Uniqueness and symmetry breaking for ground states of spinor Bose-Einstein condensates

Tuxin Li [1] **Xiao Luo** [2] **Juncheng Wei** [3] **Maoding Zhen** [4]

1 Department of Mathematics, Chinese University of Hong Kong, Shatin, NT, Hong Kong
2 School of Mathematics, Hefei University of Technology, Hefei, 230009, P. R. China
3 Department of Mathematics, Chinese University of Hong Kong, Shatin, NT
4 School of Mathematics, Hefei University of Technology, Hefei, 230009, P. R. China

Abstract

A central goal in condensed matter and modern atomic physics is the exploration of quantum phases of matter. Spinor Bose-Einstein condensates are quantum fluids that simultaneously realize superfluidity and magnetism, both of which are associated with symmetry breaking. This was explored by L.E. Sadler et al. [Nature, 2006] in ^{87}Rb spinor condensates, rapidly quenched across a quantum phase transition to a ferromagnetic state. In this paper, we provide a mathematical justification for this phenomenon by completely classifying the ground state of ferromagnetic spin-F Bose-Einstein condensates in ring traps as well as analysing its asympototic behavior on the number of atoms and total magnetization. In particular, our classification results shows the validity of single-mode approximation (SMA) phenomenon firstly observed by C.K. Law, H. Pu and N.P. Bigelow in Phys. Rev. Lett. (1998).

Keywords: Spinor Bose-Einstein condensates, Ground State, Symmetry breaking, Classification, Gross-Pitaevskii system.

AMS Subject Classification: 35J50, 35J60, 35Q40.

1 Introduction

In the early experiments, magnetic traps were used and the spin degrees of the atoms were then frozen. In 1998, by using an optical dipole trap, a spinor BEC was first produced with spin-1 ^{23}Na gases [10], where the internal spin degrees of freedom were activated. In the optical trap, particles with different hyperfine states allow different angular momentum in space, resulting in a rich variety of spin texture. Therefore, degenerate quantum spinor gases maintain both magnetism and superfluidity, and are quite promising for many fields, such as topological quantum structure, fractional quantum Hall effect [1, 6]. For a spin-F Bose condensate, there are 2F+1 hyperfine states and the spinor condensate can be described by a 2F+1 component vector wave function. For the theory of spinor BEC, we refer to [?, 6, 7, 18, 31–33].

[1]Email addresses: tuoxin@math.ubc.ca (Li).
[2]Email addresses: luoxiao@hfut.edu.cn (Luo).
[3]Email addresses: wei@math.cuhk.edu.hk (Wei).
[4]Email addresses: maodingzhen@163.com (Zhen).

In the mean field theory, a physical state of spin-3 BEC is described by 7 components of complex order parameter

$$\mathbf{\Phi}(x,t) = \big(\Phi_3(x,t), \Phi_2(x,t), \Phi_1(x,t), \Phi_0(x,t), \Phi_{-1}(x,t), \Phi_{-2}(x,t), \Phi_{-3}(x,t)\big)(x \in \mathbb{R}^d, d = 1,2,3)$$

and the time evolution of the mean field dynamics is governed by [18, 21]

$$ih\partial_t \Phi_j(x,t) = \frac{\delta E}{\delta \Phi_j^*}, \tag{1.1}$$

Φ_j^* denotes the conjugate transpose of Φ_j. Here $E_{\tau_0,\tau_1,\tau_2,\tau_3}(\mathbf{\Phi})$ is defined by

$$E_{\tau_0,\tau_1,\tau_2,\tau_3}(\mathbf{\Phi}) = \int_{\mathbb{R}^d} \left(\frac{h^2}{2m}|\nabla \mathbf{\Phi}|^2 + V|\mathbf{\Phi}|^2 + \frac{\tau_0}{2}\rho^4 + \frac{\tau_1}{2}|\mathbf{F}|^2 + \frac{\tau_2}{2}|A_{00}|^2 + \frac{\tau_3}{2}\sum_{l=-2}^{2}|A_{2l}|^2 \right) dx, \tag{1.2}$$

