\lambda_1, \lambda_2, \mu_1, \mu_2, \beta) \in \mathbb{R}^5_+ : \beta < \mu_1\beta(\lambda_2/\lambda_1))\},$$
$$\mathcal{B}_1^+ := \{b = (\lambda_1, \lambda_2, \mu_1, \mu_2, \beta) \in \mathbb{R}^5_+ : \beta > \mu_1\beta(\lambda_2/\lambda_1)\},$$

and fix parameter values $b^\pm \in \mathcal{B}_1^\pm$. The essential ingredient in the proof is the global bifurcation theorem [2, Theorem 2.2]. This deals with bifurcation of fixed points of completely continuous maps $F : \mathcal{D} \subset \mathbb{R}^n \times X \to X$ satisfying $F(b, 0) = 0$ for every b. In our case, $F : \mathbb{R}^5_+ \times X \to X$ is given by

$$F(b, \vec{v}) = \mathbb{A}(b, r(U_{b,1} + \vec{v})) - U_{b,1}.$$

Then $F(b, \vec{v}) = \vec{v}$ is equivalent to $\vec{u} := U_{b,1} + \vec{v} \in \mathbb{P}$ solving $\mathbb{A}(b, \vec{u}) = \vec{u}$. Bifurcation of fixed points from the trivial fixed point set $\mathbb{R}^5_+ \times \{0\}$ of F corresponds to bifurcation of positive fixed points of $\mathbb{A}$ from $\mathcal{T}_1$.

By Lemma 2.4 we may apply [2, Theorem 2.2] and obtain a connected set $\mathcal{S}_1 \subset \mathrm{Fix}(F)$ bifurcating from $\mathbb{R}^5_+ \times \{0\}$ with topological dimension at least 5 at every point. In order to express the global structure of $\mathcal{S}_1$ we need to introduce some notation. First, let $\check{H}^*$ denote Čech cohomology (see [15] for an excellent presentation). Then we need to add a point ∞ to $\mathbb{R}^5_+ \times X$. A neighborhood basis of ∞ consists of complements of sets $C \times B$ where $C \subset \mathbb{R}^5_+$ is compact and $B \subset X$ is bounded. For $M \subset \mathbb{R}^5_+ \times X$ we set $M^+ := M \cup \{\infty\}$. Denote by

$$\mathcal{F}_1 := \overline{\mathcal{S}}_1 \cap \mathbb{R}^5_+ \subset \mathcal{B}_1 := \{b = (\lambda_1, \lambda_2, \mu_1, \mu_2, \beta) \in \mathbb{R}^5_+ : \beta = \mu_1 \beta(\lambda_2/\lambda_1)\}$$

the set of bifurcation points of F. Then there exists a continuous map

$$e : \overline{\mathcal{S}}_1^+ / \mathcal{F}_1^+ \to \mathbb{S}^5$$

into the five-dimensional unit sphere such that the induced map in Čech cohomology

$$e^* : \check{H}^5(\mathbb{S}^5) \to \check{H}^5(\overline{\mathcal{S}}_1^+ / \mathcal{F}_1^+)$$

is not zero. Moreover, given any point $b \in \mathcal{B}_1$ there exists a path in $\mathbb{R}^5_+$ from b^- to b^+ intersecting $\mathcal{B}_1$ precisely in b. Then e has the additional property that $e|[\overline{\mathcal{S}}_1^+ \setminus \{b\}]/[\mathcal{F}_1^+ \setminus \{b\}]$ is inessential. This implies immediately that $b \in \mathcal{F}_1$, hence

$$\mathcal{F}_1 = \overline{\mathcal{S}}_1 \cap \mathbb{R}^5_+ = \mathcal{B}_1.$$

We claim that $\mathcal{S}_1$ covers $\mathcal{B}_1^-$ or $\mathcal{B}_1^+$. In order to see this we recall the construction of e from [2]. Choose a regular and injective curve $\gamma : [0,1] \to \mathbb{R}^5_+$ from b^- to b^+ and choose $\varepsilon > 0$ small such that no fixed points of F lie in the sets

$$\{(b, \vec{v}) : |b - b^\pm| < 4\varepsilon,\ 0 < \|\vec{v}\| < 4\varepsilon\}.$$

