On Spin-2 Bose-Einstein condensates: classification and dynamic behavior of ground states

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Abstract

We investigate physical states of spin-2 Bose-Einstein condensate in $\mathbb{R}^d(d = 1, 2, 3)$ in terms of spin-independent interaction τ, spin-exchange interaction τ_1 and spin-singlet interaction τ_2, two conserved quantities, the number of atoms N and the total magnetization M. We first give a complete classification of ground state solutions and show the validity of single-mode approximation (SMA) phenomenon in $\mathbb{R}^d$. In the one dimensional case, the energy functional is bounded from below on the related physical manifold, the ground states exist and are obtained as global minimizers. When $d = 2$, the energy functional is not always bounded on the related physical manifold. We give a complete classification of the existence and nonexistence of global minimizers, and the explicit thresholds of existence and nonexistence of ground state solution were obtained. In the three dimensional case, the energy functional is always unbounded on the related physical manifold, when the atoms are trapped in a harmonic potential, we prove the existence of ground states and excited states along with some precisely asymptotics. Besides, we get that the set of ground states is stable under the associated Cauchy flow while the excited state corresponds to a strongly unstable standing wave. Our results not only show some characteristics of spin-2 BEC under the effect among spin-independent interaction, spin-exchange interaction and spin-singlet interaction but also support some experimental observations as well as numerical results on spin-2 BEC. Our results are the first studies on quantitative properties of ground states for spin-2 BEC.

Keywords: Ground State, Excited State, Classification, Spin-2 Bose-Einstein condensate, Gross-Pitaevskii system.

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1 Introduction

Recent experiments on ^{23}Na condensates confined in an optical trap have stimulated extensive interest on the study of multi component spinor Bose-Einstein condensates (BECs). BECs of alkali-metal atoms

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have internal degrees of freedom which are frozen in a magnetic trap [30]. However, in an optical trap, the spin degrees of freedom to atoms are liberated enabling a rich variety of spinor BECs physics to be studied, such as various magnetic phases and spin domain formation. The possible hyperfine spins of the alkali-metal atoms are $F = 1$ and $F = 2$. The $F = 1$ BECs were first realized at MIT with ^{23}Na [51]. The $F = 2$ ^{23}Na condensate was also realized by the MIT group [22]. Since these experimental progress on spin $F = 2$ BECs, there have been several experimental studies of spin $F = 2$ systems, including investigations of their response to magnetic fields, the dynamics of multiply charged vortices, the phase separation between spin-2 and spin-1 BECs and of vortex lattice transitions.

In the mean field theory, a physical state of spin-2 BEC is described by 5 components of complex order parameter $\mathbf{\Phi}(x,t) = \big(\Phi_2(x,t), \Phi_1(x,t), \Phi_0(x,t), \Phi_{-1}(x,t), \Phi_{-2}(x,t)\big)(x \in \mathbb{R}^d)$ and the time evolution of the mean field dynamics is governed by [30, 37]

$$ih\partial_t \Phi_j(x,t) = \frac{\delta E}{\delta \Phi_j^*}, \tag{1.1}$$

Φ_j^* denotes the conjugate transpose of Φ_j. Here, $E = E_{\tau,\tau_1,\tau_2}(\mathbf{\Phi})$ is defined by

$$E_{\tau,\tau_1,\tau_2}(\mathbf{\Phi}) = \int_{\mathbb{R}^d} \left(\frac{h^2}{2m}|\nabla \mathbf{\Phi}|^2 + V|\mathbf{\Phi}|^2 + \frac{\tau}{2}\rho^4 + \frac{\tau_1}{2}|\mathbf{F}|^2 + \frac{\tau_2}{2}\mathbf{\Phi}^T A \mathbf{\Phi} \right) dx, \tag{1.2}$$

with h is the Planck constant, m is the mass of atoms and

$$\tau = \frac{4\pi h^2}{m}\frac{4a_2 + 3a_4}{7}, \ \tau_1 = \frac{4\pi h^2}{m}\frac{a_4 - a_2}{7}, \ \tau_2 = \frac{4\pi h^2}{m}\frac{7a_0 - 10a_2 + 3a_4}{7}$$

characterize the spin-independent interaction, spin-exchange interaction and spin-singlet interaction respectively with a_0, a_2, a_4 being the scattering lengths of cooling bosons. $\mathbf{F} = (F_x, F_y, F_z)^T \in \mathbb{R}^3$ is the spin vector given by

$$F_x = \mathbf{\Phi}^* f_x \mathbf{\Phi}, \ F_y = \mathbf{\Phi}^* f_y \mathbf{\Phi}, \ F_z = \mathbf{\Phi}^* f_z \mathbf{\Phi},$$

f_x, f_y, f_z are the Pauli spinor matrices

$$f_x = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & \frac{\sqrt{6}}{2} & 0 & 0 \\ 0 & \frac{\sqrt{6}}{2} & 0 & \frac{\sqrt{6}}{2} & 0 \\ 0 & 0 & \frac{\sqrt{6}}{2} & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \ f_y = i\begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & -\frac{\sqrt{6}}{2} & 0 & 0 \\ 0 & \frac{\sqrt{6}}{2} & 0 & -\frac{\sqrt{6}}{2} & 0 \\ 0 & 0 & \frac{\sqrt{6}}{2} & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \ f_z = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{pmatrix}.$$

Therefore, $F_x, \ F_y, \ F_z$ can be written explicitly as

$$F_x = \overline{\Phi}_{-2}\Phi_{-1} + \overline{\Phi}_{-1}\Phi_{-2} + \overline{\Phi}_1\Phi_2 + \overline{\Phi}_2\Phi_1 + \frac{\sqrt{6}}{2}(\overline{\Phi}_{-1}\Phi_0 + \overline{\Phi}_0\Phi_{-1} + \overline{\Phi}_0\Phi_1 + \overline{\Phi}_1\Phi_0),$$

$$F_y = i\left(\overline{\Phi}_{-2}\Phi_{-1} - \overline{\Phi}_{-1}\Phi_{-2} + \overline{\Phi}_1\Phi_2 - \overline{\Phi}_2\Phi_1 + \frac{\sqrt{6}}{2}(\overline{\Phi}_{-1}\Phi_0 - \overline{\Phi}_0\Phi_{-1} + \overline{\Phi}_0\Phi_1 - \overline{\Phi}_1\Phi_0) \right),$$

and

$$F_z = \sum_{j=-2}^{2} \big(j|\Phi_j|^2 \big).$$

Defining the matrix

$$A = \frac{1}{\sqrt{5}} \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

then $\mathbf{\Phi}^T A \mathbf{\Phi}$ can be expressed as

$$\theta(\mathbf{\Phi}) = \frac{1}{\sqrt{5}}(2\Phi_2\Phi_{-2} - 2\Phi_1\Phi_{-1} + \Phi_0^2).$$

$V = V(x)$ is a real valued function representing the trap potential and by scaling we may assume that $\frac{h^2}{2m} = 1$. From (1.1)-(1.2), in the dimensionless form, spin-2 BEC can be described by the following coupled Gross-Pitaevskii system

$$\begin{cases} i\partial_t \Phi_2(x,t) = -\Delta\Phi_2 + V(x)\Phi_2 + \tau\rho^2\Phi_2 + \tau_1(F_-\Phi_1 + 2F_z\Phi_2) + \dfrac{\tau_2}{\sqrt{5}}\theta\overline{\Phi}_{-2}, \\ i\partial_t \Phi_1(x,t) = -\Delta\Phi_1 + V(x)\Phi_1 + \tau\rho^2\Phi_1 + \tau_1\left(\dfrac{\sqrt{6}}{2}F_-\Phi_0 + F_+\Phi_2 + F_z\Phi_1\right) - \dfrac{\tau_2}{\sqrt{5}}\theta\overline{\Phi}_{-1}, \\ i\partial_t \Phi_0(x,t) = -\Delta\Phi_0 + V(x)\Phi_0 + \tau\rho^2\Phi_0 + \dfrac{\sqrt{6}}{2}\tau_1\left(F_-\Phi_{-1} + F_+\Phi_1\right) + \dfrac{\tau_2}{\sqrt{5}}\theta\overline{\Phi}_0 \\ i\partial_t \Phi_{-1}(x,t) = -\Delta\Phi_{-1} + V(x)\Phi_{-1} + \tau\rho^2\Phi_{-1} + \tau_1\left(\dfrac{\sqrt{6}}{2}F_+\Phi_0 + F_-\Phi_{-2} - F_z\Phi_{-1}\right) - \dfrac{\tau_2}{\sqrt{5}}\theta\overline{\Phi}_1, \\ i\partial_t \Phi_{-2}(x,t) = -\Delta\Phi_{-2} + V(x)\Phi_{-2} + \tau\rho^2\Phi_{-2} + \tau_1(F_+\Phi_{-1} - 2F_z\Phi_{-2}) + \dfrac{\tau_2}{\sqrt{5}}\theta\overline{\Phi}_2, \end{cases} \tag{1.3}$$

where

$$\rho^2 = \sum_{j=-2}^{2} |\Phi_j|^2, \quad F_- = F_x - iF_y = 2\overline{\Phi}_{-2}\Phi_{-1} + \sqrt{6}\overline{\Phi}_{-1}\Phi_0 + \sqrt{6}\overline{\Phi}_0\Phi_1 + 2\Phi_2\overline{\Phi}_1$$

and

$$F_+ = F_x + iF_y = \overline{F}_-.$$

Associated with (1.3) are following two conserved quantities

$$\int_{\mathbb{R}^d} \Big(\sum_{j=-2}^{2} |\Phi_j|^2 \Big) dx = N, \quad \int_{\mathbb{R}^d} \sum_{j=-2}^{2} \big(j|\Phi_j|^2 \big) dx = M.$$

Let us recall that standing wave for (1.3) is a solution of the form

$$(\Phi_2(t,x), \Phi_1(t,x), \Phi_0(t,x), \Phi_{-1}(t,x), \Phi_{-2}(t,x))$$

with

$$\Phi_2(t,x)=e^{-i(\lambda+2\mu)t}u_2(x),\quad \Phi_1(t,x)=e^{-i(\lambda+\mu)t}u_1(x),\quad \Phi_0(t,x)=e^{-i\lambda t}u_0(x),$$
$$\Phi_{-1}(t,x)=e^{-i(\lambda-\mu)t}u_{-1}(x),\quad \Phi_{-2}(t,x)=e^{-i(\lambda-2\mu)t}u_{-2}(x),$$

where $\mu,\ \lambda$ are real numbers and $\mathbf{u}=(u_2,u_1,u_0,u_{-1},u_{-2})\in H^1(\mathbb{R}^d,\mathbb{R}^5)$ satisfies the elliptic system

$$\begin{cases} -\Delta u_2+V(x)u_2+(\lambda+2\mu)u_2+\tau\rho^2u_2+\tau_1(F_xu_1+2F_zu_2)+\dfrac{\tau_2}{\sqrt{5}}\theta u_{-2}=0,\\ -\Delta u_1+V(x)u_1+(\lambda+\mu)u_1+\tau\rho^2u_1+\tau_1\left(\dfrac{\sqrt{6}}{2}F_xu_0+F_xu_2+F_zu_1\right)-\dfrac{\tau_2}{\sqrt{5}}\theta u_{-1}=0,\\ -\Delta u_0+V(x)u_0+\lambda u_0+\tau\rho^2u_0+\dfrac{\sqrt{6}}{2}\tau_1\left(F_xu_{-1}+F_xu_1\right)+\dfrac{\tau_2}{\sqrt{5}}\theta u_0=0,\\ -\Delta u_{-1}+V(x)u_{-1}+(\lambda-\mu)u_{-1}+\tau\rho^2u_{-1}+\tau_1\left(\dfrac{\sqrt{6}}{2}F_xu_0+F_xu_{-2}-F_zu_{-1}\right)-\dfrac{\tau_2}{\sqrt{5}}\theta u_1=0,\\ -\Delta u_{-2}+V(x)u_{-2}+(\lambda-2\mu)u_{-2}+\tau\rho^2u_{-2}+\tau_1(F_xu_{-1}-2F_zu_{-2})+\dfrac{\tau_2}{\sqrt{5}}\theta u_2=0,\end{cases} \tag{1.4}$$

along with the constraints

$$\int_{\mathbb{R}^d}(\sum_{j=-2}^{2}u_j^2)dx=N,\quad \int_{\mathbb{R}^d}\sum_{j=-2}^{2}(ju_j^2)dx=M. \tag{1.5}$$

where

$$F_x(\mathbf{u})=2(u_{-2}u_{-1}+u_2u_1)+\sqrt{6}(u_{-1}u_0+u_1u_0),\quad F_z(\mathbf{u})=\sum_{j=-2}^{2}(ju_j^2)$$

and

$$\rho^2(\mathbf{u})=\sum_{j=-2}^{2}u_j^2,\quad \theta(\mathbf{u})=\frac{1}{\sqrt{5}}(2u_2u_{-2}-2u_1u_{-1}+u_0^2).$$

For spin-1 BEC, the existence, asymptotic behavior and stability of solutions have been studied by many authors under certain conditions, see [15,32,38,39,42,44] and the references therein. Cao, Chern and Wei in [14] proved the existence of ground states for spin-1 BEC by minimizing the corresponding energy in one-dimensional. Kong, Wang and Zhao [38] gave the existence and detailed asymptotic behavior of ground states for spin-1 BEC with harmonic trapping potentials in two-dimensional case. Recently, in [43, 44], we developed an exhaustive analysis on standing waves with prescribed mass of physical states for spin-1 Bose-Einstein condensate in $\mathbb{R}^3$ and we give a complete description on ground states of spin-1 Bose-Einstein condensates with Ioffe-Pritchard magnetic field in $\mathbb{R}^2$ and $\mathbb{R}^3$. For numerical results on ground states and excited states of spin-1 BEC, we refer the reader to [4,15] and the reference therein.

For spin-2 BEC, Bao and Cai in [4] gave an efficient and accurate numerical method for computing ground states and dynamic behavior of spin-2 BEC based on the coupled Gross-Pitaevskii equations. The so-called single-mode approximation (SMA) phenomenon for spin-2 BEC in experimental observations [31] and numerical simulations [53] has been mentioned in [4]. To our best knowledge, mathematical theories about spin-2 BEC and the SMA phenomenon in experimental observations [31] and numerical simulations [53] has never been rigorous mathematical justifications.

In addition, in $\mathbb{R}^d(d=1,2,3)$, the study on solutions to (1.4)-(1.5) is absent in the literatures for any signs of spin-independent interaction, spin-exchange interaction and spin-singlet interaction. For spin-2 BEC, the problem becomes more difficult. One reason is that, the crossing terms in the energy functional exhibit inconsistent signs, so we can not take the absolute value of the ground state solution to obtain the non-negative property of the solutions and the Schwartz symmetrization method also does not work.

Based on these facts (two main motives) described above, in this present paper, we establish systematically mathematical theories for ground states and dynamics of spin-2 BEC. We show that rigorous mathematical justifications of these conclusions are exactly what is expected in ([4], Section 5) and show the validity of SMA phenomenon in experimental observations [31].

The working space

$$\Lambda := \Big\{(u_2,u_1,u_0,u_{-1},u_{-2})\in H^1(\mathbb{R}^d,\mathbb{R}^5)\Big| \int_{\mathbb{R}^d} V(x)\Big(\sum_{j=-2}^{2} u_j^2\Big)dx < +\infty\Big\} \tag{1.6}$$

is a Hilbert space equipped with the norm

$$\|(u_2,u_1,u_0,u_{-1},u_{-2})\|_\Lambda := \Bigg(\int_{\mathbb{R}^d}\Big(\sum_{j=-2}^{2}|\nabla u_j|^2\Big)dx + \int_{\mathbb{R}^d}\big(1+V(x)\big)\Big(\sum_{j=-2}^{2}u_j^2\Big)dx\Bigg)^{\frac{1}{2}}.$$

For any given $N>0$ and $|M|\le 2N$, we define

$$\mathcal{M} := \Big\{\mathbf{u}\in\Lambda\Big| \int_{\mathbb{R}^d}\Big(\sum_{j=-2}^{2}u_j^2\Big)dx = N,\ \int_{\mathbb{R}^d}\sum_{j=-2}^{2}(ju_j^2)dx = M\Big\},$$

then solutions to system (1.4)-(1.5) can be found as critical points of $E(\mathbf{u})$ constrained on $\mathcal{M}$ with

$$\begin{aligned}
E(\mathbf{u}) :&= \frac{1}{2}\int_{\mathbb{R}^d}\Big(\sum_{j=-2}^{2}|\nabla u_j|^2 + V(x)\rho^2\Big)dx + \frac{1}{4}\int_{\mathbb{R}^d}(\tau\rho^4 + \tau_1\mathbf{F}^2 + \tau_2\theta^2)dx \\
&= \frac{1}{2}\int_{\mathbb{R}^d}\Bigg(\sum_{j=-2}^{2}|\nabla u_j|^2 + V(x)\Big(\sum_{j=-2}^{2}u_j^2\Big)\Bigg)dx + \frac{\tau}{4}\int_{\mathbb{R}^d}\Bigg(\sum_{j=-2}^{2}u_j^2\Bigg)^2 dx \\
&\quad + \frac{\tau_1}{4}\int_{\mathbb{R}^d}\big(2(u_{-2}u_{-1}+u_2u_1)+\sqrt{6}(u_{-1}u_0+u_1u_0)\big)^2dx + \frac{\tau_1}{4}\int_{\mathbb{R}^d}\Bigg(\sum_{j=-2}^{2}(ju_j^2)\Bigg)^2dx \\
&\quad + \frac{\tau_2}{20}\int_{\mathbb{R}^d}(2u_2u_{-2}-2u_1u_{-1}+u_0^2)^2dx,
\end{aligned}$$

where $\mathbf{F}=(F_x,F_z)^T$ are real vector-valued functions.

Before introducing the main results, we recall some definitions (see also [7]):

Definition 1.1. *(i) We say that* $\mathbf{v}=(v_2,v_1,v_0,v_{-1},v_{-2})$ *is a ground state of* (1.4)-(1.5) *if*

$$E'|_{\mathcal{M}}(\mathbf{v}) = 0$$

and

$$E(\mathbf{v}) = \inf\Big\{E(\mathbf{u})\ s.t.\ E'|_{\mathcal{M}}(\mathbf{u})=0\ for\ \mathbf{u}=(u_2,u_1,u_0,u_{-1},u_{-2})\in\mathcal{M}\Big\}.$$

(ii) We say that $\mathbf{w} = (w_2, w_1, w_0, w_{-1}, w_{-2})$ *is an excited state of* (1.4)-(1.5) *if*

$$E'|_{\mathcal{M}}(\mathbf{w}) = 0$$

and

$$E(\mathbf{w}) > \inf \Big\{ E(\mathbf{u}) \ s.t.\ E'|_{\mathcal{M}}(\mathbf{u}) = 0 \ for\ \mathbf{u} = (u_2, u_1, u_0, u_{-1}, u_{-2}) \in \mathcal{M} \Big\}.$$

We emphasize that this definition is meaningful even if the energy E is unbounded from below on $\mathcal{M}$. In addition, variational problems with the energy restricted on the manifold $\mathcal{M}$ is particularly appropriate for the study of the stability properties of the ground states, as all the energy, the number of atoms N and the total magnetization M are conserved along the flow generated by (1.3).

Definition 1.2. *(i) We say that the set* $\boldsymbol{G}$ *is orbitally stable if* $\boldsymbol{G} \neq \emptyset$ *and for any* $\epsilon > 0$, *there exists a* $\delta > 0$ *such that, provided that an initial datum* $\mathbf{\Phi}(0) = (\Phi_2(0), \Phi_1(0), \Phi_0(0), \Phi_{-1}(0), \Phi_{-2}(0))$ *for* (1.3) *satisfies*

$$\inf_{(u_2,u_1,u_0,u_{-1},u_{-2}) \in \boldsymbol{G}} \big\| (u_2, u_1, u_0, u_{-1}, u_{-2}) - \mathbf{\Phi}(0) \big\|_{H^1(\mathbb{R}^d, \mathbb{C}^d)} < \delta,$$

then $(\Phi_2, \Phi_1, \Phi_0, \Phi_{-1}, \Phi_{-2})$ *is globally defined and*

$$\inf_{(u_2,u_1,u_0,u_{-1},u_{-2}) \in \boldsymbol{G}} \big\| (u_2, u_1, u_0, u_{-1}, u_{-2}) - \big(\Phi_2(t), \Phi_1(t), \Phi_0(t), \Phi_{-1}(t), \Phi_{-2}(t)\big) \big\|_{H^1(\mathbb{R}^d, \mathbb{C}^d)} < \epsilon, \ \forall\ t > 0,$$

where $\big(\Phi_2(t), \Phi_1(t), \Phi_0(t), \Phi_{-1}(t), \Phi_{-2}(t)\big)$ *is the solution to* (1.3) *corresponding to the initial condition* $\mathbf{\Phi}(0)$.

(ii) A standing wave $\big(e^{-i(\lambda+2\mu)t} u_2(x),\ e^{-i(\lambda+\mu)t} u_1(x),\ e^{-i\lambda t} u_0(x), e^{-i(\lambda-\mu)t} u_{-1}(x), e^{-i(\lambda-2\mu)t} u_{-2}(x)\big)$ *is said to be strongly unstable if for any* $\epsilon > 0$, *there exists* $\mathbf{\Phi}(0) \in H^1(\mathbb{R}^d, \mathbb{C}^5)$, *such that*

$$\big\| (u_2, u_1, u_0, u_{-1}, u_{-2}) - \mathbf{\Phi}(0) \big\|_{H^1(\mathbb{R}^d, \mathbb{C}^5)} < \epsilon,$$

and $\big(\Phi_2(t), \Phi_1(t), \Phi_0(t), \Phi_{-1}(t), \Phi_{-2}(t)\big)$ *blows-up in finite time, namely* $T_{\max} < +\infty$, *where* $T_{max} > 0$ *is the positive maximal time of existence.*

Firstly, we give some classification results for the ground state solutions of (1.4)-(1.5) in $\mathbb{R}^d (d = 1, 2, 3)$. To state our main results, we consider the following minimization problem

$$\inf_{u \in \mathcal{N}} \Big\{ \frac{1}{2} \int_{\mathbb{R}^d} (|\nabla u|^2 + V(x) u^2 + \frac{1}{4} (\tau + 4\tau_1) u^4) dx \Big\}, \tag{1.7}$$

where

$$\mathcal{N} = \Big\{ u \in H^1(\mathbb{R}^d) \big| \int_{\mathbb{R}^d} u^2 dx = N \Big\}. \tag{1.8}$$

Our main results in this aspect are the following.

Theorem 1.1. *Suppose* $\tau_1 < \frac{\tau_2}{20} < 0$ *or* $\tau_1 < 0, \tau_2 \geq 0$ *, then the ground state solution of* (1.4) *and* (1.5) *must be in the form*

$$\mathbf{u} = \pm \bigg(\frac{(2N+M)^2}{16N^2} \rho,\ \frac{(2N+M)\sqrt{4N^2 - M^2}}{8N^2} \rho,\ \frac{\sqrt{6}(4N^2 - M^2)}{16N^2} \rho, \\ \frac{(2N-M)\sqrt{4N^2 - M^2}}{8N^2} \rho,\ \frac{(2N-M)^2}{16N^2} \rho \bigg),$$

where ρ *is a solution of* (1.7)-(1.8).

Theorem 1.2. *Suppose $\tau_1 < 0$ and $\tau_2 = 0$, then the ground state solution of* (1.4)-(1.5) *must be in the form*

$$\mathbf{u} = \Big(\frac{(2N+M)^2}{16N^2}\rho,\ \frac{(2N+M)\sqrt{4N^2-M^2}}{8N^2}\rho,\ \frac{\sqrt{6}(4N^2-M^2)}{16N^2}\rho,$$
$$\frac{(2N-M)\sqrt{4N^2-M^2}}{8N^2}\rho,\ \frac{(2N-M)^2}{16N^2}\rho \Big),$$

where ρ is a solution of (1.7)-(1.8).

Remark 1.1. *To our best knowledge, this is the first theoretical result dealing with the classification of ground states for spin-2 BEC. These results not only show that spin-2 BEC has independent characteristics on the sign of spin-independent interaction, spin-exchange interaction and spin-singlet interaction, but also support the SMA phenomenon in experimental observations [31] and numerical simulations [53], that is, each component of the ground state is a multiple of one single density function. Rigorous mathematical justifications of these conclusions are exactly what is expected in ([4], Section 5).*

Remark 1.2. *The proof of above Theorems is non-trivial and very skillful, which mainly relies on the technique of mass-redistribution for the ground state. Precisely, for any $\mathbf{u} \in \mathcal{M}$, we find a special mass-redistribution $\mathbf{v} = \mathbf{b}^*\rho$ (see* (3.1) *for the definition of $\mathbf{b}^*$) of $\mathbf{u}$, that remains in $\mathcal{M}$, which has a lower total energy.*

Recall the following nonlinear equation in $\mathbb{R}^d (d = 1, 2, 3)$:

$$-\Delta u + u = u^3, \quad u \in H^1(\mathbb{R}^d), \tag{1.9}$$

from [36], there exists a unique positive solution $Q(x)$ for (1.9). By the related Pohozaev identity, we get

$$a^* := \int_{\mathbb{R}^d} |Q|^2 dx = \frac{4-d}{d} \int_{\mathbb{R}^d} |\nabla Q|^2 dx = \frac{4-d}{4} \int_{\mathbb{R}^d} |Q|^4 dx. \tag{1.10}$$

Moreover, we obtain from [24] that $Q(x)$ satisfies

$$Q(x),\ |\nabla Q(x)| = O\big(|x|^{-\frac{d-1}{2}} e^{-|x|}\big), \quad \text{as } |x| \to \infty.$$

We consider the harmonic trapped case, where the confining electromagnetic potential $V(x) = |x|^2$ in the system. Based on the fact that the characteristics of spin-2 BEC are different in 1D, 2D and 3D, we deal with them respectively. Firstly, we consider ground states of spin-2 BEC in 1D by the following minimization problem

$$m := \inf_{\mathbf{u} \in \mathcal{M}} E(\mathbf{u}), \tag{1.11}$$

Theorem 1.3. *For any $\tau,\ \tau_1,\ \tau_2 \in \mathbb{R}$, there exists a ground state solution for* (1.4)-(1.5).