with h is the Planck constant, m is the mass of atoms and τ_0, τ_1, τ_2, τ_3 characterize the spin-independent interaction, spin-exchange interaction, spin-singlet interaction and spin-quintet interaction respectively. $\mathbf{F} = (F_x, F_y, F_z)$ is the spin vector given by

$$F_x = \mathbf{\Phi}^* f_x \mathbf{\Phi}, \; F_y = \mathbf{\Phi}^* f_y \mathbf{\Phi}, \; F_z = \mathbf{\Phi}^* f_z \mathbf{\Phi},$$

f_x, f_y, f_z are the Pauli spinor matrices

$$f_x = \begin{pmatrix} 0 & \frac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 & 0 \\ \frac{\sqrt{3}}{2} & 0 & \frac{\sqrt{5}}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{\sqrt{5}}{2} & 0 & \sqrt{3} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{3} & 0 & \frac{\sqrt{5}}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{\sqrt{5}}{2} & 0 & \frac{\sqrt{3}}{2} \\ 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{2} & 0 \end{pmatrix}, f_y = i \begin{pmatrix} 0 & -\frac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 & 0 \\ \frac{\sqrt{3}}{2} & 0 & -\frac{\sqrt{5}}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{\sqrt{5}}{2} & 0 & -\sqrt{3} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 & -\sqrt{3} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{3} & 0 & -\frac{\sqrt{5}}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{\sqrt{5}}{2} & 0 & -\frac{\sqrt{3}}{2} \\ 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{2} & 0 \end{pmatrix},$$

and

$$f_z = diag(3, 2, 1, 0, -1, -2, -3).$$

Therefore, the $\mathbf{F} = (F_x, F_y, F_z)$ can be written explicitly as

$$\begin{aligned} F_x = {} & \frac{\sqrt{6}}{2}(\Phi_2^*\Phi_3 + \Phi_3^*\Phi_2 + \Phi_{-3}^*\Phi_{-2} + \Phi_{-2}^*\Phi_{-3}) + \frac{\sqrt{10}}{2}(\Phi_1^*\Phi_2 + \Phi_2^*\Phi_1 + \Phi_{-2}^*\Phi_{-1} + \Phi_{-1}^*\Phi_{-2}) \\ & + \sqrt{3}(\Phi_0^*\Phi_1 + \Phi_1^*\Phi_0 + \Phi_{-1}^*\Phi_0 + \Phi_0^*\Phi_{-1}), \end{aligned}$$

$$\begin{aligned} F_y = {} & \frac{\sqrt{6}}{2}(\Phi_2^*\Phi_3 - \Phi_3^*\Phi_2 + \Phi_{-3}^*\Phi_{-2} - \Phi_{-2}^*\Phi_{-3}) + \frac{\sqrt{10}}{2}(\Phi_1^*\Phi_2 - \Phi_2^*\Phi_1 + \Phi_{-2}^*\Phi_{-1} - \Phi_{-1}^*\Phi_{-2}) \\ & + \sqrt{3}(\Phi_0^*\Phi_1 - \Phi_1^*\Phi_0 + \Phi_{-1}^*\Phi_0 - \Phi_0^*\Phi_{-1}), \end{aligned}$$

and

$$F_z=\sum_{j=-3}^{3}\big(j|\Phi_j|^2\big).$$

Defining the matrices

$$\mathbf{A}=\frac{1}{\sqrt{7}}\begin{pmatrix}0&0&0&0&0&0&1\\0&0&0&0&0&-1&0\\0&0&0&0&1&0&0\\0&0&0&-1&0&0&0\\0&0&1&0&0&0&0\\0&-1&0&0&0&0&0\\1&0&0&0&0&0&0\end{pmatrix},\mathbf{A}_0=\frac{1}{\sqrt{7}}\begin{pmatrix}0&0&0&0&0&0&\frac{5}{2\sqrt{3}}\\0&0&0&0&0&0&0\\0&0&0&0&-\frac{\sqrt{3}}{2}&0&0\\0&0&0&\frac{2}{\sqrt{3}}&0&0&0\\0&0&-\frac{\sqrt{3}}{2}&0&0&0&0\\0&0&0&0&0&0&0\\\frac{5}{2\sqrt{3}}&0&0&0&0&0&0\end{pmatrix},$$

