Multi-spots Steady States in Two-species Keller-Segel Models with Logistic Growth: Large Chemotactic Attraction Regime

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Abstract

One of the most important findings in the study of chemotactic process is self-organized cellular aggregation, and a high volume of results are devoted to the analysis of a concentration of single species. Whereas, the multi-species case is not understood as well as the single species one. In this paper, we consider two-species chemotaxis systems with logistic source in a bounded domain $\Omega \subset \mathbb{R}^2$. Under the large chemo-attractive coefficients and one certain type of chemical production coefficient matrices, we employ the inner-outer gluing approach to construct multi-spots steady states, in which the profiles of cellular densities have strong connections with the entire solutions to Liouville systems and their locations are determined in terms of reduced-wave Green's functions. In particular, some numerical simulations and formal analysis are performed to support our rigorous studies.

1 Introduction and Main Results

In this paper, we consider the following two-species chemotaxis system with logistic growth in 2D:

$$\begin{cases} u_{1t} = \Delta u_1 - \chi_1 \nabla \cdot (u_1 \nabla v_1) + \lambda_1 u_1 (\bar{u}_1 - u_1), & x \in \Omega, t > 0, \\ u_{2t} = \Delta u_2 - \chi_2 \nabla \cdot (u_2 \nabla v_2) + \lambda_2 u_2 (\bar{u}_2 - u_2), & x \in \Omega, t > 0, \\ v_{1t} = \Delta v_1 - v_1 + a_{11} u_1 + a_{12} u_2, & x \in \Omega, t > 0, \\ v_{2t} = \Delta v_2 - v_2 + a_{21} u_1 + a_{22} u_2, & x \in \Omega, t > 0, \\ \partial_{\mathbf{n}} u_1 = \partial_{\mathbf{n}} u_2 = \partial_{\mathbf{n}} v_1 = \partial_{\mathbf{n}} v_2 = 0, & x \in \partial\Omega, t > 0, \\ u_1(x,0) = u_{10}(x),\ u_2(x,0) = u_{20}(x),\ v_1(x,0) = v_{10}(x),\ v_2(x,0) = v_{20}(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain with the smooth boundary $\partial\Omega$ in $\mathbb{R}^2$, $\mathbf{n}$ denotes the unit outer normal vector, $\chi_1 > 0$ and $\chi_2 > 0$ are chemo-attractive coefficients, $a_{ij} > 0$ with $i, j = 1, 2$ are chemical production coefficients, and initial data $(u_{10}, u_{20}, v_{10}, v_{20})$ is assumed to be smooth enough, non-negative and not identically equal to zero. Here u_1 and u_2 are cellular densities of two species; v_1 and v_2 are chemical concentrations; λ_1 and λ_2 represent intrinsic cellular growth and $\bar{u}_1$ and $\bar{u}_2$ interpret the levels of carrying capacities. Our goal in this paper is to construct multi-spots stationary solutions in (1.1) rigorously under an asymptotical limit $\chi_1, \chi_2 \to +\infty$ with $\frac{\chi_1}{\chi_2} = O(1)$.

1.1 Chemotaxis and Keller-Segel Models

Chemotaxis is a process in which uni-cellular or multi-cellular organisms direct their movements in response to chemical stimulus gradients. This mechanism is ubiquitous in pathological and physiological

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processes such as morphogenesis, wound healing, tumor growth, cell differentiation, etc. To describe the chemotactic phenomenon, E. Keller and L. Segel proposed a class of strongly coupled parabolic PDEs and some typical form reads as

$$\begin{cases} u_t = \overbrace{\Delta u}^{\text{cellular diffusivity}} - \overbrace{\chi\nabla\cdot(u\nabla v)}^{\text{chemotactic movement}} + \overbrace{f(u)}^{\text{source}}, & x\in\Omega, t>0,\\ v_t = \underbrace{\Delta v}_{\text{chemical diffusivity}} + \underbrace{g(u,v)}_{\text{chemical production/consumption}}, & x\in\Omega, t>0, \end{cases} \tag{1.2}$$

where Ω is assumed to be a bounded domain in $\mathbb{R}^N$, $N\geq 1$ or the whole space. Numerous studies are devoted to the analysis of spatial-temporal dynamics in one-population model (1.2) and we refer the readers to the survey papers [9, 10, 19, 24]. One of the most famous research results in the study of classical chemotaxis models is the so-called "chemotactic collapse" [2,6,8,11,18,20,21]. In particular, with regard to concentrated stationary solutions, it has been shown that cellular density u asymptotically converges to the linear combination of several δ-functions with some regular parts in 2D; meanwhile, the chemical concentration v converges to the finite sum of Neumann Green's functions, where the coefficients are 8π or 4π depending on the locations of δ-functions. We mention that Del pino and Wei [6] utilized the entire solution of the Liouville equation to approximate the cellular density u asymptotically.

Compared to the comprehensive study of spontaneous concentration in the single-species Keller-Segel system, the localized patterns in the multi-species system are not well-understood and a few results devoted to the pattern formation in the multi-species counterpart, see [23, 25, 26]. We point out that Wang et al. in [23] performed the bifurcation analysis around the constant steady state in the two-species competition Keller-Segel model and obtained the large chemotactic coefficient χ triggers *Turing's instability* [22]. Motivated by their results, we shall consider the "far from" Turing's regime and assume chemotactic coefficients in (1.1) are large enough, then construct the multi-spots steady states asymptotically under the singular limits of $\chi_1\to+\infty$ and $\chi_2\to+\infty$ via the gluing method.

To study the existence of non-constant steady states, we are concerned with the stationary problem of (1.1), which is

$$\begin{cases} 0=\Delta u_1-\chi_1\nabla\cdot(u_1\nabla v_1)+\lambda_1 u_1(\bar{u}_1-u_1), & x\in\Omega,\\ 0=\Delta u_2-\chi_2\nabla\cdot(u_2\nabla v_2)+\lambda_2 u_2(\bar{u}_2-u_2), & x\in\Omega,\\ 0=\Delta v_1-v_1+a_{11}u_1+a_{12}u_2, & x\in\Omega,\\ 0=\Delta v_2-v_2+a_{21}u_1+a_{22}u_2, & x\in\Omega,\\ \partial_{\mathbf{n}}u_1=\partial_{\mathbf{n}}u_2=\partial_{\mathbf{n}}v_1=\partial_{\mathbf{n}}v_2=0, & x\in\partial\Omega, \end{cases} \tag{1.3}$$

Before stating our main result, we introduce some notations and assumptions. For convenience, we set $\chi_1=\chi, \chi_2=\gamma\chi$ and $d=\frac{a_{21}}{a_{12}}\gamma^2$. Here and in the sequel, we impose the following assumptions on matrix $A=(a_{ij})_{2\times 2}$:

(H1). $(a_{ij})_{2\times 2}$ is a real, irreducible and positive matrix;

(H2). $a_{11}a_{22}-a_{12}a_{21}\gamma^2\neq 0$;

(H3). $(a_{ij})_{2\times 2}$ is a positive definite matrix.

Under the assumption (H1) and (H2), as shown in [15], we have the existence of the entire solution denote $(\Gamma_{1,\mu_1},\Gamma_{2,\mu_2})(y-\xi)$ with any $\xi\in\mathbb{R}^2$ to the following Liouville system

$$\begin{cases} \Delta\Gamma_1+b_{11}e^{\Gamma_1}+b_{12}e^{\Gamma_2}=0, & y\in\mathbb{R}^2,\\ \Delta\Gamma_2+b_{21}e^{\Gamma_1}+b_{22}e^{\Gamma_2}=0, & y\in\mathbb{R}^2, \end{cases} \tag{1.4}$$

where

$$(b_{ij})_{2\times 2} = \begin{pmatrix} a_{11} & a_{21}\gamma \\ a_{21}\gamma & a_{22}d \end{pmatrix} \tag{1.5}$$

and one further defines

$$(U_1, U_2) = (e^{\Gamma_{1,\mu_1}}, e^{\Gamma_{2,\mu_2}}). \tag{1.6}$$

To capture the global behavior of (v_1, v_2). we introduce the Neumann Green's function $G(x;\xi)$, which satisfies

$$\begin{cases} \Delta_x G - G = -\delta_\xi(x), & x \in \Omega, \\ \partial_{\mathbf{n}} G = 0, & x \in \partial\Omega. \end{cases} \tag{1.7}$$

In addition, we define the regular part of Neumann Green's function $G(x;\xi)$ as $H(x;\xi)$, which solves

$$\begin{cases} -\Delta H + H = -\frac{1}{2\pi}\log\frac{1}{|x-\xi|}, & x \in \Omega, \\ \frac{\partial H}{\partial \mathbf{n}} = \frac{1}{2\pi}\frac{(x-\xi)\cdot\mathbf{n}}{|x-\xi|^2}, & x \in \partial\Omega. \end{cases} \tag{1.8}$$

By employing the inner-outer gluing method, we extend the results shown in [14] and obtain the existence of multi-spots, which are

Theorem 1.1. *Assume that k, o are non-negative integers with $k + o \geq 1$ and $\lambda_j \bar{u}_j < \bar{C}_{j,\Omega}$, $j = 1, 2$. Then for sufficiently large $\chi_1 := \chi := \frac{1}{\varepsilon^2}$ with $\chi_2 = \gamma\chi_1$ and given positive constant γ, there exists a solution $(u_{1,\chi}, u_{2,\chi}, v_{1,\chi}, v_{2,\chi})$ to (1.3) satisfying the following form:*

$$u_{j,\chi}(x) = \sum_{k=1}^{m} c_{jk} U_{jk}\Big(\frac{x - \xi_k^\varepsilon}{\varepsilon}; \mu_k\Big) + o(1); \tag{1.9}$$

$$v_{j,\chi}(x) = \varepsilon^2 \sum_{k=1}^{m} \Big[- m_j \log\varepsilon + \Gamma_{j,\mu_{jk}}\Big(\frac{x - \xi_k^\varepsilon}{\varepsilon}\Big) + \hat{c}_{jk} H(x, \xi_k^\varepsilon) - \mu_{jk} \Big] + o(1), \tag{1.10}$$

where $m_j = \sum_{l=1}^{2} b_{jl}\sigma_l$, H is defined as the regular part of Neumann Green's function satisfying (1.8), U_{jk} and $\Gamma_{j,\mu_{jk}}$ are given by (1.6) and (1.4), respectively. Moreover, $\xi_k^\varepsilon \in \Omega$ and $\hat{c}_{jk} = 2\pi m_j$ for $k \leq o$; $\xi_k^\varepsilon \in \partial\Omega$ and $\hat{c}_{jk} = \pi m_j$ for $o < k \leq m$, where $\sigma_l := \frac{1}{2\pi}\int_{\mathbb{R}^N} e^{\Gamma_{l,\mu_{lk}}} dy$ and b_{jl} are defined in (1.5). In addition, the m -tuple $(\xi_1^\varepsilon, \cdots, \xi_m^\varepsilon)$ converges to a critical point of $\mathcal{J}_m$ as $\varepsilon \to 0$, where $\mathcal{J}_m$ is defined by

$$\mathcal{J}_m = \sum_{k=1}^{m} \bar{c}_k^2 H(x_k, x_k) + \sum_{k \neq l} \bar{c}_k \bar{c}_l G(x_k, x_l). \tag{1.11}$$

Here $\bar{c}_k = 2$ for $k \leq o$ and $\bar{c}_k = 1$ for $o < k \leq m$. In particular, the critical points of $\mathcal{J}_m$ are assumed to be non-degenerate and $\bar{C}_{j,\Omega} := \sum_{k=1}^{m} \hat{c}_{jk} C_\Omega$, where C_Ω is the positive lower bound of Green's function $G(x, y)$; $c_{jk} := \frac{2\pi\sigma_j}{\int_{\mathbb{R}^2} e^{2\Gamma_{j,\mu_{jk}}} dy}\bar{u}_j + O\Big(\frac{1}{\sqrt{\chi}}\Big)$ and μ_{jk} is determined by

$$\mu_{jk} = \hat{c}_{jk} H(\xi_k^\varepsilon, \xi_k^\varepsilon) + \sum_{l \neq k} \hat{c}_{jl} G(\xi_k^\varepsilon, \xi_l^\varepsilon).$$

Theorem 1.1 demonstrates that when the intrinsic growth rate is small, (1.3) admits infinitely many interior and boundary multi-spots under the singular limits of large χ_1 and χ_2. In Appendix A, we develop the formal construction of single interior spot, which supports our rigorous analysis for the proof of Theorem 1.1. Our main theoretical tool is the inner-outer gluing method, which has been used to study singularity formations within energy critical heat equations [3], harmonic map flows [5], Keller-Segel systems [4], etc. successfully.

2 The Choice of Ansatz and Error Computations

In this section, we shall discuss the choice of approximate solutions to (1.3). In light of the u_j-equation with $j = 1, 2$, one regards logistic source terms as perturbations and obtains

$$u_j = C_j e^{\chi_j v_j}, \tag{2.1}$$

where constant $C_j > 0$ will be determined later on. Upon substituting (2.1) into the v_j-equation in (1.3), we define $\bar{v}_j = \chi_j v_j$ and arrive at

$$\begin{cases} 0 = \Delta\bar{v}_1 - \bar{v}_1 + a_{11}C_1\chi_1 e^{\bar{v}_1} + a_{12}C_2\chi_1 e^{\bar{v}_2}, & x \in \Omega, \\ 0 = \Delta\bar{v}_2 - \bar{v}_2 + a_{21}C_1\chi_2 e^{\bar{v}_1} + a_{22}C_2\chi_2 e^{\bar{v}_2}, & x \in \Omega. \end{cases} \tag{2.2}$$

Moreover, define $C_1\chi_1 = \tilde{\varepsilon}^{m_1-2}$, $C_2\chi_2 = d\tilde{\varepsilon}^{m_2-2}$ with m_1, m_2 and d determined later on, then we have

$$\begin{cases} 0 = \Delta\bar{v}_1 - \bar{v}_1 + a_{11}\tilde{\varepsilon}^{m_1-2}e^{\bar{v}_1} + \dfrac{da_{12}}{\gamma}\tilde{\varepsilon}^{m_2-2}e^{\bar{v}_2}, & x \in \Omega, \\ 0 = \Delta\bar{v}_2 - \bar{v}_2 + a_{21}\gamma\tilde{\varepsilon}^{m_1-2}e^{\bar{v}_1} + a_{22}d\tilde{\varepsilon}^{m_2-2}e^{\bar{v}_2}, & x \in \Omega, \end{cases} \tag{2.3}$$

where $\gamma := \frac{\chi_2}{\chi_1}$. Without loss of generality, we assume that there is only one center $\xi \in \mathbb{R}^2$, then define $x - \xi = \tilde{\varepsilon}y$ and $\bar{v}_j = -m_j \log\tilde{\varepsilon} + \tilde{V}_j(y)$ with m_j determined later on to obtain from (2.3) that

$$\begin{cases} 0 = \Delta_y\tilde{V}_1 - \tilde{\varepsilon}^2\tilde{V}_1 + a_{11}e^{\tilde{V}_1} + a_{21}\gamma e^{\tilde{V}_2}, & y \in \Omega_{\tilde{\varepsilon}}, \\ 0 = \Delta_y\tilde{V}_2 - \tilde{\varepsilon}^2\tilde{V}_2 + a_{21}\gamma e^{\tilde{V}_1} + a_{22}de^{\tilde{V}_2}, & y \in \Omega_{\tilde{\varepsilon}}, \end{cases} \tag{2.4}$$

where $\Omega_{\tilde{\varepsilon}} := (\Omega - \xi)/\tilde{\varepsilon}$. By choosing d such that the coefficient matrix in (2.4) is symmetric, i.e.

$$\frac{a_{21}}{a_{12}}\gamma^2 = d, \tag{2.5}$$

we let $\tilde{\varepsilon} \to 0$ to get the limiting problem is

$$\begin{cases} 0 = \Delta_y\Gamma_1 + b_{11}e^{\Gamma_1} + b_{12}e^{\Gamma_2}, & y \in \mathbb{R}^2, \\ 0 = \Delta_y\Gamma_2 + b_{21}e^{\Gamma_1} + b_{22}e^{\Gamma_2}, & y \in \mathbb{R}^2, \end{cases} \tag{2.6}$$

where (Γ_1, Γ_2) is the leading approximation of $(\tilde{V}_1, \tilde{V}_2)$. Here

$$b_{11} := a_{11},\ b_{22} := a_{22}d,\ b_{21} = b_{12} = a_{21}\gamma. \tag{2.7}$$

Noting that $B = (b_{ij})_{2\times 2}$ is a real symmetric, irreducible, positive and invertible matrix, one utilizes the results shown in [15] to obtain there exist a family of classical solutions $(\Gamma_{1,\tilde{\mu}_1}, \Gamma_{2,\tilde{\mu}_2})$ to (2.6) such that

$$\Gamma_{j,\tilde{\mu}_j}(y) = \Gamma_j(y, \tilde{\mu}_j) + \tilde{\mu}_j \tag{2.8}$$

where $\tilde{\mu}_j$ are constants, $\hat{m} := \min\{m_1, m_2\} > 2$ and

$$\Gamma_j(y, \tilde{\mu}_j) = -m_j \log|y| + O(|y|^{2-\hat{m}}) \quad |y| \gg 1.$$

Denote $\sigma_j := \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{\Gamma_{j,\tilde{\mu}_j}}dy$, then we have m_1 and m_2 are determined by

$$m_1 = \sigma_1 b_{11} + \sigma_2 b_{12}, \quad m_2 = \sigma_1 b_{21} + \sigma_2 b_{22}. \tag{2.9}$$

Thus, for $j = 1, 2$,

$$\bar{v}_j = -m_j \log \tilde{\varepsilon} + \tilde{V}_j(y), \quad \tilde{V}_j(y) = \Gamma_{j,\tilde{\mu}_j}(y) + o(1).$$

We mention that by the blow-up analysis [15], $m_j > 2$ for all $j = 1, 2$. Moreover, for $j = 1, 2$, the leading order term of u_j is

$$u_j(x) = C_j \tilde{\varepsilon}^{-m_j} e^{\Gamma_{j,\tilde{\mu}_j}}(1 + o(1)).$$

Noting that $C_1 = \frac{\tilde{\varepsilon}^{m_1-2}}{\chi_1}$ and $C_2 = d\frac{\tilde{\varepsilon}^{m_2-2}}{\chi_2}$, we have

$$u_1 = \frac{1}{\chi_1 \tilde{\varepsilon}^2} e^{\Gamma_{1,\tilde{\mu}_1}}(1 + o(1)) := c_1 e^{\Gamma_{1,\tilde{\mu}_1}} \ \text{ and } \ u_2 = \frac{d}{\chi_2 \tilde{\varepsilon}^2} e^{\Gamma_{2,\tilde{\mu}_2}}(1 + o(1)) := c_2 e^{\Gamma_{2,\tilde{\mu}_2}}. \tag{2.10}$$

The leading part of c_j, $j = 1, 2$ is determined by the global balancing condition $\int_\Omega u_j(\bar{u}_j - u_j)dx = 0$, which implies

$$c_j + o(1) = \frac{\int_{\mathbb{R}^2} e^{\Gamma_{j,\tilde{\mu}_j}}\, dy}{\int_{\mathbb{R}^2} e^{2\Gamma_{j,\tilde{\mu}_j}}\, dy} \bar{u}_j = \frac{2\pi\sigma_j}{\int_{\mathbb{R}^2} e^{2\Gamma_{j,\tilde{\mu}_j}}\, dy} \bar{u}_j. \tag{2.11}$$

In addition, thanks to Pohozaev identity shown in [15], we find

$$4(\sigma_1 + \sigma_2) = b_{11}\sigma_1^2 + 2b_{12}\sigma_1\sigma_2 + b_{22}\sigma_2^2. \tag{2.12}$$

Combining (2.12) with (2.11), one gets (σ_1, σ_2) solves

$$\begin{cases} \dfrac{\bar{u}_1}{\bar{u}_2} \displaystyle\int_{\mathbb{R}^2} e^{2\Gamma_{2,\tilde{\mu}_2}}\, dy \sigma_1 = \dfrac{a_{12}}{a_{21}} \dfrac{\chi_1}{\chi_2} \displaystyle\int_{\mathbb{R}^2} e^{2\Gamma_{1,\tilde{\mu}_1}}\, dy \sigma_2, \\ \qquad 4(\sigma_1 + \sigma_2) = b_{11}\sigma_1^2 + 2b_{12}\sigma_1\sigma_2 + b_{22}\sigma_2^2, \end{cases} \tag{2.13}$$

where b_{ij}, $i, j = 1, 2$ are given in (2.7). Noting that $(a_{ij})_{2\times 2}$ is a positive definite matrix by assumption (H3), one has the second constraint in (2.13) is an ellipse passing through $(0, 0)$. On the other hand, the first equality in (2.13) can not be a closed curve since it cannot cross the coordinate axes by using the fact that all points must lie in the first quadrant. Therefore, the system (2.13) admits at least a positive solution (σ_1, σ_2), where the schematic diagram is shown in Figure 1.

We further define the correction term as $H_j^{\tilde{\varepsilon}}(x; \xi)$, $j = 1, 2$, which satisfies

$$\begin{cases} \Delta_x H_j^{\tilde{\varepsilon}} - H_j^{\tilde{\varepsilon}} = -m_j \log \tilde{\varepsilon} + \Gamma_{j,\tilde{\mu}_j}\Big(\dfrac{x - \xi}{\tilde{\varepsilon}}\Big), & x \in \Omega, \\ \qquad \partial_{\mathbf{n}} H_j^{\tilde{\varepsilon}} = -\partial_{\mathbf{n}} \Gamma_{j,\tilde{\mu}_j}, & x \in \partial\Omega. \end{cases} \tag{2.14}$$

In summary, we set the rough approximation of the single spot in (1.3) as

$$\begin{cases} u_j = U_j(y, \tilde{\mu}_j) := c_j e^{\Gamma_{j,\tilde{\mu}_j}}, \\ \bar{v}_j = -m_j \log \tilde{\varepsilon} + \Gamma_j(y, \tilde{\mu}_j) + H_j^{\tilde{\varepsilon}}(x; \xi), \end{cases} \tag{2.15}$$

where $y = \frac{x-\xi}{\tilde{\varepsilon}}$ and $\tilde{\varepsilon} := \frac{1}{\sqrt{c_1\chi_1}} \ll 1$. Here c_j is determined by (2.11), (σ_1, σ_2) is the solution to (2.13) and $(\Gamma_{1,\tilde{\mu}_1}, \Gamma_{2,\tilde{\mu}_2})$ solves Liouville system (2.6).