Let Σ denote the sphere of radius ε in X centered at 0. For δ small enough the δ-neighborhood of $\mathrm{tr}(\gamma) \times \Sigma = \{(\gamma(t), \vec{v}) : t \in [0,1],\ \|\vec{v}\| = \varepsilon\}$ in $\mathrm{tr}(\gamma) \times X$ is a tubular neighborhood and can be written as

$$N_\delta(\mathrm{tr}(\gamma) \times \Sigma) = \bigcup_{(t,\vec{v}) \in \mathrm{tr}(\gamma) \times \Sigma} D_\delta(t, \vec{v}) \subset \mathrm{tr}(\gamma) \times X$$

where $D_\delta(t, \vec{v})$ denotes the δ-disc in the normal space of $\mathrm{tr}(\gamma) \times \Sigma$ around $(\gamma(t), \vec{v}) \in \mathrm{tr}(\gamma) \times \Sigma$. Observe that $D_\delta(t, \vec{v})$ is a disc of dimension 5. Let $\rho : [0,1] \to [0, \delta)$ be continuous with ρ vanishing only at 0 and 1. Finally, we set

$$\mathcal{A} := \bigcup_{(t,\vec{v}) \in \mathrm{tr}(\gamma) \times \Sigma} D_{\rho(t)}(t, \vec{v}) \subset N_\delta(\mathrm{tr}(\gamma) \times \Sigma).$$

Then the map e is constructed as follows. For each $t \in [0,1]$ and $\vec{v} \in \Sigma$ let

$$p_{t,\vec{v}} : D_{\rho(t)}(t, \vec{v}) / \partial D_{\rho(t)}(t, \vec{v}) \to D^5 / \partial D^5 = \mathbb{S}^5$$

be the natural radial diffeomorphism and define

$$e : ((\mathbb{R}^5 \times X)^+ / \mathcal{F}_1^+) \setminus (\{b^-, b^+\} \times \Sigma) \to D^5 / \partial D^5 = \mathbb{S}^5$$

by

$$e(\vec{u}) := \begin{cases} p_{t,\vec{v}}(\vec{u}) & \text{if } \vec{u} \in D_{\rho(t)}(t, \vec{v}), \\ * & \text{else;} \end{cases}$$

here $* \in \mathbb{S}^5$ corresponds to ∂D^5.

Suppose there are two points $b_1^\pm \in \mathcal{B}^\pm$ such that $\mathcal{S}_1 \cap (\{b_1^-, b_1^+\} \times X) = \emptyset$. Then we claim that one can isotope $\mathcal{A}$ to a subset $\widetilde{\mathcal{A}} \subset (\mathbb{R}^5 \times X) \setminus \overline{\mathcal{S}}_1$ in such a way that $\{b^-, b^+\} \times \Sigma$ never intersects $\mathrm{Fix}(F)$ during the isotopy. This induces a homotopy of e to a map

$$e_1 : ((\mathbb{R}^5 \times X)^+/\mathcal{F}_1^+) \setminus (\{b^-, b^+\} \times \Sigma) \to D^5/\partial D^5 = \mathbb{S}^5$$

with $e_1(\overline{\mathcal{S}}_1/\mathcal{F}_1^+) = *$, implying that the induced map

$$e^* = e_1^* : \check{H}^5(\mathbb{S}^5) \to \check{H}^5(\overline{\mathcal{S}}_1^+/\mathcal{F}_1^+)$$

is trivial, a contradiction.

The isotopy is constructed as follows. Choose regular curves $\gamma^\pm$ in $\mathcal{B}_1^\pm$ from $b^\pm$ to $b_1^\pm$ so that the composition $\gamma_1 : [0,1] \to \mathbb{R}_+^5$ of the three paths $\gamma^-, \gamma, \gamma^+$ is a regular and injective curve from b_1^- to b_1^+. If ε, δ are small then the δ-neighborhood of $\mathrm{tr}(\gamma_1) \times \Sigma$ is a tubular neighborhood and $(\mathrm{tr}(\gamma^-) \cup \mathrm{tr}(\gamma^+)) \times \Sigma$ does not intersect $\mathrm{Fix}(F)$. Then one can deform $\mathcal{A}$ to a set

$$\mathcal{A}_1 := \bigcup_{(t,\vec{v}) \in \mathrm{tr}(\gamma_1) \times \Sigma} D_{\rho_1(t)}(t, \vec{v}) \subset N_\delta(\mathrm{tr}(\gamma_1) \times \Sigma)$$

with a continuous function $\rho_1 : [0,1] \to [0, \delta)$ as above. By Lemma 2.1 there exist $\sigma > 0$ and $R_0 > 0$ such that

$$\mathrm{Fix}(F) \cap (N_\sigma(\mathrm{tr}(\gamma_1)) \times X) \subset N_\sigma(\mathrm{tr}(\gamma_1)) \times \{\vec{v} : \|\vec{v}\| < R_0\}.$$