and $\mathbf{A}_l=(a_{l,jk})_{7\times 7}(l=\pm1,\pm2)$, where $a_{l,jk}$ is zero except for those $j+k=8-l$. For the simplicity of notation, we denote $\overrightarrow{a}_l=(a_{l,1(7-l)},a_{l,2(6-l)},\cdots,a_{l,(7-l)1})^T\in\mathbb{R}^{7-l}$ for $l=1,2\overrightarrow{a}_l=\big(a_{l,(1-l)7},a_{l,(2-l)6},\cdots,a_{l,7(1-l)}\big)^T\in\mathbb{R}^{7+l}$ for $l=-1,-2$ with

$$\overrightarrow{a}_{\pm1}=\frac{1}{\sqrt{7}}\left(\frac{5}{2\sqrt{3}},-\frac{\sqrt{5}}{2},\frac{1}{6},\frac{1}{6},-\frac{\sqrt{5}}{2},\frac{5}{2\sqrt{3}}\right)^T,$$

$$\overrightarrow{a}_{\pm2}=\frac{1}{\sqrt{7}}\left(\sqrt{\frac{5}{6}},-\sqrt{\frac{5}{3}},\sqrt{2},-\sqrt{\frac{5}{3}},-\frac{\sqrt{5}}{2},\sqrt{\frac{5}{6}}\right)^T.$$

Then $\mathbf{A}_{0,0}=\mathbf{\Phi}^T A\mathbf{\Phi}$ and $\mathbf{A}_{2l}=\mathbf{\Phi}^T\mathbf{A}_l\mathbf{\Phi}$ can be expressed as

$$\mathbf{A}_{0,0}=\frac{1}{\sqrt{7}}(2\Phi_3\Phi_{-3}-2\Phi_2\Phi_{-2}+2\Phi_1\Phi_{-1}-\Phi_0^2),$$

$$\mathbf{A}_{2,0}=\frac{1}{\sqrt{21}}(5\Phi_3\Phi_{-3}-3\Phi_1\Phi_{-1}+2\Phi_0^2),$$

$$\mathbf{A}_{2,\pm1}(\mathbf{\Phi})=\frac{1}{\sqrt{21}}(5\Phi_{\pm3}\Phi_{\mp2}-\sqrt{15}\Phi_{\pm2}\Phi_{\mp1}+\sqrt{2}\Phi_{\pm1}\Phi_0),$$

$$\mathbf{A}_{2,\pm2}(\mathbf{\Phi})=\frac{1}{\sqrt{21}}(\sqrt{10}\Phi_{\pm3}\Phi_{\mp1}-\sqrt{20}\Phi_{\pm2}\Phi_0+\sqrt{6}\Phi_{\pm1}^2).$$

Since $V(x)$ is a real valued function representing the trap potential and by scaling we may assume that $\frac{h^2}{2m}=1$. Associated with (1.2) are following two conserved quantities

$$\int_{\mathbb{R}^d}\Big(\sum_{j=-3}^{3}|\Phi_j|^2\Big)dx=N,\quad\int_{\mathbb{R}^d}\sum_{j=-3}^{3}\big(j|\Phi_j|^2\big)dx=M.$$

From (1.1)-(1.2), in the dimensionless form, the spin-3 BEC can be described by the following coupled Gross-Pitaevskii system,