It remains to determine the parameter $\tilde{\mu}_j$, $j = 1, 2$. In light of (1.7), similarly as shown in Lemma 2.1 of [6], we have for any $\alpha \in (0, 1)$,

$$H_j^{\tilde{\varepsilon}}(x; \xi) = \hat{c}_j H(x; \xi) - \tilde{\mu}_j + O(\tilde{\varepsilon}^\alpha), \tag{2.16}$$

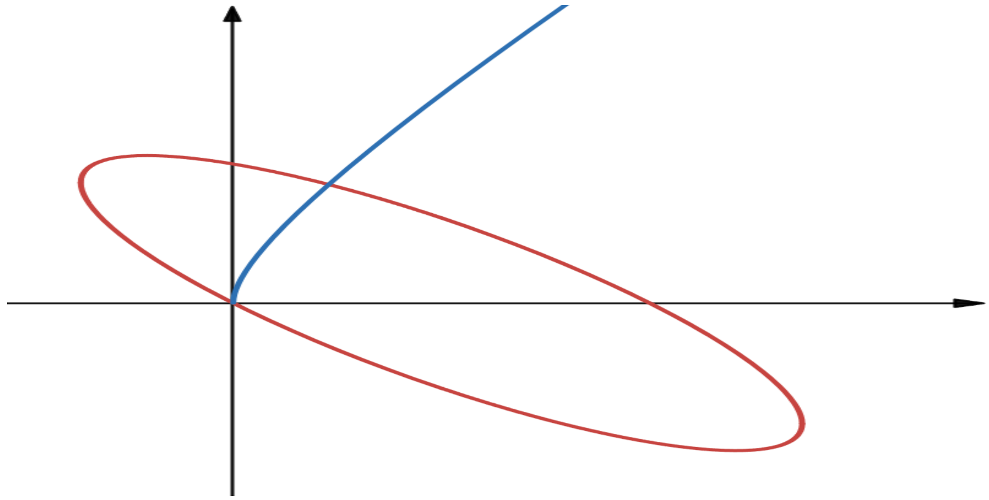

Figure 1: Schematic Diagram of (2.13)

where $\hat{c}_j = 2\pi m_j$ if $\xi \in \Omega$ and $\hat{c}_j = \pi m_j$ if $\xi \in \partial\Omega$. To guarantee the error is small, we choose

$$\tilde{\mu}_j = \hat{c}_j H(\xi, \xi), \tag{2.17}$$

where $\hat{c}_j = 2\pi m_j$ if $\xi \in \Omega$ and $\hat{c}_j = \pi m_j$ if $\xi \in \partial\Omega$. We remark that $\tilde{\varepsilon}$ depends on c_j and c_j is determined globally, which may cause the ambiguity for our subsequent analysis. To solve this issue, we define $\varepsilon = \frac{1}{\sqrt{\chi}} \ll 1$ and rewrite (2.15) as

$$\begin{cases} u_j = U_j(y; \mu_j), \\ \bar{v}_j = -m_j \log \varepsilon + \Gamma_j(y, \mu_j) + H_j^{\varepsilon}(x; \xi), \end{cases} \tag{2.18}$$

where $y := \frac{x-\xi}{\varepsilon}$, $\mu_j = \hat{c}_j H(\xi, \xi)$ and H_j^{ε} solves

$$\begin{cases} \Delta_x H_j^{\varepsilon} - H_j^{\varepsilon} = -m_j \log \varepsilon + \Gamma_{j,\mu_j}\Big(\dfrac{x-\xi}{\varepsilon}\Big), & x \in \Omega, \\ \partial_{\mathbf{n}} H_j^{\varepsilon} = -\partial_{\mathbf{n}} \Gamma_{j,\mu_j}, & x \in \partial\Omega. \end{cases} \tag{2.19}$$

Proceeding with the similar argument shown above, we set the first approximation of multi-spots with o, $0 \le o \le m$, interior bubbles to (1.3) as

$$\begin{cases} u_j = \displaystyle\sum_{k=1}^{m} U_{jk}(y - \xi_k', \mu_{jk}) := \sum_{k=1}^{m} c_{jk} e^{\Gamma_{j,\mu_{jk}}(y-\xi_k')}, \\ \bar{v}_j = \displaystyle\sum_{k=1}^{m} \Big(- m_j \log \varepsilon + \Gamma_{jk}(y - \xi_k', \mu_{jk}) + H_{jk}^{\varepsilon}(x; \xi_k) \Big), \end{cases} \tag{2.20}$$

where $m \ge 1$, $\varepsilon := \frac{1}{\sqrt{\chi}}$, $y := \frac{x}{\varepsilon}$, $\xi_k' := \frac{\xi_k}{\varepsilon}$. In particular, μ_{jk}, $j = 1, 2$ are determined by

$$\mu_{jk} = \hat{c}_{jk} H(\xi_k; \xi_k) + \sum_{l \neq k} \hat{c}_{jl} G(\xi_k; \xi_l), \tag{2.21}$$

where $G(x; \xi)$ is given by (1.7) and $\hat{c}_{jk} = 2\pi m_j$ if $k \le o$; $\hat{c}_{jk} = \pi m_j$ if $o < k \le m$. For the simplicity of our notation, we define

$$U_{jk}(y - \xi_k') = U_{jk}(y - \xi_k', \mu_{jk}) \ \text{ and } \ \Gamma_{jk}(y - \xi_k') = \Gamma_{jk}(y - \xi_k', \mu_{jk}), . \tag{2.22}$$

then rewrite the first approximation solution as

$$\begin{cases} u_j = \sum_{k=1}^{m} U_{jk}(y-\xi'_k), \\ \bar{v}_j = \sum_{k=1}^{m} \Big(-m_j \log\varepsilon + \Gamma_{jk}(y-\xi'_k) + H^{\varepsilon}_{jk}(x;\xi_k) \Big), \end{cases} \tag{2.23}$$

where H^{ε}_{jk} solves (2.19) with ξ replaced by ξ_k.

Next, we compute the error generated by (2.23). Noting that (1.3) can be reduced as the following two-coupled equations:

$$\begin{cases} S_1(u_1,u_2) = \Delta u_1 + \nabla\cdot((\Delta_x-1)^{-1}(a_{11}u_1+a_{12}u_2)) + \lambda_1 u_1(\bar{u}_1-u_1) = 0, \\ S_2(u_1,u_2) = \Delta u_2 + \nabla\cdot((\Delta_x-1)^{-1}(a_{21}u_1+a_{22}u_2)) + \lambda_2 u_2(\bar{u}_2-u_2) = 0. \end{cases} \tag{2.24}$$

Then in the region $|x-\xi_k|<\delta\varepsilon$ with constant $\delta>0$, we calculate to get

$$\begin{cases} \nabla_x u_j = \frac{1}{\varepsilon}\nabla U_{jk}(y-\xi'_k) + \sum_{l\neq k}\frac{1}{\varepsilon}\nabla U_{jl}(y-\xi'_l) + o(1), \\ \nabla_x \bar{v}_j = \frac{1}{\varepsilon}\nabla\Gamma_{jk}(y-\xi'_k) + \nabla\tilde{H}^{\varepsilon}_{jk}(x,\xi_k) + o(1), \end{cases} \tag{2.25}$$

where

$$\tilde{H}^{\varepsilon}_{jk}(x,\xi_k) = H^{\varepsilon}_{jk}(x;\xi_k) + \sum_{l\neq k}\Big(\Gamma_{jk}(y-\xi_l) + H^{\varepsilon}_{jl}(x;\xi_l)\Big). \tag{2.26}$$

Moreover, one has

$$\begin{cases} \Delta_x u_j = \frac{1}{\varepsilon^2}\Delta U_{jk}(y-\xi'_k) + \sum_{l\neq k}\frac{1}{\varepsilon^2}\Delta U_{jl}(y-\xi'_l) + o(1), \\ \Delta_x \bar{v}_j = \frac{1}{\varepsilon^2}\Delta\Gamma_{jk}(y-\xi'_k) + \Delta\tilde{H}^{\varepsilon}_{jk}(x,\xi_k) + o(1), \end{cases} \tag{2.27}$$

$$\begin{aligned} \nabla u_j\cdot\nabla\bar{v}_j =& \frac{1}{\varepsilon^2}\nabla U_{jk}(y-\xi'_k)\cdot\nabla\Gamma_{jk}(y-\xi'_k) + \frac{1}{\varepsilon}\nabla U_{jk}(y-\xi'_k)\cdot\nabla\tilde{H}^{\varepsilon}_{jk}(x,\xi_k) \\ &+ \frac{1}{\varepsilon^2}\nabla\Gamma_{jk}(y-\xi'_k)\cdot\sum_{l\neq k}\nabla U_{jl}(y-\xi'_l) + \frac{1}{\varepsilon}\sum_{l\neq k}\nabla U_{jl}(y-\xi'_l)\cdot\nabla\tilde{H}^{\varepsilon}_{jk}(x,\xi_k), \end{aligned} \tag{2.28}$$

and

$$\begin{aligned} u_j\Delta\bar{v}_j =& \frac{1}{\varepsilon^2}U_{jk}(y-\xi'_k)\cdot\Delta\Gamma_{jk}(y-\xi'_k) + \frac{1}{\varepsilon^2}\sum_{l\neq k}U_{jl}(y-\xi'_l)\Delta\Gamma_{jk}(y-\xi'_k) \\ &+ U_{jk}(y-\xi'_k)\cdot\Delta\tilde{H}^{\varepsilon}_{jk}(x,\xi_k) + \sum_{l\neq k}U_{jl}(y-\xi'_l)\cdot\Delta\tilde{H}^{\varepsilon}_{jk}(x,\xi_k). \end{aligned} \tag{2.29}$$

Then, one finds for $j = 1, 2$,

$$
\begin{aligned}
S_j(\mathbf{u}) =& \frac{1}{\varepsilon^2}[\Delta U_{jk}(y-\xi_k') - \nabla U_{jk}(y-\xi_k') \cdot \nabla \Gamma_{jk}(y-\xi_k') - U_{jk}(y-\xi_k') \cdot \Delta \Gamma_{jk}(y-\xi_k')] \\
& - \Big[\frac{1}{\varepsilon}\sum_{l\neq k} \nabla U_{jl}(y-\xi_l') \cdot \nabla \tilde{H}_{jk}^{\varepsilon}(x,\xi_k) + \sum_{l\neq k} U_{jl}(y-\xi_l') \cdot \Delta \tilde{H}_{jk}^{\varepsilon}(x,\xi_k)\Big] \\
& - \Big[\frac{1}{\varepsilon}\nabla U_{jk}(y-\xi_k') \cdot \nabla \tilde{H}_{jk}^{\varepsilon}(x,\xi_k) + \frac{1}{\varepsilon^2}\nabla \Gamma_{jk}(y-\xi_k') \cdot \sum_{l\neq k} \nabla U_{jl}(y-\xi_l') \\
& - \frac{1}{\varepsilon^2}\sum_{l\neq k} U_{jl}(y-\xi_l')\Delta \Gamma_{j,\mu_{jk}}(y-\xi_k') + U_{jk}(y-\xi_k') \cdot \Delta \tilde{H}_{jk}^{\varepsilon}(x,\xi_k)\Big] \\
& + \sum_{l\neq k} \frac{1}{\varepsilon^2}\Delta U_{jl}(y-\xi_l') + \lambda_j U_{jk}(y-\xi_k')\big(\bar{u} - U_{jk}(y-\xi_k')\big) \\
& + \lambda_j \sum_{l\neq k} U_{jl}(y-\xi_l')\big(\bar{u}_j - U_{jk}(y-\xi_k') - \sum_{l\neq k} U_{jl}(y-\xi_l')\big) \\
& + \lambda_j U_{jk}(y-\xi_k') \sum_{l\neq k} U_{jl}(y-\xi_l') + o(1),
\end{aligned}
\tag{2.30}
$$

where $\mathbf{u} = (u_1, u_2)^T$. Similarly as shown in [14], we observe from (2.30) that the main contribution term in the error is

$$
\bar{I}_{j1} := \frac{1}{\varepsilon}\nabla U_{jk}(y-\xi_k') \cdot \nabla \tilde{H}_{jk}^{\varepsilon}(x,\xi_k) + U_{jk}(y-\xi_k') \cdot \Delta \tilde{H}_{jk}^{\varepsilon}(x,\xi_k), \tag{2.31}
$$

and $\bar{I}_{j1} = O(\frac{1}{\varepsilon})$. Then it follows that the leading error of $\varepsilon^2 S_j(\mathbf{u})$ is $O(\varepsilon)$. To further study $\bar{I}_{j1}$ given in (2.31), we use a single interior spot as an example to illustrate our idea. In fact, we expand

$$
\nabla \tilde{H}_j^{\varepsilon}(x;\xi) = \hat{c}_j \nabla H(\xi;\xi) + \hat{c}_j \nabla^2 H(\xi;\xi)(x-\xi) + O(\varepsilon^{\alpha}), \tag{2.32}
$$

where (2.16) has been used. Since $x - \xi = \varepsilon y$, we have that $\varepsilon \nabla \cdot (U_j \nabla H(\xi))$ dominates. To balance this, we adjust location ξ such that $\nabla H(\xi_0) = 0$, which implies the principal term ξ_0 of ξ is a critical point of $\mathcal{J}_m$ defined by (1.11) with $m = 1$. Similarly, for the multi-spots, we adjust $(\xi_1, \cdots, \xi_m)$ such that the leading term is governed by the critical point of (1.11).

To eliminate the error generated from the logistic growth, we define the second approximation of $(u_j, \bar{v}_j)$ as

$$
\begin{cases}
u_j = \sum_{k=1}^{m} \Big(U_{jk}(y-\xi_k') + \varepsilon^2 \phi_{jk}(y-\xi_k')\Big), \\
\bar{v}_j = \sum_{k=1}^{m} \Big(-m_j \log\varepsilon + \Gamma_{jk}(y-\xi_k') + H_{jk}^{\varepsilon}(x;\xi_k)\Big) + \varepsilon^2\Big(\psi_{jk}(y-\xi_k') + \mathcal{H}_{jk}^{\varepsilon}(x,\xi_k)\Big),
\end{cases}
\tag{2.33}
$$

where (ϕ_{jk}, ψ_{jk}), $j = 1, 2$, $k = 1, \cdots, m$ are the next order term and $\mathcal{H}_{jk}^{\varepsilon}$ are correction terms of ψ_{jk} satisfying

$$
\begin{cases}
\Delta_x \mathcal{H}_{jk}^{\varepsilon}(x,\xi_k) - \mathcal{H}_{jk}^{\varepsilon}(x,\xi_k) = -\psi_{jk}, & x \in \Omega, \\
\partial_{\mathbf{n}} \mathcal{H}_{jk}^{\varepsilon} = -\partial_{\mathbf{n}} \psi_{jk}, & x \in \partial\Omega.
\end{cases}
\tag{2.34}
$$

For simplicity of notation, we drop "k" and use ψ_j, ϕ_j and U_j to replace ψ_{jk}, ϕ_{jk} and $U_{jk}(y-\xi_k')$. To balance the error generated by logistic source, we choose $(\phi_1, \phi_2, \psi_1, \psi_2)$ as a solution to

$$\begin{cases} \nabla(U_1 \nabla g_1) + \lambda_1 U_1(\bar{u}_1 - U_1) = 0, \\ \nabla(U_2 \nabla g_2) + \lambda_2 U_2(\bar{u}_2 - U_2) = 0, \\ g_1 = \dfrac{\phi_1}{U_1} - \psi_1; \quad g_2 = \dfrac{\phi_2}{U_2} - \psi_2, \\ \Delta\psi_1 + a_{11}\chi_1\phi_1 + a_{12}\chi_1\phi_2 = 0, \\ \Delta\psi_2 + a_{21}\chi_2\phi_1 + a_{22}\chi_2\phi_2 = 0. \end{cases} \tag{2.35}$$

Next, we solve (2.35) and first define $h_j := -\lambda_j U_j(\bar{u}_j - U_j)$. Then by applying the first integral method, one gets

$$g_j = \int_r^{\infty} \frac{1}{\rho U_j(\rho)} \int_0^{\rho} h_j(s) s \, ds \, d\rho, \tag{2.36}$$

where we have used

$$\int_{\mathbb{R}^2} h_j \, dy = 0.$$

Thus, one has for $j = 1, 2$,

$$g_j \sim \langle r \rangle^{2-\delta_j}, \tag{2.37}$$

where $\delta_j > 0$ is small enough. In light of $\tilde{g}_j := U_j g_j \sim \langle r \rangle^{-(m_j - 2 + \delta_j)}$, we find from the variation-of-parameters formula that there exists (ψ_1, ψ_2) such that

$$\psi_j = O(\log|y|), \text{ for } |y| \gg 1. \tag{2.38}$$

Invoking $\phi_j = U_j g_j + U_j \psi_j$, we further obtain that there exists $\phi_j \sim \langle r \rangle^{-(m_j-2+\delta_j)}$, where $m_j > 2$ and $\delta_j > 0$ is small enough.

By using $(u_{j1}, \bar{v}_{j1})$ defined in (2.33) as the ansatz, we shall perform the error computation and establish the inner and outer systems satisfied by the remainder term (φ_j, w_j). To this end, we write the solution $(u_j, \bar{v}_j)$ to (1.3) as

$$\begin{cases} u_j = \displaystyle\sum_{k=1}^{m} \Big[U_{jk}\Big(\frac{x-\xi_k}{\varepsilon}\Big) + \varepsilon^2 \phi_{jk}\Big(\frac{x-\xi_k}{\varepsilon}\Big) \Big] + \varphi_j\Big(\frac{x}{\varepsilon}\Big), \\ \bar{v}_j = \displaystyle\sum_{k=1}^{m} \Big[\Big(-m_j \log\varepsilon + \Gamma_{jk}\Big(\frac{x-\xi_k}{\varepsilon}\Big) + H^{\varepsilon}_{jk}(x;\xi_k) \Big) + \varepsilon^2 \Big(\psi_{jk}\Big(\frac{x-\xi_k}{\varepsilon}\Big) + \mathcal{H}^{\varepsilon}_{jk}(x,\xi_k) \Big) \Big] + w_j\Big(\frac{x}{\varepsilon}\Big), \end{cases} \tag{2.39}$$

where and in the sequel we rewrite ξ_k^{ε} as ξ_k for the simplicity of notations. Then, we compute the error term to get

$$\nabla u_j = \sum_{k=1}^{m} \Big[\frac{1}{\varepsilon} \nabla U_{jk}(y - \xi_k') + \varepsilon \nabla \phi_{jk}(y - \xi_k') \Big] + \frac{1}{\varepsilon} \nabla_y \varphi_j(y), \tag{2.40}$$

$$\nabla \bar{v}_j = \sum_{k=1}^{m} \Big[\frac{1}{\varepsilon} \nabla \Gamma_{jk}(y - \xi_k') + \nabla H^{\varepsilon}_{jk}(\varepsilon y, \xi_k') + \varepsilon \nabla \psi_{jk}(y - \xi_k') + \varepsilon^2 \nabla \mathcal{H}^{\varepsilon}_{jk}(\varepsilon y, \xi_k') \Big] + \frac{1}{\varepsilon} \nabla_y w_j(y), \tag{2.41}$$

$$\Delta u_j = \sum_{k=1}^{m} \Big[\frac{1}{\varepsilon^2} \Delta U_{jk}(y - \xi_k') + \Delta \phi_{jk}(y - \xi_k') \Big] + \frac{1}{\varepsilon^2} \Delta_y \varphi_j(y), \tag{2.42}$$

and

$$\Delta\bar{v}_j = \sum_{k=1}^{m}\Big[\frac{1}{\varepsilon^2}\Delta\Gamma_{jk}(y-\xi_k') + \Delta H_{jk}^{\varepsilon}(\varepsilon y,\xi_k') + \Delta\psi_{jk}(y-\xi_k') + \varepsilon^2\Delta\mathcal{H}_{jk}^{\varepsilon}(\varepsilon y,\xi_k')\Big] + \frac{1}{\varepsilon^2}\Delta_y w_j(y). \tag{2.43}$$