Now it is easy to isotope $\mathcal{A}_1$ into the complement of $\overline{\mathcal{S}}_1$. Choose a continuous map $R : [0,1] \to [1, 2R_0/\varepsilon]$ with $R(0) = R(1) = 1$ and $R(t) = 2R_0/\varepsilon$ for $t \in [\tau, 1-\tau]$ such that

$$\begin{aligned}\widetilde{\mathcal{A}} &:= \bigcup_{(t,\vec{v}) \in \mathrm{tr}(\gamma_1) \times \Sigma} D_{\rho_1(t)}(t, R(t)\vec{v}) \\ &\subset (N_\sigma(\{b_1^-, b_1^+\}) \times X) \cup (N_\sigma(\mathrm{tr}(\gamma_1)) \times \{\vec{v} : \|\vec{v}\| \ge R_0\}).\end{aligned}$$

Then the isotopy is induced by the isotopy of $\mathrm{tr}(\gamma_1) \times \Sigma$ given by $H(\gamma_1(t), \vec{v}, s) := (\gamma_1(t), (1 - s + sR(t))\vec{v})$.

This shows that $\mathcal{S}_1$ covers $\mathcal{B}_1^-$ or $\mathcal{B}_1^+$ and the same holds for the corresponding fixed point set

$$\widetilde{\mathcal{S}}_1 := \{(b, U_{b,1} + \vec{v}) : (b, \vec{v}) \in \mathcal{S}_1\} \tag{2.8}$$

of the original map $\mathbb{A}$.

Now we get back to fixed points of $\mathbb{A}$ and change notation writing $\mathcal{S}_1$ instead of $\widetilde{\mathcal{S}}_1$. In an analogous way one obtains a set $\mathcal{S}_2 \subset \mathcal{S}$ of fixed points of $\mathbb{A}$ bifurcating from

$$\mathcal{B}_2 := \{b = (\lambda_1, \lambda_2, \mu_1, \mu_2, \beta) \in \mathbb{R}_+^5 : \beta = \mu_2 \beta(\lambda_1/\lambda_2)\},$$

that is,

$$\overline{\mathcal{S}}_2 \cap \mathcal{T}_2 = \{(b, U_{b,2}) : b \in \mathcal{B}_2\},$$

and covering

$$\mathcal{B}_2^- := \{b = (\lambda_1, \lambda_2, \mu_1, \mu_2, \beta) \in \mathbb{R}_+^5 : \beta < \mu_2 \beta(\lambda_1/\lambda_2))\}$$

or

$$\mathcal{B}_2^+ := \{b = (\lambda_1, \lambda_2, \mu_1, \mu_2, \beta) \in \mathbb{R}_+^5 : \beta > \mu_2 \beta(\lambda_1/\lambda_2)\}.$$

It remains to prove that $\mathcal{S}_1 \cup \mathcal{S}_2$ covers $(\mathcal{B}_1^- \cap \mathcal{B}_2^-) \cup (\mathcal{B}_1^+ \cap \mathcal{B}_2^+)$. This is a consequence of the nonexistence theorem [6, Theorem 1.5]. If $\lambda_1 = \lambda_2$ and $\mu_1 \neq \mu_2$ then (1.2), (1.3) does not have any positive solution if

$$\min\{\mu_1\beta(\lambda_2/\lambda_1), \mu_2\beta(\lambda_1/\lambda_2))\} \le \beta \le \max\{\mu_1\beta(\lambda_2/\lambda_1), \mu_2\beta(\lambda_1/\lambda_2))\}.$$

This means that there exist parameters lying between $\mathcal{B}_1$ and $\mathcal{B}_2$ that are not covered by $\mathcal{S}_1 \cup \mathcal{S}_2$ and it implies that $\mathcal{S}_1 \cup \mathcal{S}_2$ covers $(\mathcal{B}_1^- \cap \mathcal{B}_2^-) \cup (\mathcal{B}_1^+ \cap \mathcal{B}_2^+)$. □

We conclude this note by computing the critical groups in $\mathbb{P}$ of the semipositive solutions and the critical groups at infinity. As a consequence we obtain a second proof for the existence of positive solutions in the parameter range given in Theorem 1.1. Let $\eta_b(t, \vec{u})$ be the flow on X defined by the vector field $\nabla_b E_b$, i.e.

$$\frac{d}{dt}\eta_b(t, \vec{u}) = -\nabla_b E_b(\eta_b(t, \vec{u})), \quad \eta_b(0, \vec{u}) = \vec{u} \in X = \mathbb{H}_r^1 \times \mathbb{H}_r^1.$$

Proposition 2.5. *The cone $\mathbb{P}$ is invariant under η_b in the sense that $\eta(t, \vec{u}) \in \mathbb{P}$ for all $\vec{u} \in \mathbb{P}$ and $t > 0$.*

Proof. By the form (2.2) of $\nabla_b E_b$ the claim follows from the fact that $\mathbb{A}_b(\vec{u}) \in \mathbb{P}$ for all $\vec{u} \in \mathbb{P}$; see [9, I.6.2]. □

As before we shall work on $\mathbb{P}$. For $a \in \mathbb{R}$ define

$$E_b^a = \{(u_1, u_2) \in X : E_b(u_1, u_2) \le a\}.$$

We shall use $\widetilde{E}_b$ to denote the restriction of E_b to $\mathbb{P}$, i.e.,

$$\widetilde{E}_b^a = \{(u_1, u_2) \in \mathbb{P} : E_b(u_1, u_2) \le a\}.$$

We also set $\mathbb{S}^\infty := \{\vec{u} \in X : \|\vec{u}\| = 1\}$.