Theorem 1.4. *Suppose $\tau < 4\tau_1 < 0$ and $\tau_2 = 0$, then there exists a nonnegative ground state solution for* (1.4)-(1.5). *Moreover, u_j is radial symmetric and strictly decreasing in $|x|$ for $j = 2, 1, 0, -1, -2$.*

Remark 1.3. *When $\tau < 0$, each hyperfine state is confined to form a spike due to self-attractive interaction. Theorem 1.4 characterize its normalized shape, which is symmetric about origin and decreasing in $|x|$. In this case, the spin-exchange interaction is also attractive, the five hyperfine states overlap to each other with peaks at origin. The condition $\tau < 4\tau_1 < 0$ should only be a technical condition.*

Remark 1.4. *Theorem 1.3 and Theorem 1.4 show that the ground state solution is nonnegative and radial symmetric when $\tau_2 = 0$, but we don't know whether the ground state solution is nonnegative or radial symmetric when $\tau_2 \neq 0$. Indeed, in general case, we can not take the absolute value of the ground state solution to obtain the non-negative property of the solutions and the Schwartz symmetrization method also does not work.*

Remark 1.5. *When $V(x) \equiv 0$, similar to [14], we can show that there has no ground state solution for system* (1.4)-(1.5) *if $\tau \geq 0$, $\tau_1 \geq 0$ and $\tau_2 \geq 0$. In this case, the spin-independent interaction, spin-exchange interaction and spin-singlet interaction are all repulsive, the atoms can not be confined, there is no nontrivial ground state solution. This indicates that the introduction of an external harmonic trapping potential enriches the solutions set of system* (1.4)-(1.5) *and also shows the influence of the trapping potential term $|x|^2u$ on system* (1.4)-(1.5) *is important.*

In the one dimensional case, the energy functional is bounded from below on $\mathcal{M}$, the ground states exist and they are obtained as global minimizers. However, the energy functional is not always bounded on $\mathcal{M}$ when $d = 2$. Hence, we give the explicit thresholds for existence and nonexistence of ground state solution depending on the number of atoms N. We consider ground states of spin-2 BEC in 2D by the following minimization problem

$$m(N) := \inf_{\mathbf{u}\in\mathcal{M}} E(\mathbf{u}).$$

Setting

$$N^* := -\frac{5a^*}{5\tau + 20\tau_1 + \tau_2}, \quad N^{**} := -\frac{a^*}{\tau + 4\tau_1},$$

then we have the following result.

Theorem 1.5. *Let $\tau < 0$ and $\tau_1 < 0$,*
(i) if $\tau_2 < 0$, then $m(N)$ has at least one minimizer for $0 < N < N^*$, while $m(N)$ has no minimizer for $N > N^{**}$;
(ii) if $\tau_2 \geq 0$, then $m(N)$ has at least one minimizer for $0 < N < N^{**}$, while $m(N)$ has no minimizer for $N \geq N^{**}$;
(iii) for any minimizer $\mathbf{u} = (u_2, u_1, u_0, u_{-1}, u_{-2}) \in \mathcal{M}(N)$ of $m(N)$, there holds

$$\left\| \mathbf{u} - \left(l_2 e^{-\frac{x^2}{2}}, l_1 e^{-\frac{x^2}{2}}, l_0 e^{-\frac{x^2}{2}}, l_{-1} e^{-\frac{x^2}{2}}, l_{-2} e^{-\frac{x^2}{2}} \right) \right\|_{\Lambda}^2 = O(N), \quad \text{as } N \to 0^+, \tag{1.12}$$

where

$$l_i = \frac{1}{\pi} \int_{\mathbb{R}^2} u_i e^{-\frac{x^2}{2}} dx, \quad \text{for } i = 2, 1, 0, -1, -2.$$

The above existence and nonexistence results mainly rely on the Gagliardo-Nirenberg type inequality given in Lemma 2.2. The proof of existence and nonexistence for ground state solution is nontrivial. By carefully and skilfully choosing test functions, we can obtain the explicit thresholds of existence and nonexistence for ground state solution. To obtain the uniform lower bound of $m(N)$, we have to make an accurate estimate of the coupling terms in the energy functional. By compact embedding, the existence of ground state can be obtained more easily. In the proof (iii) of Theorem 1.5, we mainly rely on the good properties of eigenvalues and eigenvectors for the harmonic oscillator $-\Delta + |x|^2$ operator, as well as the accurate estimation of the ground state energy.

Remark 1.6. *Theorem 1.5 shows that the ground state solution behaves like the first eigenvector of the harmonic oscillator $-\Delta+|x|^2$ for small N. For attractive spin-singlet interaction case $\tau_2 \geq 0$, Theorem 1.5 gives a complete classification for the existence and nonexistence of global minimizers. Moreover, the explicit thresholds for existence and nonexistence of ground state solution were obtained.*

Remark 1.7. *Theorem 1.5 gives the existence and nonexistence of ground states along with qualitative properties describing extinction of atoms, of planar spin-2 BEC. Particularly, for the repulsive spin-singlet interaction case $\tau_2 < 0$, if $N > N^{**}$, then $m(N)$ has no minimizer. Note that $N^{**} > N^*$, it remains open that whether there exists a minimizer for $m(N)$ when $N^* \leq N \leq N^{**}$. Precisely, on one hand, we don't know that in this case whether $m(N)$ is well defined. On the other hand, it seems difficult to find a suitable test function to prove that $m(N) = -\infty$, due to the competitions among the spin-independent interaction term, spin-exchange interaction term and spin-singlet interaction term. We believe it is interesting to fulfill this gap.*

In the following, C and C' are indiscriminately used to denote various absolutely positive constants. $a \sim b$ means that $Cb \leq a \leq C'b$. Next, qualitative properties of ground states in 2D are analysed.

Theorem 1.6. *Let $\tau < 0$, $\tau_1 < 0$, $\tau_2 \geq 0$, $N_n \nearrow N^{**}$ as $n \to \infty$ and $\mathbf{u_n} = (u_{2n}, u_{1n}, u_{0n}, u_{-1}, u_{-2}) \in \mathcal{M}(N_n)$ be a minimizer of $m(N_n)$. We have*

$$m(N_n) \sim (N^* - N_n)^{\frac{1}{2}}, \text{ as } n \to \infty. \tag{1.13}$$

In addition, $\mathbf{u_n}$ satisfies

$$\begin{cases} \lim\limits_{n\to\infty} \varepsilon_n u_{2n}(\varepsilon_n x + \tilde{z}_{2n}) = \sqrt{\frac{N^{**}}{a^*}} \frac{(2N^{**}+M)^2}{16(N^{**})^2} Q(x), \\ \lim\limits_{n\to\infty} \varepsilon_n u_{1n}(\varepsilon_n x + \tilde{z}_{1n}) = \sqrt{\frac{N^{**}}{a^*}} \frac{(2N^{**}+M)\sqrt{4(N^{**})^2 - M^2}}{8(N^{**})^2} Q(x), \\ \lim\limits_{n\to\infty} \varepsilon_n u_{0n}(\varepsilon_n x + \tilde{z}_{0n}) = \sqrt{\frac{N^{**}}{a^*}} \frac{\sqrt{6}(4(N^{**})^2 - M^2)}{16(N^{**})^2} Q(x), \\ \lim\limits_{n\to\infty} \varepsilon_n u_{-1n}(\varepsilon_n x + \tilde{z}_{-1n}) = \sqrt{\frac{N^{**}}{a^*}} \frac{(2N^{**}-M)\sqrt{4(N^{**})^2 - M^2}}{8(N^{**})^2} Q(x), \\ \lim\limits_{n\to\infty} \varepsilon_n u_{-2n}(\varepsilon_n x + \tilde{z}_{-2n}) = \sqrt{\frac{N^{**}}{a^*}} \frac{(2N^{**}-M)^2}{16(N^{**})^2} Q(x), \end{cases} \tag{1.14}$$

where $\tilde{z}_{in}$ $(i = 2, 1, 0, -1, -2)$ is the unique maximum point of u_{in} with

$$\lim_{n\to\infty} \left|\frac{\tilde{z}_{in} - \tilde{z}_{jn}}{\varepsilon_n}\right| = 0 \ (i, j = 2, 1, 0, -1, -2, i \neq j), \quad \lim_{n\to\infty} |\tilde{z}_{in}| = 0$$

and

$$\varepsilon_n = C\big(N^* - N_n\big)^{\frac{1}{4}}. \tag{1.15}$$

Remark 1.8. *Theorem 1.6 shows that for the attractive spin-singlet interaction $\tau_2 \geq 0$, any minimizer $\mathbf{u}$ of $m(N)$ in the case of $N \nearrow N^{**}$ is nontrivial. These results also support the SMA phenomenon in experimental observations [31] and numerical simulations [53], that is, each component of the ground state is a multiple of one single density function. Rigorous mathematical justifications of these conclusions are exactly what is expected in ([4], Section 5).*

Global minimizers obtained in Theorem 1.5 are obvious ground states for (1.4)-(1.5) in $\mathbb{R}^2$. However, the functional $E(\mathbf{u})$ is no longer bounded from below on $\mathcal{M}$ in the 3D case. Hence, the global minimization method does not work. Instead, we consider a local minimization problem. Motivated by [3], in order to get ground states, for any $r>0$ and $N \leq \frac{r}{3}$, we consider the following local minimization problem

$$m_N^r := \inf_{\mathbf{u}=(u_2,u_1,u_0,u_{-1},u_{-2})\in\mathcal{M}\cap B(r)} E(\mathbf{u}),$$

where

$$B(r) := \Big\{\mathbf{u}=(u_2,u_1,u_0,u_{-1},u_{-2})\in\Lambda \Big| \|\mathbf{u}\|_{\Lambda}^2 \leq r\Big\}$$

and

$$\|\mathbf{u}\|_{\Lambda}^2 := \int_{\mathbb{R}^3}\Big(\big(\sum_{j=-2}^{2}|\nabla u_j|^2\big)+|x|^2\big(\sum_{j=-2}^{2}u_j^2\big)\Big)dx.$$

Our main result in this aspect is the following.

Theorem 1.7. *Suppose $\tau\leq 0$, $\tau_1\leq 0$ and $\tau_2\leq 0$, then*
(i) for any $r>0$, m_N^r has a minimizer if $N\leq\frac{r}{3}$;
(ii) for any $r>0$, there exists $N_0=N_0(r)<\frac{r}{3}$, such that for $0<N\leq N_0$, each minimizer of m_N^r is a critical point of $E(\mathbf{u})$ restricted to $\mathcal{M}$. Moreover, there exists $N_1\in(0,N_0]$ small enough, such that for $0<N<N_1$, the minimizer of m_N^r is a ground state of (1.4)-(1.5) on $\mathcal{M}$;
(iii) suppose $\mathbf{u_N}=(u_{2N},u_{1N},u_{0N},u_{-1N},u_{-2N})\in\mathcal{M}\cap B(r)$ is a minimizer of m_N^r, then

$$\frac{m_N^r}{N}\to\frac{3}{2},\qquad \frac{\|\mathbf{u_N}\|_{\Lambda}^2}{N}\to 3,\quad \text{as } N\to 0^+.$$

Further,

$$\|\mathbf{u_N}-(l_{20}\Psi_0,l_{10}\Psi_0,l_{00}\Psi_0,l_{-10}\Psi_0,l_{-20}\Psi_0)\|_{\Lambda}^2=O(N^2),$$

where Ψ_0 is the unique normalized positive eigenvector of the harmonic oscillator $-\Delta+|x|^2$ and

$$l_{i0}=\int_{\mathbb{R}^3}u_{iN}\Psi_0 dx,\ \text{for } i=2,1,0,-1,-2;$$

(iv) for $r>0$ and $0<N\leq N_0$, denote

$$\mathcal{M}_N^r := \Big\{\mathbf{u}\in\mathcal{M}\cap B(r)\Big|E(\mathbf{u})=m_N^r\Big\},$$

then the set $\mathcal{M}_N^r$ is stable under the flow associated with problem (1.3).

Based on the ground states obtained in Theorem 1.7, we are able to get an excited state.

Theorem 1.8. *Suppose $\tau\leq 0$, $\tau_1\leq 0$ and $\tau_2\leq 0$, then*
(i) for any $r>0$ and $0\leq M\leq 2N\leq 2N_0$, there exists an excited state $\hat{\mathbf{u}}=(\hat{u}_2,\hat{u}_1,\hat{u}_0,\hat{u}_{-1},\hat{u}_{-2})$ of (1.4) on $\mathcal{M}$, with some $\hat{\mu},\hat{\lambda}\in\mathbb{R}$ as Lagrange multipliers;
(ii) the corresponding standing wave

$$\Big(e^{-i(\hat{\lambda}+2\hat{\mu})t}\hat{u}_2, e^{-i(\hat{\lambda}+\hat{\mu})t}\hat{u}_1, e^{-i\hat{\lambda}t}\hat{u}_0, e^{-i(\hat{\lambda}-\hat{\mu})t}\hat{u}_{-1}, e^{-i(\hat{\lambda}-2\hat{\mu})t}\hat{u}_{-2}\Big)$$

is strongly unstable.

Remark 1.9. *Theorem 1.8 together with Theorem 1.7 yield the multiplicity of standing waves for problem* (1.3) *and correspond to the numerical results established in [4]. Theorem 1.7 (iii) shows that the standing waves of problem* (1.3) *associated to the set* $\mathcal{M}_N^r$ *behave like the first eigenvector of the harmonic oscillator* $-\Delta + |x|^2$ *for small* N. *In 3D, the authors in [47] described that for an inhomogeneous condensate, however, if the nonlinearity is relatively weak, the spatial localization provided by an external trap potential can stabilize the condensate against collapse, our results are consistent with this phenomenon. Theorem 1.3, Theorem 1.5 and Theorem 1.7 show that the characteristics of spin-2 BEC are different in 1D, 2D and 3D. These phenomena are consistent with the numerical simulation results in [4].*

Remark 1.10. *Compared with spin-1 BEC, the results on vanishing phenomenon of spin-2 BEC are much less in the literature. In our previous work [43], for spin-1 BEC in* $\mathbb{R}^2$, *we prove that for the ferromagnetic case, if* $|M| \in [0, N)$, *any minimizer* $\mathbf{u}$ *of* $m(N)$ *in the case of* $N \nearrow N^*$ *is nontrivial. While for the antiferromagnetic case, when* $M = 0$, *the minimizers* $\mathbf{u}$ *of* $m(N)$ *must be semi-trivial as* $N \nearrow N^*$. *We believe that theoretically proving similar qualitative and quantitative properties of spin-2 BEC are interesting and challenging problems. In the following work, we will focus on these issues.*

Notations. In the paper, we use the following notations. $L^p = L^p(\mathbb{R}^d)$ with norm $\|\cdot\|_{L^p(\mathbb{R}^d)} = \|\cdot\|_{L^p}$, $H^1(\mathbb{R}^d)$ is the usual Sobolev space and $H^1(\mathbb{R}^d, \mathbb{R}^5) = (H^1(\mathbb{R}^d))^5$ and $L^p(\mathbb{R}^d, \mathbb{R}^5) = (L^p(\mathbb{R}^d))^5$ are the vector-valued functions spaces.

The paper is organized as follows. In Section 2, we introduce some preliminary results. In Section 3, we prove Theorem 1.1 and Theorem 1.2. In section 4, we deal with the 1D case and prove Theorem 1.3 as well as Theorem 1.4. In section 5, we consider 2D case and prove Theorem 1.5 and Theorem 1.6. Finally, Theorem 1.7 and Theorem 1.8 will be proved in Section 6.

2 Preliminaries

In this section, we give some preliminaries which are useful for the rest of the paper. First, we give a compact embedding result.

Lemma 2.1. *(Pankov [46]) The embedding* $\Lambda \hookrightarrow L^p(\mathbb{R}^d, \mathbb{R}^5)$ *is compact for any* $p \in [2, \frac{2d}{d-2})$, *where* Λ *is defined in* (1.6).

For any $u \in H^1(\mathbb{R}^d)(d = 1, 2, 3)$, by Lemma 2.4 in [8], u satisfies the classical Gagliardo-Nirenberg inequality

$$\int_{\mathbb{R}} |u|^4 dx \leq C\Big(\int_{\mathbb{R}} |u'|^2 dx\Big)^{\frac{1}{2}} \cdot \Big(\int_{\mathbb{R}} |u|^2 dx\Big)^{\frac{3}{2}}, \tag{2.1}$$

$$\int_{\mathbb{R}^2} u^4 dx \leq \frac{2}{a^*} \int_{\mathbb{R}^2} |\nabla u|^2 dx \cdot \int_{\mathbb{R}^2} u^2 dx \tag{2.2}$$

and

$$\int_{\mathbb{R}^3} u^4 dx \leq \frac{4\sqrt{3}}{9a^*}\Big(\int_{\mathbb{R}^3} |\nabla u|^2 dx\Big)^{\frac{3}{2}} \cdot \Big(\int_{\mathbb{R}^3} u^2 dx\Big)^{\frac{1}{2}}, \tag{2.3}$$

where a^* is defined in (1.10).

For any $(u_2, u_1, u_0, u_{-1}, u_{-2}) \in H^1(\mathbb{R}^d, \mathbb{R}^5)$, there also holds the similar inequality.

Lemma 2.2. *For $\mathbf{u} \in H^1(\mathbb{R}^d, \mathbb{R}^5)$, there holds*

$$\int_{\mathbb{R}} (\sum_{j=-2}^{2} |u_j|^2)^2 dx \leq C\Big(\int_{\mathbb{R}} (\sum_{j=-2}^{2} |u_j'|^2) dx\Big)^{\frac{1}{2}} \cdot \Big(\int_{\mathbb{R}} (\sum_{j=-2}^{2} |u_j|^2) dx\Big)^{\frac{3}{2}}, \tag{2.4}$$

$$\int_{\mathbb{R}^2} \big(\sum_{j=-2}^{2} u_j^2\big)^2 dx \leq \frac{2}{a^*} \int_{\mathbb{R}^2} \big(\sum_{j=-2}^{2} |\nabla u_j|^2\big) dx \cdot \int_{\mathbb{R}^2} \big(\sum_{j=-2}^{2} u_j^2\big) dx \tag{2.5}$$

and

$$\int_{\mathbb{R}^3} \big(\sum_{j=-2}^{2} u_j^2\big)^2 dx \leq C_* \Big(\int_{\mathbb{R}^3} \big(\sum_{j=-2}^{2} |\nabla u_j|^2\big) dx\Big)^{\frac{3}{2}} \cdot \Big(\int_{\mathbb{R}^3} \big(\sum_{j=-2}^{2} u_j^2\big) dx\Big)^{\frac{1}{2}}, \tag{2.6}$$

where $C_* = \frac{4\sqrt{3}}{9a^*}$. Moreover, up to translations and suitable scalings, equality (2.5) *holds only at*

$$\begin{cases} u_2(x) = Q(x) \cos \varphi_1, \\ u_1(x) = Q(x) \sin \varphi_1 \cos \varphi_2, \\ u_0(x) = Q(x) \sin \varphi_1 \sin \varphi_2 \cos \varphi_3, \\ u_{-1}(x) = Q(x) \sin \varphi_1 \sin \varphi_2 \sin \varphi_3 \cos \varphi_4, \\ u_{-2}(x) = Q(x) \sin \varphi_1 \sin \varphi_2 \sin \varphi_3 \sin \varphi_4, \end{cases} \tag{2.7}$$

for $\varphi_j \in [0, \frac{\pi}{2})$ $(j = 2, 1, 0, -1, -2)$ and $Q(x)$ is the unique positive solution to (1.9).

Proof. We only prove the 2D case, the proof of (2.4) and (2.6) are similar, we omit the details here. Consider the minimization problem:

$$k := \inf_{(0,0,0,0,0) \neq \mathbf{u} \in H^1(\mathbb{R}^2, \mathbb{R}^5)} K(\mathbf{u}), \tag{2.8}$$

where

$$K(\mathbf{u}) = \frac{\int_{\mathbb{R}^2} \big(\sum\limits_{j=-2}^{2} |\nabla u_j|^2\big) dx \cdot \int_{\mathbb{R}^2} \big(\sum\limits_{j=-2}^{2} u_j^2\big) dx}{\int_{\mathbb{R}^2} \big(\sum\limits_{j=-2}^{2} u_j^2\big)^2 dx}.$$

To obtain (2.6), it is sufficient to show $k = \frac{a^*}{2}$. Let $Q(x)$ be the unique positive solution to (1.9) and set

$$(u_2, u_1, u_0, u_{-1}, u_{-2}) = \Big(\frac{Q}{\sqrt{5}}, \frac{Q}{\sqrt{5}}, \frac{Q}{\sqrt{5}}, \frac{Q}{\sqrt{5}}, \frac{Q}{\sqrt{5}}\Big),$$

then by (1.10),

$$K(\mathbf{u}) = \frac{\int_{\mathbb{R}^2} |\nabla Q|^2 dx \cdot \int_{\mathbb{R}^2} Q^2 dx}{\int_{\mathbb{R}^2} Q^4 dx} = \frac{a^*}{2}.$$

By direct calculation, for arbitrary $(u_1, u_2, u_0, u_{-1}, u_{-2}) \in H^1(\mathbb{R}^2, \mathbb{R}^5)$, there holds

$$\left|\nabla \sqrt{\sum_{j=-2}^{2} u_j^2}\right|^2 \leq \sum_{j=-2}^{2} |\nabla u_j|^2,$$

therefore, by (2.3),

$$
\begin{aligned}
K(\mathbf{u}) &\geq \frac{\int_{\mathbb{R}^2}(|\nabla\sqrt{\sum_{j=-2}^{2} u_j^2}|^2)dx \cdot \int_{\mathbb{R}^2}\big(\sum_{j=-2}^{2} u_j^2\big)dx}{\int_{\mathbb{R}^2}\big(\sum_{j=-2}^{2} u_j^2\big)^2 dx} \\
&= \frac{\int_{\mathbb{R}^2}\big(|\nabla\sqrt{\sum_{j=-2}^{2} u_j^2}|^2\big)dx \cdot \int_{\mathbb{R}^2}\big(\sqrt{\sum_{j=-2}^{2} u_j^2}\big)^2 dx}{\int_{\mathbb{R}^2}\big(\sqrt{\sum_{j=-2}^{2} u_j^2}\big)^4 dx} \geq \frac{a^*}{2}.
\end{aligned}
$$

Thus, $k = \frac{a^*}{2}$. Similar to [21], we conclude that to find the minimizer of (2.8) is equivalent to the ground state of the following system:

$$
\begin{cases}
-\Delta u_2 + u_2 = u_2^3 + \big(u_1^2 + u_0^2 + u_{-1}^2 + u_{-2}^2\big)u_2, \\
-\Delta u_1 + u_1 = u_1^3 + \big(u_2^2 + u_0^2 + u_{-1}^2 + u_{-2}^2\big)u_1, \\
-\Delta u_0 + u_0 = u_0^3 + \big(u_2^2 + u_1^2 + u_{-1}^2 + u_{-2}^2\big)u_0, \\
-\Delta u_{-1} + u_{-1} = u_{-1}^3 + \big(u_2^2 + u_1^2 + u_0^2 + u_{-2}^2\big)u_{-1}, \\
-\Delta u_{-2} + u_{-2} = u_{-2}^3 + \big(u_2^2 + u_1^2 + u_0^2 + u_{-1}^2\big)u_{-2},
\end{cases} \tag{2.9}
$$

Moreover, we have

$$
\int_{\mathbb{R}^2}\big(\sum_{j=-2}^{2}|\nabla u_j|^2\big)dx = \int_{\mathbb{R}^2}\big(\sum_{j=-2}^{2} u_j^2\big)dx = \frac{1}{2}\int_{\mathbb{R}^2}\big(\sum_{j=-2}^{2} u_j^2\big)^2 dx.
$$

Then similar to the arguments in [19] for three components system, the ground state of (2.9) is of the form (2.7), hence equality (2.5) holds only for the ground state of (2.9). □

Finally, we give the pure point spectrum and the associated eigenvectors for harmonic oscillator $-\Delta + |x|^2$, which is useful for us to study the qualitative properties of solutions for $m(N)$.

Lemma 2.3. *([1]) The pure point spectrum of the harmonic oscillator $-\Delta + |x|^2$ is*

$$
\sigma(-\Delta + |x|^2) = \big\{\xi_k = d + 2k,\ k \in \mathbb{N}\big\},
$$

and the corresponding eigenvectors are given by Hermite functions (denoted by Ψ_k, associated to ξ_k), which form an orthogonal basis of $L^2(\mathbb{R}^d)$. Particularly, the first eigenvector is $\Psi_0 = \frac{1}{\pi^{\frac{d}{4}}}e^{-\frac{x^2}{2}}$ and further Ψ_0 satisfies the Pohozaev identity:

$$
(d-2)\int_{\mathbb{R}^d}|\nabla\Psi_0|^2dx + (d+2)\int_{\mathbb{R}^d}|x|^2\Psi_0^2dx = d^2\int_{\mathbb{R}^d}\Psi_0^2dx,
$$

which follows that

$$
\int_{\mathbb{R}^d}|\nabla\Psi_0|^2dx = \int_{\mathbb{R}^d}|x|^2\Psi_0^2dx = \frac{d}{2}. \tag{2.10}
$$

3 Proofs of Theorem 1.1 and Theorem 1.2

The proofs of Theorem 1.1 and Theorem 1.2 mainly rely on a principle, that the mass-redistribution for n-tuple of real-valued functions will decrease the kinetic energy. We now introduce the definition and properties for the mass-redistribution.