Upon substituting (2.39)–(2.43) into (2.24), one finds

$$0 = \Delta u_j - \nabla u_j\cdot\nabla\bar{v}_j - u_j\cdot\Delta\bar{v}_j + \lambda_j u_j(\bar{u}_j - u_j) = L_j[\varphi_1,\varphi_2] + \sum_{l=1}^{7} I_{jl}, \tag{2.44}$$

where

$$L_j[\varphi_1,\varphi_2] = -\Delta\varphi_j + \nabla\cdot(P_j\nabla w_j) + \nabla\cdot(\varphi_j\nabla Q_j), \tag{2.45}$$

$$P_j = \sum_{k=1}^{m} U_{jk}(y-\xi_k') \text{ and } Q_j = \sum_{k=1}^{m}\Gamma_{jk}(y-\xi_k') \tag{2.46}$$

and I_{jl}, $l = 1,\cdots,7$ are defined as

$$I_{j1} = -\frac{1}{\varepsilon^2}\sum_{k=1}^{m}\sum_{l\neq k} U_{jk}(y-\xi_k')\Delta\Gamma_{jl}(y-\xi_l'), \tag{2.47}$$

$$\begin{aligned} I_{j2} = &-\frac{1}{\varepsilon^2}\sum_{k=1}^{m}\sum_{l\neq k}\nabla U_{jk}(y-\xi_k')\cdot\nabla\Big(\Gamma_{jl}(y-\xi_l') + \varepsilon H_{jk}^{\varepsilon}(\varepsilon y,\xi_k)\Big)\\ &-\frac{1}{\varepsilon}\sum_{k=1}^{m}\nabla U_{jk}(y-\xi_k')\cdot\nabla H_{jk}^{\varepsilon}(\varepsilon y,\xi_k) + \sum_{k=1}^{m}\sum_{l=1}^{m} U_{jk}(y-\xi_k')\cdot\Delta H_{jl}^{\varepsilon}(\varepsilon y,\xi_l), \end{aligned} \tag{2.48}$$

$$\begin{aligned} I_{j3} = -\sum_{k=1}^{m}\sum_{l\neq k}\Big(&\nabla U_{jk}(y-\xi_k')\cdot\nabla\psi_{jl}(y-\xi_l') + \nabla\phi_{jk}(y-\xi_k')\cdot\nabla\Gamma_{jl}(y-\xi_l')\\ &+ U_{jk}(y-\xi_k')\cdot\Delta\psi_{jl}(y-\xi_l') + \phi_{jk}(y-\xi_k')\cdot\Delta\Gamma_{jl}(y-\xi_l')\Big), \end{aligned} \tag{2.49}$$

$$\begin{aligned} I_{j4} = -\sum_{k=1}^{m}\sum_{l=1}^{m}\Big(&\varepsilon\nabla U_{jk}(y-\xi_k')\cdot\nabla\mathcal{H}_{jl}^{\varepsilon}(\varepsilon y,\xi_k) + \varepsilon\nabla\phi_{jk}(y-\xi_k')\cdot\nabla H_{jl}^{\varepsilon}(\varepsilon y,\xi_l)\\ &+ \varepsilon^2 U_{jk}(y-\xi_k')\cdot\Delta\mathcal{H}_{jl}^{\varepsilon}(\varepsilon y,\xi_l) + \varepsilon^2\phi_{jk}(y-\xi_k')\cdot\Delta H_{jl}^{\varepsilon}(\varepsilon y,\xi_l)\Big), \end{aligned} \tag{2.50}$$

$$\begin{aligned} I_{j5} = \sum_{k=1}^{m}\sum_{l=1}^{m}\Big(&\varepsilon^2\nabla\phi_{jk}(y-\xi_k')\cdot(\nabla\psi_{jl}(y-\xi_l') + \varepsilon\nabla\mathcal{H}_{jl}^{\varepsilon}(\varepsilon y,\xi_l))\\ &+ \varepsilon^2\phi_{jk}(y-\xi_k')\cdot(\Delta\psi_{jl}(y-\xi_l') + \varepsilon^2\Delta\mathcal{H}_{jl}^{\varepsilon}(\varepsilon y,\xi_l))\Big), \end{aligned} \tag{2.51}$$

$$
\begin{aligned}
I_{j6} =& -\frac{1}{\varepsilon}\nabla\varphi_j\cdot\sum_{k=1}^{m}\nabla H_{jk}^{\varepsilon}(\varepsilon y,\xi_k)-\varphi_j\sum_{k=1}^{m}\Delta H_{jk}^{\varepsilon}(\varepsilon y,\xi_k)\\
&-\frac{1}{\varepsilon}\nabla\varphi_j\cdot\Big(\varepsilon\sum_{k=1}^{m}\nabla\psi_{jk}(y-\xi_k')+\varepsilon^2\nabla\mathcal{H}_{jk}^{\varepsilon}(\varepsilon y,\xi_k)\Big)\\
&-\varphi_j\Big(\sum_{k=1}^{m}\Delta\psi_{jk}(y-\xi_k')+\varepsilon^2\Delta\mathcal{H}_{jk}^{\varepsilon}(\varepsilon y,\xi_k)\Big)\\
&-\nabla w_j\cdot\sum_{k=1}^{m}\nabla\phi_{jk}(y-\xi_k')-\Delta w_j\cdot\sum_{k=1}^{m}\phi_{jk}(y-\xi_k')\\
&-\frac{1}{\varepsilon^2}(\nabla\varphi_j\cdot\nabla w_j+\varphi_j\Delta w_j)
\end{aligned}
\tag{2.52}
$$

and

$$
\begin{aligned}
I_{j7} =& \lambda_j\Big(\varepsilon^2\sum_{k=1}^{m}\phi_{jk}\Big(\frac{x-\xi_k}{\varepsilon}\Big)+\varphi_j\Big)\Big(\bar{u}_j-\sum_{k=1}^{m}\Big[U_{jk}\Big(\frac{x-\xi_k}{\varepsilon}\Big)+\varepsilon^2\phi_{jk}\Big(\frac{x-\xi_k}{\varepsilon}\Big)\Big]-\varphi_j\Big)\\
&-\lambda_j\sum_{k=1}^{m}U_{jk}\Big(\frac{x-\xi_k}{\varepsilon}\Big)\Big(\sum_{k=1}^{m}\varepsilon^2\phi_{jk}\Big(\frac{x-\xi_k}{\varepsilon}\Big)+\varphi_j\Big)+\lambda_j\sum_{k=1}^{m}U_{jk}\Big(\frac{x-\xi_k}{\varepsilon}\Big)\sum_{l\neq k}\Big(\bar{u}_j-U_{jl}\Big(\frac{x-\xi_l}{\varepsilon}\Big)\Big).
\end{aligned}
\tag{2.53}
$$

We summarize the computation above to obtain $\boldsymbol{\varphi} := (\varphi_1,\varphi_2)$ solves

$$
\begin{cases}
L_j[\boldsymbol{\varphi}] = \varepsilon^2\sum_{l=1}^{7}I_{jl}(\boldsymbol{\varphi},\mathbf{P}), & \text{in } \Omega_\varepsilon,\\
0 = \Delta w_1 + a_{11}\chi_1\varphi_1 + a_{12}\chi_1\varphi_2, & \text{in } \Omega_\varepsilon,\\
0 = \Delta w_2 + a_{21}\chi_2\varphi_1 + a_{22}\chi_2\varphi_2, & \text{in } \Omega_\varepsilon,\\
\frac{\partial w_1}{\partial\boldsymbol{\nu}} = \frac{\partial w_2}{\partial\boldsymbol{\nu}} = \frac{\partial\varphi_1}{\partial\boldsymbol{\nu}} = \frac{\partial\varphi_2}{\partial\boldsymbol{\nu}}, & \text{on } \partial\Omega_\varepsilon,
\end{cases}
\tag{2.54}
$$

where $j = 1,2$ and

$$
\mathbf{P} = (c_{11},\cdots,c_{1m},c_{21},\cdots,c_{2m},\xi_1,\cdots,\xi_m).
\tag{2.55}
$$

The subsequent sections are devoted to the solvability of (2.54) and the existence of solution $(\varphi_1,\varphi_2,w_1,w_2)$. To this end, we decompose the domain Ω as inner and outer regions. Correspondingly, the solution (φ_1,φ_2) is decomposed as the combination of inner and outer solutions. Firstly, we establish the linear theory in the inner region, which is shown in Section 3.

3 Inner Linear Theory

In this section, we consider the inner region $|x-\xi_k| < \delta$ with constant $\delta > 0$, where ξ_k denotes the location of the k-th spot. Noting that ε is small enough, one has the inner region $\{y\in\mathbb{R}^2 : |y| < \frac{\delta}{\varepsilon}\}$ approximates the whole space $\mathbb{R}^2$. Then, we define the stretched variable $y = \frac{x-\xi_k}{\varepsilon}$ and the k-th inner operator $L_k^{\mathrm{inn}}[\varphi_1,\varphi_2] := (L_{1k}^{\mathrm{inn}}[\varphi_1], L_{2k}^{\mathrm{inn}}[\varphi_2])^T$ as

$$
L_{jk}^{\mathrm{inn}}[\varphi_j] := -\Delta_y\varphi_j + \nabla\cdot(U_j\nabla_y w_j) + \nabla\cdot(\varphi_j\nabla_y\log U_j),\ y\in\mathbb{R}^2,
\tag{3.1}
$$

where $w_j = (-\Delta_y)^{-1}(b_{j1}\varphi_1 + b_{j2}\varphi_2)$ and $j = 1,2$. Here and in the subsequent analysis, we drop "k" and c_{jk} in U_{jk} given by (2.20) since in each inner region, the form of U_{jk} is the same and c_{jk} is a constant.

We remark that $c_{jk}\varphi_j$ here is equal to the original one, where c_{jk} is incorporated in (3.1). Similarly as shown in [14], when the location of the spot is in the interior of domain Ω, the inner problem is formulated as

$$L_{jk}^{\text{inn}}[\varphi_j] + h_j = 0 \ \text{ in } \ \mathbb{R}^2, \tag{3.2}$$

where $\mathbf{h} := (h_1, h_2)^T$ denotes the error. We introduce the intermediate variables g_1 and g_2 to simplify the inner problem (3.2) as

$$\begin{cases} \nabla_y \cdot (U_1 \nabla g_1) = h_1, \quad g_1 = \dfrac{\varphi_1}{U_1} - w_1, & y \in \mathbb{R}^2, \\ \nabla_y \cdot (U_2 \nabla g_2) = h_2, \quad g_2 = \dfrac{\varphi_2}{U_2} - w_2, & y \in \mathbb{R}^2, \\ \Delta_y w_1 + b_{11} U_1 w_1 + b_{11} U_1 g_1 + b_{12} U_2 w_2 + b_{12} U_2 g_2 = 0, & y \in \mathbb{R}^2 \\ \Delta_y w_2 + b_{21} U_1 w_1 + b_{21} U_1 g_1 + b_{22} U_2 w_2 + b_{22} U_2 g_2 = 0, & y \in \mathbb{R}^2, \end{cases} \tag{3.3}$$

where b_{ij}, $i = 1, 2$, $j = 1, 2$ are given in (2.7). System (3.3) can be regarded as the coupling of divergence form equations and the linearized Liouville systems. We first consider the non-degeneracy of operator $\nabla_y \cdot (U_j \nabla g_j)$ to U_j and obtain

Lemma 3.1. *The bounded solution space to the following problem*

$$\begin{cases} \nabla \cdot \Big(U_j \nabla \Big(\dfrac{\tilde{g}_j}{U_j} \Big) \Big) = 0, & y \in \mathbb{R}^2, \\ \tilde{g}_j \in H_0^2(\mathbb{R}^2), & |\tilde{g}_j| = O(1)(1 + r)^{-\sigma - 2} \end{cases} \tag{3.4}$$

is one-dimensional and spanned by the nontrivial kernel U_j, where $\tilde{g}_j = U_j g_j$, $j = 1, 2$ and $\sigma > 0$ is a small constant.

Proof. Thanks to the definition of $\tilde{g}_j$ shown in (3.3), we have g_j satisfies

$$\nabla \cdot (U_j \nabla g_j) = 0, \quad y \in \mathbb{R}^2. \tag{3.5}$$

Denote g_+ and η as

$$g_{j+} = \begin{cases} g_j, & g_j > 0, \\ 0, & g_j \leq 0, \end{cases} \quad \text{and} \quad \eta = \begin{cases} 1, & y \in B_R(0), \\ 0, & y \in \mathbb{R}^2 \backslash B_{2R}(0), \end{cases}$$

where $R > 0$ is a constant. Upon multiplying (3.5) by $g_{j+}^N \eta^2$ with N being determined later on, we integrate it over $\mathbb{R}^2$ to get

$$0 = \int_{\mathbb{R}^2} \nabla \cdot (U_j \nabla g_j) g_{j+}^N \eta^2 dy = - \int_{\mathbb{R}^2} (U_j \nabla g_j) \cdot \nabla (g_{j+}^N \eta^2) dy,$$

which implies

$$0 = \int_{\mathbb{R}^2} U_j \nabla g_j \cdot \nabla (g_{j+}^N) \eta^2 dy + \int_{\mathbb{R}^2} (U_j \nabla g_j) \cdot g_{j+}^N \nabla (\eta^2) dy.$$

Noting that the support of η is $B_{2R}(0)$, one further has

$$\int_{B_{2R}(0)} U_j \nabla g_j \cdot \nabla (g_{j+}^N) \eta^2 dy = - \int_{\mathbb{R}^2} (U_j \nabla g_j) \cdot g_{j+}^N \nabla (\eta^2) dy.$$

Then we utilize the integration by parts to get

$$\frac{4N}{(N+1)^2}\int_{B_{2R}(0)} U_j\big|\nabla g_{j+}^{\frac{N+1}{2}}\big|^2 dy = \frac{1}{N+1}\int_{B_{2R}(0)} g_{j+}^{N+1}\nabla\cdot(U_j\nabla\eta^2)dy.$$

Therefore,

$$\int_{B_{2R}(0)} U_j\big|\nabla g_{j+}^{\frac{N+1}{2}}\big|^2 dy \leq C\int_{B_{2R}(0)} |g_{j+}|^{N+1}\cdot|\nabla\cdot(U_j\nabla\eta^2)|dy, \tag{3.6}$$

where $C>0$ is some large constant. Since g_j satisfies $|g_j|\leq C_1\frac{1}{U_j}(1+r)^{-\sigma-2}$ for small $\sigma>0$ and some constant $C_1>0$, one has from (3.6) that

$$\int_{B_{2R}(0)} U_j\big|\nabla_y g_{j+}^{\frac{N+1}{2}}\big|^2 dy \leq C_2(2R)^2\Big(\frac{1}{U_j^N}\Big|_{|y|=2R}\Big)\frac{1}{(1+2R)^{(N+1)\sigma+2N+2}}, \tag{3.7}$$

where $C_2>0$ is a constant. Then we obtain for some positive constant C_3,

$$4R^2\Big(\frac{1}{U_j^N}\Big|_{|y|=2R}\Big)\leq C_3(1+2R)^{m_jN+2}.$$

By choosing $N=\frac{\sigma}{2(m_j-\sigma-2)}$, one uses (3.7) to get

$$\int_{B_{2R}(0)} U_j\big|\nabla_y(g_{j+}^{\frac{N+1}{2}})|^2 dy \leq C_2C_3(1+2R)^{m_jN+2}\frac{1}{(1+2R)^{(N+1)\sigma+2N+2}}\to 0 \text{ as } R\to\infty,$$

where σ can be chosen small enough. Thus, we obtain $g_{j+}\equiv C_4$ for $y_m\in\mathbb{R}^2$, where $C_4>0$ is a constant. Proceeding with the similar argument for $g_{j-}^N\eta^2$, we obtain $g_{j-}\equiv C_5$ for some constant $C_5>0$. This completes the proof of the lemma. □

Next, we analyze the non-degeneracy of the linearized operators of Liouville systems. Define

$$\mathcal{L}[w_1,w_2]=\Big(\Delta w_1+\sum_{j=1,2} b_{1j}U_jw_j, \Delta w_1+\sum_{j=1,2} b_{2j}U_jw_j\Big)^T.$$

We have its adjoint operator is

$$\mathcal{L}^*[w_1,w_2]=\Big(\Delta w_1+U_1\sum_{j=1,2} b_{1j}w_j, \Delta w_1+U_2\sum_{j=1,2} b_{2j}w_j\Big)^T.$$

For convenience, we denote the following transform

$$(\log U^1,\log U^2)=\Big(\sum_{j=1,2} b^{1j}\log U_j, \sum_{j=1,2} b^{2j}\log U_j\Big), \tag{3.8}$$

where $(b^{ij})_{2\times 2}$ is the inverse matrix of $(b_{ij})_{2\times 2}$. For the kernel of $\mathcal{L}(w_1,w_2)$, we have the following lemma.

Lemma 3.2. *Assume that $(w_1,w_2)^T$ satisfies*

$$\begin{cases}\mathcal{L}[w_1,w_2]=0 \quad in \quad \mathbb{R}^2,\\ |w_j(y)|\leq C(1+|y|^\tau), \quad for\ some\ \tau\in[0,1)\ with\ j=1,2,\end{cases} \tag{3.9}$$

then we have

$$(w_1,w_2)^T\in span\Big\{(\partial_1\log U_1,\partial_1\log U_2)^T,(\partial_2\log U_1,\partial_2\log U_2)^T,(\partial_3\log U_1,\partial_3\log U_2)^T\Big\}, \tag{3.10}$$

where $\partial_1\log U_j=\partial_{x_1}\log U_j$, $\partial_2\log U_j=\partial_{x_2}\log U_j$ and $\partial_3\log U_j=r(\log U_j)'(r)+2$ for $j=1,2$.

Proof. See the proof of Theorem 2.1 in [16] and Lemma 3.1 in [15]. □

In addition, for the kernel of adjoint operator $\mathcal{L}^*$, we have the following result.

Lemma 3.3. *Assume* $(w_1, w_2)^T$ *is a solution to*

$$\begin{cases} \mathcal{L}^*[w_1, w_2] = 0 \quad in \quad \mathbb{R}^2, \\ |w_j(z)| \le C(1 + |z|^\tau), \quad with \quad j = 1, 2, \end{cases} \tag{3.11}$$

for some $\tau \in [0, 1)$*, then we have*

$$(w_1, w_2)^T \in span\Big\{(\partial_1 \log U^1, \partial_1 \log U^2)^T, (\partial_2 \log U^1, \partial_2 \log U^2)^T, (\partial_3 \log U^1, \partial_3 \log U^2)^T\Big\}, \tag{3.12}$$

where U^1 *and* U^2 *are defined in (3.8).*

Proof. See the proof of Corollary 5.2 in [12]. □

With the aid of Lemma 3.1 and Lemma 3.3, one finds

Lemma 3.4. *Suppose that* h_j *satisfy*

$$\int_{\mathbb{R}^2} h_j(y)dy = 0, \int_{\mathbb{R}^2} h_j(y)y_l dy = 0 \;\; for \;\; j = 1, 2, \; l = 1, 2, \tag{3.13}$$

then we have for any $\|h_j\|_{4+\sigma} < \infty$ *with* $\sigma \in (0, 1)$*, there exist a solution* $\boldsymbol{\varphi} := (\varphi_1, \varphi_2)^T = \boldsymbol{\mathcal{T}_{in}}[h_1, h_2]$ *to* (3.3) *such that*

$$\|\varphi_j\|_{2+\sigma} \le C_j \|h_j\|_{4+\sigma}, \tag{3.14}$$

where $\boldsymbol{\mathcal{T}_{in}}[h_1, h_2]$ *is a continuous linear operator from the Banach space* $C^* \times C^*$ *of all functions* $(h_1, h_2)^T$ *in* $L^\infty \times L^\infty$ *for which* $\|h_1\|_{4+\sigma} + \|h_2\|_{4+\sigma} < +\infty$ *into* $L^\infty \times L^\infty$.