$$\begin{cases} -\Delta u_{\pm 3}+V(x)u_{\pm 3}+(\lambda\pm 3\mu)u_{\pm 3}+\tau\rho^2 u_{\pm 3}+\tau_1\left(\frac{\sqrt{6}}{2}F_x u_{\pm 2}\pm 3F_z u_{\pm 3}\right) \\ \quad +\frac{\tau_2}{\sqrt{7}}\mathbf{A}_{0,0}(\mathbf{u})u_{\mp 3}+\frac{5\tau_3}{2\sqrt{21}}\mathbf{A}_{2,0}(\mathbf{u})u_{\mp 3}+\frac{5\tau_3}{2\sqrt{21}}\mathbf{A}_{2,\pm 1}(\mathbf{u})u_{\mp 2}+\frac{\sqrt{10}\tau_3}{2\sqrt{21}}\mathbf{A}_{2,\pm 2}(\mathbf{u})u_{\mp 1}=0, \\ -\Delta u_{\pm 2}+V(x)u_{\pm 2}+(\lambda\pm 2\mu)u_{\pm 2}+\tau\rho^2 u_{\pm 2}+\tau_1\left(\frac{\sqrt{10}}{2}F_x u_{\pm 1}+\frac{\sqrt{6}}{2}F_x u_{\pm 3}\pm 2F_z u_{\pm 2}\right) \\ \quad -\frac{\tau_2}{\sqrt{7}}\mathbf{A}_{0,0}(\mathbf{u})u_{\mp 2}-\frac{\sqrt{20}\tau_3}{2\sqrt{21}}\mathbf{A}_{2,\pm 2}(\mathbf{u})u_0=0, \\ -\Delta u_{\pm 1}+V(x)u_{\pm 1}+(\lambda\pm\mu)u_{\pm 1}+\tau\rho^2 u_{\pm 1}+\tau_1\left(\frac{\sqrt{6}}{2}F_x u_0+\frac{\sqrt{10}}{2}F_x u_{\pm 2}+\sqrt{3}F_x u_0\pm F_z u_{\pm 1}\right) \\ \quad +\frac{\tau_2}{\sqrt{7}}\mathbf{A}_{0,0}(\mathbf{u})u_{\mp 1}-\frac{3\tau_3}{2\sqrt{21}}\mathbf{A}_{2,0}(\mathbf{u})u_{\mp 1}-\frac{\sqrt{15}\tau_3}{2\sqrt{21}}\mathbf{A}_{2,\mp 1}(\mathbf{u})u_{\mp 2} \\ \quad +\frac{\sqrt{2}\tau_3}{\sqrt{21}}\mathbf{A}_{2,\pm 1}(\mathbf{u})u_0+\frac{\sqrt{6}\tau_3}{2\sqrt{21}}\mathbf{A}_{2,\pm 2}(\mathbf{u})u_{\pm 1}=0, \\ -\Delta u_0+V(x)u_0+\lambda u_0+\tau\rho^2 u_0+\sqrt{3}\tau_1\left(F_x u_{-1}+F_x u_1\right)-\frac{\tau_2}{\sqrt{7}}\mathbf{A}_{0,0}(\mathbf{u})u_0+\frac{2\tau_3}{\sqrt{21}}\mathbf{A}_{2,0}(\mathbf{u})u_0=0, \end{cases} \tag{1.3}$$

along with the constraints

$$\int_{\mathbb{R}^d}(\sum_{j=-3}^{3}u_j^2)dx=N,\quad \int_{\mathbb{R}^d}\sum_{j=-3}^{3}(ju_j^2)dx=M, \tag{1.4}$$

where μ,λ are real numbers,

$$F_x(\mathbf{u})=\sqrt{6}(u_2u_3+u_{-3}u_{-2})+\sqrt{10}(u_1u_2+u_{-1}u_{-2}+)+2\sqrt{3}(u_0u_1+u_{-1}u_0),$$

$$F_z(\mathbf{u})=\sum_{j=-3}^{3}(ju_j^2),\quad \rho^2(\mathbf{u})=\sum_{j=-3}^{3}u_j^2,\ \ \mathbf{A}_{0,0}(\mathbf{u})=\frac{1}{\sqrt{7}}(2u_3u_{-3}-2u_2u_{-2}+2u_1u_{-1}-u_0^2),$$

$$\mathbf{A}_{2,0}(\mathbf{u})=\frac{1}{\sqrt{21}}(5u_3u_{-3}-3u_1u_{-1}+2u_0^2),\quad \mathbf{A}_{2,\pm 1}(\mathbf{u})=\frac{1}{\sqrt{21}}(5u_{\pm 3}u_{\mp 2}-\sqrt{15}u_{\pm 2}u_{\mp 1}+\sqrt{2}u_{\pm 1}u_0),$$

$$\mathbf{A}_{2,\pm 2}(\mathbf{u})=\frac{1}{\sqrt{21}}(\sqrt{10}u_{\pm 3}u_{\mp 1}-\sqrt{20}u_{\pm 2}u_0+\sqrt{6}u_{\pm 1}^2).$$