Definition 3.1. *[40] Let $\mathbf{f} = (f_1, f_2, \cdots, f_n) \in H^1(\mathbb{R}^d, \mathbb{R}^n)$ be an n-tuple of real-valued functions and $\mathbf{g} = (g_1, g_2, \cdots, g_m)$ be an m-tuple of nonnegative functions. We say $\mathbf{g}$ is a mass-redistribution of $\mathbf{f}$, if $g_l^2 = \sum_{k=1}^n b_{lk} f_k^2$ for each l, where $b_{lk} \geq 0$ are constants and $\sum_{k=1}^n b_{lk} = 1$ for each k.*

Proposition 3.1. *[40] For any mass-redistribution $\mathbf{g}$ of $\mathbf{f}$, we have*
(i) $|\mathbf{g}| = |\mathbf{f}|$;
(ii) $|\nabla \mathbf{g}| \leq |\nabla \mathbf{f}|$. Moreover, $|\nabla \mathbf{g}| = |\nabla \mathbf{f}|$ if and only if $f_j \nabla f_k = f_k \nabla f_j$ for each $k \neq j$ with $b_{lj} b_{lk} \neq 0$ for at least one l.

Suppose $b_j \geq 0$ ($j = 2, 1, 0, -1, -2$) and $\mathbf{b} = (b_2, b_1, b_0, b_{-1}, b_{-2})$, we consider the maximization problem

$$\max_{\mathbf{b} \in \mathcal{B}} Q(\mathbf{b})$$

where

$$Q(\mathbf{b}) = \big(2(b_{-2}b_{-1} + b_2 b_1) + \sqrt{6}(b_{-1}b_0 + b_1 b_0)\big)^2 + \Big(\sum_{j=-2}^{2} (j b_j^2)\Big)^2,$$

and

$$\mathcal{B} = \Big\{ \mathbf{b} \in \mathbb{R}^5 \,|\, b_j \geq 0 \text{ and } \mathbf{b} \text{ satisfies } \sum_{j=-2}^{2} b_j^2 = 1,\ \sum_{j=-2}^{2} (j b_j^2) = \frac{M}{N} \Big\}.$$

Lemma 3.1. *Assume $|M| \leq 2N$, then there exists a unique $\mathbf{b}^* = (b_2^*, b_1^*, b_0^*, b_{-1}^*, b_{-2}^*) \in \mathcal{B}$, such that*

$$\max_{\mathbf{b} \in \mathcal{B}} Q(\mathbf{b}) = Q(\mathbf{b}^*) = 4,$$

where

$$\begin{aligned} b_2^* &= \frac{(2N+M)^2}{16N^2}, \quad b_0^* = \frac{\sqrt{6}(4N^2 - M^2)}{16N^2}, \quad b_{-2}^* = \frac{(2N-M)^2}{16N^2}, \\ b_1^* &= \frac{(2N+M)\sqrt{4N^2 - M^2}}{8N^2}, \quad b_{-1}^* = \frac{(2N-M)\sqrt{4N^2 - M^2}}{8N^2}. \end{aligned} \tag{3.1}$$

Proof. By direct calculations, for any $\mathbf{b} \in \mathcal{B}$, we have

$$\begin{aligned} &4\Big(\sum_{j=-2}^{2} (b_j^2)\Big)^2 - Q(\mathbf{b}) \\ &= 2(b_0 b_{-1} - \sqrt{6} b_1 b_{-2})^2 + (b_1 b_{-1} - 4 b_2 b_{-2})^2 + (2b_0^2 - 3 b_1 b_{-1})^2 \\ &+ (\sqrt{3} b_1^2 - 2\sqrt{2} b_0 b_2)^2 + (\sqrt{3} b_{-1}^2 - 2\sqrt{2} b_0 b_{-2})^2 + 2(b_0 b_1 - \sqrt{6} b_2 b_{-1})^2 \geq 0. \end{aligned} \tag{3.2}$$

Hence, if $\mathbf{b}$ satisfies the following algebra system

$$\begin{cases} b_0b_{-1}-\sqrt{6}b_1b_{-2}=0,\\ b_1b_{-1}-4b_2b_{-2}=0,\\ 2b_0^2-3b_1b_{-1}=0,\\ \sqrt{3}b_1^2-2\sqrt{2}b_0b_2=0,\\ \sqrt{3}b_{-1}^2-2\sqrt{2}b_0b_{-2}=0,\\ b_0b_1-\sqrt{6}b_2b_{-1}=0,\end{cases} \tag{3.3}$$

then

$$Q(\mathbf{b})=4\big(\sum_{j=-2}^{2}(b_j^2)\big)^2=4.$$

By solving above algebraic system directly and using $\sum_{j=-2}^{2}b_j^2=1$ and $\sum_{j=-2}^{2}(jb_j^2)=\frac{M}{N}$, we have

$$\max_{\mathbf{b}\in\mathcal{B}}Q(\mathbf{b})=Q(\mathbf{b}^*)=4,$$

where $\mathbf{b}^*$ satisfies (3.1). $\square$

Proof of Theorem 1.1. Let $\mathbf{u}\in\mathcal{M}$ be a ground state of (1.4)-(1.5) and $\rho=(\sum_{j=-2}^{2}u_j^2)^{\frac{1}{2}}$, we claim that the mass-redistribution $\mathbf{b}^*\rho=(b_2^*\rho,b_1^*\rho,b_0^*\rho,b_{-1}^*\rho,b_{-2}^*\rho)$ of $\mathbf{u}$ is also a ground state of (1.4)-(1.5). Indeed, by direct calculations, we have

$$\theta^2(\mathbf{b}^*\rho)=0. \tag{3.4}$$

Moreover, by (3.2) and direct calculations, we get

$$\begin{aligned}&\mathbf{F}^2(\mathbf{b}^*\rho)-\mathbf{F}^2(\mathbf{u})=4\big(\sum_{j=-2}^{2}(u_j^2)\big)^2-\mathbf{F}^2(\mathbf{u})\\ &=2(u_0u_{-1}-\sqrt{6}u_1u_{-2})^2+(2u_0^2-3u_1u_{-1})^2+(u_1u_{-1}-4u_2u_{-2})^2\\ &\quad+(\sqrt{3}u_1^2-2\sqrt{2}u_0u_2)^2+(\sqrt{3}u_{-1}^2-2\sqrt{2}u_0u_{-2})^2+2(u_0u_1-\sqrt{6}u_2u_{-1})^2\\ &\geq(u_1u_{-1}-4u_2u_{-2})^2-4(u_0u_1-\sqrt{6}u_2u_{-1})(u_0u_{-1}-\sqrt{6}u_1u_{-2})\\ &\quad+(2u_0^2-3u_1u_{-1})^2+2(\sqrt{3}u_1^2-2\sqrt{2}u_0u_2)(\sqrt{3}u_{-1}^2-2\sqrt{2}u_0u_{-2})=20\theta^2(\mathbf{u}).\end{aligned} \tag{3.5}$$

Thus, for $\tau_1<\frac{\tau_2}{20}<0$ or $\tau_1<0,\tau_2\geq0$, we obtain

$$\tau_1\big(\mathbf{F}^2(\mathbf{b}^*\rho)-\mathbf{F}^2(\mathbf{u})\big)+\tau_2\big(\theta^2(\mathbf{b}^*\rho)-\theta^2(\mathbf{u})\big)\leq(20\tau_1-\tau_2)\theta^2(\mathbf{u})\leq0 \tag{3.6}$$

and further $E(\mathbf{b}^*\rho)\leq E(\mathbf{u})$. Thus $\mathbf{b}^*\rho$ is also a ground state. Consequently,

$$\big(\sum_{j=-2}^{2}|\nabla u_j|^2\big)-\big(\sum_{j=-2}^{2}|\nabla(b_j^*\rho)|^2\big)=0,\quad \frac{\tau_1}{4}\big(\mathbf{F}^2(\mathbf{b}^*\rho)-\mathbf{F}^2(\mathbf{u})\big)+\frac{\tau_2}{4}\big(\theta^2(\mathbf{b}^*\rho)-\theta^2(\mathbf{u})\big)=0.$$

Since $\tau_1<\frac{\tau_2}{20}<0$ or $\tau_1<0,\tau_2\geq0$, we get from (3.6) that $\theta^2(\mathbf{u})=0$, which yields

$$\mathbf{F}^2(\mathbf{b}^*\rho)=\mathbf{F}^2(\mathbf{u}).$$

Therefore $\mathbf{u}=\pm\mathbf{b}^*\rho$ by Lemma 3.1. $\square$

Proof of Theorem 1.2. Define

$$\mathcal{A} := \Big\{u \in \mathcal{M} |\ u_j \geq 0, j = 2, 1, 0, -1, -2\Big\}$$

and

$$\mathcal{G} := \Big\{\mathbf{u} \in \mathcal{A} |\ E(\mathbf{u}) = \inf_{\mathbf{v}\in\mathcal{A}} E(\mathbf{v}) = \inf_{\mathbf{v}\in\mathcal{M}} E(\mathbf{v})\Big\}.$$

If $\mathbf{u} \in \mathcal{M}$, then $\mathbf{b}^*\rho \in \mathcal{M}$ is a mass-redistribution of $\mathbf{u}$. By Proposition 3.1, we obtain

$$\frac{1}{2}\int_{\mathbb{R}^d} (|\nabla\rho|^2 + V(x)\rho^2 + \frac{1}{2}(\tau + 4\tau_1)\rho^4)dx = E(\mathbf{b}^*\rho) \leq E(\mathbf{u}). \tag{3.7}$$

Thus a solution of (1.7)-(1.8) gives a solution to (1.11).

On the other hand, if $\mathbf{v} = (v_2, v_1, v_0, v_{-1}, v_{-2})$ is a ground state of (1.4)-(1.5), then $|\mathbf{v}| \in \mathcal{G}$. We claim $\mathbf{w} := |\mathbf{v}| = \mathbf{b}^*\rho$. Indeed,

$$E(\mathbf{w}) - E(\mathbf{b}^*\rho) = \frac{1}{2}\int_{\mathbb{R}^d} \Big(\big(\sum_{j=-2}^{2} |\nabla w_j|^2\big) - \big(\sum_{j=-2}^{2} |\nabla(b_j^*\rho)|^2\big)\Big)dx - \frac{\tau_1}{4}\int_{\mathbb{R}^d} (4\rho^4 - F_x^2 - F_z^2)dx \geq 0.$$

Since $\mathbf{w} \in \mathcal{G}$, we obtain that

$$\big(\sum_{j=-2}^{2} |\nabla w_j|^2\big) - \big(\sum_{j=-2}^{2} |\nabla(b_j^*\rho)|^2\big) = 0, \quad 4\rho^4 - F_x^2 - F_z^2 = 0. \tag{3.8}$$

When $|M| \neq 2N$, by Proposition 3.1 and the first equality of (3.8), we get $w_j \nabla w_k = w_k \nabla w_j$ for $j \neq k$. Noting $\mathbf{w} \in \mathcal{M}$, then at least one $w_j > 0$ ($j = 2, 1, 0, -1, -2$), without loss of generality, we assume $w_0 > 0$, then we have

$$\nabla\Big(\frac{w_2}{w_0}\Big) = \nabla\Big(\frac{w_1}{w_0}\Big) = \nabla\Big(\frac{w_{-1}}{w_0}\Big) = \nabla\Big(\frac{w_{-2}}{w_0}\Big) = 0.$$

Then there are some $c_j \geq 0$ such that $w_j = c_j w_0$ for $j = 2, 1, -1, -2$. Together with the second equality of (3.8), we get $\mathbf{w} = \mathbf{b}^*\rho$. When $|M| = 2N$, the conclusion is obvious. It yields that

$$\frac{1}{2}\int_{\mathbb{R}^d} (|\nabla\rho|^2 + V(x)\rho^2 + \frac{1}{2}(\tau + 4\tau_1)\rho^4)dx = E(\mathbf{b}^*\rho) = E(|\mathbf{v}|) \leq E(\mathbf{v}) = \min_{\mathbf{u}\in\mathcal{M}} E(\mathbf{u}).$$

Thus, it is easy to see that ρ is a solution to (1.7)-(1.8). Therefore, we complete the proof. $\square$

4 The 1D case

To prove Theorem 1.3 and Theorem 1.4, we consider the minimization problem

$$m := \inf_{\mathbf{u}\in\mathcal{M}} E(\mathbf{u}).$$

Proof of Theorem 1.3. Let $\{\mathbf{u_n}\} = \{(u_{2n}, u_{1n}, u_{0n}, u_{-1n}, u_{-2n})\} \subset \mathcal{M}$ be a minimizing sequence of m, by (2.4), we deduce that

$$\int_{\mathbb{R}} \big(\sum_{j=-2}^{2} u_{jn}^2\big)^2 dx \leq C\Big(\int_{\mathbb{R}} \big(\sum_{j=-2}^{2} |u'_{jn}|^2\big)dx\Big)^{\frac{1}{2}} \cdot N^{\frac{3}{2}}. \tag{4.1}$$

For any $\tau,\ \tau_1,\ \tau_2 \in \mathbb{R}$, we argue from the following eight cases to show that $\{\mathbf{u_n}\}$ is a bounded sequence in Λ.

Case 1: $\tau \geq 0,\ \tau_1 \geq 0$ and $\tau_2 \geq 0$. From the definition of $E(\mathbf{u_n})$, we obtain

$$E(\mathbf{u_n}) \geq \frac{1}{2}\int_{\mathbb{R}}(\sum_{j=-2}^{2}|u'_{jn}|^2)dx + \frac{1}{2}\int_{\mathbb{R}}|x|^2(\sum_{j=-2}^{2}u_{jn}^2)dx.$$

Case 2: $\tau \leq 0,\ \tau_1 \geq 0$ and $\tau_2 \geq 0$. By (4.1) and the Cauchy's inequality, we get

$$\begin{aligned} E(\mathbf{u_n}) &\geq \frac{1}{2}\int_{\mathbb{R}}(\sum_{j=-2}^{2}|u'_{jn}|^2)dx + \frac{1}{2}\int_{\mathbb{R}}|x|^2(\sum_{j=-2}^{2}u_{jn}^2)dx + \frac{\tau}{4}\int_{\mathbb{R}}(\sum_{j=-2}^{2}u_{jn}^2)^2dx \\ &\geq \frac{1}{2}\int_{\mathbb{R}}(\sum_{j=-2}^{2}|u'_{jn}|^2)dx + \frac{1}{2}\int_{\mathbb{R}}|x|^2(\sum_{j=-2}^{2}u_{jn}^2)dx + \frac{\tau}{4}C\Big(\epsilon\int_{\mathbb{R}}(\sum_{j=-2}^{2}|u'_{jn}|^2)dx + C(\epsilon)N^3\Big). \end{aligned}$$

Case 3: $\tau \geq 0,\ \tau_1 \leq 0$ and $\tau_2 \geq 0$. By (3.2), we have

$$\mathbf{F}^2(\mathbf{u_n}) \leq 4\Big(\sum_{j=-2}^{2}u_{jn}^2\Big)^2, \tag{4.2}$$

which yields from (4.1) and the Cauchy's inequality that

$$\begin{aligned} E(\mathbf{u_n}) &\geq \frac{1}{2}\int_{\mathbb{R}}(\sum_{j=-2}^{2}|u'_{jn}|^2)dx + \frac{1}{2}\int_{\mathbb{R}}|x|^2(\sum_{j=-2}^{2}u_{jn}^2)dx + \tau_1\int_{\mathbb{R}}(\sum_{j=-2}^{2}u_{jn}^2)^2dx \\ &\geq \frac{1}{2}\int_{\mathbb{R}}(\sum_{j=-2}^{2}|u'_{jn}|^2)dx + \frac{1}{2}\int_{\mathbb{R}}|x|^2(\sum_{j=-2}^{2}u_{jn}^2)dx + \tau_1 C\Big(\epsilon\int_{\mathbb{R}}(\sum_{j=-2}^{2}|u'_{jn}|^2)dx + C(\epsilon)N^3\Big). \end{aligned}$$

Case 4: $\tau \geq 0,\ \tau_1 \geq 0$ and $\tau_2 \leq 0$. By direct calculations, we get

$$5\theta^2(\mathbf{u_n}) = (2u_{2n}u_{-2n} - 2u_{1n}u_{-1n} + u_{0n}^2)^2 \leq \Big(\sum_{j=-2}^{2}u_{jn}^2\Big)^2, \tag{4.3}$$

which follows that

$$\begin{aligned} E(\mathbf{u_n}) &\geq \frac{1}{2}\int_{\mathbb{R}}(\sum_{j=-2}^{2}|u'_{jn}|^2)dx + \frac{1}{2}\int_{\mathbb{R}}|x|^2(\sum_{j=-2}^{2}u_{jn}^2)dx + \frac{\tau_2}{20}\int_{\mathbb{R}}(\sum_{j=-2}^{2}u_{jn}^2)^2dx \\ &\geq \frac{1}{2}\int_{\mathbb{R}}(\sum_{j=-2}^{2}|u'_{jn}|^2)dx + \frac{1}{2}\int_{\mathbb{R}}|x|^2(\sum_{j=-2}^{2}u_{jn}^2)dx + \frac{\tau_2}{20}C\Big(\epsilon\int_{\mathbb{R}}(\sum_{j=-2}^{2}|u'_{jn}|^2)dx + C(\epsilon)N^3\Big). \end{aligned}$$

Case 5: $\tau < 0,\ \tau_1 \leq 0$ and $\tau_2 \geq 0$. By (4.1) and (4.2), we obtain

$$\begin{aligned} E(\mathbf{u_n}) &\geq \frac{1}{2}\int_{\mathbb{R}}(\sum_{j=-2}^{2}|u'_{jn}|^2)dx + \frac{1}{2}\int_{\mathbb{R}}|x|^2(\sum_{j=-2}^{2}u_{jn}^2)dx + \Big(\frac{\tau}{4}+\tau_1\Big)\int_{\mathbb{R}}(\sum_{j=-2}^{2}u_{jn}^2)^2dx \\ &\geq \frac{1}{2}\int_{\mathbb{R}}(\sum_{j=-2}^{2}|u'_{jn}|^2)dx + \frac{1}{2}\int_{\mathbb{R}}|x|^2(\sum_{j=-2}^{2}u_{jn}^2)dx + \Big(\frac{\tau}{4}+\tau_1\Big)C\Big(\epsilon\int_{\mathbb{R}}\sum_{j=-2}^{2}|u'_{jn}|^2dx + C(\epsilon)N^3\Big). \end{aligned}$$

Case 6: $\tau < 0,\ \tau_1 \geq 0$ and $\tau_2 \leq 0$. By (4.2) and (4.3), we have

$$\begin{aligned}E(\mathbf{u_n}) &\geq \frac{1}{2}\int_{\mathbb{R}}(\sum_{j=-2}^{2}|u'_{jn}|^2)dx + \frac{1}{2}\int_{\mathbb{R}}|x|^2(\sum_{j=-2}^{2}u_{jn}^2)dx + \left(\frac{\tau}{4}+\frac{\tau_2}{20}\right)\int_{\mathbb{R}}(\sum_{j=-2}^{2}u_{jn}^2)^2dx \\ &\geq \frac{1}{2}\int_{\mathbb{R}}(\sum_{j=-2}^{2}|u'_{jn}|^2)dx + \frac{1}{2}\int_{\mathbb{R}}|x|^2(\sum_{j=-2}^{2}u_{jn}^2)dx + \left(\frac{\tau}{4}+\frac{\tau_2}{20}\right)C\left(\epsilon\int_{\mathbb{R}}\sum_{j=-2}^{2}|u'_{jn}|^2dx + C(\epsilon)N^3\right).\end{aligned}$$

Case 7: $\tau \geq 0,\ \tau_1 \leq 0$ and $\tau_2 \leq 0$. By (4.1) and (4.2), we obtain

$$\begin{aligned}E(\mathbf{u_n}) &\geq \frac{1}{2}\int_{\mathbb{R}}(\sum_{j=-2}^{2}|u'_{jn}|^2)dx + \frac{1}{2}\int_{\mathbb{R}}|x|^2(\sum_{j=-2}^{2}u_{jn}^2)dx + \left(\tau_1+\frac{\tau_2}{20}\right)\int_{\mathbb{R}}(\sum_{j=-2}^{2}u_{jn}^2)^2dx \\ &\geq \frac{1}{2}\int_{\mathbb{R}}(\sum_{j=-2}^{2}|u'_{jn}|^2)dx + \frac{1}{2}\int_{\mathbb{R}}|x|^2(\sum_{j=-2}^{2}u_{jn}^2)dx + \left(\tau_1+\frac{\tau_2}{20}\right)C\left(\epsilon\int_{\mathbb{R}}\sum_{j=-2}^{2}|u'_{jn}|^2dx + C(\epsilon)N^3\right).\end{aligned}$$

Case 8: $\tau < 0,\ \tau_1 \leq 0$ and $\tau_2 \leq 0$. By (4.1)-(4.3), we get

$$\begin{aligned}E(\mathbf{u_n}) &\geq \frac{1}{2}\int_{\mathbb{R}}(\sum_{j=-2}^{2}|u'_{jn}|^2)dx + \frac{1}{2}\int_{\mathbb{R}}|x|^2(\sum_{j=-2}^{2}u_{jn}^2)dx + \left(\frac{\tau}{4}+\tau_1+\frac{\tau_2}{20}\right)\int_{\mathbb{R}}(\sum_{j=-2}^{2}u_{jn}^2)^2dx \\ &\geq \frac{1}{2}\int_{\mathbb{R}}(\sum_{j=-2}^{2}|u'_{jn}|^2)dx + \frac{1}{2}\int_{\mathbb{R}}|x|^2(\sum_{j=-2}^{2}u_{jn}^2)dx \\ &\quad + \left(\frac{\tau}{4}+\tau_1+\frac{\tau_2}{20}\right)C\left(\epsilon\int_{\mathbb{R}}\sum_{j=-2}^{2}|u'_{jn}|^2dx + C(\epsilon)N^3\right).\end{aligned} \tag{4.4}$$

For Case 8, from the definition of $E(\mathbf{u_n})$ and (4.4), there holds

$$\begin{aligned}&\int_{\mathbb{R}}(\sum_{j=-2}^{2}|u'_{jn}|^2)dx \leq 2m - 2\left(\frac{\tau}{4}+\tau_1+\frac{\tau_2}{20}\right)C\left(\epsilon\int_{\mathbb{R}}\sum_{j=-2}^{2}|u'_{jn}|^2dx + C(\epsilon)N^3\right) + o_n(1) \\ &\leq 2m - 2\left(\frac{\tau}{4}+\tau_1+\frac{\tau_2}{20}\right)C\epsilon\int_{\mathbb{R}}(\sum_{j=-2}^{2}|u'_{jn}|^2)dx + C(\epsilon)N^3 + o_n(1),\end{aligned}$$

where $o_n(1) \to 0$ as $n \to \infty$. Choosing ϵ sufficiently small, we get $\{\mathbf{u_n}\}$ is a bounded sequence in Λ. By the compact embedding theorem, there exists a minimizer denoted as $\mathbf{u} = (u_2, u_1, u_0, u_{-1}, u_{-2}) \in \mathcal{M}$ for m. Therefore, we have obtained the conclusion when $\tau < 0,\ \tau_1 \leq 0$ and $\tau_2 \leq 0$. The other cases can be proved similarly, we omit the details here. □

Proof of Theorem 1.4 . Let $\{\mathbf{u_n}\} = \{(u_{2n}, u_{1n}, u_{0n}, u_{-1n}, u_{-2n})\} \subset \mathcal{M}$ be a minimizing sequence of m. Since $\tau_2 = 0$, there holds

$$E(|u_{2n}|, |u_{1n}|, |u_{0n}|, |u_{-1n}|, |u_{-2n}|) \leq E(\mathbf{u_n}).$$

Thus, without loss of generality, we may assume $u_{jn} \geq 0$ for $j = 2, 1, 0, -1, -2$.