Proof. Similarly as shown in [14], we perform Fourier projection and obtain the k-th mode of (3.3) as follows

$$\begin{cases} \dfrac{d^2\tilde{g}_{1k}}{dr^2} + \dfrac{1}{r}\dfrac{d\tilde{g}_{1k}}{dr} - \dfrac{k^2}{r^2}\tilde{g}_{1k} + (\log U_1)_r \dfrac{d\tilde{g}_{1k}}{dr} + U_1 \tilde{g}_{1k} = h_{1k}, \\ \dfrac{d^2\tilde{g}_{2k}}{dr^2} + \dfrac{1}{r}\dfrac{d\tilde{g}_{2k}}{dr} - \dfrac{k^2}{r^2}\tilde{g}_{2k} + (\log U_2)_r \dfrac{d\tilde{g}_{2k}}{dr} + U_2 \tilde{g}_{2k} = h_{2k}, \\ \dfrac{d^2 w_{1k}}{dr^2} + \dfrac{1}{r}\dfrac{dw_{1k}}{dr} - \dfrac{k^2}{r^2} w_{1k} + b_{11}U_1 w_{1k} + b_{11}\tilde{g}_{1k} + b_{12}U_2 w_{2k} + b_{12}\tilde{g}_{2k} = 0, \\ \dfrac{d^2 w_{2k}}{dr^2} + \dfrac{1}{r}\dfrac{dw_{2k}}{dr} - \dfrac{k^2}{r^2} w_{2k} + b_{11}U_1 w_{2k} + b_{11}\tilde{g}_{1k} + b_{12}U_2 w_{2k} + b_{12}\tilde{g}_{2k} = 0, \end{cases} \tag{3.15}$$

where we define $U_j g_j = \tilde{g}_j$ and

$$\tilde{g}_j = \sum_{k=0}^{\infty} \tilde{g}_{jk}(r)e^{ik\theta}, \;\; h_j = \sum_{k=0}^{\infty} h_j(r)e^{ik\theta}, \;\; w_j = \sum_{k=0}^{\infty} w_{jk}(r)e^{ik\theta}, \;\; j = 1, 2. \tag{3.16}$$

First of all, we consider the 0-th mode in (3.15), which is

$$\begin{cases} \dfrac{d^2\tilde{g}_{10}}{dr^2} + \dfrac{1}{r}\dfrac{d\tilde{g}_{10}}{dr} + (\log U_1)_r \dfrac{d\tilde{g}_{10}}{dr} + U_1 \tilde{g}_{10} = h_{10}, \\ \dfrac{d^2\tilde{g}_{20}}{dr^2} + \dfrac{1}{r}\dfrac{d\tilde{g}_{20}}{dr} + (\log U_2)_r \dfrac{d\tilde{g}_{20}}{dr} + U_2 \tilde{g}_{20} = h_{20}, \\ \dfrac{d^2 w_{10}}{dr^2} + \dfrac{1}{r}\dfrac{dw_{10}}{dr} + b_{11}U_1 w_{10} + b_{12}U_2 w_{20} = f_{10}, \\ \dfrac{d^2 w_{20}}{dr^2} + \dfrac{1}{r}\dfrac{dw_{20}}{dr} + b_{21}U_1 w_{10} + b_{22}U_2 w_{20} = f_{20}, \end{cases} \tag{3.17}$$

where

$$f_{j0} := -b_{j1}\tilde{g}_{10} - b_{j2}\tilde{g}_{20}, \;\; j = 1, 2. \tag{3.18}$$

We choose the solution to the $\tilde{g}_j$-equation in (3.17) as

$$\tilde{g}_j = U_j g_{j0}, \;\; g_{j0} = \int_r^\infty \frac{1}{\rho U_j(\rho)} \int_0^\rho h_{j0}(s) s \, ds \, d\rho. \tag{3.19}$$

By using the mass condition in (3.13), one further has

$$0 = \int_{\mathbb{R}^2} h_j \, dy = \int_0^{2\pi} \int_0^\infty \sum_{k=0}^\infty h_{jk} e^{ik\theta} r \, dr \, d\theta = 2\pi \int_0^\infty h_{j0}(r) r \, dr.$$

Then it follows

$$g_{j0} \sim \langle r \rangle^{m_j - 2 - \sigma}, \;\; \tilde{g}_{j0} \sim \langle r \rangle^{-2-\sigma}, \;\; \text{for } \sigma > 0 \text{ small enough.} \tag{3.20}$$

Next, we shall solve (w_{10}, w_{20}) in (3.17) via the variation-of-parameter method. To begin with, we focus on the following homogeneous problem

$$\begin{cases} \dfrac{d^2 w_{10}}{dr^2} + \dfrac{1}{r}\dfrac{dw_{10}}{dr} + b_{11} U_1 w_{10} + b_{12} U_2 w_{20} = 0, \\ \dfrac{d^2 w_{20}}{dr^2} + \dfrac{1}{r}\dfrac{dw_{20}}{dr} + b_{11} U_1 w_{10} + b_{12} U_2 w_{20} = 0. \end{cases} \tag{3.21}$$

By using Lemma 2.1 of [15], we have there exist two linearly independent solution pairs $\mathbf{Z}_j = (Z_{j1}, Z_{j2})^T$ with $j = 1, 2$ of (3.21), which satisfy

$$Z_{j1} = O(\log(1 + r)), \quad Z_{j2} = O(\log(1 + r)). \tag{3.22}$$

We further rewrite the equation (3.21) as

$$\mathcal{L}_0[\boldsymbol{W}_0] = \boldsymbol{f}_0, \tag{3.23}$$

where $\boldsymbol{W}_0 = (w_{10}, w_{20})^T$, $\boldsymbol{f}_0 = (f_{10}, f_{20})^T$,

$$\mathcal{L}_0 = \text{diag}(\Delta_r, \Delta_r) + \boldsymbol{A}, \;\; \mathcal{L}_0^* = \text{diag}(\Delta_r, \Delta_r) + \boldsymbol{A^T}, \tag{3.24}$$

and

$$\boldsymbol{A} = \begin{pmatrix} b_{11} U_1 & b_{12} U_2 \\ b_{21} U_1 & b_{22} U_2 \end{pmatrix}. \tag{3.25}$$

Next, we are concerned with the following homogeneous adjoint problem

$$\mathcal{L}_0^*[\mathbf{Z}_j^*] = 0, \tag{3.26}$$

where $\mathcal{L}_0^*$ is defined in (3.24). We claim that (3.26) admits two linear independent solution pairs $\mathbf{Z}_j^* = (Z_{j1}^*, Z_{j2}^*)$ with $j = 1, 2$ and they satisfy

$$Z_{j1}^* = O(\log(1 + r)) \text{ and } Z_{j2}^* = O(\log(1 + r)). \tag{3.27}$$

To show this, we rewrite (3.26) as

$$\begin{cases} \dfrac{d^2 w_1}{dr^2} + \dfrac{1}{r}\dfrac{dw_1}{dr} + b_{11} U_1 w_1 + b_{21} U_1 w_2 = 0, \\ \dfrac{d^2 w_2}{dr^2} + \dfrac{1}{r}\dfrac{dw_2}{dr} + b_{12} U_2 w_1 + b_{22} U_2 w_2 = 0. \end{cases} \tag{3.28}$$

By using the transform $\hat{w}_j(r) = w_j(e^r)$ with $j = 1, 2$, one finds

$$\frac{d^2\hat{w}_j}{dr^2} + \Big(\sum_{k=1,2} a_{kj}\hat{w}_k\Big)U_j(e^r)e^{2r} = 0. \tag{3.29}$$

Noting that $U_j(r) \sim \langle r\rangle^{-m_j}$ with $m_j > 2$, we follow the similar argument shown in Lemma 2.1 of [15] and obtain that there exists a solution (w_1, w_2) satisfying

$$w_j = O(\log(1 + r)),\ \ j = 1, 2,$$

which proves our claim. Now, we are ready to solve the inhomogeneous problem (3.23). In fact, by applying the integration by parts, one gets

$$\begin{aligned} &\int_{B_r(0)} \mathcal{L}_0(\boldsymbol{W}_0)\cdot \mathbf{Z}_j^*\,dx - \int_{B_r(0)} \mathcal{L}_0^*(\mathbf{Z}_j^*)\cdot \boldsymbol{W}_0\,dx \\ &= \int_{\partial B_r(0)} \frac{\partial \boldsymbol{W}_0}{\partial \boldsymbol{\nu}}\cdot \mathbf{Z}_j^*\,dS - \int_{\partial B_r(0)} \frac{\partial \mathbf{Z}_j^*}{\partial \boldsymbol{\nu}}\cdot \boldsymbol{W}_0\,dS,\ \ j = 1, 2, \end{aligned} \tag{3.30}$$

where $\mathbf{Z}_j^* := (Z_{j1}^*, Z_{j2}^*)^T$ are given in (3.27). It follows that $\boldsymbol{W}_0$ satisfies the following first order ODEs

$$\boldsymbol{W}_0' = H(r)\boldsymbol{W}_0 + \hat{\boldsymbol{f}}_0, \tag{3.31}$$

where

$$H := \begin{pmatrix} Z_{11}^* & Z_{12}^* \\ Z_{21}^* & Z_{22}^* \end{pmatrix}^{-1} \begin{pmatrix} \frac{dZ_{11}^*}{dr} & \frac{dZ_{12}^*}{dr} \\ \frac{dZ_{21}^*}{dr} & \frac{dZ_{22}^*}{dr} \end{pmatrix},\ \ \hat{\boldsymbol{f}}_0 := \frac{1}{2\pi r}\begin{pmatrix} Z_{11}^* & Z_{12}^* \\ Z_{21}^* & Z_{22}^* \end{pmatrix}^{-1} \begin{pmatrix} \int_{B_r}(f_{10}Z_{11}^* + f_{20}Z_{12}^*)\,dx \\ \int_{B_r}(f_{10}Z_{21}^* + f_{20}Z_{22}^*)\,dx \end{pmatrix}.$$

In light of (3.20), one finds

$$\int_{B_r(0)} (f_{10}Z_{j1}^* + f_{20}Z_{j2}^*)\,dx \sim \langle r\rangle^{-\sigma}, \tag{3.32}$$

where $\sigma > 0$ is small enough. Thus, we have

$$\hat{\boldsymbol{f}}_0 \sim o\Big(\frac{1}{r}\Big).$$

Therefore, we choose a solution to (3.31) as

$$\boldsymbol{W}_0 = t_1\mathbf{Z}_1 + t_2\mathbf{Z}_2,$$

where $t_j = O(\log(1 + r))$, $\mathbf{Z}_1 = (Z_{11}, Z_{12})^T$ and $\mathbf{Z}_2 = (Z_{21}, Z_{22})^T$ given by (3.22) are the fundamental solutions of (3.31). Moreover, we have the mode 0 of the φ_j–component in (3.3) exists and satisfies

$$\varphi_{j0} = U_j w_{j0} + \tilde{g}_{j0} \sim \langle r\rangle^{-2-\sigma},\ \ j = 1, 2,$$

where $\sigma > 0$ is small enough.

We next focus on the mode $k \geq 1$ in (3.15). For the $\tilde{g}_j$-equation in (3.15), we define

$$\mathcal{L}_{jk}[z] := -\frac{d^2z}{dr^2} - \frac{1}{r}\frac{dz}{dr} + \frac{k^2}{r^2}z - (\log U_j)_r\frac{dz}{dr} - U_j z, \tag{3.33}$$

and construct the barrier function

$$z_{jk} := C_{jk}\|h_j\|_{4+\sigma}(1 + r)^{-2-\sigma}, \tag{3.34}$$

where σ is small enough. Then, we compute to get

$$\mathcal{L}_{jk}[z_{jk}] + h_{jk} = \frac{C_{jk}\|h_j\|_{4+\sigma}}{(1+r)^{4+\sigma}}\Big[-(2+\sigma)(3+\sigma) + \frac{1+r}{r}(2+\sigma) + \frac{k^2}{r^2}(1+r)^2 - U_{j0}(1+r)^2 - (\log U_j)_r(1+r)(2+\sigma)\Big] + h_{jk}.$$

Since

$$(\log U_j)_r \sim -\frac{m_j}{r}, \text{ for } r \gg 1,\ m_j > 2, \tag{3.35}$$

we choose positive constant R large enough such that

$$\mathcal{L}_{jk}[z_{jk}] + h_k > 0 \ \text{ for } \ r > R.$$

In addition, with the fixed $R > 0$, we set a large constant $C_{jk} > 0$ to obtain

$$\frac{C_{jk}\|h_j\|_{4+\sigma}}{(1+R)^{2+\sigma}} - \max_{y\in B_R(0)} \tilde{g}_{jk} > 0, \tag{3.36}$$

where $\tilde{g}_{jk}$ is bounded in $B_R(0)$. Then, we apply the maximum principle to get

$$\tilde{g}_{jk} \le \frac{C_{jk}\|h_j\|_{4+\sigma}}{(1+r)^{2+\sigma}}. \tag{3.37}$$

Similarly, we have

$$\mathcal{L}_{jk}[-z_{jk}] + h_{jk} < 0 \ \text{ for } \ r > R,$$

which implies

$$-\tilde{g}_{jk} \le \frac{C_{jk}\|h_j\|_{4+\sigma}}{(1+r)^{2+\sigma}}, \tag{3.38}$$

where the maximum principle was used.

One collects (3.37) and (3.38) to get

$$\|\tilde{g}_{jk}\|_{L^\infty(\mathbb{R}^2)} \le \frac{C_{jk}\|h_j\|_{4+\sigma}}{(1+r)^{2+\sigma}} \quad \text{in} \quad \mathbb{R}^2. \tag{3.39}$$

On the other hand, the existence of $\tilde{g}_{jk}$ follows from Fredholm alternative theorem since Lemma 3.1 implies mode $k \ge 1$ of $\tilde{g}_{jk}$-equation in (3.15) does not admit any nontrivial kernel.

For the w_{jk}-equation with $k \ge 1$ in (3.15), we first consider mode 1 and focus on the following equation

$$\begin{cases} \dfrac{d^2 w_{11}}{dr^2} + \dfrac{1}{r}\dfrac{dw_{11}}{dr} - \dfrac{1}{r^2}w_{11} + b_{11}U_1 w_{11} + b_{12}U_2 w_{21} = f_{11}, \\ \dfrac{d^2 w_{21}}{dr^2} + \dfrac{1}{r}\dfrac{dw_{21}}{dr} - \dfrac{1}{r^2}w_{21} + b_{21}U_1 w_{11} + b_{22}U_2 w_{21} = f_{21}, \end{cases} \tag{3.40}$$

where

$$f_{j1} := -b_{j1}\tilde{g}_{11} - b_{j2}\tilde{g}_{21}, \ j = 1, 2. \tag{3.41}$$

With the aid of Lemma 3.3, we find (3.40) only admits one bounded kernel $(\partial_r \log U_1, \partial_r \log U_2)^T$. Next, we shall show the existence of (w_{11}, w_{21}) to (3.40). To this end, we follow the argument shown in the proof of Lemma 2.3 in [1] and define

$$X_\alpha := \Big\{ \boldsymbol{u} \in L^2_{\text{loc}}(\mathbb{R}^2) \times L^2_{\text{loc}}(\mathbb{R}^2) \Big| \int_{\mathbb{R}^2} (1 + r^{2+\alpha})|\boldsymbol{u}|^2\, dx < +\infty \Big\}, \tag{3.42}$$

$$Y_\alpha := \left\{ \boldsymbol{u} \in W^{2,2}_{\text{loc}}(\mathbb{R}^2) \times W^{2,2}_{\text{loc}}(\mathbb{R}^2) \middle| \int_{\mathbb{R}^2} (1 + r^{2+\alpha})|\Delta \boldsymbol{u}|^2 + \frac{|\boldsymbol{u}|^2}{(1 + r^{2+\alpha})}\, dx < +\infty \right\} \tag{3.43}$$

for some $\alpha > 0$. Moreover, we denote

$$\mathcal{L}_1 = \text{diag}\Big(\Delta_r - \frac{1}{r^2}, \Delta_r - \frac{1}{r^2}\Big) + \boldsymbol{A}, \tag{3.44}$$

with $\boldsymbol{A}$ defined by (3.25) and consider

$$\mathcal{L}_1 : Y_\alpha \to X_\alpha. \tag{3.45}$$

As shown in [1], one finds $\mathcal{L}_1$ is a bounded linear operator and has a closed range in X_α for $\alpha \in (0, \frac{1}{2})$. It follows that X_α can be decomposed as

$$X_\alpha = \text{Im}\mathcal{L}_1 \oplus (\text{Im}\mathcal{L}_1)^\perp. \tag{3.46}$$

As a consequence, let $\boldsymbol{\phi} \in (\text{Im}\mathcal{L}_1)^\perp$, then we have $(\mathcal{L}_1[\boldsymbol{W}], \boldsymbol{\phi})_{X_\alpha} = 0$, $\forall \boldsymbol{W} \in Y_\alpha$, or equivalently,

$$(\mathcal{L}_1[\boldsymbol{W}], \boldsymbol{\varphi})_{L^2(\mathbb{R}^2)} = 0, \tag{3.47}$$

where $\boldsymbol{\varphi} = (1 + r^{2+\alpha})\boldsymbol{W}$. Thus, $\mathcal{L}_1^* \boldsymbol{\varphi} = 0$ in $\mathbb{R}^2$ with

$$\int_{\mathbb{R}^2} \frac{|\boldsymbol{\varphi}|^2(x)}{1 + |x|^{2+\alpha}} dx < +\infty.$$

Then we apply Green's formula to get

$$|\boldsymbol{\varphi}(z)| = O(1 + \log|z|).$$

With this, by using Lemma 3.3, we have $\boldsymbol{\varphi} \in \text{span}\{(\partial_r \log U^1, \partial_r \log U^2)^T\}$. Thus,

$$(\text{Im}\mathcal{L}_1)^\perp \subseteq \text{span}\{(\partial_r \log U_1, \partial_r \log U_2)^T\}.$$

In addition, we use the integration by parts and the fact that $\partial_r \log U_j \to 0$ as $r \to +\infty$ to get

$$\text{span}\{(\partial_r \log U^1, \partial_r \log U^2)^T\} \subseteq (\text{Im}\mathcal{L}_1)^\perp.$$

Therefore, we obtain

$$(\text{Im}\mathcal{L}_1) = \text{span}\{(\partial_r \log U^1, \partial_r \log U^2)^T\}^\perp. \tag{3.48}$$

Next, we claim that

$$\int_{\mathbb{R}^2} (\boldsymbol{f}_1 \cdot \boldsymbol{Z}^*) dy = 0, \tag{3.49}$$

where $\boldsymbol{Z}^* := (\partial_r \log U^1, \partial_r \log U^2)^T$ given in (3.8) and $\boldsymbol{f}_1 := (f_{11}, f_{21})^T$ defined in (3.41). Indeed, by testing y_i, $i = 1, 2$ against the g_j-equation in (3.3), one has

$$\int_{\mathbb{R}^2} \nabla \cdot (U_j \nabla g_j) y_i dy = \int_{\mathbb{R}^2} h_j y_i dy = 0. \tag{3.50}$$

Moreover, the left hand side of (3.50) can be written as

$$\int_{\mathbb{R}^2} \nabla \cdot (U_j \nabla g_j) y_i dy = -\int_{\mathbb{R}^2} U_j \nabla g_j \cdot e_i \text{dy} = \int_{\mathbb{R}^2} g_j U_j \nabla \log U_j \cdot e_i \text{dy}, \tag{3.51}$$

where $e_1 = (1,0)$ and $e_2 = (0,1)$. For $i = 1$, we further calculate to get

$$\begin{aligned}\int_{\mathbb{R}^2} g_j U_j \nabla \log U_j \cdot e_1 dy &= \int_0^{2\pi}\int_0^{\infty}\sum_{j=1}^{\infty} g_{jk}(r)e^{ik\theta}U_j(r)(\partial_r \log U_j)(r) r\cos\theta drd\theta \\ &= \pi\int_0^{\infty} g_{j1}(r)U_j(r)(\partial_r \log U_j)(r)rdr.\end{aligned} \tag{3.52}$$

Collecting (3.50), (3.51) and (3.52), one obtains

$$\int_0^{\infty} g_{j1}(r)U_j(r)(\partial_r \log U_j)rdr = 0.$$

Then we invoking the definition of $\mathbf{Z}^*$ in Lemma 3.3 and (3.41) to finish the proof of our claim. Thanks to (3.49), we apply Fredholm alternative theorem to get that there exists a solution $\boldsymbol{W}_1 \in Y_\alpha$ to problem (3.40).

Next, we derive the estimate of $\boldsymbol{W}_1$. Define

$$z_{j1} = \bar{C}_{j1}\|h_j\|_{4+\sigma}(1+r)^{\sigma} + \delta r,\ \ j = 1,2,$$

and

$$\bar{\mathcal{L}}_{11}[z] = -\frac{d^2}{dr^2}z - \frac{1}{r}\frac{d}{dr}z + \frac{1}{r^2}z - b_{11}U_1 z - b_{12}U_2 w_{21}, \tag{3.53}$$

$$\bar{\mathcal{L}}_{21}[z] = -\frac{d^2}{dr^2}z - \frac{1}{r}\frac{d}{dr}z + \frac{1}{r^2}z - b_{21}U_1 w_{11} - b_{22}U_2 z, \tag{3.54}$$

where $\bar{C}_{j1} > 0$ are some large constants and $\delta > 0$ is a small constant. Then we compute to get

$$\begin{aligned}\bar{\mathcal{L}}_{j1}[z_{j1}] + f_{j1} =& \bar{C}_{j1}\|h_j\|_{4+\sigma}(1+r)^{\sigma-2}\Big[\sigma(1-\sigma) - \frac{1}{r}\sigma(1+r) + \frac{1}{r^2}(1+r)^2 - b_{jj}U_j(1+r)^2\Big] \\ &- 2b_{ij,i\neq j}U_i w_{i1} + f_{j1} - b_{11}\delta U_1 r.\end{aligned} \tag{3.55}$$

Noting that $(w_{11}, w_{21})^T \in X_\alpha$ and $U_j \sim -\frac{m_j}{r}$ for r large and $m_j > 2$, we choose $\bar{C}_{j1} > 0$ and $R_j > 0$ large enough to obtain

$$\bar{\mathcal{L}}_{j1}[z_{j1}] + f_{j1} > 0,\ \text{for } r > R_j.$$

With the fixed $R_j > 0$, we further set a large constant $\bar{C}_{j1} > 0$ to get

$$\bar{C}_{j1}\|h_j\|_{4+\sigma}(1+R)^{\sigma} - \max_{y\in B_R(0)} w_{j1} > 0, \tag{3.56}$$

where w_{j1} is bounded in $B_R(0)$. By using the maximum principle on annulus $B_{\tilde{R}_j}(0)\backslash B_{R_j}(0)$, we have

$$w_{j1} \leq \bar{C}_{j1}\|h_j\|_{4+\sigma}(1+r)^{\sigma}, \tag{3.57}$$

where we let $\tilde{R}_j \to +\infty$ then $\delta \to 0$. Similarly, we apply the maximum principle into $-w_{j1}$ and compute to get

$$|w_{j1}| \leq \bar{C}_{j1}\|h_j\|_{4+\sigma}(1+r)^{\sigma}, \tag{3.58}$$

where (3.57) was used.

It remains to analyze mode $k \geq 2$ of the linearization of the w_j-equations with $j = 1,2$ shown in (3.3). As stated in Lemma 3.3, there is not any nontrivial kernel to the mode-k equations with $k \geq 2$, which satisfies $w_j \leq C(1+|z|^{\tau})$ for some $\tau \in [0,1)$. Similarly as above, we consider

$$\mathcal{L}_k : Y_\alpha \to X_\alpha, \tag{3.59}$$

where

$$\mathcal{L}_k := \mathrm{diag}\Big(\Delta_r - \frac{k^2}{r^2}, \Delta_r - \frac{k^2}{r^2}\Big) + \boldsymbol{A} \tag{3.60}$$

with $\boldsymbol{A}$ defined by (3.25). It can be shown that $\mathcal{L}_k$ is a bounded linear operator and has a closed range in X_α for $\alpha \in (0, \frac{1}{2})$. Thus, X_α can be decomposed as

$$X_\alpha = \mathrm{Im}\mathcal{L}_k \oplus (\mathrm{Im}\mathcal{L}_k)^\perp. \tag{3.61}$$

Similarly, we prove that $(\mathrm{Im}\mathcal{L}_k)^\perp = \emptyset$ by using Lemma 3.3. Then we apply the Fredholm alternative theorem to obtain there exists a solution $\boldsymbol{W}_k := (w_{1k}, w_{2k})^T$ satisfying

$$\begin{cases} \dfrac{d^2 w_{1k}}{dr^2} + \dfrac{1}{r}\dfrac{dw_{1k}}{dr} - \dfrac{k^2}{r^2} w_{1k} + b_{11} U_1 w_{1k} + b_{12} U_2 w_{2k} = f_{1k}, \\ \dfrac{d^2 w_{2k}}{dr^2} + \dfrac{1}{r}\dfrac{dw_{2k}}{dr} - \dfrac{k^2}{r^2} w_{2k} + b_{21} U_1 w_{1k} + b_{22} U_2 w_{2k} = f_{2k}. \end{cases} \tag{3.62}$$

We next establish the estimate of $\boldsymbol{W}_k \in Y_\alpha$. To this end, we define

$$\bar{\mathcal{L}}_{jk}[z] = -\frac{d^2}{dr^2} z - \frac{1}{r}\frac{d}{dr} + \frac{k^2}{r^2} z - b_{jj} U_j z - b_{i\neq j} U_i w_{ik}, \ \ j = 1, 2, \tag{3.63}$$

and

$$z_{jk} := \bar{C}_{jk} \|h_j\|_{4+\sigma} (1 + r)^\sigma + \delta_k r, \tag{3.64}$$

where constant $\sigma > 0$, $\delta_k > 0$ are small and $\bar{C}_{jk} > 0$ is large. By the direct computation, we use the maximum principle, choose $\bar{C}_{jk} > 0$ large enough and take $\delta_k \to 0$ such that

$$|w_{jk}| \leq \bar{C}_{jk} \|h_j\|_{4+\sigma} (1 + r)^\sigma, \tag{3.65}$$

where $\sigma > 0$ is sufficiently small. The existence of $(w_{1k}, w_{2k})^T$ to (3.62) directly follows from the invertibility of $\mathcal{L}_k$.