Proposition 2.6. *For any $M > 0$, we have $\widetilde{E}_b^{-M} \simeq \mathbb{S}^\infty \cap \mathbb{P}$, i.e., $\mathbb{S}^\infty \cap \mathbb{P}$ is homotopy equivalent to $\widetilde{E}_b^{-M}$.*

Proof. For $\vec{u} \in \mathbb{P}$ and $t > 0$ with $E_b(t\vec{u}) < 0$ we have

$$\frac{d}{dt}E_b(t\vec{u}) = t\|\vec{u}\|_b^2 - \int_{\mathbb{R}^n} f_b(t\vec{u})\vec{u} = \frac{4}{t}E_b(t\vec{u}) - t\|\vec{u}\|_b^2 < 0.$$

Now fixing $M > 0$, for every $\vec{u} \in \mathbb{P} \cap \mathbb{S}^\infty$ there exists a unique $T(\vec{u}) > 0$ such that

$$E_b(T(\vec{u})\vec{u}) = -M.$$

From this and the implicit function theorem, $T(\vec{u}) \in C(\mathbb{P} \cap \mathbb{S}^\infty, \mathbb{R})$. It follows that the map

$$\mathbb{P} \cap \mathbb{S}^\infty \to \{\vec{u} \in \mathbb{P} : E_b(\vec{u}) = -M\}, \quad \vec{u} \mapsto T(\vec{u})\vec{u},$$

is a homeomorphism with inverse $\vec{v} \mapsto \vec{v}/\|\vec{v}\|_b$, and that the map

$$\widetilde{E}_b^{-M} \to \mathbb{P} \cap \mathbb{S}^\infty, \quad \vec{v} \mapsto \vec{v}/\|\vec{v}\|_b,$$

is a homotopy equivalence. □

We shall make use of the concept of critical groups for isolated critical points, as defined in [9]. Given an isolated critical point $\vec{u}_0$ with $E_b(\vec{u}_0) = c$, define for $k \in \mathbb{N}_0$,

$$C_k(E_b, \vec{u}_0) = H_k(E_b^c, E_b^c \setminus \{\vec{u}_0\})$$

where H_* denotes singular homology with coefficients in a field $\mathbb{F}$. Since $\mathbb{P}$ is invariant under the negative gradient flow of E_b we may also consider the critical groups of isolated critical points in $\mathbb{P}$ relative to the cone $\mathbb{P}$ (see [9]),

$$C_k(\widetilde{E}_b, \vec{u}_0) = H_k(E_b^c \cap \mathbb{P}, (E_b^c \cap \mathbb{P}) \setminus \{\vec{u}_0\}).$$

Proposition 2.7. *$H_k(\mathbb{P} \cap \mathbb{S}^\infty) = \delta_{k0}\mathbb{F}$ for $k \in \mathbb{Z}$.*

Proof. Fix a $\vec{\varphi} \in \mathbb{P} \cap \mathbb{S}^\infty$ with $\varphi_1, \varphi_2 > 0$ in $\mathbb{R}^n$. Then for any $\vec{u} \in \mathbb{P} \cap \mathbb{S}^\infty$, $\|s\vec{\varphi} + (1-s)\vec{u}\| \neq 0$. Define $\tau : [0,1] \times \mathbb{P} \cap \mathbb{S}^\infty \to \mathbb{P} \cap \mathbb{S}^\infty$ by

$$\tau(s, \vec{u}) = \frac{s\vec{\varphi} + (1-s)\vec{u}}{\|s\vec{\varphi} + (1-s)\vec{u}\|},$$

which shows that $\mathbb{P}\cap\mathbb{S}^\infty$ is contractible to a point. Then the conclusion follows. □

Proposition 2.8. (i) *$H_k(\mathbb{P}, \widetilde{E}_b^{-1}) = 0$, $k \in \mathbb{Z}$.*
(ii) *For $\delta > 0$ small enough, $H_k(\widetilde{E}_b^\delta, \widetilde{E}_b^{-1}) = \delta_{k0}\mathbb{F}$, $k \in \mathbb{Z}$.*

Proof. (i) follows from Propositions 2.7 and 2.6.