For $\mathbf{u} = (u_2, u_1, u_0, u_{-1}, u_{-2})$ with $0 \le u_j \in H^1(\mathbb{R})$, $j = 2, 1, 0, -1, -2$, we denote its Schwartz symmetrization by $\mathbf{u}^* = (u_2^*, u_1^*, u_0^*, u_{-1}^*, u_{-2}^*)$. From [41], we obtain

$$
\begin{cases}
\int_{\mathbb{R}} |(u_j^*)'|^2 dx \le \int_{\mathbb{R}} |u_j'|^2 dx, \quad j = 2, 1, 0, -1, -2, \\
\int_{\mathbb{R}} (u_j^*)^2 dx = \int_{\mathbb{R}} u_j^2 dx, \quad \int_{\mathbb{R}} (u_j^*)^4 dx = \int_{\mathbb{R}} u_j^4 dx, \quad j = 2, 1, 0, -1, -2, \\
\int_{\mathbb{R}} (u_j^*)^2 (u_l^*)^2 dx \ge \int_{\mathbb{R}} u_j^2 u_l^2 dx, \quad j, l = 2, 1, 0, -1, -2, \\
\int_{\mathbb{R}} u_j^* u_l^* (u_i^*)^2 dx \ge \int_{\mathbb{R}} u_j u_l u_i^2 dx, \quad i, j, l = 2, 1, 0, -1, -2.
\end{cases}
\tag{4.5}
$$

Since $\tau < 4\tau_1 < 0$, we get

$$
\begin{aligned}
&(\tau - 2\tau_1) \int_{I_k} u_{2n}^2 u_{-1n}^2 dx + (\tau - 4\tau_1) \int_{I_k} u_{2n}^2 u_{-2n}^2 dx \\
&+ (\tau - \tau_1) \int_{I_k} u_{1n}^2 u_{-1n}^2 dx + (\tau - 2\tau_1) \int_{I_k} u_{1n}^2 u_{-2n}^2 dx \\
\ge\, &(\tau - 2\tau_1) \int_{I_k} (u_{2n}^*)^2 (u_{-1n}^*)^2 dx + (\tau - 4\tau_1) \int_{I_k} (u_{2n}^*)^2 (u_{-2n}^*)^2 dx \\
&+ (\tau - \tau_1) \int_{I_k} (u_{1n}^*)^2 (u_{-1n}^*)^2 dx + (\tau - 2\tau_1) \int_{I_k} (u_{1n}^*)^2 (u_{-2n}^*)^2 dx,
\end{aligned}
$$

then it follows $\mathbf{u}_\mathbf{n}^* = (u_{2n}^*, u_{1n}^*, u_{0n}^*, u_{-1n}^*, u_{-2n}^*) \in \mathcal{M}$ and

$$
E(\mathbf{u}_\mathbf{n}^*) \le E(\mathbf{u}_\mathbf{n}).
$$

Hence, we assume u_{jn} are non-negative, even and non-increasing for $j = 2, 1, 0, -1, -2$. Then following some similar arguments as in the proof of Theorem 1.3, we can get the existence of minimizers for m in $\mathcal{M}$. $\square$

5 The 2D case

When $\tau < 0$, $\tau_1 \le 0$ and $\tau_2 \le 0$, for any $\mathbf{u} \in \mathcal{M}$, by the definition of $N^* := -\frac{5a^*}{5\tau + 20\tau_1 + \tau_2}$ and (2.5), we obtain

$$
\begin{aligned}
E(\mathbf{u}) &\ge \frac{1}{2} \int_{\mathbb{R}^2} (\sum_{j=-2}^{2} |\nabla u_j|^2) dx + \frac{1}{2} \int_{\mathbb{R}^2} |x|^2 (\sum_{j=-2}^{2} u_j^2) dx + \Big(\frac{\tau}{4} + \tau_1 + \frac{\tau_2}{20}\Big) \int_{\mathbb{R}^2} (\sum_{j=-2}^{2} u_j^2)^2 dx \\
&= \frac{1}{2} \int_{\mathbb{R}^2} (\sum_{j=-2}^{2} |\nabla u_j|^2) dx + \frac{1}{2} \int_{\mathbb{R}^2} |x|^2 (\sum_{j=-2}^{2} u_j^2) dx - \frac{a^*}{4N^*} \int_{\mathbb{R}^2} (\sum_{j=-2}^{2} u_j^2)^2 dx \\
&\ge \frac{1}{2} \int_{\mathbb{R}^2} (\sum_{j=-2}^{2} |\nabla u_j|^2) dx + \frac{1}{2} \int_{\mathbb{R}^2} |x|^2 (\sum_{j=-2}^{2} u_j^2) dx - \frac{a^*}{4N^*} \cdot \frac{2N}{a^*} \int_{\mathbb{R}^2} (\sum_{j=-2}^{2} |\nabla u_j|^2) dx \\
&= \frac{1}{2N^*} (N^* - N) \int_{\mathbb{R}^2} (\sum_{j=-2}^{2} |\nabla u_j|^2) dx + \frac{1}{2} \int_{\mathbb{R}^2} |x|^2 (\sum_{j=-2}^{2} u_j^2) dx,
\end{aligned}
\tag{5.1}
$$

when $\tau<0,\ \tau_1\le 0$ and $\tau_2\ge 0$, for any $\mathbf{u}\in\mathcal{M}$, by the definition of $N^{**}:=-\frac{a^*}{\tau+4\tau_1}$ and (2.5), we get

$$\begin{aligned}E(\mathbf{u})&\ge \frac{1}{2}\int_{\mathbb{R}^2}(\sum_{j=-2}^{2}|\nabla u_j|^2)dx+\frac{1}{2}\int_{\mathbb{R}^2}|x|^2(\sum_{j=-2}^{2}u_j^2)dx+\Big(\frac{\tau}{4}+\tau_1\Big)\int_{\mathbb{R}^2}(\sum_{j=-2}^{2}u_j^2)^2dx\\&=\frac{1}{2}\int_{\mathbb{R}^2}(\sum_{j=-2}^{2}|\nabla u_j|^2)dx+\frac{1}{2}\int_{\mathbb{R}^2}|x|^2(\sum_{j=-2}^{2}u_j^2)dx-\frac{a^*}{4N^{**}}\int_{\mathbb{R}^2}(\sum_{j=-2}^{2}u_j^2)^2dx\\&=\frac{1}{2N^{**}}(N^{**}-N)\int_{\mathbb{R}^2}(\sum_{j=-2}^{2}|\nabla u_j|^2)dx+\frac{1}{2}\int_{\mathbb{R}^2}|x|^2(\sum_{j=-2}^{2}u_j^2)dx.\end{aligned}\tag{5.2}$$

Proof of Theorem 1.5. (i) Suppose $\tau_2<0$. Let $\{\mathbf{u_n}\}\subset\mathcal{M}$ be a minimizing sequence of $m(N)$, then by (5.1), $\{\mathbf{u_n}\}$ is bounded in Λ if $0<N<N^*$. Applying Lemma 2.1, there exists $\mathbf{w}=(w_2,w_1,w_0,w_{-1},w_{-2})\in H$, such that up to a subsequence, as $n\to+\infty$,

$$\begin{cases}\mathbf{u_n}\rightharpoonup\mathbf{w}, & \text{in } H.\\ \mathbf{u_n}\to\mathbf{w}, & \text{in } L^t(\mathbb{R}^2,\mathbb{R}^5),\ \forall t\in[2,+\infty).\\ \mathbf{u_n}\to\mathbf{w}, & a.e. \text{ in } \mathbb{R}^2.\end{cases}$$

Then $\mathbf{w}\in\mathcal{M}$. Further, by the lower semi-continuity of the norm, there holds

$$m(N)\le I(\mathbf{w})\le\varliminf_{n\to\infty}I(\mathbf{u_n})=m(N).$$

It yields $I(\mathbf{w})=m(N)$, that is, $\mathbf{w}\in\mathcal{M}$ is a minimizer of $m(N)$ for any $N\in(0,N^*)$.

Next, we show that there has no minimizer for $m(N)$ when $N>N^{**}$ by carefully and skilfully choosing some proper test functions. For $\sigma>0$, we define $\mathbf{\Phi}=(\Phi_2,\Phi_1,\Phi_0,\Phi_{-1},\Phi_{-2})$ as

$$\begin{aligned}&\Phi_2(x):=\frac{(2N+M)^2}{16N^{\frac{3}{2}}\sqrt{a^*}}\sigma Q(\sigma x),\quad \Phi_{-2}(x)=\frac{(2N-M)^2}{16N^{\frac{3}{2}}\sqrt{a^*}}\sigma Q(\sigma x),\\&\Phi_1(x):=\frac{(2N+M)\sqrt{4N^2-M^2}}{8N^{\frac{3}{2}}\sqrt{a^*}}\sigma Q(\sigma x),\quad \Phi_{-1}(x):=\frac{(2N-M)\sqrt{4N^2-M^2}}{8N^{\frac{3}{2}}\sqrt{a^*}}\sigma Q(\sigma x),\\&\Phi_0(x):=\frac{\sqrt{6}(4N^2-M^2)}{16N^{\frac{3}{2}}\sqrt{a^*}}\sigma Q(\sigma x),\end{aligned}\tag{5.3}$$

where $Q(x)$ is the unique positive solution of equation (1.9). From the definition of $\mathbf{b}^*$ in Lemma 3.1, it is obvious that $\mathbf{\Phi}\in\mathcal{M}$. By direct calculations, we get

$$\frac{1}{2}\int_{\mathbb{R}^2}(\sum_{j=-2}^{2}|\nabla\Phi_j|^2)dx-\frac{a^*}{4N}\int_{\mathbb{R}^2}(\sum_{j=-2}^{2}\Phi_j^2)^2dx=\frac{1}{2}\cdot N\sigma^2-\frac{a^*}{4N}\cdot\frac{2N^2\sigma^2}{a^*}=0,$$

$$\int_{\mathbb{R}^2}|x|^2(\sum_{j=-2}^{2}|\Phi_j|^2)dx=\frac{N}{a^*}\int_{\mathbb{R}^2}|x|^2\sigma^2Q^2(\sigma x)dx=\frac{N\sigma^{-2}}{a^*}\int_{\mathbb{R}^2}|x|^2Q^2(x)dx,$$

$$\Big(\frac{a^*}{4N}+\frac{\tau}{4}+\tau_1\Big)\int_{\mathbb{R}^2}(\sum_{j=-2}^{2}\Phi_j^2)^2dx=\Big(\frac{a^*}{4N}+\frac{\tau}{4}+\tau_1\Big)\cdot\frac{2N^2\sigma^2}{a^*},$$

$$\frac{\tau_1}{4}\int_{\mathbb{R}^2}\big(2(\Phi_{-2}\Phi_{-1}+\Phi_2\Phi_1)+\sqrt{6}(\Phi_{-1}\Phi_0+\Phi_1\Phi_0)\big)^2dx+\frac{\tau_1}{4}\int_{\mathbb{R}^2}\Big(\sum_{j=-2}^{2}(j\Phi_j^2)\Big)^2dx$$

$$=\tau_1\int_{\mathbb{R}^2}\Big(\sum_{j=-2}^{2}\Phi_j^2\Big)^2dx$$

and

$$\frac{\tau_2}{20}\int_{\mathbb{R}^2}(2\Phi_2\Phi_{-2}-2\Phi_1\Phi_{-1}+\Phi_0^2)^2dx=0.$$

It follows that

$$\begin{aligned}
E(\mathbf{\Phi})=&\frac{1}{2}\int_{\mathbb{R}^2}\Big(\sum_{j=-2}^{2}|\nabla\Phi_j|^2\Big)dx-\frac{a^*}{4N}\int_{\mathbb{R}^2}(\sum_{j=-2}^{2}\Phi_j^2)^2dx-\tau_1\int_{\mathbb{R}^2}\Big(\sum_{j=-2}^{2}\Phi_j^2\Big)^2dx\\
&+\frac{1}{2}\int_{\mathbb{R}^2}\Big(|x|^2\sum_{j=-2}^{2}\Phi_j^2\Big)dx+\Big(\frac{a^*}{4N}+\frac{\tau}{4}+\tau_1\Big)\int_{\mathbb{R}^2}\Big(\sum_{j=-2}^{2}\Phi_j^2\Big)^2dx\\
&+\frac{\tau_1}{4}\int_{\mathbb{R}^2}\big(2(\Phi_{-2}\Phi_{-1}+\Phi_2\Phi_1)+\sqrt{6}(\Phi_{-1}\Phi_0+\Phi_1\Phi_0)\big)^2dx+\frac{\tau_1}{4}\int_{\mathbb{R}^2}\Big(\sum_{j=-2}^{2}(j\Phi_j^2)\Big)^2dx\\
&+\frac{\tau_2}{20}\int_{\mathbb{R}^2}\big(2\Phi_2\Phi_{-2}-2\Phi_1\Phi_{-1}+\Phi_0^2\big)^2dx\\
=&\frac{N\sigma^{-2}}{2a^*}\int_{\mathbb{R}^2}|x|^2Q^2(x)dx+\Big(\frac{a^*}{4N}+\frac{\tau}{4}+\tau_1\Big)\frac{2N^2}{a^*}\sigma^2.
\end{aligned}\tag{5.4}$$

We conclude that for any $\sigma>0$,

$$\begin{aligned}
m(N)\le&\frac{N\sigma^{-2}}{2a^*}\int_{\mathbb{R}^2}|x|^2Q^2(x)dx+\bigg(\frac{N}{2}+\Big(\frac{\tau}{4}+\tau_1\Big)\frac{2N^2}{a^*}\bigg)\sigma^2\\
=&\frac{N\sigma^{-2}}{2a^*}\int_{\mathbb{R}^2}|x|^2Q^2(x)dx+\frac{N}{2N^{**}}(N^{**}-N)\sigma^2.
\end{aligned}\tag{5.5}$$

If $N>N^{**}$, let $\sigma\to\infty$ in (5.5), then $m(N)\to-\infty$. Thus, there has no minimizer for $m(N)$.

(ii) Suppose $\tau_2\ge 0$. The existence of minimizer when $0<N<N^{**}$ can be obtained similarly as (i) by (5.2). Moreover, the non-existence arguments in (i) is also valid for $\tau_2\ge0$ when $N>N^{**}$. Hence, it is sufficient to show there has no minimizer for $m(N)$ if $N=N^{**}$ in this case.

First, we claim

$$\lim_{N\nearrow N^{**}}m(N)=0.\tag{5.6}$$

On the one hand, when $N\in(0,N^{**})$, we obtain from (5.2) that $E(\mathbf{u})\ge 0$ for any $\mathbf{u}\in\mathcal{M}$, which implies $\lim\limits_{N\nearrow N^{**}}m(N)\ge0$. On the other hand, taking

$$\sigma=\Big(\frac{N^{**}\int_{\mathbb{R}^2}|x|^2Q^2(x)dx}{a^*(N^{**}-N)}\Big)^{\frac14},$$

we get

$$m(N)\le N\cdot\Big(\frac{\int_{\mathbb{R}^2}|x|^2Q^2(x)dx\cdot(N^{**}-N)}{a^*N^{**}}\Big)^{\frac12}\to0,\quad\text{as } N\nearrow N^{**},\tag{5.7}$$

that is, $\lim\limits_{N\nearrow N^{**}} m(N)\le 0$. Thus, we have obtained (5.6).

Next, we argue by contradiction to show that there has no minimizer for $m(N^{**})$. Suppose $\boldsymbol{u^*}=(u_2^*,u_1^*,u_0^*,u_{-1}^*,u_{-2}^*)$ is a minimizer of $m(N^{**})$. From the proof of (5.2), we have

$$E(\boldsymbol{u^*})\ge \frac{1}{2}\int_{\mathbb{R}^2}(\sum_{j=-2}^{2}|\nabla u_j^*|^2)dx+\frac{1}{2}\int_{\mathbb{R}^2}|x|^2(\sum_{j=-2}^{2}|u_j^*|^2)dx-\frac{a^*}{4N}\int_{\mathbb{R}^2}\Big(\sum_{j=-2}^{2}(u_j^*)^2\Big)^2dx\ge 0.$$

Together with (5.7), we get $m(N^{**})=0$. As a consequence,

$$\int_{\mathbb{R}^2}(\sum_{j=-2}^{2}|\nabla u_j^*|^2)dx=\frac{a^*}{2N}\int_{\mathbb{R}^2}\Big(\sum_{j=-2}^{2}(u_j^*)^2\Big)^2dx \tag{5.8}$$

and

$$\int_{\mathbb{R}^2}|x|^2(\sum_{j=-2}^{2}|u_j^*|^2)dx=0. \tag{5.9}$$

From (5.8), $\boldsymbol{u^*}$ is an optimal function of the Gagliardo-Nirenberg inequality (2.5). By Lemma 2.2, $\boldsymbol{u^*}$ can be formed as a scaling of $Q(x)$. However, this contradicts to (5.9). Therefore, there has no minimizer for $m(N^{**})$ and we complete the proof. □

Finally, we prove (iii) of Theorem 1.5. Before that, we give an estimate for the least energy $m(N)$.

Lemma 5.1. *Suppose $\tau<0$, $\tau_1\le 0$ and $\tau_2<0$ (resp. $\tau_2\ge 0$), then there holds $m(N)<N$, for $N\in(0,N^*)$ (resp. $N\in(0,N^{**})$).*

Proof. If $\tau_2<0$, since $\Big(0,\sqrt{\frac{N+M}{2}}\Psi_0,0,\sqrt{\frac{N-M}{2}}\Psi_0,0\Big)\in\mathcal{M}$, we get from (2.10),

$$\begin{aligned}m(N)&=\inf_{\mathbf{u}\in\mathcal{M}}E(\mathbf{u})\le E\Bigg(0,\sqrt{\frac{N+M}{2}}\Psi_0,0,\sqrt{\frac{N-M}{2}}\Psi_0,0\Bigg)\\&<\frac{N}{2}\int_{\mathbb{R}^2}\Big(|\nabla\Psi_0|^2+|x|^2{\Psi_0}^2\Big)dx=N.\end{aligned}$$

If $\tau_2\ge 0$, let

$$\begin{aligned}&\widetilde{\Phi}_2(x):=\frac{(2N+M)^2}{16N^{\frac{3}{2}}}\Psi_0,\quad \widetilde{\Phi}_{-2}(x)=\frac{(2N-M)^2}{16N^{\frac{3}{2}}}\Psi_0,\quad \widetilde{\Phi}_0(x):=\frac{\sqrt{6}(4N^2-M^2)}{16N^{\frac{3}{2}}}\Psi_0\\&\widetilde{\Phi}_1(x):=\frac{(2N+M)\sqrt{4N^2-M^2}}{8N^{\frac{3}{2}}}\Psi_0,\quad \widetilde{\Phi}_{-1}(x):=\frac{(2N-M)\sqrt{4N^2-M^2}}{8N^{\frac{3}{2}}}\Psi_0,\end{aligned}$$

then $\Big(\widetilde{\Phi}_2(x),\widetilde{\Phi}_1(x),\widetilde{\Phi}_0(x),\widetilde{\Phi}_{-1}(x),\widetilde{\Phi}_{-2}(x)\Big)\in\mathcal{M}$. Further, we get from (2.10) that

$$\begin{aligned}m(N)&=\inf_{\mathbf{u}\in\mathcal{M}}E(\mathbf{u})\le E\Big(\widetilde{\Phi}_2(x),\widetilde{\Phi}_1(x),\widetilde{\Phi}_0(x),\widetilde{\Phi}_{-1}(x),\widetilde{\Phi}_{-2}(x)\Big)\\&<\frac{N}{2}\int_{\mathbb{R}^2}\Big(|\nabla\Psi_0|^2+|x|^2{\Psi_0}^2\Big)dx=N.\end{aligned}$$

□

Proof of Theorem 1.5. (iii) Set $l_{ik}=\int_{\mathbb{R}^2}u_i\Psi_k dx$ for $i=2,1,0,-1,-2$, then

$$\mathbf{u}=\Big(\sum_{k=0}^{\infty}l_{2k}\Psi_k,\sum_{k=0}^{\infty}l_{1k}\Psi_k,\sum_{k=0}^{\infty}l_{0k}\Psi_k,\sum_{k=0}^{\infty}l_{-1k}\Psi_k,\sum_{k=0}^{\infty}l_{-2k}\Psi_k\Big).$$

Moreover, we conclude

$$\begin{aligned}N&=\|(u_2,u_1,u_0,u_{-1},u_{-2})\|_{L^2}^2=\sum_{k=0}^{\infty}(l_{2k}^2+l_{1k}^2+l_{0k}^2+l_{-1k}^2+l_{-2k}^2)\|\Psi_k\|_{L^2}^2\\&=\sum_{k=0}^{\infty}(l_{2k}^2+l_{1k}^2+l_{0k}^2+l_{-1k}^2+l_{-2k}^2)\end{aligned}\tag{5.10}$$

and

$$\|\mathbf{u}\|_{\Lambda}^2=\sum_{k=0}^{\infty}(l_{2k}^2+l_{1k}^2+l_{0k}^2+l_{-1k}^2+l_{-2k}^2)\|\Psi_k\|_{\Lambda}^2=\sum_{k=0}^{\infty}\xi_k(l_{2k}^2+l_{1k}^2+l_{0k}^2+l_{-1k}^2+l_{-2k}^2).$$

Denote $M_0:=\frac{1}{2N^*}(N^*-N)\in\big(0,\frac{1}{2}\big)$, then we get

$$\begin{aligned}m(N)=E(\mathbf{u})&\geq M_0A(\mathbf{u})+\frac{1}{2}\int_{\mathbb{R}^2}|x|^2\mathbf{u}^2dx\geq M_0\|\mathbf{u}\|_{\Lambda}^2\\&=M_0\cdot\sum_{k=0}^{\infty}\xi_k(l_{2k}^2+l_{1k}^2+l_{0k}^2+l_{-1k}^2+l_{-2k}^2)\\&=M_0\cdot\sum_{k=0}^{\infty}(\xi_k-\xi_0)(l_{2k}^2+l_{1k}^2+l_{0k}^2+l_{-1k}^2+l_{-2k}^2)\\&\quad+M_0\cdot\sum_{k=0}^{\infty}\xi_0(l_{2k}^2+l_{1k}^2+l_{0k}^2+l_{-1k}^2+l_{-2k}^2).\end{aligned}$$

By Lemma 5.1 and (5.10), we have

$$\begin{aligned}&(\xi_1-\xi_0)\sum_{k=1}^{\infty}(l_{2k}^2+l_{1k}^2+l_{0k}^2+l_{-1k}^2+l_{-2k}^2)\leq\sum_{k=1}^{\infty}(\xi_k-\xi_0)(l_{2k}^2+l_{1k}^2+l_{0k}^2+l_{-1k}^2+l_{-2k}^2)\\&\leq\frac{m(N)}{M_0}-\sum_{k=0}^{\infty}\xi_0(l_{2k}^2+l_{1k}^2+l_{0k}^2+l_{-1k}^2+l_{-2k}^2)\leq\Big(\frac{1}{M_0}-2\Big)N,\end{aligned}$$

then

$$\sum_{k=1}^{\infty}(l_{2k}^2+l_{1k}^2+l_{0k}^2+l_{-1k}^2+l_{-2k}^2)\leq\Big(\frac{1}{M_0}-2\Big)\cdot\frac{N}{\xi_1-\xi_0}.$$

Thus

$$\begin{aligned}&\sum_{k=1}^{\infty}\xi_k(l_{2k}^2+l_{1k}^2+l_{0k}^2+l_{-1k}^2+l_{-2k}^2)\\&=\sum_{k=1}^{\infty}(\xi_k-\xi_0)(l_{2k}^2+l_{1k}^2+l_{0k}^2+l_{-1k}^2+l_{-2k}^2)+\xi_0\sum_{k=1}^{\infty}(l_{2k}^2+l_{1k}^2+l_{0k}^2+l_{-1k}^2+l_{-2k}^2)\end{aligned}$$

$$
\leq \Big(\frac{1}{M_0}-2\Big)N+\xi_0\Big(\frac{1}{M_0}-2\Big)\cdot\frac{N}{\xi_1-\xi_0}=\frac{\xi_1}{\xi_1-\xi_0}\cdot\Big(\frac{1}{M_0}-2\Big)N.
$$

For $N\to 0^+$, we can see that

$$
\begin{aligned}
&\|\boldsymbol{u}-(l_2\Psi_0,l_1\Psi_0,l_0\Psi_0,l_{-1}\Psi_0,l_{-2}\Psi_0)\|_{\dot\Lambda}^2\\
=&\Big\|\Big(\sum_{k=1}^{\infty}l_{2k}\Psi_k,\sum_{k=1}^{\infty}l_{1k}\Psi_k,\sum_{k=1}^{\infty}l_{0k}\Psi_k,\sum_{k=1}^{\infty}l_{-1k}\Psi_k,\sum_{k=1}^{\infty}l_{-2k}\Psi_k\Big)\Big\|_{\dot\Lambda}^2\\
=&\sum_{k=1}^{\infty}\xi_k(l_{2k}^2+l_{1k}^2+l_{0k}^2+l_{-1k}^2+l_{-2k}^2)=O(N)
\end{aligned}
$$

and

$$
\|\boldsymbol{u}-(l_2\Psi_0,l_1\Psi_0,l_0\Psi_0,l_{-1}\Psi_0,l_{-2}\Psi_0)\|_{L^2}^2=\sum_{k=1}^{\infty}(l_{2k}^2+l_{1k}^2+l_{0k}^2+l_{-1k}^2+l_{-2k}^2)=O(N).
$$

Therefore, it is obvious the conclusion holds and we complete the proof. □

5.1 Proof of Theorem 1.6

Assume $\tau<0,\ \tau_1<0$, $\tau_2\geq 0$ and $N_n\nearrow N^{**}$ as $n\to\infty$, let $\mathbf{u_n}=(u_{2n},u_{1n},u_{0n},u_{-1n},u_{-2n})\in\mathcal{M}(N_n)$ be a minimizer for $m(N_n)$. Then $\mathbf{u_n}$ satisfies the following Euler-Lagrange system

$$
\begin{cases}
-\Delta u_{2n}+|x|^2u_{2n}+(\lambda_n+2\mu_n)u_{2n}+\tau\rho^2u_{2n}+\tau_1(F_xu_{1n}+2F_zu_{2n})+\frac{\tau_2}{\sqrt5}\theta u_{-2n}=0,\\
-\Delta u_{1n}+|x|^2u_{1n}+(\lambda_n+\mu_n)u_{1n}+\tau\rho^2u_{1n}+\tau_1\Big(\frac{\sqrt6}{2}F_xu_{0n}+F_xu_{2n}+F_zu_{1n}\Big)-\frac{\tau_2}{\sqrt5}\theta u_{-1n}=0,\\
-\Delta u_{0n}+|x|^2u_{0n}+\lambda_nu_{0n}+\tau\rho^2u_{0n}+\frac{\sqrt6}{2}\tau_1\Big(F_xu_{-1n}+F_xu_{1n}\Big)+\frac{\tau_2}{\sqrt5}\theta u_{0n}=0,\\
-\Delta u_{-1n}+|x|^2u_{-1n}+(\lambda_n-\mu_n)u_{-1n}+\tau\rho^2u_{-1n}+\tau_1\Big(\frac{\sqrt6}{2}F_xu_{0n}+F_xu_{-2n}-F_zu_{-1n}\Big)-\frac{\tau_2}{\sqrt5}\theta u_{1n}=0,\\
-\Delta u_{-2n}+|x|^2u_{-2n}+(\lambda_n-2\mu_n)u_{-2n}+\tau\rho^2u_{-2n}+\tau_1(F_xu_{-1n}-2F_zu_{-2n})+\frac{\tau_2}{\sqrt5}\theta u_{2n}=0,
\end{cases}
\tag{5.11}
$$

where λ_n and μ_n are the corresponding Lagrange multipliers. By (5.2), we have

$$
\begin{aligned}
E(\mathbf{u_n})=&\frac12\int_{\mathbb{R}^2}\Big(\sum_{j=-2}^{2}|\nabla u_{jn}|^2\Big)dx-\frac{a^*}{4N^{**}}\int_{\mathbb{R}^2}\Big(\sum_{j=-2}^{2}u_{jn}^2\Big)^2dx-\tau_1\int_{\mathbb{R}^2}\Big(\sum_{j=-2}^{2}u_{jn}^2\Big)^2dx\\
&+\frac12\int_{\mathbb{R}^2}|x|^2\Big(\sum_{j=-2}^{2}u_{jn}^2\Big)dx+\frac{\tau_1}{4}\int_{\mathbb{R}^2}\big(2(u_{-2n}u_{-1n}+u_{2n}u_{1n})+\sqrt6(u_{-1n}u_{0n}+u_{1n}u_{0n})\big)^2dx\\
&+\frac{\tau_1}{4}\int_{\mathbb{R}^2}\Big(\sum_{j=-2}^{2}(ju_{jn}^2)\Big)^2dx+\frac{\tau_2}{20}\int_{\mathbb{R}^2}\big(2u_{2n}u_{-2n}-2u_{1n}u_{-1n}+u_{0n}^2\big)^2dx\\
\geq&\frac{1}{2N^{**}}(N^{**}-N_n)\int_{\mathbb{R}^2}\Big(\sum_{j=-2}^{2}|\nabla u_{jn}|^2\Big)dx+\frac12\int_{\mathbb{R}^2}|x|^2\Big(\sum_{j=-2}^{2}u_{jn}^2\Big)dx\geq 0.
\end{aligned}
\tag{5.12}
$$

Combining with the fact that $\lim\limits_{N\nearrow N^{**}} m(N) = 0$, we can see that

$$\lim_{n\to\infty}\int_{\mathbb{R}^d}\big(2u_{2n}u_{-2n}-2u_{1n}u_{-1n}+u_{0n}^2\big)^2dx=0, \tag{5.13}$$

$$\begin{aligned}&\lim_{n\to\infty}\frac{\tau_1}{4}\int_{\mathbb{R}^2}\big(2(u_{-2n}u_{-1n}+u_{2n}u_{1n})+\sqrt{6}(u_{-1n}u_{0n}+u_{1n}u_{0n})\big)^2dx\\ &+\lim_{n\to\infty}\frac{\tau_1}{4}\int_{\mathbb{R}^2}\Big(\sum_{j=-2}^{2}(ju_{jn}^2)\Big)^2dx-\lim_{n\to\infty}\tau_1\int_{\mathbb{R}^2}\Big(\sum_{j=-2}^{2}u_{jn}^2\Big)^2dx=0\end{aligned} \tag{5.14}$$

and

$$\lim_{n\to\infty}\frac{\int_{\mathbb{R}^2}(\sum\limits_{j=-2}^{2}|\nabla u_{jn}|^2)dx}{\int_{\mathbb{R}^2}(\sum\limits_{j=-2}^{2}u_{jn}^2)^2dx}=\frac{a^*}{2N^{**}}. \tag{5.15}$$

We claim that

$$\lim_{n\to\infty}\int_{\mathbb{R}^2}(\sum_{j=-2}^{2}|\nabla u_{jn}|^2)dx=+\infty.$$

We argue by contradiction. Suppose there exists a positive constant C, such that $\int_{\mathbb{R}^2}(\sum_{j=-2}^{2}|\nabla u_{jn}|^2)dx\le C$ for large n. Then $\{\mathbf{u_n}\}$ is a bounded sequence in Λ, which implies that there exist a subsequence (still denoted by $\{\mathbf{u_n}\}$) and $\mathbf{u}^*:=(u_2^*,u_1^*,u_0^*,u_{-1}^*,u_{-2}^*)$, such that as $n\to\infty$,

$$\mathbf{u_n}\to\mathbf{u}^* \text{ in } L^t(\mathbb{R}^2,\mathbb{R}^5),\ \forall t\in[2,+\infty).$$

Hence, by the weak lower semi-continuity of the norm, we get

$$0=\lim_{n\to\infty}E(\mathbf{u_n})\ge E(\mathbf{u}^*)\ge m(N^{**})=0.$$

It shows $\mathbf{u}^*$ is a minimizer of $m(N^{**})$, which contradicts to Theorem 1.5. Thus, we obtain the claim.