Recall that for $j = 1, 2$ and $k \in \mathbb{N}$,

$$\phi_{jk} = \tilde{g}_{jk} + U_j \psi_{jk}. \tag{3.66}$$

Thus, we have there exist ϕ_{jk} satisfying

$$|\phi_{jk}| \leq C_{jk} \frac{1}{(1 + r)^{2+\sigma}}, \tag{3.67}$$

where $\sigma \in (0, 1)$ small. This completes the proof of our lemma. □

In Lemma 3.4, we establish the existence and a-priori estimate of (φ_1, φ_2) to inner problem (3.3), which corresponds to the linearization around the interior spots to (1.3). Whereas, if the center is at the boundary $\partial\Omega$, we must solve the inner problem (3.3) in the half space $\mathbb{R}^2_+ = \{(y_1, y_2) \in \mathbb{R}^2 : y_2 \geq 0\}$ rather than $\mathbb{R}^2$. To this end, we define norm $\|\cdot\|_{\nu,H}$ as

$$\|h\|_{\nu,H} := \sup_{y \in \mathbb{R}^2_+} |h|(1 + |y|)^\nu, \ \ \nu > 0,$$

and develop the following solvability results.

Lemma 3.5. *Given any function h_j, $j = 1, 2$ and $\beta_j(x)$ satisfying*

$$\int_{\mathbb{R}^2_+} h_j dy - \int_{\partial\mathbb{R}^2_+} \beta_j dS_y = 0, \quad \int_{\mathbb{R}^2_+} h_j y_1 dy - \int_{\partial\mathbb{R}^2_+} \beta_j y_1 dS_y = 0, \tag{3.68}$$

and $\|h_j\|_{4+\sigma,H} < \infty$ with $\sigma \in (0, 1)$, we have the problem

$$\begin{cases} L_k^{inn}[\varphi_1, \varphi_2] = (h_1, h_2)^T & \text{in } \mathbb{R}^2_+, \\ U_j \dfrac{\partial g_j}{\partial \boldsymbol{\nu}} = \beta_j(x) & \text{on } \partial\mathbb{R}^2_+, \ j = 1, 2 \end{cases} \tag{3.69}$$

admits a solution (φ_1, φ_2) satisfying the following estimate:

$$\|\varphi_j\|_{2+\sigma,H} \le C_j \|h_j\|_{4+\sigma,H}, \tag{3.70}$$

where $C_j > 0$ is a constant and $g_j = \frac{\varphi_j}{U_j} - w_j$. Moreover, $\boldsymbol{\varphi}$ satisfies $(\varphi_1, \varphi_2)^T = \hat{\mathcal{T}}_p[h_1, h_2]$, where $\hat{\mathcal{T}}_p[h_1, h_2]$ is defined by a linear operator.

Proof. For any given (β_1, β_2) defined on $\partial\mathbb{R}^2_+ \times \partial\mathbb{R}^2_+$, we have there exists a function pair $(\tilde{\varphi}_j, \tilde{w}_j)$ such that

$$\frac{\partial \tilde{\varphi}_j}{\partial \boldsymbol{\nu}} - U_j \frac{\partial \tilde{w}_j}{\partial \boldsymbol{\nu}} = \beta_j \ \text{ on } \ \partial\mathbb{R}^2_+,$$

where $\|\tilde{\varphi}\|_{2+\sigma,H} \le C\|h_j\|_{4+\sigma,H}$, $j = 1, 2$. Then, we define $\vartheta_j := \frac{\varphi_j}{U_j} - w_j$ and find ϑ_j satisfies

$$\int_{\mathbb{R}^2_+} U_j \nabla \vartheta_j \cdot \boldsymbol{e}_1 dy = 0, \tag{3.71}$$

where $\boldsymbol{e}_1 = (1, 0)$. Now, the problem (3.69) is transformed into the following form

$$\begin{cases} \nabla \cdot (U_j \nabla \bar{g}_j) = h_j - \nabla \cdot (U_j \nabla \vartheta_j) & \text{in } \mathbb{R}^2_+ \\ U_j \dfrac{\partial \bar{g}_j}{\partial \boldsymbol{\nu}} = 0 & \text{on } \partial\mathbb{R}^2_+, \ j = 1, 2, \end{cases} \tag{3.72}$$

where $\bar{g}_j := g_j - \vartheta_j$. Define the solution of (3.72) as $(\phi_\vartheta, \psi_\vartheta)$ and

$$\bar{g}_j := \begin{cases} \bar{g}_j(y_1, y_2) & \text{for } y_2 \ge 0; \\ \bar{g}_j(y_1, -y_2) & \text{for } y_2 < 0, \end{cases}$$

then we have the equation in (3.72) is evenly extended into the whole space, which is

$$\nabla \cdot (U_j \nabla \bar{g}_j) = \tilde{h}_j \ \text{ in } \ \mathbb{R}^2, \ j = 1, 2,$$

where

$$\tilde{h}_j(y_1, y_2) = \begin{cases} h_j(y_1, y_2) - \nabla \cdot (U_j \nabla \vartheta_j)(y_1, y_2) & \text{for } y_2 \ge 0, \\ h_j(y_1, -y_2) - \nabla \cdot (U_j \nabla \vartheta_j)(y_1, -y_2) & \text{for } y_2 < 0. \end{cases}$$

It is easy to check that $\|\tilde{h}_j\|_{4+\sigma} < \infty$ due to $\|h_j\|_{4+\sigma,H} < +\infty$. The key step is the verification of the orthogonality condition. To finish it , we first obtain from the property of even function that

$$\int_{\mathbb{R}^2_-} h_j(y_1, -y_2) - \nabla \cdot (U_j \nabla \vartheta_j)(y_1, -y_2) dy = \int_{\mathbb{R}^2_+} h_j(y_1, y_2) - \nabla \cdot (U_j \nabla \vartheta_j)(y_1, y_2) dy,$$

and

$$\int_{\mathbb{R}^2_-}[h_j(y_1,-y_2)-\nabla\cdot(U_j\nabla\vartheta_j)(x_1,-x_2)]y_1dy=\int_{\mathbb{R}^2_+}[h_j(y_1,y_2)-\nabla\cdot(U_j\nabla\vartheta_j)(y_1,y_2)]y_1dy.$$

Then, by using condition (3.68), we have from the divergence Theorem that

$$\begin{aligned}\int_{\mathbb{R}^2}\tilde{h}_jdy&=2\int_{\mathbb{R}^2_+}h_j-\nabla\cdot(U_j\nabla\vartheta_j)dy=2\int_{\mathbb{R}^2_+}h_jdy-2\int_{\partial\mathbb{R}^2_+}(U_j\nabla\vartheta_j)\cdot\boldsymbol{\nu}dS_y\\&=2\int_{\mathbb{R}^2_+}h_jdy-2\int_{\partial\mathbb{R}^2_+}U_j\frac{\partial\vartheta_j}{\partial\boldsymbol{\nu}}dS_y=2\int_{\mathbb{R}^2_+}h_jdy-2\int_{\partial\mathbb{R}^2_+}\beta_jdS_y=0,\end{aligned}$$

which implies the mass condition in (3.13) holds. For the first moment condition, we utilize the integration by parts and (3.71) to get

$$\begin{aligned}\int_{\mathbb{R}^2_+}\tilde{h}_jy_1dy&=2\int_{\mathbb{R}^2_+}[h_j-\nabla\cdot(U_j\nabla\vartheta_j)]y_1dy\\&=2\int_{\mathbb{R}^2_+}h_jy_1dy-2\int_{\partial\mathbb{R}^2_+}y_1U_{j0}\nabla\vartheta_j\cdot\boldsymbol{\nu}dS_y+2\int_{\mathbb{R}^2_+}U_j\nabla\vartheta_j\cdot\boldsymbol{e}_1dy\\&=2\int_{\mathbb{R}^2_+}h_jy_1dy-2\int_{\partial\mathbb{R}^2_+}\beta_jy_1dS_y=0.\end{aligned}\tag{3.73}$$

Since $\tilde{h}_j$ is even with respect to y_1, we can easily obtain from (3.73) that $\int_{\mathbb{R}^2}\tilde{h}_jy_1dy=0$, which completes the verification of orthogonality condition (3.13). Therefore, we can utilize the results shown in Lemma 3.4 to find there exists the solution $(\tilde{\phi}_1,\tilde{\phi}_2,\tilde{\psi}_1,\tilde{\psi}_2)$ to the following system:

$$\begin{cases}-\Delta\tilde{\psi}_1=U_1\tilde{\psi}_1+U_1\tilde{g}_1, & \text{in } \mathbb{R}^2,\\ -\Delta\tilde{\psi}_2=U_2\tilde{\psi}_2+U_2\tilde{g}_2, & \text{in } \mathbb{R}^2,\\ -\Delta\tilde{\psi}_1=a_{11}\tilde{\phi}_1+a_{12}\tilde{\phi}_2 & \text{in } \mathbb{R}^2,\\ -\Delta\tilde{\psi}_2=a_{21}\tilde{\phi}_1+a_{22}\tilde{\phi}_2 & \text{in } \mathbb{R}^2.\end{cases}$$

In particular, $\tilde{\phi}_j$, $j=1,2$ satisfies the following estimate

$$|\tilde{\phi}_j|\leq C_j\frac{\|h_j\|_{4+\sigma,H}}{(1+r)^{2+\sigma}}.\tag{3.74}$$

Since $\tilde{\phi}_j$ is even, it can be defined as the even extension of ϕ_j. By using (3.74), we further show that (ϕ_1,ϕ_2) is the solution of (3.69) and satisfies

$$\|\phi_j\|_{2+\sigma,H}=\|\tilde{\phi}_j+\phi_{j0}\|_{2+\sigma,H}\leq\|\tilde{\phi}_j\|_{2+\sigma,H}+\|\phi_{j0}\|_{2+\sigma,H}\leq C_j\|h_j\|_{4+\sigma,H},$$

which completes the proof of (3.70) and this Lemma. □

In the next section, we focus on the outer problem and establish the outer linear theory.

4 Outer Linear Theory

Similarly as shown in Subsection 3.2 of [14], we first formulate the outer operator. Concerning the φ_j-equations of (1.3), we define

$$\bar{L}_j[\varphi_1,\varphi_2]=-\Delta\varphi_j+\nabla\cdot(P_j\nabla w_j)+\nabla\cdot(\varphi_j\nabla\bar{Q}_j),\tag{4.1}$$

where

$$P_j(y) = \sum_{k=1}^{m} U_{jk}(y-\xi_k') \text{ and } \bar{Q}_j(y) = \sum_{k=1}^{m}\Big(- m_j \log\varepsilon + \Gamma_{jk}(y-\xi_k') + \hat{c}_{jk} H_j^{\varepsilon}(y,\xi_k)\Big).$$

Here c_{jk} given in (2.20) is included in U_{jk}. Then, we substitute $\Delta\bar{Q}_j = \varepsilon^2\bar{Q}_j - a_{j1}P_1 - a_{j2}P_2$ into (4.1) and expand it to obtain

$$\begin{aligned}\bar{L}_j[\varphi_1,\varphi_2] =& \Delta\varphi_j - \nabla\varphi_j\cdot\nabla\bar{Q}_j - \varepsilon^2\bar{Q}_j\varphi_j \\ & - \nabla P_j\cdot\nabla w_j - P_j\Delta w_j + (a_{j1}P_1 + a_{j2}P_2)\phi_j := I_{31} + I_{32}.\end{aligned}$$

Due to the decay property of P_j, we have I_{32} is negligible in the outer region and I_{31} dominates. On the other hand, the leading term in the logistic source is $\varepsilon^2\lambda_j\bar{u}_j\phi_j$. Combining this term with I_{31}, we define the outer operator as

$$L_j^o[\varphi_1,\varphi_2] = -\Delta\varphi_j + \nabla\varphi_j\cdot\nabla\bar{Q}_j - \varepsilon^2(\lambda_j\bar{u}_j - \bar{Q}_j)\phi_j \text{ in } \Omega_\varepsilon, \tag{4.2}$$

where $\varepsilon y = x$ and $\varepsilon\xi_j' = \xi_j$. Moreover, the outer norm $\|\cdot\|_{\nu,o}$, $\nu > 0$ is defined as

$$\|h_j\|_{\nu,o} := \sup_{y\in\Omega_\varepsilon} \frac{|h_j|}{\sum_{k=1}^{m}(1+|y-\xi_k'|)^{-\nu}}.$$

We next derive a-priori estimate of outer solution (φ_1,φ_2) then prove its existence. Our results are summarized as

Lemma 4.1. *Assume that $\|h_j\|_{b+2,o} < +\infty$ and $\lambda_j\bar{u}_j < \bar{C}_j$, where $\bar{C}_j := \sum_{k=1}^{m}\hat{c}_{jk}C_\Omega$ and C_Ω is the positive lower bound of Green's function, then the problem*

$$\begin{cases} L_j^o[\varphi_1,\varphi_2] + h_j = 0 & \text{in } \Omega_\varepsilon, \\ \dfrac{\partial\varphi_j}{\partial\boldsymbol{\nu}} = 0 & \text{on } \partial\Omega_\varepsilon \end{cases} \tag{4.3}$$

admits the solution $(\varphi_1,\varphi_2)^T = \mathcal{T}_o[h_1,h_2]$ satisfying

$$\|\varphi_j\|_{b,o} \le C\|h_j\|_{b+2,o}, \tag{4.4}$$

where $b > 0$ is a constant, $C > 0$ is a constant and $\mathcal{T}_o[h_1,h_2]$ is a continuous linear mapping.

Proof. To show that a-priori estimates (4.4) satisfied by φ_1 and φ_2 hold, we can follow the argument shown in the proof of Lemma 3.3 in [14] with some slight modification and the details are omitted. The existence of (φ_1,φ_2) immediately follows from Fredholm alternative theorem. □

Remark 4.1. *As shown in [4, 14], the principal parts of outer operators in single species Keller-Segel models can be formally regarded as 6-dimensional Laplacians. Whereas, the principal term in $L_j^o[\varphi_1,\varphi_2]$, $j = 1,2$ defined by (4.2) is approximated by the $m_j + 2$-dimensional Laplacian with $m_j > 2$ since the algebraic decay rate of cellular density u is m_j.*

Lemma 4.1 demonstrates that the outer problem (4.3) admits the decay solution if source (h_1,h_2) decays fast. Next, we shall first employ Lemma 3.4 and Lemma to construct the multi-interior spots to (1.3) via the inner-outer gluing scheme. The existence of multi-boundary spots can also be shown by invoking Lemma 3.5 and the detailed discussions are exhibited in Section 5.

5 Inner-outer Gluing Procedures

This section is devoted to the construction of multiple interior spots and boundary spots. Noting that when the locations of spots are at the boundary $\partial\Omega$, the inner problem near each spot has to be solved in the half space $\mathbb{R}^2_+$ and it is necessary to use the straighten transformation, which may cause some difficulty. Thus, we divide our following discussions into two cases: construction of multi-interior spots and construction of multi-boundary spots.

5.1 Construction of Interior Spots

To begin with, we give some preliminary notation and definitions. Recall that the inner and outer norms $\|\cdot\|_{\nu,k}$ and $\|\cdot\|_{b,o}$ are given by

$$\|h\|_{\nu,k} := \sup_{y\in\mathbb{R}^2} |h(y)|(1+|y-\xi_k'|)^{\nu} \ \text{ and } \ \|h\|_{\nu,o} := \sup_{x\in\Omega} \frac{|h|}{\sum_{k=1}^m (1+|x-\xi_k'|)^{-\nu}}. \tag{5.1}$$

In addition, we denote $\delta' = \inf_{j\neq k}|\xi_j-\xi_k|$ and the cut-off functions as $\eta_j := \eta(|y-\xi_k'|) > 0$, where

$$\eta(r) := \begin{cases} 1, & r \le \frac{\delta}{\varepsilon}; \\ 0, & r > \frac{2\delta}{\varepsilon} \end{cases} \tag{5.2}$$

and δ is a fixed number.

With the aid of (5.1) and (5.2), we now decompose (ϕ_1,ϕ_2) and (w_1,w_2) into the following form

$$\begin{aligned} &\varphi_j = \sum_{k=1}^m \varepsilon^{\gamma_1} c_{jk}\varphi_{jk}(y)\eta_k + \varepsilon^{\gamma_2}\varphi_j^o, \quad w_{jk} = (-\Delta)^{-1}(b_{j1}\varphi_{1k}+b_{j2}\varphi_{2k}) \\ &w_j^o = \varepsilon^{\gamma_2}(-\Delta+\varepsilon^2)^{-1}(a_{j1}\varphi_1^o + a_{j2}\varphi_2^o), \quad w_{jk}' = (-\Delta+\varepsilon^2)^{-1}(b_{j1}\varphi_{1k}\eta_k + b_{j2}\varphi_{2k}\eta_k), \end{aligned} \tag{5.3}$$

where $\gamma_1 > 0$ and $\gamma_2 > 0$ will determined later on. In light of (2.45) and (2.54), we find for $j = 1, 2$,

$$L_j(\varphi_j) = \varepsilon^{\gamma_1}\sum_{k=1}^m L_j[\varphi_{jk}\eta_k] + \varepsilon^{\gamma_2}L_j[\varphi_j^o] = \varepsilon^2\sum_{k=1}^5 I_{jk}(\mathbf{P}) + \varepsilon^2 I_6(\boldsymbol{\varphi},\mathbf{P}) + \varepsilon^2 I_7(\boldsymbol{\varphi},\mathbf{P}).$$

Invoking the definitions of L_{jk}^{inn} and L_j^o given by (3.1) and (4.2), respectively, we have

$$\begin{aligned} &\varepsilon^{\gamma_1}\sum_{k=1}^m c_{jk}L_{jk}^{\text{inn}}[\varphi_{jk}\eta_k] + \varepsilon^{\gamma_2}L_j^o[\varphi_1^o,\varphi_2^o] \\ =&\varepsilon^2 h_j(\boldsymbol{\varphi},\mathbf{P}) - \varepsilon^{\gamma_2}\nabla\cdot(P_j\nabla\bar{w}_j^o) + \varepsilon^{\gamma_2}\Delta\bar{Q}_j\varphi_j^o - \varepsilon^{2+\gamma_2}\lambda_j\bar{u}_j\varphi_j^o \\ &+\varepsilon^{1+\gamma_2}\sum_{k=1}^m \nabla_y\cdot(\varphi_j^o\nabla_x H_{jk}^{\varepsilon}(\varepsilon y,\xi_k)) + \varepsilon^{\gamma_1+1}\Big(\sum_{k=1}^m \varphi_{jk}\nabla\eta_k\Big)\cdot\Big(\sum_{j=1}^m \nabla H_{jk}^{\varepsilon}(\varepsilon y,\xi_k)\Big) \\ &-\varepsilon^{\gamma_1}\sum_{k=1}^m\sum_{l\neq k}[\nabla\cdot(U_{jl}\nabla\bar{w}_{jk}) + \nabla\cdot(\varphi_{jk}\nabla\Gamma_{jl})\eta_k] - \varepsilon^{\gamma_1}\sum_{k=1}^m\sum_{l\neq k}\varphi_{jk}\nabla\eta_k\cdot\nabla\Gamma_{jl}, \end{aligned} \tag{5.4}$$

where $h_j(\boldsymbol{\varphi},\mathbf{P}) := \sum_{k=1}^7 I_{jk}(\varphi_{11},\cdots,\varphi_{1m},\varphi_{21},\cdots,\varphi_{2m},\varphi^o,\mathbf{P})$ and $\bar{w}_j^o(y) = w_j^o(\varepsilon y)$, $\bar{w}_{jk}'(y) = w_{jk}(\varepsilon y)$.