(ii) follows from the fact that 0 is a strict local minimizer, that 0 is the only critical point of E in $\widetilde{E}^\delta$ for $\delta > 0$ small, and that E satisfies the Palais–Smale condition. □

Now we compute the critical groups of E_b at $U_{b,j}$, $j = 1, 2$, relative to the cone $\mathbb{P}$. Since $U_{b,1}$ is an isolated critical point of E_b in $\mathbb{P}$ if $\beta \neq \mu_1\beta(\lambda_2/\lambda_1)$, the critical groups $C_k(\widetilde{E}_b, U_{b,1})$ are defined for all $b = (\lambda_1, \lambda_2, \mu_1, \mu_2, \beta) \in \mathbb{R}_+^5$ with $\beta \neq \mu_1\beta(\lambda_2/\lambda_1)$. By [9, Theorem I.5.6] the critical groups $C_k(E_b, U_{b,1})$ can only change when β passes the critical value $\mu_1\beta(\lambda_2/\lambda_1)$. Since the cone $\mathbb{P}$ is positive invariant under the negative gradient flow of E_b for all b, the proof of [9, Theorem 5.6] yields the same result for $C_k(\widetilde{E}_b, U_{b,1})$. An analogous statement holds for $C_k(\widetilde{E}_b, U_{b,2})$, of course.

Proposition 2.9. *For $b = (\lambda_1, \lambda_2, \mu_1, \mu_2, \beta) \in \mathbb{R}_+^5$ we have*

$$C_k(\widetilde{E}_b, U_{b,1}) = \begin{cases} \delta_{k1}\mathbb{F} & \text{if } \beta < \mu_1\beta(\lambda_2/\lambda_1), \\ 0 & \text{if } \beta > \mu_1\beta(\lambda_2/\lambda_1). \end{cases}$$

An analogous result holds for $U_{b,2}$ with $\mu_1\beta(\lambda_2/\lambda_1)$ replaced by $\mu_2\beta(\lambda_1/\lambda_2)$.

Postponing the proof of Proposition 2.9 we give a second proof for the existence of positive solutions for the parameter values b from Theorem 1.1.

Corollary 2.10. *Given* $b = (\lambda_1, \lambda_2, \mu_1, \mu_2, \beta) \in \mathbb{R}^5_+$, *problem* (1.2), (1.3) *has a positive solution if*

$$\beta < \min\{\mu_1\beta(\lambda_2/\lambda_1), \mu_2\beta(\lambda_1/\lambda_2)\}$$

or

$$\beta > \max\{\mu_1\beta(\lambda_2/\lambda_1), \mu_2\beta(\lambda_1/\lambda_2)\}.$$

Proof. Assume that $\widetilde{E}_b$ has only three critical points in $\mathbb{P}$: $U_0 = 0$, $U_1 = U_{b,1}$, $U_2 = U_{b,2}$. The Morse identity implies

$$\sum_{k=0}^{\infty}(-1)^k \dim H_k(\mathbb{P}, \widetilde{E}_b^{-M}) = \sum_{j=0}^{2}\sum_{k=0}^{\infty}(-1)^k \dim C_k(\widetilde{E}_b, U_j) \tag{2.9}$$

By Proposition 2.8, the left hand side of (2.9) is 0. The right hand side is -1 when $\beta<\min\{\mu_1\beta(\lambda_2/\lambda_1), \mu_2\beta(\lambda_1/\lambda_2)$, and 1 when $\beta>\max\{\mu_1\beta(\lambda_2/\lambda_1), \mu_2\beta(\lambda_1/\lambda_2)\}$. This is a contradiction. □

The rest of the paper is devoted to

Proof of Proposition 2.9. Since the critical groups $C_q(\widetilde{E}_b, U_{b,1})$ are constant for $b \in B_1^-$ and for $b \in B_1^+$, it is sufficient to compute them for some $\beta < \mu_1\beta(\lambda_2/\lambda_1)$ and some $\beta > \mu_1\beta(\lambda_2/\lambda_1)$. In particular, we may assume that $U_{b,1}$ is a nondegenerate critical point of E_b, that is, β is not an eigenvalue of (2.5). Fixing b, for simplicity of notation we set $w_1 := w_{\lambda_1,\mu_1}$, so $U_{b,1} = (w_1, 0)$. For $(u_1, u_2) \in X$, define

$$K(u_1, u_2) = K_1(u_1) + K_2(u_2)$$

with

$$K_1(u_1) = E(U_{b,1}) + \frac{1}{2}\int(|\nabla(u_1 - w_1)|^2 + \lambda_1(u_1 - w_1)^2 - 3\mu_1 w_1^2(u_1 - w_1)^2) - \mu_1\int w_1(u_1 - w_1)^3 - \frac{\mu_1}{4}\int(u_1 - w_1)^4$$

and

$$K_2(u_2) = \frac{1}{2}\int(|\nabla u_2|^2 + \lambda_2 u_2^2 - \beta w_1^2 u_2^2).$$