Now, defining

$$\varepsilon_n:=\sqrt{N^{**}}\Big(\int_{\mathbb{R}^2}(\sum_{j=-2}^{2}|\nabla u_{jn}|^2)dx\Big)^{-\frac12}, \tag{5.16}$$

then it is easy to see that $\varepsilon_n\to0$ as $n\to\infty$.

Proof of Theorem 1.6. On the one hand, by (5.5), we get for any $\sigma>0$,

$$m(N_n)\le\frac{N\sigma^{-2}}{2a^*}\int_{\mathbb{R}^2}|x|^2Q^2(x)dx+\frac{N_n}{2N^{**}}(N^{**}-N_n)\sigma^2.$$

By (5.7), it follows that

$$\lim_{n\to\infty}\frac{m(N_n)}{(N^{**}-N_n)^{\frac12}}\le\Big(\frac{N^{**}\int_{\mathbb{R}^2}|x|^2Q^2(x)dx}{a^*}\Big)^{\frac12}. \tag{5.17}$$

On the other hand, let $\tilde{\mathbf{w}}_{\mathbf{n}} := (\tilde{w}_{2n}, \tilde{w}_{1n}, \tilde{w}_{0n}, \tilde{w}_{-1n}, \tilde{w}_{-2n})$ with $\tilde{w}_{jn}(x) := \varepsilon_n u_{jn}(\varepsilon_n x)$ $(j = 2, 1, 0, -1, -2)$, then

$$\int_{\mathbb{R}^2} (\sum_{j=-2}^{2} |\nabla \tilde{w}_{jn}|^2) dx = \varepsilon_n^2 \int_{\mathbb{R}^2} (\sum_{j=-2}^{2} |\nabla u_{jn}|^2) dx = N^{**}.$$

Moreover, from (5.15), we have

$$\lim_{n\to\infty} \frac{\int_{\mathbb{R}^2} (\sum_{j=-2}^{2} |\nabla \tilde{w}_{jn}|^2) dx}{\int_{\mathbb{R}^2} (\sum_{j=-2}^{2} \tilde{w}_{jn}^2)^2 dx} = \lim_{n\to\infty} \frac{\int_{\mathbb{R}^2} (\sum_{j=-2}^{2} |\nabla u_{jn}|^2) dx}{\int_{\mathbb{R}^2} (\sum_{j=-2}^{2} u_{jn}^2)^2 dx} = \frac{a^*}{2N^{**}}, \tag{5.18}$$

which yields that

$$\lim_{n\to\infty} \int_{\mathbb{R}^2} (\sum_{j=-2}^{2} \tilde{w}_{jn}^2)^2 dx = \frac{2(N^{**})^2}{a^*}. \tag{5.19}$$

We claim that there exist $\{y_n\} \subset \mathbb{R}^2$ and $R_0,\ \eta > 0$, such that at least one $j \in \{2, 1, 0, -1, -2\}$ satisfies $\liminf\limits_{n\to\infty} \int_{B_{R_0}(y_n)} \tilde{w}_{jn}^2 dx \geq \eta > 0$. Otherwise, suppose for any $R > 0$, there has a subsequence $\{\tilde{w}_{jn_k}\}$ $(j = 2, 1, 0, -1, -2)$, such that $\lim\limits_{k\to\infty} \sup\limits_{x\in\mathbb{R}^2} \int_{B_R(y)} \tilde{w}_{jn_k}^2 dx = 0$. Then by Lion's vanishing Lemma, we conclude that $\tilde{w}_{jn_k} \to 0$ $(j = 2, 1, 0, -1, -2)$ in $L^t(\mathbb{R}^2)$ for $t \in (2, \infty)$, which contradicts to (5.19). Hence, we obtain the claim. Now we define $\mathbf{w}_{\mathbf{n}} := (w_{2n}, w_{1n}, w_{0n}, w_{-1n}, w_{-2n})$ with

$$w_{jn}(x) := \tilde{w}_{jn}(x + y_n) = \varepsilon_n u_{jn}(\varepsilon_n x + \varepsilon_n y_n), \quad j = 2, 1, 0, -1, -2 \tag{5.20}$$

then

$$\lim_{n\to\infty} \int_{\mathbb{R}^2} (\sum_{j=-2}^{2} |\nabla w_{jn}|^2) dx = \lim_{n\to\infty} \int_{\mathbb{R}^2} (\sum_{j=-2}^{2} w_{jn}^2) dx = N^{**}$$

and

$$\lim_{n\to\infty} \int_{\mathbb{R}^2} (\sum_{j=-2}^{2} w_{jn}^2)^2 dx = \frac{2(N^{**})^2}{a^*}.$$

Moreover, there exists some $j \in \{2, 1, 0, -1, -2\}$, such that

$$\liminf_{n\to\infty} \int_{B_{R_0}(0)} |w_{jn}|^2 dx \geq \eta > 0. \tag{5.21}$$

It follows that

$$\lim_{n\to\infty} \frac{\int_{\mathbb{R}^2} (\sum_{j=-2}^{2} |\nabla w_{jn}|^2) dx \int_{\mathbb{R}^2} (\sum_{j=-2}^{2} w_{jn}^2) dx}{\int_{\mathbb{R}^2} (\sum_{j=-2}^{2} w_{jn}^2)^2 dx} = \frac{a^*}{2}. \tag{5.22}$$

By Lemma 2.2, $\{\mathbf{w}_{\mathbf{n}}\}$ is a minimizing sequence for the following minimization problem:

$$k := \inf_{(0,0,0,0,0)\neq \mathbf{u}\in H} K(\mathbf{u}),$$

where

$$K(\mathbf{u}) := \frac{\int_{\mathbb{R}^2}(\sum\limits_{j=-2}^{2}|\nabla u_{jn}|^2)dx \int_{\mathbb{R}^2}(\sum\limits_{j=-2}^{2}u_{jn}^2)dx}{\int_{\mathbb{R}^2}(\sum\limits_{j=-2}^{2}u_{jn}^2)^2dx}.$$

From Lemma 2.2, the minimizer $\mathbf{w} = (w_2, w_1, w_0, w_{-1}, w_{-2})$ must be in form

$$\begin{cases} w_2(x) = \sqrt{\dfrac{N^{**}}{a^*}}Q(x)\cos\varphi_1, \\ w_1(x) = \sqrt{\dfrac{N^{**}}{a^*}}Q(x)\sin\varphi_1\cos\varphi_2, \\ w_0(x) = \sqrt{\dfrac{N^{**}}{a^*}}Q(x)\sin\varphi_1\sin\varphi_2\cos\varphi_3, \\ w_{-1}(x) = \sqrt{\dfrac{N^{**}}{a^*}}Q(x)\sin\varphi_1\sin\varphi_2\sin\varphi_3\cos\varphi_4, \\ w_{-2}(x) = \sqrt{\dfrac{N^{**}}{a^*}}Q(x)\sin\varphi_1\sin\varphi_2\sin\varphi_3\sin\varphi_4, \end{cases}$$

for $\varphi_1, \varphi_2, \varphi_3, \varphi_4 \in [0, \frac{\pi}{2})$. Since $\int_{\mathbb{R}^2}(\sum_{j=-2}^{2}w_j^2)dx = N^{**}$, we get $\mathbf{w_n} \to \mathbf{w}$ in $L^2(\mathbb{R}^2, \mathbb{R}^5)$. Further, using the interpolation inequality, there holds $\mathbf{w_n} \to \mathbf{w}$ in $L^4(\mathbb{R}^2, \mathbb{R}^5)$. From (5.22), we obtain

$$\begin{aligned} &\frac{a^*}{2}\int_{\mathbb{R}^2}(\sum_{j=-2}^{2}w_j^2)^2dx = N^{**}\int_{\mathbb{R}^2}(\sum_{j=-2}^{2}|\nabla w_j|^2)dx \le \lim_{n\to\infty}N_n\int_{\mathbb{R}^2}(\sum_{j=-2}^{2}|\nabla w_{jn}|^2)dx \\ = &\frac{a^*}{2}\lim_{n\to\infty}\int_{\mathbb{R}^2}(\sum_{j=-2}^{2}w_{jn}^2)^2dx = \frac{a^*}{2}\int_{\mathbb{R}^2}(\sum_{j=-2}^{2}w_j^2)^2dx, \end{aligned}$$

which gives that $\lim\limits_{n\to\infty}\int_{\mathbb{R}^2}(\sum_{j=-2}^{2}|\nabla w_{jn}|^2)dx = \int_{\mathbb{R}^2}(\sum_{j=-2}^{2}|\nabla w_j|^2)dx$, that is, $\mathbf{w_n} \to \mathbf{w}$ in $H^1(\mathbb{R}^2, \mathbb{R}^5)$ as $n \to \infty$. Therefore, there exists some $x_1 \in \mathbb{R}^2$, such that

$$\begin{cases} \lim\limits_{n\to\infty} w_{2n}(x) = \sqrt{\dfrac{N^{**}}{a^*}}Q(x-x_1)\cos\varphi_1, \\ \lim\limits_{n\to\infty} w_{1n}(x) = \sqrt{\dfrac{N^{**}}{a^*}}Q(x-x_1)\sin\varphi_1\cos\varphi_2, \\ \lim\limits_{n\to\infty} w_{0n}(x) = \sqrt{\dfrac{N^{**}}{a^*}}Q(x-x_1)\sin\varphi_1\sin\varphi_2\cos\varphi_3, \\ \lim\limits_{n\to\infty} w_{-1n}(x) = \sqrt{\dfrac{N^{**}}{a^*}}Q(x-x_1)\sin\varphi_1\sin\varphi_2\sin\varphi_3\cos\varphi_4, \\ \lim\limits_{n\to\infty} w_{-2n}(x) = \sqrt{\dfrac{N^{**}}{a^*}}Q(x-x_1)\sin\varphi_1\sin\varphi_2\sin\varphi_3\sin\varphi_4, \end{cases} \tag{5.23}$$

for $\varphi_1, \varphi_2, \varphi_3, \varphi_4 \in [0, \frac{\pi}{2})$.

By direct calculations, we obtain from (5.20) that

$$\begin{aligned}&\int_{\mathbb{R}^2}|x|^2\mathbf{u_n}^2dx=\sum_{i=-2}^{2}\int_{\mathbb{R}^2}|x|^2\cdot\frac{1}{\varepsilon_n^2}\Big(w_{in}\Big(\frac{x-\varepsilon_n y_n}{\varepsilon_n}\Big)\Big)^2dx\\ =&\sum_{i=-2}^{2}\int_{\mathbb{R}^2}|\varepsilon_n x+\varepsilon_n y_n|^2w_{in}^2(x)dx=\sum_{i=-2}^{2}\varepsilon_n^2\int_{\mathbb{R}^2}\big|x+y_n+x_1\big|^2w_{in}^2(x+x_1)dx.\end{aligned}\tag{5.24}$$

We now claim $\lim\limits_{n\to\infty}|y_n|\le C$ for some positive constant C. Otherwise, suppose that $\lim\limits_{n\to\infty}\big|y_n+x_1\big|=+\infty$, then it follows from (5.24) that for arbitrary $C_1>0$, there holds $\int_{\mathbb{R}^2}|x|^2|\mathbf{u_n}|^2dx\ge C_1\varepsilon_n^2$, as $n\to\infty$. By (5.18) and (5.19), we have

$$\begin{aligned}E(\mathbf{u_n})\ge&\frac{1}{2}\int_{\mathbb{R}^2}(\sum_{j=-2}^{2}|\nabla u_{jn}|^2)dx-\frac{a^*}{4N_n}\int_{\mathbb{R}^2}(\sum_{j=-2}^{2}u_{jn}^2)^2dx+\frac{1}{2}\int_{\mathbb{R}^2}|x|^2(\sum_{j=-2}^{2}u_{jn}^2)dx\\ &+\frac{a^*}{4N_n}\int_{\mathbb{R}^2}(\sum_{j=-2}^{2}u_{jn}^2)^2dx+\Big(\frac{\tau}{4}+\tau_1\Big)\int_{\mathbb{R}^2}(\sum_{j=-2}^{2}u_{jn}^2)^2dx\\ &\ge\frac{1}{2}\int_{\mathbb{R}^2}|x|^2(\sum_{j=-2}^{2}u_{jn}^2)dx+\Big(\frac{a^*}{4N_n}+\frac{\tau}{4}+\tau_1\Big)\int_{\mathbb{R}^2}(\sum_{j=-2}^{2}u_{jn}^2)^2dx\\ &\ge\frac{1}{2}C_1\varepsilon_n^2+\Big(\frac{a^*}{2N_n}+\frac{\tau}{2}+2\tau_1\Big)\cdot\frac{(N^{**})^2\varepsilon_n^{-2}}{a^*}+o_n(1)\\ &=\frac{1}{2}C_1\varepsilon_n^2+\frac{N^{**}}{2N_n}(N^{**}-N_n)\varepsilon_n^{-2}+o_n(1),\end{aligned}$$

where $o_n(1)\to0$ as $n\to\infty$. Taking the infimum with respect to $\varepsilon_n>0$, then we conclude

$$\lim_{n\to\infty}\frac{m(N_n)}{(N^{**}-N_n)^{\frac{1}{2}}}\ge C_1^{\frac{1}{2}}.$$

However, it contradicts to (5.17). Thus, there exists $x_2\in\mathbb{R}^2$, such that

$$\lim_{n\to\infty}\big(y_n+x_1\big)=x_2,\tag{5.25}$$

which yields $\lim\limits_{n\to\infty}|y_n|\le C$. Therefore, by (5.18), (5.24) and Fatou's Lemma, we have

$$\begin{aligned}E(\mathbf{u_n})\ge&\frac{1}{2}\int_{\mathbb{R}^2}|x|^2(\sum_{j=-2}^{2}u_{jn}^2)dx+\Big(\frac{a^*}{4N_n}+\frac{\tau}{4}+\tau_1\Big)\int_{\mathbb{R}^2}(\sum_{j=-2}^{2}u_{jn}^2)^2dx\\ \ge&\frac{1}{2}\frac{N^{**}\varepsilon_n^2}{a^*}\cdot\int_{\mathbb{R}^2}|x|^2Q^2(x)dx+\frac{N^{**}}{2N_n}(N^{**}-N_n)\varepsilon_n^{-2}+o_n(1).\end{aligned}\tag{5.26}$$

Then taking

$$\varepsilon_n=\Big(\frac{a^*(N^{**}-N_n)}{N_n\int_{\mathbb{R}^2}|x|^2Q^2(x)dx}\Big)^{\frac{1}{4}},$$

we get

$$\lim_{n\to\infty}\frac{m(N_n)}{(N^{**}-N_n)^{\frac{1}{2}}}\geq\Big(\frac{N^{**}\int_{\mathbb{R}^2}|x|^2Q^2(x)dx}{a^*}\Big)^{\frac{1}{2}}.$$

Combining with (5.17), we conclude

$$m(N_n)\sim(N^{**}-N_n)^{\frac{1}{2}},\quad \text{as } n\to\infty.$$

Now, we are ready to prove the limit behavior of $\{\mathbf{u_n}\}$ as $n\to\infty$. By (5.12)-(5.15) and the fact that

$$\lim_{N\nearrow N^{**}}m(N)=0,$$

we get

$$\begin{aligned}&\frac{1}{\varepsilon_n^2}\Big\{\lim_{n\to\infty}\frac{\tau_1}{4}\int_{\mathbb{R}^2}\big(2(w_{-2n}w_{-1n}+w_{2n}w_{1n})+\sqrt{6}(w_{-1n}w_{0n}+w_{1n}w_{0n})\big)^2dx\\&+\lim_{n\to\infty}\frac{\tau_1}{4}\int_{\mathbb{R}^2}\Big(\sum_{j=-2}^{2}(jw_{jn}^2)\Big)^2dx-\lim_{n\to\infty}\tau_1\int_{\mathbb{R}^2}\Big(\sum_{j=-2}^{2}w_{jn}^2\Big)^2dx\Big\}=0.\end{aligned}$$

Since $\varepsilon_n\to0$ as $n\to\infty$, we deduce

$$\begin{aligned}&\frac{\tau_1}{4}\int_{\mathbb{R}^2}\big(2(w_{-2}w_{-1}+w_2w_1)+\sqrt{6}(w_{-1}w_0+w_1w_0)\big)^2dx\\&+\frac{\tau_1}{4}\int_{\mathbb{R}^2}\Big(\sum_{j=-2}^{2}(jw_j^2)\Big)^2dx-\tau_1\int_{\mathbb{R}^2}\Big(\sum_{j=-2}^{2}w_j^2\Big)^2dx=0.\end{aligned}\tag{5.27}$$

Denoting

$$w_j(x):=c_jQ(x-x_1),\quad j=2,1,0,-1,-2,$$

we get from (5.23) that

$$\sum_{j=-2}^{2}c_j^2=\frac{N^{**}}{a^*},\quad \sum_{j=-2}^{2}(jc_j^2)=\frac{M}{a^*}.$$

Thus $b_j:=\sqrt{\frac{a^*}{N^{**}}}c_j\in\mathcal{B}$. By (5.27) and Lemma 3.1, we obtain $b_j=b_j^*$ with $N=N^{**}$ in (3.1) for $j=2,1,0,-1,-2$. Hence, we conclude

$$\begin{cases}\lim\limits_{n\to\infty}w_{2n}(x)=\sqrt{\dfrac{N^{**}}{a^*}}\dfrac{(2N^{**}+M)^2}{16(N^{**})^2}Q(x-x_1),\\ \lim\limits_{n\to\infty}w_{1n}(x)=\sqrt{\dfrac{N^{**}}{a^*}}\dfrac{(2N^{**}+M)\sqrt{4(N^{**})^2-M^2}}{8(N^{**})^2}Q(x-x_1),\\ \lim\limits_{n\to\infty}w_{0n}(x)=\sqrt{\dfrac{N^{**}}{a^*}}\dfrac{\sqrt{6}(4(N^{**})^2-M^2)}{16(N^{**})^2}Q(x-x_1),\\ \lim\limits_{n\to\infty}w_{-1n}(x)=\sqrt{\dfrac{N^{**}}{a^*}}\dfrac{(2N^{**}-M)\sqrt{4(N^{**})^2-M^2}}{8(N^{**})^2}Q(x-x_1),\\ \lim\limits_{n\to\infty}w_{-2n}(x)=\sqrt{\dfrac{N^{**}}{a^*}}\dfrac{(2N^{**}-M)^2}{16(N^{**})^2}Q(x-x_1).\end{cases}\tag{5.28}$$

Noting that $\mathbf{u_n}$ satisfies the Euler-Lagrange system (5.11), then

$$
\begin{aligned}
&-(\lambda_n N_n+\mu_n M)\\
=&\int_{\mathbb{R}^2}\Big(\sum_{j=-2}^{2}|\nabla u_{jn}|^2\Big)dx+\int_{\mathbb{R}^2}|x|^2(\sum_{j=-2}^{2}u_{jn}^2)dx+\tau\int_{\mathbb{R}^2}\Big(\sum_{j=-2}^{2}u_{jn}^2\Big)^2dx\\
&+\tau_1\int_{\mathbb{R}^2}\big(2(u_{-2n}u_{-1n}+u_{2n}u_{1n})+\sqrt{6}(u_{-1n}u_{0n}+u_{1n}u_{0n})\big)^2dx+\tau_1\int_{\mathbb{R}^2}\Big(\sum_{j=-2}^{2}(ju_{jn}^2)\Big)^2dx\\
&+\frac{\tau_2}{5}\int_{\mathbb{R}^2}\big(2u_{2n}u_{-2n}-2u_{1n}u_{-1n}+u_{0n}^2\big)^2dx\\
=&\int_{\mathbb{R}^2}\Big(\sum_{j=-2}^{2}|\nabla u_{jn}|^2\Big)dx+\int_{\mathbb{R}^2}|x|^2(\sum_{j=-2}^{2}u_{jn}^2)dx\\
&-2\Big(\int_{\mathbb{R}^2}\Big(\sum_{j=-2}^{2}|\nabla u_{jn}|^2\Big)dx+\int_{\mathbb{R}^2}|x|^2(\sum_{j=-2}^{2}u_{jn}^2)dx-2E(\mathbf{u_n})\Big)\\
=&\,4E(\mathbf{u_n})-\int_{\mathbb{R}^2}\Big(\sum_{j=-2}^{2}|\nabla u_{jn}|^2\Big)dx-\int_{\mathbb{R}^2}|x|^2(\sum_{j=-2}^{2}u_{jn}^2)dx.
\end{aligned}
$$

Hence, by (5.13), (5.14) and (5.16), we get $\lim_{n\to\infty}\varepsilon_n^2\big(\lambda_n N_n+\mu_n M\big)=N^{**}$. By (5.11) and (5.20), $\mathbf{w_n}$ satisfies the following system

$$
\begin{cases}
-\Delta w_{2n}+\varepsilon_n^2|\varepsilon_n x+\varepsilon_n y_n|^2w_{2n}+\varepsilon_n^2(\lambda_n+2\mu_n)w_{2n}+\tau\rho^2w_{2n}\\
\qquad+\tau_1(F_xw_{1n}+2F_zw_{2n})+\dfrac{\tau_2}{\sqrt{5}}\theta w_{-2n}=0,\\
-\Delta w_{1n}+\varepsilon_n^2|\varepsilon_n x+\varepsilon_n y_n|^2w_{1n}+\varepsilon_n^2(\lambda_n+\mu_n)w_{1n}+\tau\rho^2w_{1n}\\
\qquad+\tau_1\left(\dfrac{\sqrt{6}}{2}F_xw_{0n}+F_xw_{2n}+F_zw_{1n}\right)-\dfrac{\tau_2}{\sqrt{5}}\theta w_{-1n}=0,\\
-\Delta w_{0n}+\varepsilon_n^2|\varepsilon_n x+\varepsilon_n y_n|^2w_{0n}+\varepsilon_n^2\lambda_n w_{0n}+\tau\rho^2w_{0n}+\dfrac{\sqrt{6}}{2}\tau_1\left(F_xw_{-1n}+F_xw_{1n}\right)+\dfrac{\tau_2}{\sqrt{5}}\theta w_{0n}=0,\\
-\Delta w_{-1n}+\varepsilon_n^2|\varepsilon_n x+\varepsilon_n y_n|^2w_{-1n}+\varepsilon_n^2(\lambda_n-\mu_n)w_{-1n}+\tau\rho^2w_{-1n}\\
\qquad+\tau_1\left(\dfrac{\sqrt{6}}{2}F_xw_{0n}+F_xw_{-2n}-F_zw_{-1n}\right)-\dfrac{\tau_2}{\sqrt{5}}\theta w_{1n}=0,\\
-\Delta w_{-2n}+\varepsilon_n^2|\varepsilon_n x+\varepsilon_n y_n|^2w_{-2n}+\varepsilon_n^2(\lambda_n-2\mu_n)w_{-2n}+\tau\rho^2w_{-2n}\\
\qquad+\tau_1(F_xw_{-1n}-2F_zw_{-2n})+\dfrac{\tau_2}{\sqrt{5}}\theta w_{2n}=0.
\end{cases}
\tag{5.29}
$$

If we let $\lim_{n\to\infty}\varepsilon_n^2\lambda_nN_n=N_1,\lim_{n\to\infty}\varepsilon_n^2\mu_nM=N_2$, using (5.28) and taking limit on both sides of the first equation and the fifth equation in (5.29), we can deduce that $\lim\limits_{n\to\infty}\mu_n\varepsilon_n^2=0$. Therefore

$$\lim_{n\to\infty}\lambda_n\varepsilon_n^2=1.$$

The following proof details are similar to the proof of Theorem 2 in [44], we omit it here. $\square$

6 The 3D case

In this section, we are going to investigate the existence, stability and asymptotic behavior of solutions to (1.4)-(1.5) with $V(x) = |x|^2$.