Define

$$\begin{aligned}F_j(\boldsymbol{\varphi}, \mathbf{P}) :=&\varepsilon^2 h_j(\boldsymbol{\varphi}, \mathbf{P}) - \varepsilon^{\gamma_2}\nabla\cdot(P_j\nabla\bar{w}_j^o) + \varepsilon^{\gamma_2}P_j\varphi_j^o - \varepsilon^{2+\gamma_2}\lambda_j\bar{u}_j\varphi_j^o\\ &+ \varepsilon^{1+\gamma_2}\sum_{k=1}^m \nabla_y\cdot(\varphi_j^o\nabla_x H_{jk}^\varepsilon(\varepsilon y, \xi_k))\\ &- \varepsilon^{\gamma_1}\sum_{k=1}^m\sum_{l\neq k}[\nabla\cdot(U_{jl}\nabla\bar{\psi}_{jk}) + \nabla\cdot(\varphi_{jk}\nabla\Gamma_{jl})\eta_k].\end{aligned} \tag{5.5}$$

Moreover, a simple computation shows that

$$\begin{aligned}L_{jk}^{\rm inn}[\varphi_{jk}\eta_k] =&\eta_k L_{jk}^{\rm inn}[\varphi_{jk}] - 2\nabla\varphi_{jk}\cdot\nabla\eta_k - \varphi_{jk}\Delta\eta_k + \varphi_{jk}\cdot\nabla\Gamma_{jk}(y-\xi_k')\nabla\eta_k\\ &+ \nabla\cdot(U_{jk}\nabla w_{jk}') - \nabla\cdot(U_{jk}\nabla w_{jk})\eta_k.\end{aligned} \tag{5.6}$$

Combining (5.5) with (5.6), one denotes

$$F_{jk}(\boldsymbol{\varphi}, \mathbf{P}) := F_j(\boldsymbol{\varphi}, \mathbf{P})\eta_k - \varepsilon^{\gamma_1}\nabla\cdot(U_{jk}\nabla w_{jk}') + \varepsilon^{\gamma_1}\nabla\cdot(U_{jk}\nabla w_{jk})\eta_k. \tag{5.7}$$

Moreover, we define J_j as

$$\begin{aligned}J_j(\boldsymbol{\varphi}, \mathbf{P}) =&F_j(\boldsymbol{\varphi}, \mathbf{P})(1 - \sum_{k=1}^m \eta_k^2) + \varepsilon^{\gamma_1+1}\Big(\sum_{k=1}^m \varphi_{jk}\nabla\eta_k\Big)\cdot\Big(\sum_{j=1}^m \nabla H_{jk}^\varepsilon(\varepsilon y, \xi_k)\Big)\\ &+ \varepsilon^{\gamma_1}\sum_{k=1}^m[2\nabla\varphi_{jk}\cdot\nabla\eta_k + \varphi_{jk}\Delta\eta_k - \varphi_{jk}\cdot\nabla\Gamma_{jk}(y-\xi_k')\nabla\eta_k]\\ &- \varepsilon^{\gamma_1}\sum_{k=1}^m\sum_{l\neq k}\varphi_{jk}\nabla\eta_k\cdot\nabla\Gamma_{jl}.\end{aligned} \tag{5.8}$$

Collecting (5.4), (5.7) and (5.8), we formulate the system satisfied by φ_{jk} and φ_j^o as

$$\begin{cases}L_{jk}^{\rm inn}[\varphi_{1k}, \varphi_{2k}] = \varepsilon^{-\gamma_1}F_{jk}(\boldsymbol{\varphi}, \mathbf{P}), & \text{in } \mathbb{R}^2, \quad k = 1, \cdots, m,\\ L_j^o[\varphi_1^o, \varphi_2^o] = \varepsilon^{-\gamma_2}J_j(\boldsymbol{\varphi}, \mathbf{P}), & \text{in } \Omega_\varepsilon, \quad j = 1, 2.\end{cases} \tag{5.9}$$

In order to use the solvability results stated in Lemma 3.4, we must impose the orthogonality conditions. To this end, we let compactly supported functions W_{ik} be radial with respect to ξ_k' and satisfy

$$\int_{\mathbb{R}^2} W_{0k}(y-\xi_k')dy = 1,$$

and compactly supported radial functions $W_{lk},\ l = 1, 2$ satisfy

$$\int_{\mathbb{R}^2} W_{lk}(|y-\xi_k'|)(y-\xi_k')_l dy = 1.$$

With these test functions, we modify (5.9) as the following problem

$$\begin{cases}L_{jk}^{\rm inn}[\varphi_{1k}, \varphi_{2k}] = \varepsilon^{-\gamma_1}F_{jk}(\boldsymbol{\varphi}, \mathbf{P}) - \displaystyle\sum_{l=0,1,2} m_{jlk}[\varepsilon^{-\gamma_1}F_{jk}(\boldsymbol{\varphi}, \mathbf{P})]W_{lk} \ \text{ for } \ k = 1, \cdots, m,\\ L_j^o[\varphi_1^o, \varphi_2^o] = \varepsilon^{-\gamma_2}J_j(\boldsymbol{\varphi}, \mathbf{P}), \quad j = 1, 2,\end{cases} \tag{5.10}$$

where

$$m_{j0k}[h_j] = \int_{\mathbb{R}^2} h_j dy \ \text{ and } \ m_{jlk}[h_j] = \int_{\mathbb{R}^2} h_j(y)(y-\xi_k')_l dy. \tag{5.11}$$

Thanks to Lemma 3.4 and 4.1, we find if right hand sides in (5.10) are given, then there exists a solution of

$$(\varphi_{11}, \cdots, \varphi_{1m}, \varphi_1^o, \varphi_{21}, \cdots, \varphi_{2m}, \varphi_2^o).$$

to (5.10) provided with

$$m_{jlk}[\varepsilon^{-\gamma_1} F_{jk}(\boldsymbol{\varphi}, \mathbf{P})] = 0 \ \text{ for } \ l = 0, 1, 2; \ k = 1, 2, \cdots, m; \ j = 1, 2. \tag{5.12}$$

Moreover,

$$\begin{aligned} \varphi_{jk} &= \mathcal{A}_{jk}(\varphi_{11}, \cdots, \varphi_{1m}, \varphi_1^o, \varphi_{21}, \cdots, \varphi_{2m}, \varphi_2^o, \mathbf{P}_0 + \mathbf{P}_1), \\ \varphi_j^o &= \mathcal{A}_j^o(\varphi_{11}, \cdots, \varphi_{1m}, \varphi_1^o, \varphi_{21}, \cdots, \varphi_{2m}, \varphi_2^o, \mathbf{P}_0 + \mathbf{P}_1), \\ \mathbf{P} &= \mathcal{A}_p(\varphi_{11}, \cdots, \varphi_{1m}, \varphi_1^o, \varphi_{21}, \cdots, \varphi_{2m}, \varphi_2^o, \mathbf{P}_0 + \mathbf{P}_1), \end{aligned} \tag{5.13}$$

where $\mathcal{A}_{jk}$, $\mathcal{A}_j^o$ and $\mathcal{A}_p$ are linear operators and

$$\mathbf{P} = \mathbf{P}_0 + \mathbf{P}_1 \ \text{ with } \ \mathbf{P}_0 = (c_{110}, \cdots, c_{1m0}, c_{210}, \cdots, c_{2m0}, \xi_{10}, \cdots, \xi_{m0}). \tag{5.14}$$

Then we use (5.13) and (5.14) to rewrite the solutions and the operators in the form of

$$\vec{\varphi} = (\varphi_{11}, \cdots, \varphi_{1m}, \varphi_1^o, \varphi_{21}, \cdots, \varphi_{2m}, \varphi_2^o, \mathbf{P}_0 + \mathbf{P}_1) \tag{5.15}$$

and

$$\mathcal{A}(\vec{\varphi}) = (\mathcal{A}_{11}(\vec{\varphi}), \cdots, \mathcal{A}_{1m}(\vec{\varphi}), \mathcal{A}_1^o(\vec{\varphi}), \mathcal{A}_{21}(\vec{\varphi}), \cdots, \mathcal{A}_{2m}(\vec{\varphi}), \mathcal{A}_2^o(\vec{\varphi}), \mathcal{A}_p(\vec{\varphi})). \tag{5.16}$$

We further define

$$\begin{aligned} X_k = \Big\{ \varphi \in L^\infty(\mathbb{R}^2) : \nabla\varphi \in L^\infty(\mathbb{R}^2); \|\varphi\|_{2+\sigma,k} < \infty, \\ \int_{\mathbb{R}^2} \varphi dy = 0 \ \text{ and } \ \int_{\mathbb{R}^2} \varphi(y)(y - \xi_k')_l dy = 0, l = 1, 2 \Big\}, \end{aligned}$$

$$X_o = \Big\{ \varphi \in L^\infty(\Omega_\varepsilon) : \nabla\varphi \in L^\infty(\Omega_\varepsilon), \ \|\varphi\|_{b,o} < +\infty, \ \frac{\partial\varphi}{\partial\boldsymbol{\nu}} = 0 \text{ on } \partial\Omega_\varepsilon \Big\}$$

and

$$\begin{aligned} X_p = \Big\{ (c_{11}, \cdots, c_{1m}, c_{21}, \cdots, c_{2m}, \xi_1, \cdots, \xi_m) \in \mathbb{R}^m \times \mathbb{R}^m \times (\mathbb{R}^2)^m : \\ \|\mathbf{P}\|_p = \sup_k |c_{1k}| + \sup_k |c_{2k}| + \sup_k |\xi_k| \Big\}. \end{aligned}$$

We collect them to define X as

$$X := \Big(\prod_{k=1}^m X_k \times X_o \Big)^2 \times X_p \tag{5.17}$$

equipped with the following norm

$$\|\vec{\varphi}\|_X = \sum_{j=1,2} \Big(\sum_{k=1}^m \|\varphi_{jk}\|_{2+\sigma,k} + \|\varphi_j^o\|_{b,o} \Big) + \|\mathbf{P}\|_p. \tag{5.18}$$

With (5.17) and (5.18), we shall show the existence of solution $\vec{\varphi}$ given by (5.15) to problem (5.10) in X via the fixed point theorem. First of all, we claim that for $\|\vec{\varphi}\|_X < 1$,

$$\|\mathcal{A}(\vec{\varphi})\|_X < 1, \tag{5.19}$$

where $\mathcal{A}$ is defined by (5.16). For the inner operator, by using Lemma 3.4, we find it is sufficient to prove

$$\|\varepsilon^{-\gamma_1} CF_{jk}(\boldsymbol{\varphi}, \mathbf{P})\|_{4+\sigma,k} < 1, \tag{5.20}$$

which concludes $\|\varepsilon^{-\gamma_1} CF_{jk}(\boldsymbol{\varphi}, \mathbf{P})\|_{4+\sigma,k} < 1$, where $C := \max\{C_1, C_2\}$ given in (3.14). To this end, we expand $\varepsilon^{-\gamma_1} F_{jk}(\boldsymbol{\varphi}, \mathbf{P})$ as

$$\begin{aligned}
\varepsilon^{-\gamma_1} F_{jk}(\boldsymbol{\varphi}, \mathbf{P}) =& \varepsilon^{2-\gamma_1} \sum_{l=1}^{5} I_{jl} \\
&+ \Big[\varepsilon^{2-\gamma_1} I_{j7} - \varepsilon^{2+\gamma_2-\gamma_1} \lambda_j \bar{u}_j \phi_j^o - \varepsilon^{\gamma_2-\gamma_1} \nabla \cdot (P_j \nabla w_j^o) + \varepsilon^{\gamma_2-\gamma_1} \varphi_j^o \Delta Q_j\Big] \\
&+ \Big[\varepsilon^{2-\gamma_1} I_{j6} + \varepsilon^{1+\gamma_2-\gamma_1} \sum_{k=1}^{m} \nabla_y \cdot (\varphi_j^o \nabla_x H_{jk}^{\varepsilon}(\varepsilon y, \xi_k)) \\
&- \sum_{k=1}^{m} \sum_{l \neq k} [\nabla \cdot (U_{jl} \nabla \bar{\psi}_{jk}) + \nabla \cdot (\varphi_{jk} \nabla \Gamma_{jl}) \eta_k]\Big] \\
&- \nabla \cdot (U_{jk}(y - \xi_j') \nabla (\bar{w}_{jk} - \bar{w}_{jk}')) \\
:=& II_1 + II_2 + II_3 - II_4.
\end{aligned}$$

For II_1 , we show that

$$\|II_1\|_{4+\sigma,k} = o(1), \tag{5.21}$$

and only discuss $\nabla_y U \cdot \nabla_y \Gamma$ since the others can be treat in the same way. For $y \in B_{\delta/2\varepsilon}(\xi_k')$, we have for r large,

$$\begin{aligned}
&\varepsilon^{-\gamma_1} |(1 + |y - \xi_k'|)^{4+\sigma} \nabla U_{jl}(y - \xi_l') \cdot \nabla \Gamma_{jk}(y - \xi_k')| \\
&\quad \leq \varepsilon^{-\gamma_1} C \Big|(1 + |y - \xi_k'|)^{4+\sigma} \frac{1}{(1 + |y - \xi_l'|)^{m_j+1}} \cdot \frac{1}{1 + |y - \xi_k'|}\Big| \leq C \varepsilon^{m_j - 2 - \gamma_1 - \sigma},
\end{aligned} \tag{5.22}$$

where $C > 0$ is a large constant. Taking $\sigma > 0$ and $\gamma_1 > 0$ small enough such that $m_j - 2 - \gamma_1 - \sigma > 0$ for $j = 1, 2$, we then obtain from (5.22) that

$$\|\nabla U_{jl}(y - \xi_l') \cdot \nabla \Gamma_{jk}(y - \xi_k')\|_{4+\sigma,k} = o(1).$$

The leading terms in II_2 are generated by the outer solution. We only analyze $\varepsilon^{\gamma_2-\gamma_1} \varphi_j^o \Delta Q_j$ as an example and proceed with the same argument on the others. Notice that

$$|\Delta Q_j| \leq C \sum_{k=1}^{m} (b_{j1} e^{\Gamma_{1k}} + b_{j2} e^{\Gamma_{2k}}) \leq C \sum_{k=1}^{m} \frac{1}{(1 + |y - \xi_k'|)^{\min\{m_1, m_2\}}},$$

where $C > 0$ is a constant. Therefore,

$$\varepsilon^{\gamma_2-\gamma_1} |\varphi_j^o \Delta Q_j| \leq C \varepsilon^{\gamma_2-\gamma_1} \|\varphi_j^o\|_{b,o} \sum_{k=1}^{m} \frac{1}{(1 + |y - \xi_k'|)^{\min\{m_1, m_2\}}} \cdot \sum_{k=1}^{m} (1 + |y - \xi_k'|)^{-b}, \tag{5.23}$$

where $C > 0$ is a constant. Since $\min\{m_1, m_2\} > 2 + \sigma$, we take $b > 2$ and obtain $m_j + b > 4 + \sigma$ for $j = 1, 2$. Assume that $\gamma_2 > \gamma_1$, we then obtain from (5.23) that

$$\|\varepsilon^{\gamma_1-\gamma_2} \varphi_j^o \Delta Q_j\|_{4+\sigma,k} = o(1). \tag{5.24}$$

Thus, we have

$$\|II_2\|_{4+\sigma,k} = o(1). \tag{5.25}$$

For II_4, since

$$\bar{w}_{jk} = C + O\Big(\frac{1}{|y|}\Big), \ \text{ and } \ \bar{w}'_{jk} = O(\varepsilon^{\sigma}) + C + O\Big(\frac{1}{|y|}\Big) \ \text{ as } \ |y| \to \infty,$$

we immediately get that

$$\|II_4\|_{4+\sigma,k} = o(1), \tag{5.26}$$

where the algebraic decay properties of U_{jk}, $j = 1, 2$ were used. For II_3, we focus on the worse term $\varepsilon|\nabla\varphi_{jk} \cdot \nabla\tilde{H}^{\varepsilon}_{jk}(\varepsilon y, \xi'_k)|$, where $\tilde{H}$ is defined by (2.26). The other terms in II_3 can be treated similarly as shown above. We find

$$\Big|\varepsilon\nabla\varphi_{jk} \cdot \nabla\tilde{H}^{\varepsilon}_{jk}(\varepsilon y, \xi'_k)\Big| \le \frac{C\delta}{(1 + |y - \xi'|)^{4+\sigma}},$$

where $C > 0$ is a constant. Taking δ small enough, we then obtain that

$$\|\varepsilon\nabla\varphi_{jk} \cdot \nabla\tilde{H}^{\varepsilon}_{jk}(\varepsilon y, \xi'_k)\|_{4+\sigma,k} < \frac{1}{4}.$$

Thus, by taking $\delta < \bar{C}_1$ with $\bar{C}_1$ is a constant, then we have $\|II_3\|_{4+\sigma,k} < \frac{1}{2}$. Combining this with (5.21), (5.25) and (5.26), one gets

$$\|\mathcal{A}_{jk}(\vec{\varphi})\|_{2+\sigma,k} \le \varepsilon^{-\gamma_1}\|CF_{jk}(\boldsymbol{\varphi}, \mathbf{P})\|_{4+\sigma,k} < 1, \tag{5.27}$$

which completes the proof of (5.20).

We next prove that the outer operator satisfies

$$\|\mathcal{A}^o_j(\vec{\varphi})\|_{b,o} < 1, \ \ j = 1, 2, \tag{5.28}$$

provided with $\|\vec{\varphi}\|_X < 1$. Invoking Lemma 4.1, we have

$$\|\mathcal{A}^o_j(\vec{\varphi})\|_{b,o} \le C\|\varepsilon^{-\gamma_{j2}} J_j(\boldsymbol{\varphi}, \mathbf{P})\|_{b+2,o}, \tag{5.29}$$

where $C > 0$ given in (4.4). Thus, it suffices to show that $C\|\varepsilon^{-\gamma_{j2}} J(\varphi_j, \mathbf{P})\|_{b+2,o} < 1$. Next, we are only concerned with the error terms generated by the inner solutions $(\varphi_{1k}, \varphi_{2k})$ since they are leading order ones. In fact, we have for $j = 1, 2$,

$$\begin{aligned} \varepsilon^{\gamma_1-\gamma_2}|\nabla\varphi_{jk} \cdot \nabla\eta_k| &\le C\varepsilon^{\gamma_1-\gamma_2}\frac{1}{(1 + |y - \xi'_k|)^{4+\sigma}} \\ &\le C\frac{\varepsilon^{\gamma_2-\gamma_1}}{\delta^{2(\gamma_2-\gamma_1)}}\frac{1}{(1 + |y - \xi'_k|)^{4+\sigma+2\gamma_1-2\gamma_2}}. \end{aligned}$$

By choosing $\delta > \bar{C}_2\sqrt{\varepsilon}$, $2\gamma_1 - 2\gamma_2 = -\frac{\sigma}{2}$ and $b = 2 + \frac{\sigma}{2}$, one gets

$$\varepsilon^{\gamma_1-\gamma_2}\|\nabla\varphi_{jk} \cdot \nabla\eta_k\|_{b+2,o} \le \sigma^*,$$

where σ^* is a small constant and constant $\bar{C}_2 = O(1)$ is chosen to guarantee the smallness of σ^*. Proceeding with the other error terms generated by the inner solutions in a similar way, we can indeed show that $C\|\varepsilon^{-\gamma_2} J_j(\boldsymbol{\varphi}, \mathbf{P})\|_{b+2,o} < 1$, which implies (5.28) holds.

In summary, we take $\sigma \in (0, 1)$ small enough, $\delta \in (\sqrt{\varepsilon}\bar{C}_2, \bar{C}_1)$, $b = 2 + \frac{\sigma}{2}$, $\gamma_1 = \frac{\min\{m_1,m_2\}-2-\sigma}{2}$ and $\gamma_2 = \gamma_1 + \frac{\sigma}{4}$ to get (5.19) and (5.28) hold.

For the contraction properties of inner and outer operators, we perform the same arguments shown above to derive that there exist constants $\alpha_1, \alpha_2 \in (0,1)$ such that

$$\begin{cases} \|\mathcal{A}_{jk}[\vec{\varphi}_1] - \mathcal{A}_{jk}[\vec{\varphi}_2]\|_{2+\sigma,k} \leq \alpha_1 \|\vec{\varphi}_1 - \vec{\varphi}_2\|_X; \\ \quad \|\mathcal{A}_j^o[\vec{\varphi}_1] - \mathcal{A}_j^o[\vec{\varphi}_2]\|_{2+\sigma,k} \leq \alpha_2 \|\vec{\varphi}_1 - \vec{\varphi}_2\|_X \end{cases} \tag{5.30}$$

for any $\vec{\varphi}_1, \vec{\varphi}_2 \in X$ with $\|\vec{\varphi}_1\|_X, \|\vec{\varphi}_2\|_X < 1$, where $j = 1,2$ and $k = 1, \cdots, m$ and the details are omitted.

It remains to study the position operator $\mathcal{A}_p$. As discussed in Section 2, we have $\mathbf{P}_0$ is defined as

$$\mathbf{P} = \mathbf{P}_0 + \mathbf{P}_1 \quad \text{with} \quad \mathbf{P}_0 = (c_{11}^0, \cdots, c_{1m}^0, c_{21}^0, \cdots, c_{2m}^0, \xi_1^0, \cdots, \xi_m^0), \tag{5.31}$$

where $(\xi_1^0, \cdots, \xi_m^0)$ is the critical point of $\mathcal{J}_m$. Next, we adjust $\mathbf{P}_1$ to guarantee the orthogonality conditions shown in (5.12) hold, i.e., $m_{jlk}[h] = 0$ with $l = 1,2,3$. We shall show that $\mathbf{P}_1$ is $o(1)$, which immediately implies that $\|\mathcal{A}_p\|_p$ is a contraction mapping and satisfies $\|\mathcal{A}_p\|_p < 1$.