Choosing β appropriately, we may also assume that K_2 is a nondegenerate quadratic form. It is readily checked that

$$\nabla K_1(u_1) = u_1 - w_1 - (-\Delta + \lambda_1)^{-1}(\mu_1 u_1^3 - \mu_1 w_1^3),$$
$$\nabla K_2(u_2) = u_2 - (-\Delta + \lambda_2)^{-1} g_2(\beta w_1^2 u_2).$$

They satisfy $u_1 - \nabla K_1(u_1) \geq 0$ and $u_2 - \nabla K_2(u_2) \geq 0$ for $u_1, u_2 \geq 0$. Similar to Proposition 2.5 the negative gradient flow of K_i on $\mathbb{H}_1^r$ preserves the positive cone $\mathbb{P}_1 = \{v \in \mathbb{H}_1^r : v \geq 0\}$, therefore the negative gradient flow for K preserves the cone $\mathbb{P}$ in X.

Consider $J_t(u_1, u_2) = tK(u_1, u_2) + (1-t)E_b(u_1, u_2)$ for $t \in [0, 1]$. Then using the fact that $(w_1, 0)$ is a nondegenerate critical point of E_b it is easy to check that $(w_1, 0)$ is a nondegenerate critical point of J_t for all $t \in [0, 1]$. Again by [9, Theorem I.5.6] we have $C_k(E_b, U_{b,1}) = C_k(K, U_{b,1})$ for all k. Since the positive cone $\mathbb{P}$ is invariant under the gradient flow of J_t for all $t \in [0,1]$, the proof in [9] yields $C_k(\widetilde{E}_b, U_{b,1}) = C_k(\widetilde{K}, U_{b,1})$ for all k, where similar to $\widetilde{E}_b$, $\widetilde{K}$ is the restriction of K to $\mathbb{P}$. Since K has a direct sum decomposition with respect to u_1 and u_2, by [9, Theorem I.5.5] we have $C_*(K, U_{b,1}) = C_*(K_1, w_1) \otimes C_*(K_2, 0)$. Again since the negative gradient flows of K_1 and K_2 preserve $\mathbb{P}_1$, we also have $C_*(\widetilde{K}, U_{b,1}) = C_*(\widetilde{K}_1, w_1) \otimes C_*(\widetilde{K}_2, 0)$.

In order to compute $C_k(\widetilde{K}_1, w_1)$ one shows as in Proposition 2.6 that $\widetilde{K}_1^{-M}$ is radially homotopy equivalent to $\{v \in \mathbb{P}_1 : \|v\| = 1\}$ for $M > 0$. It follows that $H_k(\mathbb{P}_1, \widetilde{K}_1^{-M}) = 0$ for all $k \in \mathbb{Z}$. The only non-negative critical points of K_1 are 0 and w_1. Since 0 is a local minimum we have $C_k(\widetilde{K}_1, 0) = \delta_{k0}\mathbb{F}$. The Morse inequality now implies

$$\begin{aligned} 0 &= \sum_{k=0}^{q} (-1)^{q-k} \dim H_k(\mathbb{P}_1, \widetilde{K}_1^{-M}) \\ &\le \sum_{k=0}^{q} (-1)^{q-k} (\dim C_k(\widetilde{K}_1, 0) + \dim C_k(\widetilde{K}_1, w_1)) \\ &= (-1)^q + \sum_{k=0}^{q} (-1)^{q-k} \dim C_k(\widetilde{K}_1, w_1) \end{aligned}$$

and therefore $C_k(\widetilde{K}_1, w_1) = \delta_{k1}\mathbb{F}$ for all k, independently of β.

In case $\beta < \mu_1\beta(\lambda_2/\lambda_1)$, 0 is a local minimum of K_2. Thus we get $C_k(\widetilde{K}_2, 0) = \delta_{k0}\mathbb{F}$, so the proposition follows in this case.