Now, we prove a local minima structure for $E(\mathbf{u})$ on $\mathcal{M}$. Define

$$\|\mathbf{u}\|_{\Lambda}^2 := \int_{\mathbb{R}^3} \Big((\sum_{j=-2}^{2} |\nabla u_j|^2) + |x|^2 (\sum_{j=-2}^{2} u_j^2) \Big) dx$$

and for any $r > 0$, let

$$B(r) := \Big\{ \mathbf{u} \in \Lambda \Big| \|\mathbf{u}\|_{\Lambda}^2 \leq r \Big\}.$$

Lemma 6.1. *Suppose $\tau \leq 0$, $\tau_1 \leq 0$ and $\tau_2 \leq 0$, then for any $r > 0$, it holds*

$$\mathcal{M} \cap B(r) \neq \emptyset, \quad \text{if } N \leq \frac{r}{3}, \tag{6.1}$$

and further $E(\mathbf{u})$ is bounded from below on $\mathcal{M} \cap B(r)$.

Proof. For any $r > 0$, by Lemma 2.3, it is easy to see that $\Big(0, \sqrt{\frac{N+M}{2}}\Psi_0, 0, \sqrt{\frac{N-M}{2}}\Psi_0, 0\Big) \in \mathcal{M}$. Moreover, if $N \leq \frac{r}{3}$,

$$\Big\| \Big(0, \sqrt{\frac{N+M}{2}}\Psi_0, 0, \sqrt{\frac{N-M}{2}}\Psi_0, 0\Big) \Big\|_{\Lambda}^2 = N \int_{\mathbb{R}^3} \Big(|\nabla \Psi_0|^2 + |x|^2 {\Psi_0}^2 \Big) dx = 3N \leq r.$$

Hence, $\Big(0, \sqrt{\frac{N+M}{2}}\Psi_0, 0, \sqrt{\frac{N-M}{2}}\Psi_0, 0\Big) \in \mathcal{M} \cap B(r)$.

For any $\mathbf{u} \in \mathcal{M} \cap B(r)$, by (2.6), (4.2) and (4.3), we get

$$\begin{aligned} E(\mathbf{u}) \geq & \frac{1}{2} \int_{\mathbb{R}^3} (\sum_{j=-2}^{2} |\nabla u_j|^2 + |x|^2 (\sum_{j=-2}^{2} u_j^2)) dx \\ & + \Big(\frac{\tau}{4} + \tau_1 + \frac{\tau_2}{20} \Big) C_* \Big(\int_{\mathbb{R}^3} (\sum_{j=-2}^{2} |\nabla u_j|^2) dx \Big)^{\frac{3}{2}} N^{\frac{1}{2}} \\ \geq & \Big(\frac{\tau}{4} + \tau_1 + \frac{\tau_2}{20} \Big) C_* \Big(\|\mathbf{u}\|_{\Lambda}^2 \Big)^{\frac{3}{2}} N^{\frac{1}{2}} \geq \Big(\frac{\tau}{4} + \tau_1 + \frac{\tau_2}{20} \Big) C_* r^{\frac{3}{2}} N^{\frac{1}{2}}. \end{aligned}$$

Therefore, we have proved that $E(\mathbf{u})$ is bounded from below on $\mathcal{M} \cap B(r)$. $\square$

For any $r > 0$ and $N \leq \frac{r}{3}$, we consider the following local minimization problem:

$$m_N^r := \inf_{\mathbf{u} \in \mathcal{M} \cap B(r)} E(\mathbf{u}).$$

By Lemma 6.1, m_N^r is well defined.

Lemma 6.2. *Suppose $\tau \leq 0$, $\tau_1 \leq 0$ and $\tau_2 \leq 0$, then for any $r > 0$, there exists $\tilde{N} = \tilde{N}(r)$, such that*

$$m_N^r = \inf_{\mathbf{u} \in \mathcal{M} \cap B(\frac{r}{2})} E(\mathbf{u}), \quad \text{for } N \leq \tilde{N}. \tag{6.2}$$

Proof. For any $r > 0$, if $\mathcal{M} \cap \big(B(r) \setminus B(\frac{r}{2})\big) = \emptyset$, then it is easy to see that (6.2) holds.

If $\mathcal{M} \cap \big(B(r) \setminus B(\frac{r}{2})\big) \neq \emptyset$, then for any $\mathbf{u} \in \mathcal{M} \cap \big(B(r) \setminus B(\frac{r}{2})\big)$ and

$$N \leq \left(\frac{1}{-16\left(\frac{\tau}{4} + \tau_1 + \frac{\tau_2}{20}\right) C_* r^{\frac{1}{2}}}\right)^2,$$

we have

$$\begin{aligned} E(\mathbf{u}) &= \frac{1}{2}\int_{\mathbb{R}^3} \left(\sum_{j=-2}^{2} |\nabla u_j|^2 + |x|^2 \sum_{j=-2}^{2} u_j^2\right) dx + \frac{\tau}{4}\int_{\mathbb{R}^3} \Big(\sum_{j=-2}^{2} u_j^2\Big)^2 dx \\ &\quad + \frac{\tau_1}{4}\int_{\mathbb{R}^3} \big(2(u_{-2}u_{-1} + u_2u_1) + \sqrt{6}(u_{-1}u_0 + u_1u_0)\big)^2 dx + \frac{\tau_1}{4}\int_{\mathbb{R}^3} \Big(\sum_{j=-2}^{2} (ju_j^2)\Big)^2 dx \\ &\quad + \frac{\tau_2}{20}\int_{\mathbb{R}^3} \big(2u_2u_{-2} - 2u_1u_{-1} + u_0^2\big)^2 dx \\ &\geq \frac{1}{2}\|\mathbf{u}\|_\Lambda^2 + \left(\frac{\tau}{4} + \tau_1 + \frac{\tau_2}{20}\right) C_* \left(\|\mathbf{u}\|_\Lambda^2\right)^{\frac{3}{2}} N^{\frac{1}{2}} \\ &\geq \frac{r}{4} + \left(\frac{\tau}{4} + \tau_1 + \frac{\tau_2}{20}\right) C_* r^{\frac{3}{2}} N^{\frac{1}{2}} \geq \frac{3}{16} r. \end{aligned}$$

It follows for any $\mathbf{u} \in \mathcal{M} \cap B(\frac{r}{4})$,

$$E(\mathbf{u}) \leq \frac{1}{2}\|\mathbf{u}\|_\Lambda^2 \leq \frac{r}{8} < \frac{3r}{16} \leq \inf_{\mathbf{u} \in \mathcal{M} \cap \left(B(r) \setminus B(\frac{r}{2})\right)} E(\mathbf{u}).$$

For any $r > 0$, by (6.1),

$$\mathcal{M} \cap B\Big(\frac{r}{4}\Big) \neq \emptyset, \quad \text{if } N \leq \frac{r}{12}.$$

Taking

$$\tilde{N} := \min\left\{\left(\frac{1}{-16\left(\frac{\tau}{4} + \tau_1 + \frac{\tau_2}{20}\right) C_* r^{\frac{1}{2}}}\right)^2, \frac{r}{12}\right\},$$

then we conclude that for $0 < N \leq \tilde{N}$,

$$m_N^r \leq \inf_{\mathbf{u} \in \mathcal{M} \cap B(\frac{r}{4})} E(\mathbf{u}) < \inf_{\mathbf{u} \in \mathcal{M} \cap \left(B(r) \setminus B(\frac{r}{2})\right)} E(\mathbf{u}). \tag{6.3}$$

Therefore, we complete the proof. $\square$

Lemma 6.3. *Suppose $\tau \leq 0$, $\tau_1 \leq 0$ and $\tau_2 \leq 0$, then for any $r > 0$, there exists $N_0 = N_0(r)$, such that for $N \leq N_0$,*

$$\inf_{\mathbf{u} \in \mathcal{M} \cap B(\frac{r}{4})} E(\mathbf{u}) < \inf_{\mathbf{u} \in \mathcal{M} \cap \left(B(r) \setminus B(\frac{r}{2})\right)} E(\mathbf{u}). \tag{6.4}$$

Proof. We first show that $\mathcal{M} \cap \big(B(r) \setminus B(\frac{r}{2})\big) \neq \emptyset$ for small N. For any $\xi > 0$ and $\mathbf{u} \in H^1(\mathbb{R}^3, \mathbb{R}^5)$, we define

$$(\xi \star \mathbf{u})(x) := e^{\frac{3\xi}{2}} \mathbf{u}(e^{\xi} x), \tag{6.5}$$

then by Lemma 6.1,

$$\mathbf{U} := (U_2, U_1, U_0, U_{-1}, U_{-2}) := \xi \star \Big(0, \sqrt{\frac{N+M}{2}}\Psi_0, 0, \sqrt{\frac{N-M}{2}}\Psi_0, 0\Big) \in \mathcal{M}.$$

By direct calculation, we get

$$\big\|\mathbf{U}\big\|_{\Lambda}^2 = e^{2\xi} N \int_{\mathbb{R}^3} |\nabla \Psi_0|^2 dx + e^{-2\xi} N \int_{\mathbb{R}^3} |x|^2 {\Psi_0}^2 dx.$$

Denoting

$$D_1 := \int_{\mathbb{R}^3} |\nabla \Psi_0|^2 dx, \quad D_2 := \int_{\mathbb{R}^3} |x|^2 {\Psi_0}^2 dx,$$

then it is obvious that

$$e^{2\xi} D_1 + e^{-2\xi} D_2 \geq 2\sqrt{D_1 D_2}.$$

Hence for any $r > 0$, if we choose

$$N \leq \frac{3r}{8\sqrt{D_1 D_2}},$$

then there exists $\xi > 0$, such that $\big\|\mathbf{U}\big\|_{\Lambda}^2 = \frac{3}{4}r$, that is $\mathbf{U} \in \mathcal{M} \cap \big(B(r) \setminus B(\frac{r}{2})\big)$. Let

$$N_0 := \min\Big\{\tilde{N}, \frac{3r}{8\sqrt{D_1 D_2}}\Big\},$$

we conclude (6.4) from the proof Lemma 6.2. □

Lemma 6.4. *Suppose $\tau \leq 0$, $\tau_1 \leq 0$ and $\tau_2 \leq 0$, then for any $r > 0$ and $0 < N \leq N_0$, there holds*

$$m_N^r < \frac{3N}{2}.$$

Proof. From the proof of Lemma 6.1, we get $\Big(0, \sqrt{\frac{N+M}{2}}\Psi_0, 0, \sqrt{\frac{N-M}{2}}\Psi_0, 0\Big) \in \mathcal{M} \cap B(r)$. Thus

$$\begin{aligned} m_N^r &= \inf_{\mathbf{u} \in \mathcal{M} \cap B(r)} E(\mathbf{u}) \leq E\Big(0, \sqrt{\frac{N+M}{2}}\Psi_0, 0, \sqrt{\frac{N-M}{2}}\Psi_0, 0\Big) \\ &< \frac{N}{2} \int_{\mathbb{R}^3} \Big(|\nabla \Psi_0|^2 + |x|^2 {\Psi_0}^2\Big) dx = \frac{3N}{2}. \end{aligned}$$

□

Defining

$$\begin{aligned} P(\mathbf{u}) :&= \int_{\mathbb{R}^3} \Big(\sum_{j=-2}^{2} |\nabla u_j|^2 - |x|^2 (\sum_{j=-2}^{2} u_j^2) \Big) dx + \frac{3}{4} \int_{\mathbb{R}^3} (\tau \rho^4 + \tau_1 \mathbf{F}^2 + \tau_2 \theta^2) dx \\ &= \int_{\mathbb{R}^3} \Big(\sum_{j=-2}^{2} |\nabla u_j|^2 - |x|^2 (\sum_{j=-2}^{2} u_j^2) \Big) dx + \frac{3\tau}{4} \int_{\mathbb{R}^3} \Big(\sum_{j=-2}^{2} u_j^2 \Big)^2 dx \\ &\quad + \frac{3\tau_1}{4} \int_{\mathbb{R}^3} \big(2(u_{-2}u_{-1} + u_2 u_1) + \sqrt{6}(u_{-1}u_0 + u_1 u_0)\big)^2 dx + \frac{3\tau_1}{4} \int_{\mathbb{R}^3} \Big(\sum_{j=-2}^{2} (j u_j^2) \Big)^2 dx \\ &\quad + \frac{3\tau_2}{20} \int_{\mathbb{R}^3} \big(2u_2 u_{-2} - 2u_1 u_{-1} + u_0^2\big)^2 dx, \end{aligned} \tag{6.6}$$

then we have

Lemma 6.5. *Suppose $\mathbf{u} \in \mathcal{M}$ is a solution of (1.4)-(1.5), then $P(\mathbf{u}) = 0$ and further $E(\mathbf{u}) > 0$.*

Proof. Since $\mathbf{u}$ is a solution of (1.4)-(1.5), we get that $\mathbf{u}$ satisfies the following Pohozaev identity

$$\begin{aligned} &\int_{\mathbb{R}^3} (\sum_{j=-2}^{2} |\nabla u_j|^2) dx + 5 \int_{\mathbb{R}^3} |x|^2 (\sum_{j=-2}^{2} u_j^2) dx + \frac{3}{2} \int_{\mathbb{R}^3} (\tau\rho^4 + \tau_1 \mathbf{F}^2 + \tau_2 \theta^2) dx \\ = &-3\Big((\lambda + 2\mu) \int_{\mathbb{R}^3} u_2^2 dx + (\lambda + \mu) \int_{\mathbb{R}^3} u_1^2 dx + \lambda \int_{\mathbb{R}^3} u_0^2 dx + (\lambda - \mu) \int_{\mathbb{R}^3} u_{-1}^2 dx + (\lambda - 2\mu) \int_{\mathbb{R}^3} u_{-2}^2 dx\Big). \end{aligned} \tag{6.7}$$

Multiplying the equations in (1.4)-(1.5) by $u_2,\ u_1,\ u_0,\ u_{-1},\ u_{-2}$ and integrating by parts respectively, we then obtain

$$\begin{aligned} &\int_{\mathbb{R}^3} (\sum_{j=-2}^{2} |\nabla u_j|^2) dx + \int_{\mathbb{R}^3} |x|^2 (\sum_{j=-2}^{2} u_j^2) dx + \int_{\mathbb{R}^3} (\tau\rho^4 + \tau_1 \mathbf{F}^2 + \tau_2 \theta^2) dx \\ = &-\Big((\lambda + 2\mu) \int_{\mathbb{R}^3} u_2^2 dx + (\lambda + \mu) \int_{\mathbb{R}^3} u_1^2 dx + \lambda \int_{\mathbb{R}^3} u_0^2 dx + (\lambda - \mu) \int_{\mathbb{R}^3} u_{-1}^2 dx + (\lambda - 2\mu) \int_{\mathbb{R}^3} u_{-2}^2 dx\Big). \end{aligned}$$

Together with (6.7), we have $P(\mathbf{u}) = 0$. It implies that

$$\begin{aligned} E(\mathbf{u}) &= \frac{1}{2} \int_{\mathbb{R}^3} \big(\sum_{j=-2}^{2} |\nabla u_j|^2 + |x|^2 (\sum_{j=-2}^{2} u_j^2) \big) dx - \frac{1}{3} \Big\{ \int_{\mathbb{R}^3} \big(\sum_{j=-2}^{2} |\nabla u_j|^2 \big) dx - \int_{\mathbb{R}^3} \big(|x|^2 (\sum_{j=-2}^{2} u_j^2) \big) dx \Big\} \\ &= \frac{1}{6} \int_{\mathbb{R}^3} \big(\sum_{j=-2}^{2} |\nabla u_j|^2 \big) dx + \frac{5}{6} \int_{\mathbb{R}^3} |x|^2 (\sum_{j=-2}^{2} u_j^2) dx > 0. \end{aligned}$$

Hence, we complete the proof. □

Proof of Theorem 1.7. (i) For any $r > 0$ and $0 < N \leq \frac{r}{3}$, suppose $\{\mathbf{u_n}\} \subset \mathcal{M} \cap B(r)$ is a minimizing sequence for m_N^r, i.e. $E(\mathbf{u_n}) \to m_N^r$ as $n \to \infty$. Then

$$\|\mathbf{u_n}\|_{\Lambda}^2 = \|\mathbf{u_n}\|_{\dot{\Lambda}}^2 + \|\mathbf{u_n}\|_{L^2}^2 \leq r + N,$$

which implies that $\{\mathbf{u_n}\}$ is bounded in Λ. Therefore, there exists $\tilde{\mathbf{u}} \in \Lambda$, such that up to a subsequence, as $n \to \infty$,

$$\begin{cases} \mathbf{u_n} \rightharpoonup \tilde{\mathbf{u}}, & \text{in } \Lambda. \\ \mathbf{u_n} \to \tilde{\mathbf{u}}, & \text{in } L^t(\mathbb{R}^3) \times L^t(\mathbb{R}^3) \times L^t(\mathbb{R}^3),\ \forall t \in [2, 2^*). \\ \mathbf{u_n} \to \tilde{\mathbf{u}}, & a.e. \text{ in } \mathbb{R}^3. \end{cases}$$

Then we get $\tilde{\mathbf{u}} \in \mathcal{M} \cap B(r)$. Further, by the lower semi-continuity of the norm in Λ, there holds

$$m_N^r \leq E(\tilde{\mathbf{u}}) \leq \lim_{n\to\infty} E(\mathbf{u_n}) = m_N^r.$$

It yields that $E(\tilde{\mathbf{u}}) = m_N^r$. Hence, m_N^r has at least one minimizer for any $r > 0$ and $N \leq \frac{r}{3}$.

(ii) For any $r > 0$ and $0 < N \leq N_0$, by (6.2), we can see that $\tilde{\mathbf{u}} \in B(\frac{r}{2})$, which follows that $\tilde{\mathbf{u}}$ stays away from the boundary of $B(r)$. Thus, $\tilde{\mathbf{u}}$ is indeed a critical point of $E(\mathbf{u})$ restricted to $\mathcal{M}$ and further $\tilde{\mathbf{u}}$ is a weak solution for (1.4)-(1.5) with some constants $\tilde{\mu},\ \tilde{\lambda}$ as Lagrange multipliers.

Next, we show that $\tilde{\mathbf{u}}$ is a ground state solution for (1.4)-(1.5) as N small by contradiction. Let $N_n := \min\{\frac{1}{n}, N_0\}$, suppose there exists $r_0 > 0$ and $\{\mathbf{v}_n\} \subset \mathcal{M}(N_n)$, such that

$$E'|_{\mathcal{M}}(\mathbf{v}_n) = 0 \quad \text{and} \quad E(\mathbf{v}_n) < m^{r_0}_{N_n}.$$

Then by Lemma 6.5, we get $P(\mathbf{v}_n) = 0$ and further by Lemma 6.4,

$$\begin{aligned} E(\mathbf{v}_n) &= \frac{1}{6}\int_{\mathbb{R}^3}\big(\sum_{j=-2}^{2}|\nabla v_{jn}|^2\big)dx + \frac{5}{6}\int_{\mathbb{R}^3}|x|^2\big(\sum_{j=-2}^{2}v_{1n}^2\big)dx \\ &< m^{r_0}_{N_n} \to 0, \quad \text{as } n\to\infty. \end{aligned}$$

It implies that

$$\|\mathbf{v}_n\|^2_{\Lambda} = \int_{\mathbb{R}^3}\big(\sum_{j=-2}^{2}|\nabla v_{jn}|^2\big)dx + \int_{\mathbb{R}^3}|x|^2\big(\sum_{j=-2}^{2}v_{1n}^2\big)dx \to 0,$$

then $\mathbf{v}_n \in \mathcal{M}(N_n)\cap B(r_0)$. We can see that $E(\mathbf{v}_n) \geq m^{r_0}_{N_n}$, which is a contradiction. Therefore, $\tilde{\mathbf{u}}$ is a ground state solution of (1.4)-(1.5).

(iii) By Lemma 6.4, for any $r > 0$ and $0 < N \leq N_0$, there holds

$$m^r_N < \frac{3N}{2}. \tag{6.8}$$

Denote

$$\mathcal{M}^r_N := \Big\{\mathbf{u}\in\mathcal{M}\cap B(r)\big|E(\mathbf{u}) = m^r_N\Big\}.$$

Suppose $\mathbf{u}_N \in \mathcal{M}^r_N$, by Lemma 6.5, we can see that

$$\begin{aligned} m^r_N &= E(\mathbf{u}_N) \\ &= \frac{1}{6}\int_{\mathbb{R}^3}\big(\sum_{j=-2}^{2}|\nabla u_{jN}|^2\big)dx + \frac{5}{6}\int_{\mathbb{R}^3}|x|^2\big(\sum_{j=-2}^{2}u_{jN}^2\big)dx \\ &\geq \frac{1}{6}\Big\{\int_{\mathbb{R}^3}\big(\sum_{j=-2}^{2}|\nabla u_{jN}|^2\big)dx + \int_{\mathbb{R}^3}|x|^2\big(\sum_{j=-2}^{2}u_{jN}^2\big)dx\Big\} \\ &= \frac{1}{6}\|\mathbf{u}_N\|^2_{\Lambda}, \end{aligned}$$

that is $\|\mathbf{u}_N\|^2_{\Lambda} \leq 6m^r_N$. Together with (6.8), we have

$$\Big(\frac{\|\mathbf{u}_N\|^2_{\Lambda}}{N}\Big)^{\frac{3}{2}} \leq \Big(\frac{6m^r_N}{N}\Big)^{\frac{3}{2}} < \Big(\frac{6\cdot\frac{3N}{2}}{N}\Big)^{\frac{3}{2}} = 27. \tag{6.9}$$

Then by (2.6), (4.2) and (4.3), we get

$$\begin{aligned} &-\frac{1}{N}\Bigg\{\frac{\tau}{4}\int_{\mathbb{R}^3}\Big(\sum_{j=-2}^{2}u_{jN}^2\Big)^2dx + \frac{\tau_2}{20}\int_{\mathbb{R}^3}\big(2u_{2N}u_{-2N} - 2u_{1N}u_{-1N} + u_{0N}^2\big)^2dx \\ &+\frac{\tau_1}{4}\int_{\mathbb{R}^3}\big(2(u_{-2N}u_{-1N} + u_{2N}u_{1N}) + \sqrt{6}(u_{-1N}u_{0N} + u_{1N}u_{0N})\big)^2dx + \frac{\tau_1}{4}\int_{\mathbb{R}^3}\Big(\sum_{j=-2}^{2}(ju_{jN}^2)\Big)^2dx\Bigg\} \end{aligned}$$

$$
\begin{aligned}
&\leq -\Big(\frac{\tau}{4}+\tau_1+\frac{\tau_2}{20}\Big)C_*\Big(\|\mathbf{u}_N\|_{\dot\Lambda}^2\Big)^{\frac{3}{2}}N^{-\frac{1}{2}} = -\Big(\frac{\tau}{4}+\tau_1+\frac{\tau_2}{20}\Big)C_*\Big(\frac{\|\mathbf{u}_N\|_{\dot\Lambda}^2}{N}\Big)^{\frac{3}{2}}N\\
&< -27\Big(\frac{\tau}{4}+\tau_1+\frac{\tau_2}{20}\Big)C_*N \to 0, \quad \text{as } N\to 0^+,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&\lim_{N\to 0^+}\frac{1}{N}\bigg\{\frac{\tau}{4}\int_{\mathbb{R}^3}\Big(\sum_{j=-2}^{2}u_{jN}^2\Big)^2dx + \frac{\tau_2}{20}\int_{\mathbb{R}^3}\big(2u_{2N}u_{-2N}-2u_{1N}u_{-1N}+u_{0N}^2\big)^2dx\\
&+\frac{\tau_1}{4}\int_{\mathbb{R}^3}\big(2(u_{-2N}u_{-1N}+u_{2N}u_{1N})+\sqrt{6}(u_{-1N}u_{0N}+u_{1N}u_{0N})\big)^2dx + \frac{\tau_1}{4}\int_{\mathbb{R}^3}\Big(\sum_{j=-2}^{2}(ju_{jN}^2)\Big)^2dx\bigg\} = 0.
\end{aligned} \tag{6.10}
$$

Since $I'|_{\mathcal{M}}(\mathbf{u}_N)=0$, there exist two sequences $\{\lambda_N\},\ \{\mu_N\}\subset\mathbb{R}$, such that

$$
\begin{aligned}
&-(\lambda_N N+\mu_N M)\\
&=\int_{\mathbb{R}^3}\Big(\sum_{j=-2}^{2}|\nabla u_{jN}|^2\Big)dx + \int_{\mathbb{R}^3}|x|^2\Big(\sum_{j=-2}^{2}u_{jN}^2\Big)dx\\
&\quad+\bigg\{\tau\int_{\mathbb{R}^3}\Big(\sum_{j=-2}^{2}u_{jN}^2\Big)^2dx + \frac{\tau_2}{5}\int_{\mathbb{R}^3}\big(2u_{2N}u_{-2N}-2u_{1N}u_{-1N}+u_{0N}^2\big)^2dx\\
&\quad+\tau_1\int_{\mathbb{R}^3}\big(2(u_{-2N}u_{-1N}+u_{2N}u_{1N})+\sqrt{6}(u_{-1N}u_{0N}+u_{1N}u_{0N})\big)^2dx + \tau_1\int_{\mathbb{R}^3}\Big(\sum_{j=-2}^{2}(ju_{jN}^2)\Big)^2dx\bigg\}.
\end{aligned} \tag{6.11}
$$