We first consider the multi-interior spots case. Focusing on the mass condition and the k-th inner region, we find that the leading one in the right hand side of (5.10) is $\int_{\Omega_\varepsilon} f(U_{jk})\eta_k dy$ for $j = 1,2$ with $f_j(u) = u(\bar{u}_j - u)$. Then we calculate to find

$$\begin{aligned}
\int_{\Omega_\varepsilon} f(U_{jk})\eta_k dy &= \int_{\Omega_\varepsilon} c_{jk} e^{\Gamma_{j\mu_{jk}}} \Big(\bar{u}_j - c_{jk} e^{\Gamma_{j\mu_{jk}}}\Big)\eta_k dy + O(\varepsilon^2) \\
&= \int_{B_{2\delta/\varepsilon}(\xi_k')} (c_{jk}^0 + c_{jk}^1) e^{\Gamma_{j\mu_{jk}}} \Big[\bar{u}_j - (c_{jk}^0 + c_{jk}^1) e^{\Gamma_{j\mu_{jk}}}\Big] dy + O(\varepsilon^2) \\
&= \int_{\mathbb{R}^2} c_{jk}^0 e^{\Gamma_{j\mu_{jk}}} (\bar{u}_j - c_{jk}^0 e^{\Gamma_{j\mu_{jk}}}) dy \\
&\quad + \int_{B_{2\delta/\varepsilon}(\xi_k')} (c_{jk}^0 + c_{jk}^1) e^{\Gamma_{j\mu_{jk}}} \Big[\bar{u}_j - (c_{jk}^0 + c_{jk}^1) e^{\Gamma_{j\mu_{jk}}}\Big] dy - \int_{\mathbb{R}^2} c_{jk}^0 e^{\Gamma_{j\mu_{jk}}} (\bar{u}_j - c_{jk}^0 e^{\Gamma_{j\mu_{jk}}}) dy \\
&\quad + O(\varepsilon^2) \\
&= \int_{\mathbb{R}^2} c_{jk}^0 e^{\Gamma_{j\mu_{jk}}} (\bar{u}_j - c_{jk}^0 e^{\Gamma_{j\mu_{jk}}}) dy + O(1) c_{jk}^1 + O(\varepsilon^2),
\end{aligned}$$

where c_{jk}^0 is chosen such that

$$\int_{\mathbb{R}^2} c_{jk}^0 e^{\Gamma_{j\mu_{jk}}} (\bar{u}_j - c_{jk}^0 e^{\Gamma_{j\mu_{jk}}}) dy = 0.$$

Now, we take $c_{jk}^0 = O(\varepsilon^2)$ to guarantee that the mass condition holds.

Next, we verify the first-moment orthogonality condition. In fact, the leading term is $\sum_{n=1}^m \sum_{k=1}^m \nabla_x \cdot (U_{jk} \nabla_x (\Gamma_{jn} + H_{jn}^\varepsilon)) - \nabla_x \cdot (U_{jk} \nabla \Gamma_{jk})$. We expand it and obtain for $\iota = 1,2$,

$$\begin{aligned}
&\sum_{n=1}^m \sum_{k=1}^m \varepsilon \int_{\Omega_\varepsilon} \nabla_y \cdot \Big(U_{jk}(y - \xi_k') \nabla H_{jn}^\varepsilon(\varepsilon y, \xi_n')\Big)(y - \xi_k')_\iota \eta_k(y) \mathrm{d}y \\
&+ \sum_{n=1} \sum_{k \neq n} \int_{\Omega_\varepsilon} \nabla_y \cdot (U_{jk}(y - \xi_k') \nabla_y (\Gamma_{jn}(y - \xi_n') + H_{jn}^\varepsilon))(y - \xi_k')_\iota \eta_k \mathrm{d}y \\
&= \sum_{n=1}^m \sum_{k=1}^m \varepsilon \int_{\Omega_\varepsilon} U_{jk} \nabla H_{jn}^\varepsilon \cdot \boldsymbol{e}_\iota \eta_k(y) \mathrm{d}y + \sum_{n=1}^m \sum_{k \neq n}^m \int_{\Omega_\varepsilon} U_{jk} \nabla (\Gamma_{jn} + H_{jn}^\varepsilon) \cdot \boldsymbol{e}_\iota \eta_k(y) dy \\
&+ \sum_{n=1}^m \sum_{k=1}^m \varepsilon \int_{\Omega_\varepsilon} U_{jk} \nabla H_{jn}^\varepsilon \cdot (y - \xi_k')_\iota \nabla \eta_k(y) dy + \sum_{n=1}^m \sum_{k \neq n}^m \int_{\Omega_\varepsilon} U_{jk} \nabla (\Gamma_{jn} + H_{jn}^\varepsilon) \cdot (y - \xi_k')_\iota \nabla \eta_k(y) dy \\
&:= I\bar{I}I_A + I\bar{I}I_B.
\end{aligned} \tag{5.32}$$

We estimate $I\bar{I}I_A$ and $I\bar{I}I_B$ term by term given in (5.32). For $I\bar{I}I_B$, we have from the decay property of U_{jk} and $\nabla\Gamma_{jn}$ that $|I\bar{I}I_B| = O(\varepsilon^{m_j})$ with $m_j > 2$. For $I\bar{I}I_A$, similarly as shown in (2.32), we expand

$$\nabla H^{\varepsilon}_{jn}(\varepsilon y, \xi'_k) \cdot \boldsymbol{e}_\iota = \hat{c}_{jn}\partial_{x_\iota} H(\xi_k, \xi_k) + \varepsilon\hat{c}_{jn}(\partial^2_{x_\iota} H)(\xi_k, \xi_k) \cdot (y - \xi'_k) + O(\varepsilon^{\alpha}),\ \iota = 1, 2.$$

where $\alpha \in (0, 1)$. Similarly, we have

$$\nabla(\Gamma_{jn} + H^{\varepsilon}_{jn})(\varepsilon y, \xi'_k) \cdot \boldsymbol{e}_\iota = \hat{c}_{jn}\partial_{x_\iota} G(\xi_k, \xi_k) + \varepsilon\hat{c}_{jn}\partial^2_{x_\iota} G(\xi_k, \xi_k)(y - \xi'_k) + O(\varepsilon^{\alpha}),$$

where G satisfies (1.7). Upon substituting the two expansions into $I\bar{I}I_A$ in (5.32), we further obtain that

$$\begin{aligned} I\bar{I}I_A &= \varepsilon\pi^2\sigma_j^2 c_{j0}\partial_{x_\iota}\Big(\sum_{j=1}^{m}\bar{c}_k^2 H(\xi_k, \xi_k) + \sum_{j\neq l}\bar{c}_k\bar{c}_l G(\xi_k, \xi_l)\Big) + O(\varepsilon^{\alpha+1}) \\ &= \varepsilon\pi^2\sigma_j^2 c_{j0}\partial_{x_\iota}\mathcal{J}_m + O(\varepsilon^{\alpha+1}), \end{aligned} \tag{5.33}$$

where $\mathcal{J}_m$ is defined by (1.11), σ_j and c_{j0} are given in (2.9) and (2.11), respectively. Noting that $(\xi^0_1, \cdots, \xi^0_m)$ is a m-tuple critical point of $\mathcal{J}_m$ and $\mathcal{J}_m$ has a non-degenerate property, we further expand

$$\partial_{x_\iota} H(\xi_k, \xi_k) = \partial_{x_\iota} H(\xi^0_k, \xi^0_k) + \partial^2_{x_\iota} H(\xi^0_k, \xi^0_k)\xi^\iota_{k1} + O(|\xi_{k1}|^2),$$

and

$$\partial_{x_\iota} G(\xi_k, \xi_k) = \partial_{x_\iota} G(\xi^0_k, \xi^0_k) + \partial^2_{x_\iota} G(\xi^0_k, \xi^0_k)\xi^\iota_{k1} + O(|\xi_{k1}|^2).$$

Upon substituting them into (5.33), we apply the fact that $(\xi^0_1, \cdots, \xi^0_m)$ is a critical point of $\mathcal{J}_m$ to conclude

$$\xi^\iota_{k1} = O(\varepsilon^{\bar{\alpha}}),\ \ \iota = 1, 2,\ k = 1, \cdots, m,$$

where we have used that $|I\bar{I}I_B| = o(\varepsilon^{m_j})$ $0 < \bar{\alpha} < 1$ and $\bar{\alpha} \approx 1$. Since ξ_{k0} is the critical point of $\bar{H}_k$ and $\mathcal{J}_m$ has the non-degenerate property, $I\bar{I}I_A$ can be written as

$$I\bar{I}I_A = \varepsilon\xi^\iota_{k1}\partial^2_{x_\iota}\bar{H}_k(\xi_{k0}, \xi_{k0}) + O(\varepsilon^2) + O(|\xi_{k1}|^2)\ \text{ for }\ \iota = 1, 2.$$

We remark that it is straightforward to verify the other terms, e.g. $\sum_{n=1}\sum_{k=1}\nabla_x \cdot (\phi_{jk}\nabla_x(\Gamma_{jn} + H_n)) - \nabla_x \cdot (\phi_{jk}\nabla\Gamma_{jk})$, in the divergence form operator are negligible. Now, we complete the proof of our claim that when $\|\phi\|_X < 1$, $\|\mathcal{A}(\phi)\|_X < 1$ and $\mathcal{A}_p(\vec{\phi})$ is a contraction mapping.

Define $\mathcal{B}$ as

$$\mathcal{B} = \{\varphi \in X : \|\varphi\|_X < 1\}.$$

Thanks to (5.30), we have

$$\mathcal{A}(\mathcal{B}) \subset \mathcal{B},\ \text{and}\ \|\mathcal{A}(\vec{\varphi_1}) - \mathcal{A}(\vec{\varphi_2})\|_X \leq \frac{2}{3}\|\vec{\varphi_1} - \vec{\varphi_2}\|_X,\ \ \forall\varphi_1, \varphi_2 \in \mathcal{B}.$$

It follows that there exists a solution such that $\vec{\varphi} = \mathcal{A}\vec{\varphi}$. Now, we constructed the multi-interior spots rigorously and next focus on the multi-boundary spots.

5.2 Construction of Boundary Spots

In this subsection, we are concerned with the existence of multi-boundary spots. Firstly, we introduce the transformation to straighten the boundary. Define the graph $\rho(x_1)$ as $(x_1, x_2) = (x_1, \rho(x_1))$ with $\rho(0) = \rho'(x)$, then for $k = 1, \cdots, m$, we transform (y_1, y_2) into

$$z_{1,k} = y_1 - \xi'_{k,1},\ \ z_{2,k} = y_2 - \xi'_{k,2} - \frac{1}{\varepsilon}\rho(\varepsilon(y - \xi'_{k,1})), \tag{5.34}$$

where $y_1 = x_1/\varepsilon$ and $y_2 = x_2/\varepsilon$. Moreover, we denote operator P_{ρ,ξ'_k} such that for any function $\omega(y_1, y_2)$,

$$P_{\rho,\xi'_k}\omega(y_1, y_2) = \omega(z_{k,1}, z_{k,2}). \tag{5.35}$$

Under the transformation shown above, we have the Laplacian operator and Neumann boundary operator become

$$\begin{cases} \Delta_y \omega = \Delta_{z,k}\omega + (\rho'(\varepsilon z_{1,k}))^2 \partial^2_{z_{2,k}}\omega - 2\rho'(\varepsilon z_{1,k})\partial_{z_{1,k},z_{2,k}}\omega - \varepsilon\rho''(\varepsilon z_{1,k})\partial_{z_{2,k}}\omega, \\ \sqrt{1+(\rho'(\varepsilon z_{1,k}))^2}\dfrac{\partial \omega}{\partial \boldsymbol{\nu}} = \rho'(\varepsilon z_{1,k})\partial_{z_{1,k}}\omega - [1+(\rho'(\varepsilon z_{1,k}))^2]\partial_{z_{2,k}}\omega. \end{cases} \tag{5.36}$$

Without confusing the reader, we replace $(z_{k,1}, z_{k,2})$ by (z_1, z_2) for the simplicity of notations, then obtain

$$\Delta_y\omega = \Delta_z\omega + (\rho''(0))^2\varepsilon^2 z_1^2 \partial_{z_2 z_2}\omega - 2\rho''(0)\varepsilon z_1 \partial_{z_1 z_2}\omega - \varepsilon\rho''(0)\partial_{z_2}\omega + O(\varepsilon^2) \tag{5.37}$$

and

$$\begin{aligned} \nabla_y\omega_1 \cdot \nabla_y\omega_2 =& \nabla_z\omega_1 \cdot \nabla_z\omega_2 + \frac{\partial\omega_1}{\partial z_2}\cdot\frac{\partial\omega_2}{\partial z_2}(\rho''(0))^2\varepsilon^2 z_1^2 \\ & - \Big(\frac{\partial\omega_1}{\partial z_1}\cdot\frac{\partial\omega_2}{\partial z_2} + \frac{\partial\omega_1}{\partial z_2}\cdot\frac{\partial\omega_2}{\partial z_1}\Big)\rho''(0)\varepsilon z_1 + O(\varepsilon^2), \end{aligned} \tag{5.38}$$

where we have used the following expansions of ρ and ρ'

$$\rho(\varepsilon z_1) = \frac{1}{2}\rho''(0)\varepsilon^2 z_1^2 + O(\varepsilon^2) \ \text{ and } \ \rho'(\varepsilon z_1) = \rho''(0)\varepsilon z_1 + O(\varepsilon^3). \tag{5.39}$$

Due to the presence of extra terms in (5.37) and (5.38), we expect that there exist many new terms in the error generated by the ansatz of boundary spots compared to interior ones. Whereas, we shall show they are all higher order terms and enjoy good decay estimates while performing the fixed point argument.

Before formulating the inner problem in the half space $\mathbb{R}^2_+$, we introduce the following cut-off function η_H

$$\eta_{H,k}(z) = 1 \ \text{ for } \ z \in \bar{\mathbb{R}}^2_+ \cap \bar{B}_{\delta/\varepsilon}(\xi'_k) \ \text{ and } \ \eta_{H,k} = 0 \ \text{ for } \ z \in \mathbb{R}^2 \cap B^c_{2\delta/\varepsilon}(\xi'_k). \tag{5.40}$$

Invoking (5.40), (5.37) and (5.38), we define the new error function as

$$\begin{aligned} N^\rho_{jk} =& (\rho'(\varepsilon z_1))^2\Big[\frac{\partial^2(\varphi_{H,jk}\eta_{H,k})}{\partial z_2^2} - \Big(\frac{\partial U_{jk}}{\partial z_2}\cdot\frac{\partial \bar{w}_{H,jk}}{\partial z_2} + \frac{\partial \bar{w}_{H,jk}}{\partial z_2^2}\cdot U_{jk} \\ & + \frac{\partial(w_{H,jk}\eta_{H,k})}{\partial z_2}\cdot\frac{\partial\Gamma_{jk}}{\partial z_2} + \frac{\partial^2\Gamma_{jk}}{\partial z_2^2}\cdot(\varphi_{H,jk}\eta_{H,k})\Big)\Big] \\ & - (\rho'(\varepsilon z_1))\Big[\frac{\partial(\varphi_{H,jk}\eta_{H,k})}{\partial z_1\partial z_2} - \Big(\frac{\partial U_{jk}}{\partial z_1}\cdot\frac{\partial\bar{w}_{H,jk}}{\partial z_2} + \frac{\partial U_{jk}}{\partial z_2}\cdot\frac{\partial\bar{w}_{H,jk}}{\partial z_1}\Big) - \frac{\partial^2\bar{w}_{H,jk}}{\partial z_1\partial z_2}\cdot U_{jk} \\ & - \Big(\frac{\partial(\varphi_{H,jk}\eta_{H,k})}{\partial z_1}\cdot\frac{\partial\Gamma_{jk}}{\partial z_2} + \frac{\partial\Gamma_{jk}}{\partial z_2}\cdot\frac{\partial(\varphi_{H,jk}\eta_{H,k})}{\partial z_1}\Big) - \frac{\partial^2\Gamma_{jk}}{\partial z_1\partial z_2}(\varphi_{H,jk}\eta_{H,k})\Big] \\ & - \varepsilon\rho''(\varepsilon z_1)\Big[\frac{\partial(\varphi_{H,jk}\eta_{H,k})}{\partial z_2} - U_{jk}\frac{\partial\bar{w}_{H,jk}}{\partial z_2} - (\varphi_{H,jk}\eta_{H,k})\frac{\partial\Gamma_{jk}}{\partial z_2}\Big], \end{aligned} \tag{5.41}$$

where $\bar{w}_{H,jk} = (-\Delta+\varepsilon^2)^{-1}[(a_{j1}\varphi_{H,1k}+a_{j2}\varphi_{H,2k})\eta_{H,k}]$. Define $\hat{\mathbf{P}}_1$ and $\hat{\mathbf{P}}_2$ as the first and second coordinates of ξ, then we set the parameter vector $\mathbf{P}_H$ as

$$\mathbf{P}_H = (\mathbf{c}, \hat{\mathbf{P}}_{H_1}, \hat{\mathbf{P}}_{H_2}) = (\mathbf{c}, \hat{\mathbf{P}}_1, \hat{\mathbf{P}}_2 - \rho(\hat{\mathbf{P}}_1)). \tag{5.42}$$

With the aid of (5.41) and (5.42), we find the inner equation given in (5.9) becomes

$$L^{\text{inn}}_{jk}[\varphi_{1k},\varphi_{2k}] = \left(N^{\rho}_{jk} + \varepsilon^{-\gamma_1} F_{jk}(P_{\rho,\xi'_k}(\boldsymbol{\varphi}),\mathbf{P}_H)\right)\eta_{H,k} := F_{H,jk}(P_{\rho,\xi'_k}(\boldsymbol{\varphi}),\mathbf{P}_H)\eta_{H,k}, \tag{5.43}$$

where $(z_1, z_2) \in \mathbb{R}^2_+$ and $F_{jk}(\boldsymbol{\varphi},\mathbf{P})$ is given by (5.7).

Before estimating $F_{H,jk}$ in (5.43), we define $\xi'_{H,k} = (\xi'_{k,1}, \xi'_{k,2} - \frac{1}{\varepsilon}\rho(\varepsilon\xi'_{k,1}))$ and the inner norm in the half space as

$$\|h\|_{\nu,H,k} := \sup_{z\in\mathbb{R}^2_+} |h|(1+|z|)^{\nu}. \tag{5.44}$$

Moreover, we denote space $X_{k,H}$, $X_{o,H}$ and $X_{p,H}$ the same as X_k, X_o and X_p except that $\mathbb{R}^2$ and $\|\cdot\|_{2+\sigma,k}$ are replaced by $\mathbb{R}^2_+$ and $\|\cdot\|_{2+\sigma,H,k}$. In addition, we define the norm and inner solution for boundary spots as $\|\cdot\|_X$ and $\vec{\varphi}_{H,k}$.

Next, we discuss the new error N^{ρ}_{jk}, where the straighten operator P_{ρ,ξ'_k} is involved. As shown in (5.41), the worse term is

$$\begin{aligned}&(\rho'(\varepsilon z_1))^2\frac{\partial^2(\varphi_{H,jk}\eta_{H,k})}{\partial z_2^2} - (\rho'(\varepsilon z_1))\frac{\partial^2(\varphi_{H,jk}\eta_{H,k})}{\partial z_1\partial z_2} - \varepsilon\rho''(\varepsilon z_1)\frac{\partial(\varphi_{H,jk}\eta_{H,k})}{\partial z_2}\\ &= (\rho''(0))^2(\varepsilon z_1)^2\frac{\partial^2(\varphi_{H,jk}\eta_{H,k})}{\partial z_2^2} - \rho''(0)\varepsilon z_1\frac{\partial^2(\varphi_{H,jk}\eta_{H,k})}{\partial z_1\partial z_2} - \varepsilon\rho''(0)\frac{\partial(\varphi_{H,ik}\eta_{H,k})}{\partial z_2} + O(\varepsilon^2).\end{aligned} \tag{5.45}$$

Since $|z| < \delta$ for some constant $\delta > 0$, we can chose $\delta > 0$ small enough such that

$$\left\|(\rho'(\varepsilon z_1))^2\frac{\partial^2(\varphi_{H,jk}\eta_{H,k})}{\partial z_2^2} - \rho'(\varepsilon z_1)\frac{\partial^2(\varphi_{H,jk}\eta_{H,k})}{\partial z_1\partial z_2} - \varepsilon\rho''(\varepsilon z_1)\frac{\partial(\varphi_{H,jk}\eta_{H,k})}{\partial z_2}\right\|_{4+\sigma,H,k} < \sigma_1, \tag{5.46}$$

where $\sigma_1 > 0$ is a small constant. For the other terms in N^{ρ}_{jk}, we analyze in a similar way and show that

$$\|N^{\rho}_{jk}\eta_{H,k}\|_{4+\sigma,H,k} \le \sigma_2, \tag{5.47}$$

where $\sigma_2 > 0$ is a small constant.

For the contraction property of $\mathcal{A}_H(\vec{\varphi_H})$, we follow the argument shown in Section 5 of [14] and obtain the desired conclusion. Now, we shall check the orthogonality condition exhibited in Lemma 3.5, which is equivalent to study $\mathcal{A}_{p,H}(\vec{\phi}_H)$. It remains to check orthogonality conditions shown in (3.68). Noting that the leading term in $F_{H,jk}$ is $\int_{\mathbb{R}^+_2} f(U)\eta_{H,k}dz$, we perform the similar argument shown in Subsection 5.1 to get

$$\varepsilon^2\int_{\mathbb{R}^2_+} f(U)\eta_{H,k}dz = O(\varepsilon^2)c_{jk1} + O(\varepsilon^4), \tag{5.48}$$

where c_{jk1} is the error of $c_{jk} = c^0_{jk} + c_{jk1}$ with c^0_{jk} given by (2.11).

For the first-moment orthogonality given in (3.68), we follow the same procedure shown in Section 5 of [14] to derive $\xi_{H,jk1} = O(\varepsilon^{\bar{\alpha}})$ with $\bar{\alpha} < 1$ but close to 1, where the details are omitted. we point out that $\xi_{H,jk2} \equiv 0$ since the centre of boundary spot is located at the boundary. Since $\mathbf{P}_H = o(1)$, we have $\mathcal{A}_p(\vec{\phi})$ is a contraction mapping. Then by following the same argument shown in the end of Subsection 5.1, we find the existence of the remainder term $(\varphi_{1,H}, \varphi_{2,H})$ to boundary spots. This completes the proof of Theorem 1.1.

6 Numerical Studies and Discussion

This section is devoted to the numerical simulation for the emergence of spot patterns in system (1.1). The error threshold is set as $\epsilon = 0.01$ and the maximal time is $t = 2000$.