It remains to consider the case $\beta > \mu_1\beta(\lambda_2/\lambda_1)$. We claim that in this case $C_k(\widetilde{K}_2, 0) = 0$ for all k, which implies the proposition. To that end, we first notice that the first eigenvalue of $K_2''(0)$ is negative corresponding to a positive eigenfunction $\psi \in \mathbb{P}_1$. Given u_2 we write a direct sum decomposition $u_2 = \alpha\psi + v + w$ where v lies in the subspace V spanned by the eigenfunctions corresponding to negative eigenvalues different from the first one, and w lies in the subspace W spanned by the eigenfunctions corresponding to positive eigenvalues. Then we choose a neighborhood O of 0 of the form $O = [-\delta_0, \delta_0]\psi \times B_{\delta_0}(V \oplus W)$. We need to compute $H_*(K_2^0 \cap O \cap \mathbb{P}, (K_2^0 \cap O \cap \mathbb{P}) \setminus \{0\})$. We observe that if $u_2 \in \mathbb{P}$, then $\alpha \ge 0$, and $\alpha = 0$ if and only if $v = w = 0$. Note that if $u_2 = \alpha\psi + v + w$ is in $K_2^0 \cap O \cap \mathbb{P}$, so is $\alpha\psi + v + sw$ for $s \in [0,1]$, because $K_2(u_2) = \frac{1}{2}\langle K_2''(0)u_2, u_2\rangle$. Now we use s as a deformation parameter to deform $K_2^0 \cap O \cap \mathbb{P}$ to $K_2^0 \cap ([-\delta_0, \delta_0]\psi \times B_{\delta_0}(V)) \cap \mathbb{P}$. Similarly, if $u_2 = \alpha\psi + v$ is in $K_2^0 \cap ([-\delta_0, \delta_0]\psi \times B_{\delta_0}(V)) \cap \mathbb{P}$, so is $\alpha\psi + sv$ for $s \in [0,1]$. Thus we may deform $K_2^0 \cap ([-\delta_0, \delta_0]\psi \times B_{\delta_0}(V)) \cap \mathbb{P}$ to $K_2^0 \cap ([-\delta_0, \delta_0]\psi \cap \mathbb{P} = [0, \delta_0]\psi$. This homotopy deforms $(K_2^0 \cap O \cap \mathbb{P}) \setminus \{0\}$ to $(0, \delta_0]\psi$, which proves the claim.

The computation for $U_{b,2}$ is the same. □

This is a revised version of our preprint [7]. Since then there have been several interesting papers containing results related to ours: see [3, 4, 6, 14, 25, 29] for existence of ground and bound state solutions, in particular [3, 6, 25, 29] giving an existence result for β in the large, a result in similar spirit to ours; and [23, 28, 33, 34] for multiplicity results and semiclassical states. The results and the two approaches we used here are different from the existing techniques in the above papers, and they give a different perspective into the problems which should be useful in further investigations.

Acknowledgments

The second author thanks the Institute of Mathematical Sciences at the Chinese University of Hong Kong for their hospitality during his stay. The research of the third author is partially supported by an Earmarked Grant from RGC of Hong Kong.

References

[1] N. Akhmediev and A. Ankiewicz, *Partially coherent solitons on a finite background.* Phys. Rev. Lett. **82** (1999) 2661, 1–4.

[2] J. C. Alexander and S. S. Antman, *Global and local behavior of bifurcating multidimensional continua of solutions for multiparameter nonlinear eigenvalue problems.* Arch. Ration. Mech. Anal. **76** (1981), 339–354.

[3] A. Ambrosetti and E. Colorado, *Bound and ground states of coupled nonlinear Schrödinger equations.* C. R. Math. Acad. Sci. Paris **342** (2006), 453–458.

[4] A. Ambrosetti and E. Colorado, *Standing waves of some coupled nonlinear Schrödinger equations.* J. London Math. Soc. **75** (2007), 67–82.

[5] T. Bartsch, Z. Liu and T. Weth, Sign *changing solutions to superlinear Schrödinger equations.* Comm. Partial Differential Equations **29** (2004), 25–42.

[6] T. Bartsch and Z.-Q. Wang, *Note on ground states of nonlinear Schrödinger systems.* J. Partial Differential Equations **19** (2006), 200–207.

[7] T. Bartsch, Z.-Q. Wang and J.-C. Wei, *Bound states for a coupled Schrödinger system.* Preprint, 2005.

[8] J. Busca and B. Sirakov, *Symmetry results for semilinear elliptic systems in the whole space.* J. Differential Equations **163** (2000), 41–56.

[9] K. C. Chang, *Infinite Dimensional Morse Theory and Multiple Solution Problems.* Progr. Nonlinear Differential Equations Appl. 6, Birkhäuser, Boston, 1993.

[10] K. W. Chow, *Periodic solutions for a system of four coupled nonlinear Schrödinger equations.* Phys. Lett. A **285** (2001), 319–326.

[11] D. N. Christodoulides, T. H. Coskun, M. Mitchell and M. Segev, *Theory of incoherent self-focusing in biased photorefractive media.* Phys. Rev. Lett. **78** (1997), 646 (1–4).

[12] M. Conti, L. Merizzi and S. Terracini, *On the existence of many solutions for a class of superlinear elliptic systems.* J. Differential Equations **167** (2000), 357–387.