By Lemma 2.3 with $d=3$, for any $\mathbf{u}\in\Lambda$, there holds

$$
\int_{\mathbb{R}^3}\Big(\sum_{j=-2}^{2}|\nabla u_j|^2 + |x|^2\sum_{j=-2}^{2}u_j^2\Big)dx \geq 3\int_{\mathbb{R}^3}\Big(\sum_{j=-2}^{2}u_j^2\Big)dx. \tag{6.12}
$$

Then by (6.10) and (6.12), we obtain

$$
-\lim_{N\to 0^+}\frac{\lambda_N N+\mu_N M}{N}\geq 3. \tag{6.13}
$$

By (6.8) and (6.11), we can see that

$$
\begin{aligned}
&-(\lambda_N N+\mu_N M)\\
&=2E(\mathbf{u}_N)+\frac{1}{2}\bigg\{\tau\int_{\mathbb{R}^3}\Big(\sum_{j=-2}^{2}u_{jN}^2\Big)^2dx + \frac{\tau_2}{5}\int_{\mathbb{R}^3}\big(2u_{2N}u_{-2N}-2u_{1N}u_{-1N}+u_{0N}^2\big)^2dx\\
&\quad+\tau_1\int_{\mathbb{R}^3}\big(2(u_{-2N}u_{-1N}+u_{2N}u_{1N})+\sqrt{6}(u_{-1N}u_{0N}+u_{1N}u_{0N})\big)^2dx + \tau_1\int_{\mathbb{R}^3}\Big(\sum_{j=-2}^{2}(ju_{jN}^2)\Big)^2dx\bigg\}\\
&\leq 2E(u_N)=2m_N^r<2\cdot\frac{3}{2}N=3N.
\end{aligned}
$$

Hence, together with (6.13), we get $\lim_{N\to 0^+}\frac{\lambda_N N+\mu_N M}{N}=-3$. Further, we can deduce from (6.10) and (6.11) that

$$\lim_{N\to 0^+}\frac{\|\mathbf{u}_N\|_{\Lambda}^2}{N}=\lim_{N\to 0^+}\frac{2E(\mathbf{u}_N)}{N}=\lim_{N\to 0^+}\frac{2m_N^r}{N}=3.$$

Next, we show as $N\to 0^+$, there holds

$$\|\mathbf{u}_N-(l_{20}\Psi_0,l_{10}\Psi_0,l_{00}\Psi_0,l_{-10}\Psi_0,l_{-20}\Psi_0)\|_{\Lambda}^2=O(N^2), \tag{6.14}$$

where $l_{j0}=\int_{\mathbb{R}^3}u_{jN}\Psi_0dx$ for $j=2,1,0,-1,-2$. Set $l_{jk}=\int_{\mathbb{R}^3}u_{jN}\Psi_kdx$ for $j=2,1,0,-1,-2$, we have

$$\mathbf{u}_N=\Big(\sum_{k=0}^{\infty}l_{2k}\Psi_k,\sum_{k=0}^{\infty}l_{1k}\Psi_k,\sum_{k=0}^{\infty}l_{0k}\Psi_k,\sum_{k=0}^{\infty}l_{-1k}\Psi_k,\sum_{k=0}^{\infty}l_{-2k}\Psi_k\Big).$$

Moreover, we can conclude

$$N=\|\mathbf{u}_N\|_{L^2}^2=\sum_{k=0}^{\infty}\Big(\sum_{j=-2}^{2}l_{jk}^2\Big)\|\Psi_k\|_{L^2}^2=\sum_{k=0}^{\infty}\Big(\sum_{j=-2}^{2}l_{jk}^2\Big) \tag{6.15}$$

and

$$\|\mathbf{u}_N\|_{\Lambda}^2=\sum_{k=0}^{\infty}\Big(\sum_{j=-2}^{2}l_{jk}^2\Big)\|\Psi_k\|_{\Lambda}^2=\sum_{k=0}^{\infty}\xi_k\Big(\sum_{j=-2}^{2}l_{jk}^2\Big).$$

By (2.6), (4.2), (4.3) and (6.9), we get

$$\begin{aligned}
&m_N^r=E(\mathbf{u}_N)\\
&=\frac12\int_{\mathbb{R}^3}(\sum_{j=-2}^{2}|\nabla u_{jN}|^2)dx+\int_{\mathbb{R}^3}|x|^2(\sum_{j=-2}^{2}u_{jN}^2)dx\\
&\quad+\frac14\Big\{\tau\int_{\mathbb{R}^3}\Big(\sum_{j=-2}^{2}u_{jN}^2\Big)^2dx+\frac{\tau_2}{5}\int_{\mathbb{R}^3}\big(2u_{2N}u_{-2N}-2u_{1N}u_{-1N}+u_{0N}^2\big)^2dx\\
&\quad+\tau_1\int_{\mathbb{R}^3}\big(2(u_{-2N}u_{-1N}+u_{2N}u_{1N})+\sqrt{6}(u_{-1N}u_{0N}+u_{1N}u_{0N})\big)^2dx+\tau_1\int_{\mathbb{R}^3}\Big(\sum_{j=-2}^{2}(ju_{jN}^2)\Big)^2dx\Big\}\\
&\geq\frac12\|\mathbf{u}_N\|_{\Lambda}^2+\Big(\frac{\tau}{4}+\tau_1+\frac{\tau_2}{20}\Big)C_*\|\mathbf{u}_N\|_{\Lambda}^3N^{\frac12}\geq\frac12\|\mathbf{u}_N\|_{\Lambda}^2+27\Big(\frac{\tau}{4}+\tau_1+\frac{\tau_2}{20}\Big)C_*N^2\\
&=\frac12\sum_{k=0}^{\infty}(\xi_k-\xi_0)\Big(\sum_{j=-2}^{2}l_{jk}^2\Big)+\frac12\sum_{k=0}^{\infty}\xi_0\Big(\sum_{j=-2}^{2}l_{jk}^2\Big)+27\Big(\frac{\tau}{4}+\tau_1+\frac{\tau_2}{20}\Big)C_*N^2.
\end{aligned}$$

Then by (6.8) and (6.15), we have

$$\begin{aligned}
&(\xi_1-\xi_0)\sum_{k=1}^{\infty}\Big(\sum_{j=-2}^{2}l_{jk}^2\Big)\leq\sum_{k=1}^{\infty}(\xi_k-\xi_0)\Big(\sum_{j=-2}^{2}l_{jk}^2\Big)\\
&\leq-54\Big(\frac{\tau}{4}+\tau_1+\frac{\tau_2}{20}\Big)C_*N^2-3N+2m_N^r<-54\Big(\frac{\tau}{4}+\tau_1+\frac{\tau_2}{20}\Big)C_*N^2,
\end{aligned}$$

which yields that

$$\sum_{k=1}^{\infty}\Big(\sum_{j=-2}^{2} l_{jk}^2\Big) \le \frac{-54\Big(\frac{\tau}{4}+\tau_1+\frac{\tau_2}{20}\Big)C_*N^2}{\xi_1-\xi_0}.$$

Thus

$$\begin{aligned}\sum_{k=1}^{\infty}\xi_k\Big(\sum_{j=-2}^{2} l_{jk}^2\Big) &= \sum_{k=1}^{\infty}(\xi_k-\xi_0)\Big(\sum_{k=-2}^{2} l_{jk}^2\Big)+\xi_0\sum_{k=1}^{\infty}\Big(\sum_{k=-2}^{2} l_{jk}^2\Big)\\ &\le -54\Big(\frac{\tau}{4}+\tau_1+\frac{\tau_2}{20}\Big)C_*N^2+\xi_0\cdot\frac{-54\Big(\frac{\tau}{4}+\tau_1+\frac{\tau_2}{20}\Big)C_*N^2}{\xi_1-\xi_0}\\ &= -54\frac{\xi_1}{\xi_1-\xi_0}\cdot\Big(\frac{\tau}{4}+\tau_1+\frac{\tau_2}{20}\Big)C_*N^2.\end{aligned}$$

For $N\to 0^+$, we can see that

$$\begin{aligned}&\|\mathbf{u}_N-(l_{20}\Psi_0,l_{10}\Psi_0,l_{00}\Psi_0,l_{-10}\Psi_0,l_{-20}\Psi_0)\|_{\dot\Lambda}^2\\ &=\Big\|\Big(\sum_{k=1}^{\infty}l_{2k}\Psi_k,\sum_{k=1}^{\infty}l_{1k}\Psi_k,\sum_{k=1}^{\infty}l_{0k}\Psi_k,\sum_{k=1}^{\infty}l_{-1k}\Psi_k,\sum_{k=1}^{\infty}l_{-2k}\Psi_k\Big)\Big\|_{\dot\Lambda}^2\\ &=\sum_{k=1}^{\infty}\xi_k\Big(\sum_{j=-2}^{2}l_{jk}^2\Big)=O(N^2)\end{aligned}$$

and

$$\begin{aligned}&\|\mathbf{u}_N-(l_{20}\Psi_0,l_{10}\Psi_0,l_{00}\Psi_0,l_{-10}\Psi_0,l_{-20}\Psi_0)\|_{L^2}^2\\ &=\Big\|\Big(\sum_{k=1}^{\infty}l_{2k}\Psi_k,\sum_{k=1}^{\infty}l_{1k}\Psi_k,\sum_{k=1}^{\infty}l_{0k}\Psi_k,\sum_{k=1}^{\infty}l_{-1k}\Psi_k,\sum_{k=1}^{\infty}l_{-2k}\Psi_k\Big)\Big\|_{L^2}^2\\ &=\sum_{k=1}^{\infty}\Big(\sum_{j=-2}^{2}l_{jk}^2\Big)=O(N^2).\end{aligned}$$

Therefore, it is obvious that (6.14) holds. We complete the proof of (iii) in Theorem 1.7. □

Next, we will show that the set $\mathcal{M}_N^r$ is orbitally stable under the flow of (1.3). To this end, we need the following global well-posedness result.

Lemma 6.6. *For any $r>0$, $\mathbf{u}(\mathbf{0}):=\big(u_2(0),u_1(0),u_0(0),u_{-1}(0),u_{-2}(0)\big)$ in Λ be such that $\|\mathbf{u}(\mathbf{0})\|_{\dot\Lambda}^2\le r$. Then there exists $N_0=N_0(r)>0$ sufficiently small such that for all $0<N<N_0$, if $\mathbf{u}(\mathbf{0})\in\mathcal{M}$, then the corresponding solution to* (1.3) *exists globally in time.*

Proof. The proof is based on the following continuity argument: Let $I\subset\mathbb{R}$ be a time interval and $X: I\to[0,+\infty)$ be a continuous function satisfying $X(t)\le a+b(X(t))^{\theta}$, for every $t\in I$ and some constants $a,\ b,\ \theta>0$. Assume that $X(t_0)\le 2a$ for some $t_0\in I$ and $b<2^{-\theta}a^{1-\theta}$, then we have $X(t)\le 2a$ for every $t\in I$. By the uncertainty principle (see e.g. [54]), we get

$$\int_{\mathbb{R}^3}\Big(\sum_{j=-2}^{2}u_j^2(0)\Big)dx\le\frac{2}{3}\Big(\int_{\mathbb{R}^3}\Big(\sum_{j=-2}^{2}|\nabla u_j(0)|^2\Big)dx\Big)^{\frac12}\Big(\int_{\mathbb{R}^3}|x|^2\Big(\sum_{j=-2}^{2}u_j^2(0)\Big)dx\Big)^{\frac12}\le\|\mathbf{u}(\mathbf{0})\|_{\dot\Lambda}^2\le r.$$

By (2.6), (4.2) and (4.3), we have

$$\big|E(\mathbf{u(0)})\big| \le \frac{1}{2}\|(\mathbf{u(0)})\|_{\Lambda}^2 - \Big(\frac{\tau}{4}+\tau_1+\frac{\tau_2}{20}\Big)C_*\Big(\int_{\mathbb{R}^3}\big(\sum_{j=-2}^{2}|\nabla u_j(0)|^2\big)dx\Big)^{\frac{3}{2}}N^{\frac{1}{2}} \le C(r),$$

for some constant $C(r)$ depending only on r. Let $\mathbf{u(t)} := (u_2(t), u_1(t), u_0(t), u_{-1}(t), u_{-2}(t))$, by the conservations of mass and energy, we have

$$\|\mathbf{u(t)}\|_{\Lambda}^2 \le 2|E(\mathbf{u(t)})| - 2\Big(\frac{\tau}{4}+\tau_1+\frac{\tau_2}{20}\Big)C_*\Big(\int_{\mathbb{R}^3}\big(\sum_{j=-2}^{2}|\nabla u_j(t)|^2\big)dx\Big)^{\frac{3}{2}}N^{\frac{1}{2}}. \tag{6.16}$$

Set $X(t) = \|\mathbf{u(t)}\|_{\Lambda}^2$, $a = 2|E(\mathbf{u(t)})| + \frac{1}{2}\|\mathbf{u(0)}\|_{\Lambda}^2$ and

$$b = -2\Big(\frac{\tau}{4}+\tau_1+\frac{\tau_2}{20}\Big)C_*N^{\frac{1}{2}},$$

then we see from (6.16) that $X(t) \le a + b(X(t))^{\frac{3}{2}}$ for all t in the existence time interval. Since $X(0) = \|\mathbf{u(0)}\|_{\Lambda}^2 \le 2a$ and a is bounded from above by some constant depending only on r, for N is sufficiently small, we can set $b < 2^{-\frac{3}{2}}a^{-\frac{1}{2}}$. Applying the above continuity argument, we obtain $X(t) \le 2a$ for all t in the existence time. This shows that for N is sufficiently small depending only on r, the corresponding solution to (1.3) and (1.5) has bounded norm. Then local theory implies that the solution exists globally in time. □

Proof of Theorem 1.7. (iv) If $\mathbf{u(0)} = (u_2(0), u_1(0), u_0(0), u_{-1}(0), u_{-2}(0)) \in \mathcal{M}$ with $\|\mathbf{u(0)}\|_{\Lambda}^2 \le r$, then the corresponding solution to (1.3) exists globally in time by Lemma 6.6. Suppose that there exists $\epsilon_0 > 0$, a sequence of initial data $\mathbf{u_n(0},\cdot) = \big(u_{2n}(0,\cdot), u_{1n}(0,\cdot), u_{0n}(0,\cdot), u_{-1n}(0,\cdot), u_{-2n}(0,\cdot)\big) \subset \Lambda$ and a sequence $\{t_n\} \subset \mathbb{R}$, such that the solution $\mathbf{u_n} = (u_{2n}, u_{1n}, u_{0n}, u_{-1n}, u_{-2n})$ of problem (1.3) with initial data $\mathbf{u_n(0},\cdot)$ satisfies

$$\inf_{\mathbf{u}=(u_2,u_1,u_0,u_{-1},u_{-2})\in\mathcal{M}_N^r} \big\|\mathbf{u} - \mathbf{u_n(0},\cdot)\big\|_{\Lambda} < \frac{1}{n} \tag{6.17}$$

and

$$\inf_{\mathbf{u}=(u_2,u_1,u_0,u_{-1},u_{-2})\in\mathcal{M}_N^r} \big\|\mathbf{u} - \mathbf{u_n(t_n},\cdot)\big\|_{\Lambda} > \epsilon_0, \tag{6.18}$$

where $\mathbf{u_n(t_n},\cdot) = \big(u_{2n}(t_n,\cdot), u_{1n}(t_n,\cdot), u_{0n}(t_n,\cdot), u_{-1n}(t_n), u_{-2n}(t_n)\big)$.

Without loss of generality, we may assume $\big\{\mathbf{u_n(0},\cdot)\big\} \subset \mathcal{M}$, we claim that $\big\{\mathbf{u_n(t_n},\cdot)\big\} \subset B(r)$. Indeed, if $\big\{\mathbf{u_n(t_n},\cdot)\big\} \subset \Lambda\backslash B(r)$, then by the continuity, there exists $\bar{t}_n \in [0, t_n]$ such that $\big\{\mathbf{u_n(\bar{t}_n},\cdot)\big\} \subset \partial B(r)$. Hence by the conversation laws of the energy and mass (see [12]), Lemma 6.2 and (6.4), we see that

$$I\big(\mathbf{u_n(0},\cdot)\big) = I\big(\big\{\mathbf{u_n(\bar{t}_n},\cdot)\big\}\big) \ge \inf_{\mathbf{u}\in\mathcal{M}\cap\left(B(r)\backslash B(\frac{r}{2})\right)} E(\mathbf{u}) > \inf_{\mathbf{u}\in\mathcal{M}\cap B(\frac{r}{4})} E(\mathbf{u}) \ge m_N^r,$$

which contradicts to (6.17). Then $\big\{\mathbf{u_n(t_n},\cdot)\big\}$ is a minimizing sequence of m_N^r. Similarly to the proof of Theorem 1.7 (i), there exists $\mathbf{v} = (v_2, v_1, v_0, v_{-1}, v_{-2}) \in \mathcal{M}_N^r$ such that $\mathbf{u_n(t_n},\cdot) \to \mathbf{v}$ as $n \to \infty$ in Λ, which contradicts to (6.18). Therefore, we complete the proof of (iv) in Theorem 1.7. □

7 Proof of Theorem 1.8

For any $r > 0$ and $0 < N \leq N_0$, suppose $\tilde{\mathbf{u}} \in \mathcal{M} \cap B(r)$ is the solution of (1.4)-(1.5) obtained in Theorem 1.7 (ii), then we see that $\tilde{\mathbf{u}} \in B(\frac{r}{2})$. Let $\xi \star \mathbf{u}$ be the operator defined in (6.5), by direct calculations, we get

$$\int_{\mathbb{R}^3} |(\xi \star \mathbf{u})|^2 dx = \int_{\mathbb{R}^3} \Big(\sum_{j=-2}^{2} u_j^2 \Big) dx$$

and

$$E(\xi \star \mathbf{u}) = \frac{1}{2} e^{2\xi} \int_{\mathbb{R}^3} \Big(\sum_{j=-2}^{2} |\nabla u_j|^2 \Big) dx + \frac{1}{2} e^{-2\xi} \int_{\mathbb{R}^3} |x|^2 \rho^2 dx + \frac{1}{4} e^{3\xi} \int_{\mathbb{R}^3} (\tau \rho^4 + \tau_1 \mathbf{F}^2 + \tau_2 \theta^2) dx.$$

Then for any $\mathbf{u} \in \mathcal{M}$, there holds $\lim_{\tau \to +\infty} E\big(\xi \star \mathbf{u}\big) = -\infty$. Hence there exists a large $\xi_1 > 0$, such that

$$\big\| \big(\xi_1 \star \tilde{\mathbf{u}}\big) \big\|_{\dot{\Lambda}}^2 > r \quad \text{and} \quad E\big(\xi \star \tilde{\mathbf{u}}\big) < 0.$$

We now define a path as

$$\Gamma := \Big\{ g \in C\big([0,1], \mathcal{M}\big) \big|\ g(0) = \tilde{\mathbf{u}},\ g(1) = \xi_1 \star \tilde{\mathbf{u}} \Big\},$$

then for $t \in [0,1]$, it is easy to see that $g(t) := \big((1-t) + t\xi_1\big) \star \tilde{\mathbf{u}} \in \Gamma$, that is $\Gamma \neq \emptyset$. Hence, the minimax value

$$\sigma := \inf_{g \in \Gamma} \max_{t \in [0,1]} E\big(g(t)\big)$$

is well defined. Further, we can deduce

$$\sigma > \max \Big\{ E(\tilde{u}), E\big(\xi_1 \star \tilde{u}\big) \Big\} > 0. \tag{7.1}$$

Indeed, for any $g \in \Gamma$, we have $g(0) = \tilde{u} \in B(\frac{r}{2})$ and $g(1) = \xi_1 \star \tilde{u}$ with $\|(\xi_1 \star \tilde{u})\|_{\dot{\Lambda}}^2 > r$, then there exists $t_0 \in (0,1)$, such that $g(t_0) \in \partial B(r)$. Then by (6.2) and (6.4), we get

$$\begin{aligned} &\max_{t \in [0,1]} E\big(g(t)\big) \geq E\big(g(t_0)\big) \geq \inf_{\mathbf{u} \in \mathcal{M} \cap \left(B(r) \setminus B(\frac{r}{2})\right)} E(\mathbf{u}) \\ &> \inf_{\mathbf{u} \in \mathcal{M} \cap B(\frac{r}{4})} E(\mathbf{u}) \geq \inf_{\mathbf{u} \in \mathcal{M} \cap B(\frac{r}{2})} E(\mathbf{u}) \\ &= m_N^r = E(\tilde{u}) > 0 > E\big(\xi_1 \star \tilde{u}\big), \end{aligned}$$

which implies (7.1).

Defining the Pohozaev manifold of system (1.4) as

$$\mathcal{P} := \Big\{ \mathbf{u} \in \mathcal{M} \big|\ P(\mathbf{u}) = 0 \Big\}, \tag{7.2}$$

where $P(\mathbf{u})$ is the corresponding Pohozaev identity of system (1.4) defined in (6.6), then we get following Lemmas. The proofs are similar to Lemma Lemma 3.5 in [44], we omit the details here.

Lemma 7.1. *$E(\mathbf{u})$ is bounded from below and coercive on $\mathcal{P}$. Moreover, there exists a positive constant C, such that $E(\mathbf{u}) \geq C$ for any $\mathbf{u} \in \mathcal{P}$.*

Lemma 7.2. *$\mathcal{P}$ is a C^1 submanifold in $\mathcal{M}$ with codimension 3.*

Lemma 7.3. *Suppose $\tau \le 0$, $\tau_1 \le 0$ and $\tau_2 \le 0$, then for any $r > 0$ and $0 < N \le N_0$, there exists a bounded Palais-Smale sequence $\{\mathbf{u}_n\}$ for E restricted to $\mathcal{M}$ at level σ. In addition,*

$$
\begin{aligned}
P(\mathbf{u}_n) = &\int_{\mathbb{R}^3} \Big(\sum_{j=-2}^{2} |\nabla u_{jn}|^2 - |x|^2 (\sum_{j=-2}^{2} u_{jn}^2) \Big) dx + \frac{3\tau}{4} \int_{\mathbb{R}^3} \Big(\sum_{j=-2}^{2} u_{jn}^2 \Big)^2 dx \\
&+ \frac{3\tau_1}{4} \int_{\mathbb{R}^3} \big(2(u_{-2n}u_{-1n} + u_{2n}u_{1n}) + \sqrt{6}(u_{-1n}u_{0n} + u_{1n}u_{0n}) \big)^2 dx + \frac{3\tau_1}{4} \int_{\mathbb{R}^3} \Big(\sum_{j=-2}^{2} (j u_{jn}^2) \Big)^2 dx \\
&+ \frac{3\tau_2}{20} \int_{\mathbb{R}^3} \big(2u_{2n}u_{-2n} - 2u_{1n}u_{-1n} + u_{0n}^2 \big)^2 dx = o_n(1), \quad \text{as } n \to \infty.
\end{aligned}
$$

where $o_n(1) \to 0$ as $n \to \infty$.