Figure 2 presents the numerical profile of single boundary spot solution we constructed in Theorem 1.1, which is a snapshot with $t \approx 2000$ in the spatial-temporal dynamics of (1.1). Here the chemotactic coefficients $\chi_1 = \chi_2$ are large enough. As shown in the figures, (u_1, u_2), located at a corner of the rectangle Ω, enjoys the fast decay property in the "far-from" corner region; while (v_1, v_2) follows the shape of (u_1, u_2) mildly but has a global structure, also is positively bounded from below. These characteristics well match our theoretical analysis shown above. We remark that the profiles of u_1 and u_2 are distinct due to the variation of matrix coefficients given in $A = (a_{ij})_{i,j=1,2}$. In addition, the locations of u_1 and u_2 are the same since all coefficients in A are positive.

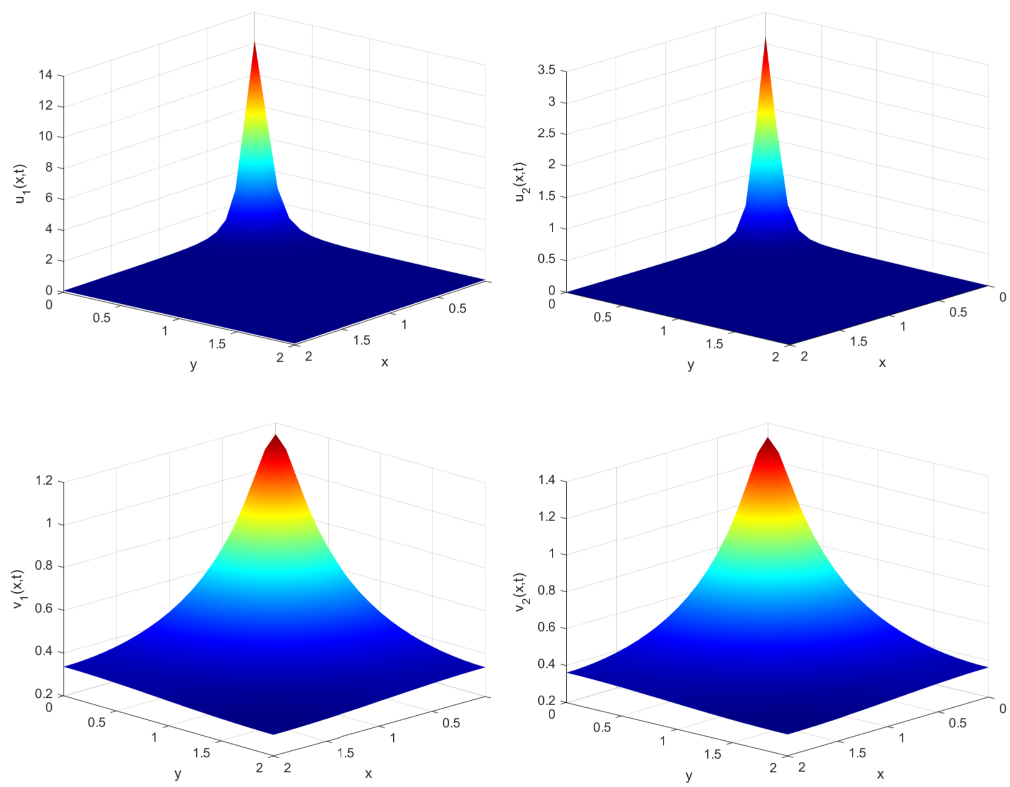

Figure 2: The numerical profile of a single boundary spot steady state obtained by using FLEXPDE7 [7] to (1.1) with $\Omega = (0, 2) \times (0, 2)$, where the rest parameters are set as $\chi_1 = \chi_2 = 8.5$, $\lambda_1 = \lambda_2 = 0.5$, $\bar{u}_1 = 2$, $\bar{u}_2 = 1$, $a_{11} = 2$, $a_{12} = 1$, $a_{21} = 2$ and $a_{22} = 3$. Here the initial data are chosen as $u_{10} = u_{20} = 6e^{-10(x^2+y^2)}+0.1$ and $v_{10} = v_{20} = 2e^{-10(x^2+y^2)}+0.1$. The numerical solution is captured by approximating the time-dependent system (1.1) with $t = 2000$.

Figure 3 illustrates that the general form of (1.1) may admit the stable single interior spots in other regimes, which are different from the large chemotactic movement. In addition, the half profiles of solutions shown in Figure 3 are non-monotone with respect to radius r.

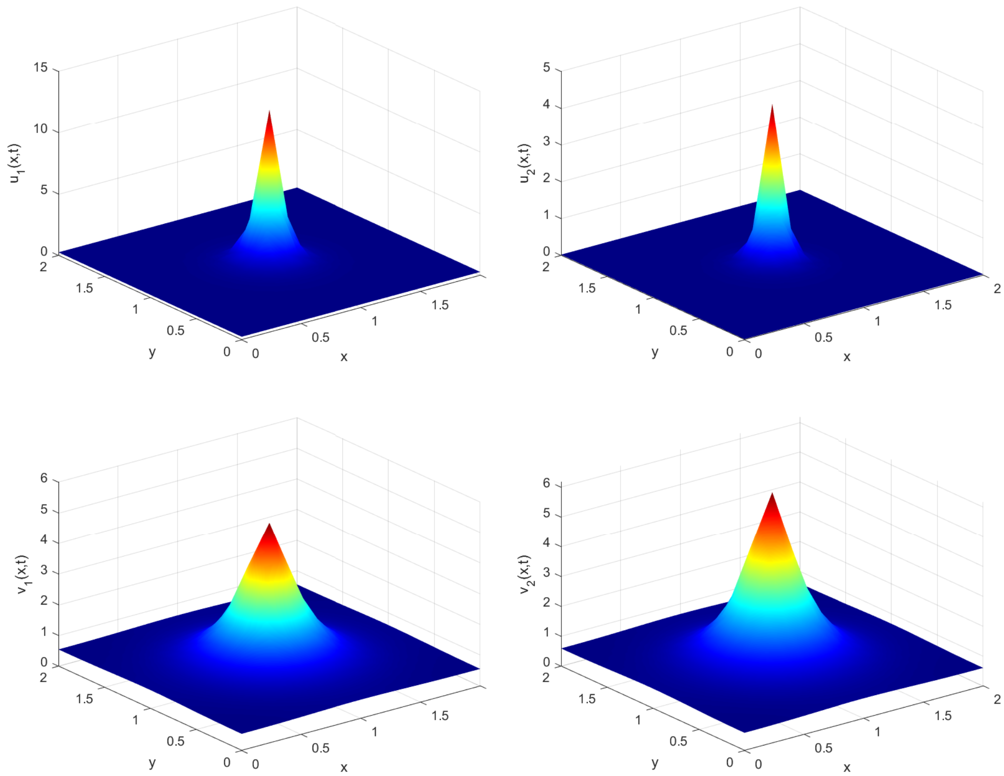

Figure 3: The numerical profile of a single interior spot to (1.3) with $\Omega = (0,2)\times(0,2)$, where the rest parameters are the same except $\chi_1 = \chi_2 = 1$ and $d_{v1} = d_{v2} = 0.05$. Here we incorporate chemical self-diffusion rates d_{v1} and d_{v2} in the v_1-equation and v_2-equation of (1.3). In particular, the initial data are chosen as $u_{10} = u_{20} = 6e^{-10[(x-1)^2+(y-1)^2]} + 0.1$ and $v_{10} = v_{20} = 2e^{-10[(x-1)^2+(y-1)^2]} + 0.1$. The numerical results suggest that the single interior spot in (1.3) is locally stable.

Figure 4 demonstrates that when the coefficient matrix A in (1.3) is not positive, (1.3) admits the spot steady states for sufficient large χ_1 and χ_2, in which cellular densities u_1 and u_2 are located at different points in $\bar{\Omega}$. Also, the facts $a_{11} \neq a_{21}$ and $a_{12} \neq a_{22}$ in matrix A trigger the formation of spots, where u_1 and u_2 do not share the same localized structure.

6.1 Discussion

In this paper, we study the localized pattern formation in (1.1) under the asymptotical limits of χ_1, $\chi_2 \to +\infty$. Our main goal is to extend the results shown in [14] to the multi-species Keller-Segel model counterpart by employing the inner-outer gluing method. Imposing some assumptions on the coefficient matrix A such that A is positive and irreducible, i.e. the interactions are both attractive, we show (1.3) admits the multi-spots given by (1.9) and (1.10). Compared to the core problem given in [14], the core equation governing the profile of solution to (1.3) is still strongly coupled, which may cause the difficulty while establishing the linear theories especially in the inner region. To overcome this, we borrow the ideas shown in [15–17] and develop the inner linear theories stated in Section 3, where the extensive analysis of bounded kernels to the linearized Liouville system is crucial. The main restriction

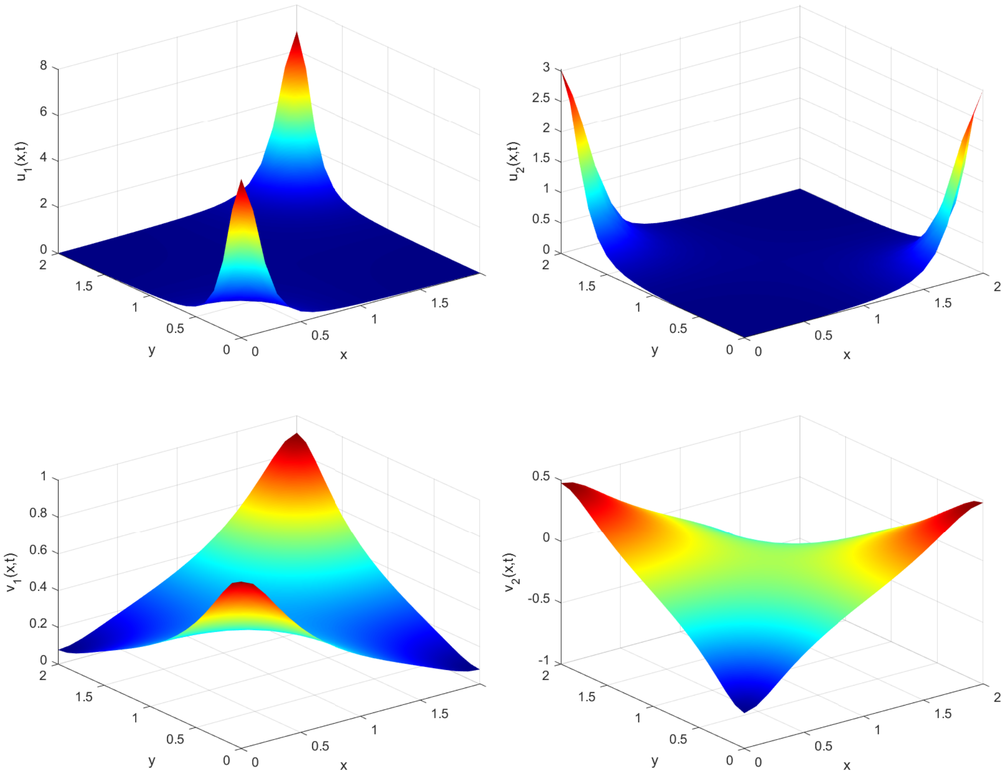

Figure 4: The profiles of two double boundary spots to (1.3) with $\Omega = (0,2) \times (0,2)$. Here the rest parameters are the same as those stated in Figure 2 except that $a_{12} = -1$ and $a_{21} = -2$. These numerical findings suggest that the locations of concentrated cellular densities u_1 and u_2 are different when the interaction coefficients are negative.

in their results is the assumption that the coefficient matrix is positive. As stated in Theorem 1.1, the profiles of cellular densities u_1 and u_2 are determined by the entire solutions e^{Γ_1} and e^{Γ_2} solving (1.4). Unlike the governing profile shown in [14], we do not have the explicit forms of Γ_1 and Γ_2 defined by (1.4). Whereas, we can still establish linear theories and perform the fixed point argument since the relations between algebraic decay rates of e^{Γ_1}, e^{Γ_2} and their total mass are well understood.

We would like to point out some intriguing research directions that deserve exploration in the future. As discussed above, we only consider that the coefficient matrix is positive. As shown in Figure 4, when this assumption is not satisfied, system (1.3) admit new types of concentrated patterns when χ_1 and χ_2 are large enough, where the locations of are u_1 and u_2 are at alternative corners. The theoretical analysis for the existence of these stationary solutions is challenging but worthwhile. Figure 3 reveals that the logistic multi-species Keller-Segel model under the small chemical diffusivity regime admits stable interior spots, which can not be detected under the large chemotactic movement regime. Investigating the relevant pattern formation is open and presents an intriguing direction for future research. It seems some ideas shown in [13] devoted to the single-species Keller-Segel model with logistic growth are beneficial.

7 Acknowledgments

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Appendix A Formal Construction of Spots

In Appendix A, we shall employ the matched asymptotic analysis to reconstruct the multi-spots (1.9) and (1.10), which are complementary of our rigorous analysis. Without loss of generality, we only consider the single interior spot case.

First of all, let $\bar{v}_1 = \chi_1 v_1$ and $\bar{v}_2 = \chi_2 v_2$ in (1.3), we have

$$\begin{cases} 0 = \Delta u_1 - \nabla\cdot(u_1\nabla\bar{v}_1) + \lambda_1 u_1(\bar{u}_1 - u_1), & x\in\Omega,\\ 0 = \Delta u_2 - \nabla\cdot(u_2\nabla\bar{v}_2) + \lambda_2 u_2(\bar{u}_2 - u_2), & x\in\Omega,\\ 0 = \Delta\bar{v}_1 - \bar{v}_1 + a_{11}\chi_1 u_1 + a_{12}\chi_1 u_2, & x\in\Omega,\\ 0 = \Delta\bar{v}_2 - \bar{v}_2 + a_{21}\chi_2 u_1 + a_{22}\chi_2 u_2, & x\in\Omega,\\ \partial_{\mathbf{n}} u_1 = \partial_{\mathbf{n}} u_2 = \partial_{\mathbf{n}} v_1 = \partial_{\mathbf{n}} v_2 = 0, & x\in\partial\Omega. \end{cases} \tag{1.1}$$

Let $\chi_1 = \frac{1}{\varepsilon^2} \gg 1$ with $\chi_2 = \gamma\chi_1$ and in the inner region, we introduce

$$y = \frac{x-\xi}{\varepsilon},\ U_i(y) = \bar{u}_i(x),\ \bar{V}_i(y) = \bar{v}_i(x),\ i = 1,2,$$

where ξ denotes the center.

Then, we expand

$$U_i(y) = U_{i0} + \varepsilon^2 U_{i1} + \cdots, \ \bar{V}_i(y) = \bar{V}_{i0} + \varepsilon^2 \bar{V}_{i1} + \cdots,$$

and obtain from (1.1) that the leading order equation is

$$\begin{cases} 0 = \Delta U_{10} - \nabla \cdot (U_{10} \nabla \bar{V}_{10}), & y \in \mathbb{R}^2, \\ 0 = \Delta U_{20} - \nabla \cdot (U_{20} \nabla \bar{V}_{20}), & y \in \mathbb{R}^2, \\ 0 = \Delta \bar{V}_{10} + a_{11} U_{10} + a_{12} U_{20}, & y \in \mathbb{R}^2, \\ 0 = \Delta \bar{V}_{20} + a_{21} \gamma U_{10} + a_{22} \gamma U_{20}, & y \in \mathbb{R}^2. \end{cases} \tag{1.2}$$

Then we let $\nu := -\frac{1}{\log \varepsilon}$, further expand for $i = 1, 2$,

$$U_{i0} = U_{i0,0} + \nu U_{i0,1} + \cdots, \ \bar{V}_{i0} = \frac{1}{\nu} \bar{V}_{i0,0} + \bar{V}_{i0,1} + \cdots.$$

Upon substituting the expansion into (1.2), one has $\bar{V}_{i0,0} = D_{i0}$ with constant D_{i0} determined later on and

$$\begin{cases} 0 = \Delta U_{10,0} - \nabla \cdot (U_{10,0} \nabla \bar{V}_{10,1}), & y \in \mathbb{R}^2, \\ 0 = \Delta U_{20,0} - \nabla \cdot (U_{20,0} \nabla \bar{V}_{20,1}), & y \in \mathbb{R}^2, \\ 0 = \Delta \bar{V}_{10,1} + a_{11} U_{10} + a_{12} U_{20}, & y \in \mathbb{R}^2, \\ 0 = \Delta \bar{V}_{20,1} + a_{21} \gamma U_{10} + a_{22} \gamma U_{20}, & y \in \mathbb{R}^2. \end{cases} \tag{1.3}$$

Then, we find $U_{i0,0} = C_{i,0} e^{\bar{V}_{i0,1}}$ with $i = 1, 2$ and

$$\begin{cases} 0 = \Delta \bar{V}_{10,1} + a_{11} C_{1,0} e^{\bar{V}_{10,1}} + a_{12} C_{2,0} e^{\bar{V}_{20,1}}, & y \in \mathbb{R}^2, \\ 0 = \Delta \bar{V}_{20,1} + a_{21} \gamma C_{1,0} e^{\bar{V}_{10,1}} + a_{22} \gamma C_{2,0} e^{\bar{V}_{20,1}}, & y \in \mathbb{R}^2. \end{cases} \tag{1.4}$$

Let $\tilde{y} = \sqrt{C_{1,0}} y$, then (1.4) becomes

$$\begin{cases} 0 = \Delta \bar{V}_{10,1} + a_{11} e^{\bar{V}_{10,1}} + \frac{a_{12} C_{2,0}}{C_{1,0}} e^{\bar{V}_{20,1}}, & \tilde{y} \in \mathbb{R}^2, \\ 0 = \Delta \bar{V}_{20,1} + a_{21} \gamma e^{\bar{V}_{10,1}} + a_{22} \gamma \frac{C_{2,0}}{C_{1,0}} e^{\bar{V}_{20,1}}, & \tilde{y} \in \mathbb{R}^2. \end{cases} \tag{1.5}$$

Assume that

$$a_{12} C_{20} = a_{21} \gamma C_{10}, \tag{1.6}$$

then as shown in [15], we have (1.5) admits a family of solution pair $(\bar{V}_{10,1}, \bar{V}_{20,1})$ depending on (μ_1, μ_2). It follows that

$$U_{i0,0}(y) = C_{i,0} e^{\bar{V}_{i0,1}(\tilde{y})}, \ i = 1, 2. \tag{1.7}$$

Moreover, the integral constraints imply for $i = 1, 2$,

$$C_{i,0} = \frac{\bar{u}_i \int_{\mathbb{R}^2} e^{\bar{V}_{i0,1}} d\tilde{y}}{\int_{\mathbb{R}^2} e^{2\bar{V}_{i0,1}} d\tilde{y}}.$$

Of concern (1.6), we find the following condition is assume to hold

$$\frac{\int_{\mathbb{R}^2} e^{\bar{V}_{10,1}} \, d\tilde{y} \bar{u}_1}{\int_{\mathbb{R}^2} e^{\bar{V}_{20,1}} \, d\tilde{y} \bar{u}_2} = \frac{a_{12}}{a_{21}} \frac{\chi_1}{\chi_2} \frac{\int_{\mathbb{R}^2} e^{2\bar{V}_{10,1}} \, d\tilde{y}}{\int_{\mathbb{R}^2} e^{2\bar{V}_{20,1}} \, d\tilde{y}}.$$

In addition, we have by Pohozaev identity that

$$4(\sigma_1+\sigma_2)=b_{11}\sigma_1^2+2b_{12}\sigma_1\sigma_2+b_{22}\sigma_2^2,$$

where b_{ij}, $i,j=1,2$ are given by (2.7) and

$$\sigma_i=\frac{1}{2\pi}\int e^{\bar{V}_{i0,1}}d\tilde{y},\ \ i=1,2.$$

In the outer region, we approximate u_1 and u_2 as Dirac-delta functions to obtain for $i=1,2$,

$$\Delta\bar{v}_1-\bar{v}_1\sim-\Big(a_{11}\int_{\mathbb{R}^2}U_{10,0}d\tilde{y}+a_{12}\int_{\mathbb{R}^2}U_{20,0}d\tilde{y}\Big)\delta(x-\xi),$$

and

$$\Delta\bar{v}_2-\bar{v}_2\sim-\Big(a_{21}\gamma\int_{\mathbb{R}^2}U_{10,0}d\tilde{y}+a_{22}\gamma\int_{\mathbb{R}^2}U_{20,0}d\tilde{y}\Big)\delta(x-\xi),$$

where $U_{10,0}$ and $U_{20,0}$ are defined by (1.7). Hence,

$$\bar{v}_i=2\pi m_iG(x;\xi), \tag{1.8}$$

where m_i are given in (2.9) and G is the Neumann reduced-wave Green's function satisfying

$$\begin{cases}\Delta G-G=-\delta(x-\xi), & x\in\Omega,\\ \partial_{\mathbf{n}}G=0, & x\in\partial\Omega.\end{cases} \tag{1.9}$$

Finally, we match the inner and outer solutions to determine parameters D_{i0},μ_i with $i=1,2$. Recall that as $\tilde{y}\to+\infty$, the inner solution satisfies

$$\bar{V}_{i0}\sim\frac{1}{\nu}D_{i0}+\bar{\mu}_i+m_i\log|\tilde{y}|.$$

On the other hand, the far-field behavior of the outer solution is

$$\bar{v}_i\sim2\pi m_i\Big[-\frac{1}{2\pi}\log|x-\xi|+H(\xi,\xi)+\nabla H(\xi,\xi)\cdot(x-\xi)\Big].$$

Since $\frac{\tilde{y}}{\sqrt{C_{1,0}}}=\frac{x-\xi}{\varepsilon}$, we match the inner and outer solutions implies

$$D_{i0}=m_i,\ \ \mu_i=2\pi m_iH(\xi,\xi),\ \ \nabla H(\xi,\xi)=0,$$

where $\mu_i:=\bar{\mu}_i-m_i\log\sqrt{C_{1,0}}$.