Solutions to system (1.3)-(1.4) can be found as critical points of $E(\mathbf{u})$ constrained on $\mathcal{M}$,

$$E(\mathbf{u}):=\frac{1}{2}\int_{\mathbb{R}^d}(\sum_{j=-3}^{3}|\nabla u_j|^2+V(x)\rho^2)dx+\frac{1}{4}\int_{\mathbb{R}^d}(\tau_0\rho^4+\tau_1\mathbf{F}^2+\tau_2\mathbf{A}_{0,0}^2+\sum_{j=-2}^{2}\tau_3\mathbf{A}_{2,j}^2),$$

where $\mathbf{F}=(F_x,F_z)$ are real vector-valued functions,

$$\mathcal{M}:=\left\{\mathbf{u}\in\Lambda\Big|\int_{\mathbb{R}^d}(\sum_{j=-3}^{3}u_j^2)dx=N,\ \int_{\mathbb{R}^d}\sum_{j=-3}^{3}(ju_j^2)dx=M\right\}.$$

Here the working space

$$\Lambda := \left\{ (u_3, u_2, u_1, u_0, u_{-1}, u_{-2}, u_{-3}) \in (H^1(\mathbb{R}^d))^7 \big| \int_{\mathbb{R}^d} V(x) \big(\sum_{j=-3}^{3} u_j^2 \big) dx < +\infty \right\} \tag{1.5}$$

is a Hilbert space equipped with the norm

$$\|(u_3, u_2, u_1, u_0, u_{-1}, u_{-2}, u_{-3})\|_\Lambda := \left(\int_{\mathbb{R}^d} \big(\sum_{j=-3}^{3} |\nabla u_j|^2 \big) dx + \int_{\mathbb{R}^d} \big(1 + V(x)\big) \big(\sum_{j=-3}^{3} u_j^2 \big) dx \right)^{\frac{1}{2}}.$$

Before introducing the main results, we recall a definition (see also [4]):

Definition. *We say that* $(v_3, v_2, v_1, v_0, v_{-1}, v_{-2}, v_{-3})$ *is a ground state of* (1.3)-(1.4) *if*

$$E'|_{\mathcal{M}}(v_3, v_2, v_1, v_0, v_{-1}, v_{-2}, v_{-3}) = 0$$

and

$$\begin{aligned} &E(v_3, v_2, v_1, v_0, v_{-1}, v_{-2}, v_{-3}) \\ &= \inf \big\{ E(u_3, u_2, u_1, u_0, u_{-1}, u_{-2}, u_{-3}) \ s.t. \ E'|_{\mathcal{M}}(u_3, u_2, u_1, u_0, u_{-1}, u_{-2}, u_{-3}) = 0 \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{and } (u_3, u_2, u_1, u_0, u_{-1}, u_{-2}, u_{-3}) \in \mathcal{M} \big\}. \end{aligned}$$

We emphasize that this definition is meaningful even if the energy E is unbounded from below on $\mathcal{M}$. In addition, variational problems with the energy restricted on the manifold $\mathcal{M}$ is particularly appropriate for the study of the stability properties of the ground states.

Up to now, various spinor condensates including spin 1 or 2 ^{87}Rb condensate [1], spin-1 ^{23}Na condensate [10] and spin-2 ^{7}Li condensate [6] have been achieved in experiments. In this growing research direction, mathematical models and analysis as well as numerical simulation have been playing an important role in understanding the theoretical part of spinor BEC and predicting and guiding the experiments. However, there are few results regarding the mathematical theory of spinor Bose-Einstein condensates.

Recently, in [25, 27], we developed an exhaustive analysis on standing waves with prescribed mass of physical states for spin-1 Bose-Einstein condensate in $\mathbb{R}^3$ and we give a complete description on ground states of spin-1 Bose-Einstein condensates with Ioffe-Pritchard magnetic field in $\mathbb{R}^2$ and $\mathbb{R}^3$. In addition, in [26], we gave a complete classification of ground state solutions and show the validity of single-mode approximation (SMA) phenomenon in $\mathbb{R}^d$. We also presented the thresholds for the existence and nonexistence of ground state, and analyzed the asymptotic behavior of the ground state at the thresholds. For more results on ground states and excited states of spinor Bose-Einstein condensates, we refer the reader to [3, 8, 9, 19, 22, 23, 27, 34, 35] and the reference therein.