[13] E. N. Dancer, *Positivity of maps and applications.* In: Topological Nonlinear Analysis, M. Matzeu and A. Vignoli (eds.), Birkhäuser, Boston, 1995, 303–340.

[14] E. N. Dancer, J.-C. Wei and T. Weth, *A priori bounds versus multiple existence of positive solutions for a nonlinear Schrödinger system.* Preprint, 2007.

[15] A. Dold, *Lectures on Algebraic Topology.* Grundlehren Math. Wiss. 200, Springer, Berlin, 1980.

[16] B. D. Esry, C. H. Greene, J. P. Burke Jr. and J. L. Bohn, *Hartree–Fock theory for double condensates.* Phys. Rev. Lett. **78** (1997), 3594–3597.

[17] B. Gidas, *Symmetry property and isolated singularities of positive solutions of nonlinear elliptic equations.* In: Nonlinear Partial Differential Equations in Engineering and Applied Sciences, Lecture Notes in Pure Appl. Math. 54, Dekker, New York, 1980, 255–273.

[18] F. T. Hioe, *Solitary waves for N coupled nonlinear Schrödinger equations.* Phys. Rev. Lett. **82** (1999), 1152–1155.

[19] F. T. Hioe and T. S. Salter, *Special set and solutions of coupled nonlinear Schrödinger equations.* J. Phys. A Math. Gen. **35** (2002), 8913–8928.

[20] T. Kanna and M. Lakshmanan, *Exact soliton solutions, shape changing collisions, and partially coherent solitons in coupled nonlinear Schrödinger equations.* Phys. Rev. Lett. **86** (2001), 5043 (1–4).

[21] T.-C. Lin and J.-C. Wei, *Ground state of N coupled nonlinear Schrödinger equations in $\mathbb{R}^n$, $n \leq 3$.* Comm. Math. Phys. **255** (2005), 629–653.

[22] T.-C. Lin and J.-C. Wei, *Spikes in two coupled nonlinear Schrödinger equations.* Ann. Inst. H. Poincaré Anal. Non Linéaire **22** (2005), 403–439.

[23] T.-C. Lin and J.-C. Wei, *Spikes in two-component systems of nonlinear Schrödinger equations with trapping potentials.* J. Differential Equations **229** (2006), 538–569.

[24] Z. Liu, F. A. van Heerden and Z.-Q. Wang, *Nodal type bound states of Schrödinger equations via invariant set and minimax methods.* J. Differential Equations **214** (2005), 358–390.

[25] L. A. Maia, E. Montefusco and B. Pellacci, *Positive solutions for a weakly coupled nonlinear Schrödinger system.* J. Differential Equations **229** (2006), 743–767.

[26] M. Mitchell, Z. Chen, M. Shih, and M. Segev, *Self-trapping of partially spatially incoherent light.* Phys. Rev. Lett. **77** (1996), 490–493.

[27] W.-M. Ni and I. Takagi, *Locating the peaks of least energy solutions to a semilinear Neumann problem.* Duke Math. J. **70** (1993), 247–281.

[28] A. Pomponio, *Coupled nonlinear Schrödinger systems with potentials.* J. Differential Equations **227** (2006), 258–281.

[29] B. Sirakov, *Least energy solitary waves for a system of nonlinear Schrödinger equations in $\mathbb{R}^n$.* Comm. Math. Phys. **271** (2007), 199–221.

[30] E. Timmermans, *Phase separation of Bose–Einstein condensates.* Phys. Rev. Lett. **81** (1998), 5718–721.

[31] Z.-Q. Wang, *On a superlinear elliptic equation.* Ann. Inst. H. Poincaré Anal. Non Linéaire **8** (1991), 43–57.

[32] M. Winter and J.-C. Wei, *Critical threshold and stability of cluster solutions for large reaction-diffusion systems in R^1.* SIAM J. Math. Anal. **33** (2002), 1058–1089.

[33] J.-C. Wei and T. Weth, *Nonradial symmetric bound states for a system of two coupled Schrödinger equations.* Rend. Lincei Mat. Appl. **18** (2007), 279–293.

[34] J.-C. Wei and T. Weth, *Radial solutions and phase separation in a system of two coupled Schrödinger equations.* Arch. Ration. Mech. Anal., to appear.

Thomas Bartsch
Mathematisches Institut
Universität Giessen
35392 Giessen, Germany
e-mail: Thomas.Bartsch@math.uni-giessen.de

Zhi-Qiang Wang
Department of Mathematics and Statistics
Utah State University
Logan, UT 84322, USA
e-mail: wang@math.usu.edu

Juncheng Wei
Department of Mathematics
The Chinese University of Hong Kong
Shatin, Hong Kong
e-mail: wei@math.cuhk.edu.hk

To access this journal online:
www.birkhauser.ch/jfpta