Proof. The existence of Palais-Smale sequence $\{\mathbf{u_n}\}$ for E at level σ with $P(\mathbf{u_n}) = o_n(1)$ is similar to the proof of Proposition 3.1 in [44], we omit the details here. We only show $\{\mathbf{u_n}\} \subset \mathcal{M}$ is bounded in Λ. Indeed, direct calculation gives

$$
\begin{aligned}
E(\mathbf{u_n}) &= E(\mathbf{u_n}) - \frac{1}{3} P(\mathbf{u_n}) + o_n(1) \\
&= \frac{1}{6} \int_{\mathbb{R}^3} \Big(\sum_{j=-2}^{2} |\nabla u_{jn}|^2 \Big) dx + \frac{5}{6} \int_{\mathbb{R}^3} \Big(|x|^2 (\sum_{j=-2}^{2} u_{jn}^2) \Big) dx + o_n(1).
\end{aligned}
$$

Since $\{\mathbf{u_n}\} \subset \mathcal{M}$ and $E(\mathbf{u_n}) \to \sigma$ as $n \to \infty$, then we get the boundedness of $\{\mathbf{u_n}\}$ in Λ. Therefore, we complete the proof. $\square$

Lemma 7.4. *Suppose $\tau \le 0$, $\tau_1 \le 0$ and $\tau_2 \le 0$, for any $r > 0$ and $0 < N \le N_0$, let $\{\mathbf{u_n}\} \subset \mathcal{M}$ be the Palais-Smale sequence obtained in Proposition 7.3, then there exists $\hat{\mathbf{u}} \in \mathcal{M}$, such that $\mathbf{u_n} \to \hat{\mathbf{u}}$ is strongly in Λ as $n \to \infty$.*

Proof. By Lemma 2.1 and Proposition 7.3, there exists $\hat{\mathbf{u}} = (\hat{u}_2, \hat{u}_1, \hat{u}_0, \hat{u}_{-1}, \hat{u}_{-2}) \in \Lambda$, such that up to a subsequence, as $n \to +\infty$,

$$
\begin{cases}
\mathbf{u_n} \rightharpoonup \hat{\mathbf{u}}, & \text{in } \Lambda. \\
\mathbf{u_n} \to \hat{\mathbf{u}}, & \text{in } L^t(\mathbb{R}^3, \mathbb{R}^5), \ \forall t \in [2, 2^*). \\
\mathbf{u_n} \to \hat{\mathbf{u}}, & a.e. \text{ in } \mathbb{R}^3.
\end{cases} \tag{7.3}
$$

Since $E'\big|_{\mathcal{M}}(\mathbf{u_n}) \to 0$, then there exist two sequences $\{\lambda_n\}$, $\{\mu_n\} \subset \mathbb{R}$, such that

$$
\begin{aligned}
&\int_{\mathbb{R}^3} \sum_{j=-2}^{2} (\nabla u_{jn} \nabla \phi_j) dx + \int_{\mathbb{R}^3} |x|^2 \Big(\sum_{j=-2}^{2} (u_{jn}\phi_j) \Big) dx + A(\mathbf{u_n}, \boldsymbol{\phi}) + B(\mathbf{u_n}, \boldsymbol{\phi}) + C(\mathbf{u_n}, \boldsymbol{\phi}) \\
&= (\lambda_n + 2\mu_n) \int_{\mathbb{R}^3} u_{2n}\phi_2 dx + (\lambda_n + \mu_n) \int_{\mathbb{R}^3} u_{1n}\phi_1 dx + \lambda_n \int_{\mathbb{R}^3} u_{0n}\phi_0 dx \\
&\quad + (\lambda_n - \mu_n) \int_{\mathbb{R}^3} u_{-1n}\phi_{-1} dx + (\lambda_n - 2\mu_n) \int_{\mathbb{R}^3} u_{-2n}\phi_{-2} dx + o_n(1),
\end{aligned} \tag{7.4}
$$

where

$$A(\mathbf{u_n}, \boldsymbol{\phi}) = \tau \int_{\mathbb{R}^3} \sum_{k=-2}^{2} \Big(u_{kn}^2 \sum_{j=-2}^{2} (u_{jn}\phi_j) \Big) dx,$$

$$\begin{aligned}
B(\mathbf{u_n}, \boldsymbol{\phi}) &= \tau_1 \int_{\mathbb{R}^3} (4u_{2n}^3\phi_2 + u_{1n}^3\phi_1 + u_{-1n}^3\phi_{-1} + 4u_{-2n}^3\phi_{-2}) dx \\
&+ \tau_1 \int_{\mathbb{R}^3} (4u_{2n}u_{1n}^2\phi_2 + 4u_{1n}u_{2n}^2\phi_1 + 4u_{-2n}u_{-1n}^2\phi_{-2} + 4u_{-1n}u_{-2n}^2\phi_{-1}) dx \\
&- \tau_1 \int_{\mathbb{R}^3} (2u_{2n}u_{-1n}^2\phi_2 + 2u_{-1n}u_{2n}^2\phi_{-1} + 4u_{2n}u_{-2n}^2\phi_2 + 4u_{-2n}u_{2n}^2\phi_{-2}) dx \\
&- \tau_1 \int_{\mathbb{R}^3} (u_{1n}u_{-1n}^2\phi_1 + u_{-1n}u_{1n}^2\phi_{-1} + 2u_{1n}u_{-2n}^2\phi_1 + 2u_{-2n}u_{1n}^2\phi_{-2}) dx \\
&+ \tau_1 \int_{\mathbb{R}^3} (3u_{-1n}u_{0n}^2\phi_{-1} + 3u_{0n}u_{-1n}^2\phi_0 + 3u_{1n}u_{0n}^2\phi_1 + 3u_{0n}u_{1n}^2\phi_0) dx \\
&+ \tau_1 \int_{\mathbb{R}^3} (2u_{-1n}u_{2n}u_{1n}\phi_{-2} + 2u_{-2n}u_{2n}u_{1n}\phi_{-1} + 2u_{-2n}u_{-1n}u_{1n}\phi_2 + 2u_{-2n}u_{-1n}u_{2n}\phi_1) dx \\
&+ \tau_1 \int_{\mathbb{R}^3} (3u_{1n}u_{0n}^2\phi_{-1} + 3u_{-1n}u_{0n}^2\phi_1 + 6u_{-1n}u_{1n}u_{0n}\phi_0) dx \\
&+ \tau_1 \int_{\mathbb{R}^3} (\sqrt{6}u_{0n}u_{-1n}^2\phi_{-2} + \sqrt{6}u_{-2n}u_{-1n}^2\phi_0 + 2\sqrt{6}u_{-2n}u_{-1n}u_{0n}\phi_{-1}) dx \\
&+ \tau_1 \int_{\mathbb{R}^3} (\sqrt{6}u_{-1n}u_{0n}u_{1n}\phi_{-2} + \sqrt{6}u_{-2n}u_{0n}u_{1n}\phi_{-1} + \sqrt{6}u_{-2n}u_{-1n}u_{1n}\phi_0 + \sqrt{6}u_{-2n}u_{-1n}u_{0n}\phi_1) dx \\
&+ \tau_1 \int_{\mathbb{R}^3} (\sqrt{6}u_{1n}u_{-1n}u_{0n}\phi_2 + \sqrt{6}u_{2n}u_{-1n}u_{0n}\phi_1 + \sqrt{6}u_{2n}u_{1n}u_{0n}\phi_{-1} + \sqrt{6}u_{2n}u_{1n}u_{-1n}\phi_0) dx \\
&+ \tau_1 \int_{\mathbb{R}^3} (\sqrt{6}u_{0n}u_{1n}^2\phi_2 + \sqrt{6}u_{2n}u_{1n}^2\phi_0 + 2\sqrt{6}u_{2n}u_{0n}u_{1n}\phi_1) dx
\end{aligned}$$

and

$$\begin{aligned}
C(\mathbf{u_n}, \boldsymbol{\phi}) &= \frac{\tau_2}{5} \int_{\mathbb{R}^3} \Big(2u_{2n}u_{-2n}^2\phi_2 + 2u_{-2n}u_{2n}^2\phi_{-2} + 2u_{1n}u_{-1n}^2\phi_1 + 2u_{-1n}u_{1n}^2\phi_{-1} + u_{0n}^3\phi_0) dx \\
&- \frac{\tau_2}{5} \int_{\mathbb{R}^3} \Big(2u_{-2n}u_{1n}u_{-1n}\phi_2 + 2u_{2n}u_{1n}u_{-1n}\phi_{-2} + 2u_{1n}u_{2n}u_{-2n}\phi_{-1} + 2u_{-1n}u_{2n}u_{-2n}\phi_1) dx \\
&+ \frac{\tau_2}{5} \int_{\mathbb{R}^3} \Big(u_{-2n}u_{0n}^2\phi_2 + u_{2n}u_{0n}^2\phi_{-2} + 2u_{2n}u_{-2n}u_{0n}\phi_0) dx \\
&- \frac{\tau_2}{5} \int_{\mathbb{R}^3} \Big(u_{-1n}u_{0n}^2\phi_1 + u_{1n}u_{0n}^2\phi_{-1} + 2u_{1n}u_{-1n}u_{0n}\phi_0) dx,
\end{aligned}$$

for any $\boldsymbol{\phi} = (\phi_2, \phi_1, \phi_0, \phi_{-1}, \phi_{-2}) \in \Lambda$. Since $\{\mathbf{u_n}\} \subset \mathcal{M}$ is bounded in Λ by Proposition 7.3, taking $\boldsymbol{\phi} = \mathbf{u_n}$ in (7.4), then it is easy to see that $\{\mu_n\}$, $\{\lambda_n\}$ are two bounded sequences in $\mathbb{R}$. Suppose that $\mu_n \to \hat{\mu}$, $\lambda_n \to \hat{\lambda}$ as $n \to \infty$. Choosing $\boldsymbol{\phi} = \mathbf{u_n} - \hat{\mathbf{u}}$ in (7.4), then we get

$$\begin{aligned}
&\int_{\mathbb{R}^3} \sum_{j=-2}^{2} \big(\nabla u_{jn} \cdot \nabla (u_{jn} - \hat{u}_j)\big) dx + \int_{\mathbb{R}^3} |x|^2 \Big(\sum_{j=-2}^{2} u_{jn}(u_{jn} - \hat{u}_j)\Big) dx \\
&+ A'(\mathbf{u_n}, \mathbf{u_n} - \hat{\mathbf{u}}) + B'(\mathbf{u_n}, \mathbf{u_n} - \hat{\mathbf{u}}) + C'(\mathbf{u_n}, \mathbf{u_n} - \hat{\mathbf{u}}) \qquad (7.5)
\end{aligned}$$

$$
\begin{aligned}
&= (\lambda_n + 2\mu_n)\int_{\mathbb{R}^3} u_{2n}(u_{2n}-\hat{u}_2)dx + (\lambda_n+\mu_n)\int_{\mathbb{R}^3} u_{1n}(u_{1n}-\hat{u}_1)dx + \lambda_n \int_{\mathbb{R}^3} u_{0n}(u_{0n}-\hat{u}_0)dx \\
&\quad + (\lambda_n - \mu_n)\int_{\mathbb{R}^3} u_{-1n}(u_{-1n}-\hat{u}_{-1})dx + (\lambda_n - 2\mu_n)\int_{\mathbb{R}^3} u_{-2n}(u_{-2n}-\hat{u}_{-2})dx + o_n(1),
\end{aligned}
$$

where A', B', C' are obtained by replacing $\boldsymbol{\phi}$ with $\mathbf{u_n} - \hat{\mathbf{u}}$ in $A(\mathbf{u_n}, \boldsymbol{\phi}),\ B(\mathbf{u_n}, \boldsymbol{\phi}),\ C(\mathbf{u_n}, \boldsymbol{\phi})$.

By (7.3), we get $\hat{\mathbf{u}}$ satisfies (1.4). Thus using $\mathbf{u_n} - \hat{\mathbf{u}}$ as a text function in (1.4), we then obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^3} \sum_{j=-2}^{2} \big(\nabla \hat{u}_j \cdot \nabla (u_{jn} - \hat{u}_j)\big)dx + \int_{\mathbb{R}^3} |x|^2 \big(\sum_{j=-2}^{2} \hat{u}_j (u_{jn} - \hat{u}_j)\big)dx \\
&\quad + A''(\hat{\mathbf{u}}, \mathbf{u_n} - \hat{\mathbf{u}}) + B''(\hat{\mathbf{u}}, \mathbf{u_n} - \hat{\mathbf{u}}) + C''(\hat{\mathbf{u}}, \mathbf{u_n} - \hat{\mathbf{u}}) \\
&= (\hat{\lambda} + 2\hat{\mu})\int_{\mathbb{R}^3} \hat{u}_2(u_{2n}-\hat{u}_2)dx + (\hat{\lambda}+\hat{\mu})\int_{\mathbb{R}^3} \hat{u}_1(u_{1n}-\hat{u}_1)dx + \hat{\lambda}\int_{\mathbb{R}^3} \hat{u}_0(u_{0n}-\hat{u}_0)dx \\
&\quad + (\hat{\lambda} - \hat{\mu})\int_{\mathbb{R}^3} \hat{u}_{-1}(u_{-1n}-\hat{u}_{-1})dx + (\hat{\lambda} - 2\hat{\mu})\int_{\mathbb{R}^3} \hat{u}_{-2}(u_{-2n}-\hat{u}_{-2})dx,
\end{aligned} \tag{7.6}
$$

where A'', B'', C'' are obtained by replacing $\mathbf{u_n}$ with $\hat{\mathbf{u}}$ in $A(\mathbf{u_n}, \mathbf{u_n} - \hat{\mathbf{u}}),\ B(\mathbf{u_n}, \mathbf{u_n} - \hat{\mathbf{u}}),\ C(\mathbf{u_n}, \mathbf{u_n} - \hat{\mathbf{u}})$.

Together with (7.3), (7.5), (7.6), we can see that

$$
\int_{\mathbb{R}^3} \big(\sum_{j=-2}^{2} |\nabla (u_{jn} - \hat{u}_j)|^2\big)dx + \int_{\mathbb{R}^3} |x|^2 \Big(\sum_{j=-2}^{2} |u_{1n} - \hat{u}_1|^2\Big)dx = o_n(1),
$$

which gives

$$
\int_{\mathbb{R}^3} \big(\sum_{j=-2}^{2} |\nabla u_{jn}|^2\big)dx \to \int_{\mathbb{R}^3} \big(\sum_{j=-2}^{2} |\nabla \hat{u}_j|^2\big)dx, \quad \text{as } n \to \infty
$$

and

$$
\int_{\mathbb{R}^3} |x|^2 \Big(\sum_{j=-2}^{2} u_{1n}^2\Big)dx \to \int_{\mathbb{R}^3} |x|^2 \Big(\sum_{j=-2}^{2} \hat{u}_1^2\Big)dx, \quad \text{as } n \to \infty.
$$

Therefore, we get the strong convergence of $\mathbf{u_n} \to \hat{\mathbf{u}}$ in Λ as $n \to \infty$ and we complete the proof. $\square$

By some similar arguments as the proof of Lemma 3.10 in [44], we have

Lemma 7.5. *Let* $\hat{m} = \inf_{\mathbf{u} \in \mathcal{P}} E(\mathbf{u})$*, then*

$$
P(\mathbf{u}) < 0 \Rightarrow P(\mathbf{u}) \le E(\mathbf{u}) - \hat{m}.
$$

Proof of Theorem 1.8. (i) By Lemma 7.3 and Lemma 7.4, $\hat{\mathbf{u}} \in \mathcal{M}$ is a mountain pass type solution to (1.4)-(1.5). Moreover, by (7.1),

$$
\sigma > \max\big\{E(\tilde{\mathbf{u}}), E\big(\xi_1 \star \tilde{\mathbf{u}}\big)\big\} > 0.
$$

Therefore, $\hat{\mathbf{u}} \in \mathcal{M}$ is indeed an excited state solution to (1.4)-(1.5).

(ii) Let $\hat{\mathbf{u}}$ be the solution to (1.4)-(1.5) obtained in Theorem 1.8 (i). For any $\xi > 0$, denote $\hat{\mathbf{u}}_\xi := \xi \star \hat{\mathbf{u}}$, then $P(\hat{\mathbf{u}}_\xi) < 0$ since $\hat{\mathbf{u}} \in \mathcal{P}$. Let $\mathbf{\Phi}^\xi = (\Phi_2^\xi, \Phi_1^\xi, \Phi_0^\xi, \Phi_{-1}^\xi, \Phi_{-2}^\xi)$ be the solution of system (1.3) with initial datum $\hat{\mathbf{u}}_\xi$ defined on the maximal interval $(T_{\min}, T_{\max})$. By the continuity of P, provided $|t|$ is sufficiently small we have $P(\mathbf{\Phi}^\xi(t)) < 0$. Therefore, by Lemma 7.5 and recalling that the energy is conserved along trajectories of system (1.3), we have

$$P(\mathbf{\Phi}^\xi(t)) \leq E(\mathbf{\Phi}^\xi(t)) - \hat{m} = E(\hat{\mathbf{u}}_\xi) - \hat{m} =: -\delta < 0.$$

For any such t, by continuity again we infer that $P(\mathbf{\Phi}^\xi(t)) < -\delta < 0$ for every $t \in (T_{\min}, T_{\max})$. To obtain a contradiction, we define

$$f_\xi(t) := \int_{\mathbb{R}^3} |x|^2 \Big(\sum_{j=-2}^{2} |\Phi_j^\xi(t,x)|^2 \Big) dx,$$

then

$$f_\xi'(t) = 2 \sum_{j=-2}^{2} \mathrm{Im} \int_{\mathbb{R}^3} |x|^2 \overline{\Phi_j^\xi}(t,x) i\partial_t \Phi_j^\xi(t,x) dx = 4 \sum_{j=-2}^{2} \mathrm{Im} \int_{\mathbb{R}^3} \overline{\Phi_j^\xi}(t,x) x \cdot \nabla \Phi_j^\xi(t,x) dx.$$

Indeed, we use the fact that

$$\sum_{j=-2}^{2} \mathrm{Im} \int_{\mathbb{R}^3} |x|^2 \overline{\Phi_j^\xi} (|x|^2 \Phi_j^\xi + \tau \rho^2 \Phi_j^\xi) dx = 0, \quad \sum_{j=-2, j\neq 0}^{2} 2j\tau_1 \mathrm{Im} \int_{\mathbb{R}^3} |x|^2 \overline{\Phi_j^\xi} F_z \Phi_j^\xi dx = 0,$$

$$\tau_1 \mathrm{Im} \int_{\mathbb{R}^3} |x|^2 (F_- \overline{\Phi_2^\xi} \Phi_1^\xi + F_+ \overline{\Phi_1^\xi} \Phi_2^\xi) dx = 0,$$

$$\frac{\sqrt{6}}{2} \tau_1 \mathrm{Im} \int_{\mathbb{R}^3} |x|^2 (F_- \overline{\Phi_1^\xi} \Phi_0^\xi + F_+ \overline{\Phi_0^\xi} \Phi_1^\xi) dx = 0,$$

$$\frac{\sqrt{6}}{2} \tau_1 \mathrm{Im} \int_{\mathbb{R}^3} |x|^2 (F_- \overline{\Phi_0^\xi} \Phi_{-1}^\xi + F_+ \overline{\Phi_{-1}^\xi} \Phi_0^\xi) dx = 0,$$

$$\tau_1 \mathrm{Im} \int_{\mathbb{R}^3} |x|^2 (F_- \overline{\Phi_{-1}^\xi} \Phi_{-2}^\xi + F_+ \overline{\Phi_{-2}^\xi} \Phi_{-1}^\xi) dx = 0$$

and

$$\frac{\tau_2}{\sqrt{5}} \mathrm{Im} \int_{\mathbb{R}^3} |x|^2 \theta (2\overline{\Phi_2^\xi \Phi_{-2}^\xi} - 2\overline{\Phi_{-1}^\xi \Phi_{-1}^\xi} + \overline{\Phi_0^\xi}^2) dx = 0.$$

Thus

$$f_\xi''(t) = 4 \sum_{j=-2}^{2} \mathrm{Im} \int_{\mathbb{R}^3} \Big(\partial_t \overline{\Phi_j^\xi}(t,x) x \cdot \nabla \Phi_j^\xi(t,x) + \overline{\Phi_j^\xi}(t,x) x \cdot \nabla \partial_t \Phi_j^\xi(t,x) \Big) dx.$$

Since

$$4 \sum_{j=-2}^{2} \mathrm{Im} \int_{\mathbb{R}^3} \overline{\Phi_j^\xi}(t,x) x \cdot \nabla \partial_t \Phi_j^\xi(t,x) dx = 4 \sum_{j=-2}^{2} \sum_{k=1}^{3} \mathrm{Im} \int_{\mathbb{R}^3} \overline{\Phi_j^\xi}(t,x) x_k \cdot \partial_k \partial_t \Phi_j^\xi(t,x) dx$$

$$= -4 \sum_{j=-2}^{2} \sum_{k=1}^{3} \mathrm{Im} \int_{\mathbb{R}^3} \partial_t \Phi_j^\xi(t,x) \partial_k (\overline{\Phi_j^\xi}(t,x) x_k) dx$$

$$= -4\Big(\sum_{j=-2}^{2} \mathrm{Im}\int_{\mathbb{R}^3} \partial_t\Phi_j^\xi(t,x)x\cdot\triangledown\overline{\Phi_j^\xi}(t,x)dx + 3\sum_{j=-2}^{2} \mathrm{Im}\int_{\mathbb{R}^3} \partial_t\Phi_j^\xi(t,x)\overline{\Phi_j^\xi}(t,x)dx\Big),$$

we have

$$\begin{aligned} f_\xi''(t) &= 4\sum_{j=-2}^{2} \mathrm{Im}\int_{\mathbb{R}^3} \Big(\partial_t\overline{\Phi_j^\xi}(t,x)x\cdot\triangledown\Phi_j^\xi(t,x) + \overline{\Phi_j^\xi}(t,x)x\cdot\triangledown\partial_t\Phi_j^\xi(t,x)\Big)dx \\ &= -4\Big(\sum_{j=-2}^{2} \mathrm{Im}\int_{\mathbb{R}^3} \partial_t\Phi_j^\xi(t,x)2x\cdot\triangledown\overline{\Phi_j^\xi}(t,x)dx + 3\sum_{j=-2}^{2} \mathrm{Im}\int_{\mathbb{R}^3} \partial_t\Phi_j^\xi(t,x)\overline{\Phi_j^\xi}(t,x)dx\Big) \\ &= 4\Big(\sum_{j=-2}^{2} \mathrm{Re}\int_{\mathbb{R}^3} i\partial_t\Phi_j^\xi(t,x)2x\cdot\triangledown\overline{\Phi_j^\xi}(t,x)dx + 3\sum_{j=-2}^{2} \mathrm{Re}\int_{\mathbb{R}^3} i\partial_t\Phi_j^\xi(t,x)\overline{\Phi_j^\xi}(t,x)dx\Big). \end{aligned}$$

Since $i\partial_t\Phi_j^\xi(t,x)$ satisfies (1.3) and $\mathrm{Re}(u\nabla\overline{u}) = \frac{1}{2}\nabla|u|^2$, through some lengthy and basic calculations, we have

$$\begin{aligned} f_\xi''(t) &= 8\int_{\mathbb{R}^3}\Big(\sum_{j=-2}^{2}|\nabla\Phi_j^\xi|^2 - |x|^2\sum_{j=-2}^{2}|\Phi_j^\xi|^2\Big)dx + 6\tau\int_{\mathbb{R}^3}\Big(\sum_{j=-2}^{2}|\Phi_j^\xi|^2\Big)^2dx \\ &+ 6\tau_1\int_{\mathbb{R}^3}\big(2(\Phi_{-2}^\xi\Phi_{-1}^\xi + \Phi_2^\tau\Phi_1^\xi) + \sqrt{6}(\Phi_{-1}^\xi\Phi_0^\xi + \Phi_1^\xi\Phi_0^\xi)\big)^2dx + 6\tau_1\int_{\mathbb{R}^3}\Big(\sum_{j=-2}^{2}(j|\Phi_j^\xi|^2)\Big)^2dx \\ &+ \frac{6\tau_2}{5}\int_{\mathbb{R}^3}\big(2\Phi_2^\xi\Phi_{-2}^\xi - 2\Phi_1^\xi\Phi_{-1}^\xi + (\Phi_0^\xi)^2\big)^2dx. \end{aligned}$$

Indeed,

$$\begin{aligned} &8\sum_{j=-2}^{2}\mathrm{Re}\int_{\mathbb{R}^3}(-\Delta\Phi_j^\xi(t,x))x\cdot\triangledown\overline{\Phi_j^\xi}(t,x)dx + 12\sum_{j=-2}^{2}\mathrm{Re}\int_{\mathbb{R}^3}(-\Delta\Phi_j^\xi(t,x))\overline{\Phi_j^\xi}(t,x)dx \\ &= 8\sum_{j=-2}^{2}\mathrm{Re}\int_{\mathbb{R}^3}(-div(\nabla\Phi_j^\xi(t,x))x\cdot\triangledown\overline{\Phi_j^\xi}(t,x)dx + 12\sum_{j=-2}^{2}\mathrm{Re}\int_{\mathbb{R}^3}(-div(\nabla\Phi_j^\xi(t,x))\overline{\Phi_j^\xi}(t,x)dx \\ &= 8\sum_{j=-2}^{2}\mathrm{Re}\int_{\mathbb{R}^3}\big(\nabla\Phi_j^\xi(t,x)\nabla(x\cdot\triangledown\overline{\Phi_j^\xi}(t,x)\big)dx + 12\sum_{j=-2}^{2}\mathrm{Re}\int_{\mathbb{R}^3}(\nabla\Phi_j^\xi(t,x)\nabla\overline{\Phi_j^\xi}(t,x)dx \\ &= 8\sum_{j=-2}^{2}\int_{\mathbb{R}^3}|\nabla\Phi_j^\xi(t,x)|^2dx \end{aligned}$$

and

$$\begin{aligned} &8\sum_{j=-2}^{2}\mathrm{Re}\int_{\mathbb{R}^3}|x|^2\Phi_j^\xi x\cdot\triangledown\overline{\Phi_j^\xi}(t,x)dx + 12\sum_{j=-2}^{2}\mathrm{Re}\int_{\mathbb{R}^3}|x|^2\Phi_j^\xi\overline{\Phi_j^\xi}(t,x)dx \\ &= -8\sum_{j=-2}^{2}\int_{\mathbb{R}^3}|\Phi_j^\xi|^2 div(|x|^2\frac{x}{2})dx + 12\sum_{j=-2}^{2}\int_{\mathbb{R}^3}|x|^2|\Phi_j^\xi|^2dx \end{aligned}$$

$$
\begin{aligned}
&= -20\sum_{j=-2}^{2}\int_{\mathbb{R}^3}|x|^2|\Phi_j^\xi|^2dx + 12\sum_{j=-2}^{2}\int_{\mathbb{R}^3}|x|^2|\Phi_j^\xi|^2dx \\
&= -8\sum_{j=-2}^{2}\int_{\mathbb{R}^3}|x|^2|\Phi_j^\xi|^2dx.
\end{aligned}
$$

The integrals of other terms can be calculated similarly, we omit the details here. Since

$$
\begin{aligned}
P(\mathbf{\Phi}^\xi(t)) = &\int_{\mathbb{R}^3}\Big(\sum_{j=-2}^{2}|\nabla\Phi_j^\xi|^2 - |x|^2\sum_{j=-2}^{2}|\Phi_j^\xi|^2\Big)dx + \frac{3\tau}{4}\int_{\mathbb{R}^3}\Big(\sum_{j=-2}^{2}|\Phi_j^\xi|^2\Big)^2dx \\
&+ \frac{3\tau_1}{4}\int_{\mathbb{R}^3}\big(2(\Phi_{-2}^\xi\Phi_{-1}^\xi + \Phi_2^\xi\Phi_1^\xi) + \sqrt{6}(\Phi_{-1}^\xi\Phi_0^\xi + \Phi_1^\xi\Phi_0^\xi)\big)^2dx + \frac{3\tau_1}{4}\int_{\mathbb{R}^3}\Big(\sum_{j=-2}^{2}(j|\Phi_j^\xi|^2)\Big)^2dx \\
&+ \frac{3\tau_2}{20}\int_{\mathbb{R}^3}\big(2\Phi_2^\xi\Phi_{-2}^\xi - 2\Phi_1^\xi\Phi_{-1}^\xi + (\Phi_0^\xi)^2\big)^2dx,
\end{aligned}
$$

we have $f_\xi''(t) = 8P(\mathbf{\Phi}^\xi(t)) < -8\delta < 0$, and as a consequence

$$
0 \le f_\xi(t) \le -\delta t^2 + O(t), \quad \text{for all } t \in (-T_{\min}, T_{\max}).
$$

Since the right hand side becomes negative for $|t|$ sufficiently large, it is necessary that both $T_{\min}$ and $T_{\max}$ are bounded. This proves that, for a sequence of initial data arbitrarily close to $\hat{\mathbf{u}}$, we have blow-up in finite time, which implies the instability. Therefore, we complete the proof. □

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