To our best knowledge, mathematical theories about uniqueness, symmetry breaking for ground states of spinor Bose-Einstein condensates and the SMA phenomenon in experimental observations [19] and numerical simulations [35] has never been rigorous mathematical justifications. As a continuation of our previous work [26, 27], the main purpose of this paper is to provide a mathematical justification for the symmetry breaking phenomenon explored by L.E. Sadler et al. [Nature, 2006] in ^{87} Rb spinor condensates and show the validity of SMA phenomenon in experimental observations [19].

Firstly, we give a complete classification results for ground states of (1.3)-(1.4) in $\mathbb{R}^d (d = 1, 2, 3)$. To state our main results, we consider the following minimization problem

$$\inf_{u \in \mathcal{N}} \left\{ \frac{1}{2} \int_{\mathbb{R}^d} (|\nabla u|^2 + V(x)u^2 + \left(\frac{\tau_0}{4} + \frac{9\tau_1}{4} + \frac{\tau_2}{28} \right) u^4) dx \right\}, \tag{1.6}$$

where

$$\mathcal{N} = \left\{ u \in H^1(\mathbb{R}^d) \Big| \int_{\mathbb{R}^d} u^2 dx = N \right\}. \tag{1.7}$$

Let W_{τ_0,τ_1,τ_2} be a positive solution of (1.6) and define

$$\begin{aligned} (H1): &\quad \tau_1 \le \frac{\tau_2}{63} \le 0,\ \tau_3 \ge 0, \\ (H2): &\quad \tau_1 \le 0,\ \tau_2 \ge 0,\ \tau_3 \ge 0. \end{aligned}$$

Our main results in this aspect are the following

Theorem 1.1. *Let $|M| \le 3N$ and one of $(H1), (H2)$ hold. Then the ground state of* (1.3)-(1.4) *must be the form*

$$\Bigg(u_3 = \pm \frac{(3N+M)^3}{216N^3} W_{\tau_0,\tau_1,0},\ u_{-3} = \pm \frac{(3N-M)^3}{216N^3} W_{\tau_0,\tau_1,0},\ u_0 = \pm\sqrt{20} \frac{(9N^2-M^2)^{\frac{3}{2}}}{216N^3} W_{\tau_0,\tau_1,0},$$
$$u_2 = \pm\sqrt{6} \frac{(3N+M)^{\frac{5}{2}}(3N-M)^{\frac{1}{2}}}{216N^3} W_{\tau_0,\tau_1,0},\ u_{-2} = \pm\sqrt{6} \frac{(3N+M)^{\frac{1}{2}}(3N-M)^{\frac{5}{2}}}{216N^3} W_{\tau_0,\tau_1,0},$$
$$u_1 = \pm\sqrt{15} \frac{(3N+M)^2(3N-M)}{216N^3} W_{\tau_0,\tau_1,0},\ u_{-1} = \pm\sqrt{15} \frac{(3N+M)(3N-M)^2}{216N^3} W_{\tau_0,\tau_1,0} \Bigg).$$

In particularly, the ground state must be positive if $\tau_1 < 0, \tau_2 = \tau_3 = 0$.

Remark 1.1. *To our best knowledge, this is the first systematically mathematical theories for ground states and dynamics of spin-3 BEC as well as the first theoretical result dealing with the classification of ground states for spin-3 BEC. These results not only show that spin-3 BEC has independent characteristics on the sign of spin-independent interaction, spin-exchange interaction, spin-singlet interaction and spin-quintet interaction, but also support the SMA phenomenon in experimental observations [19] and numerical simulations [35], that is, each component of the ground state is a multiple of one single density function. Rigorous mathematical justifications of these conclusions are exactly what is expected in ([3], Section 5).*

The proof of Theorem 1.1 is non-trivial and very skillful, which mainly relies on the technique of mass-redistribution for the ground state. Precisely, for any $\mathbf{u} \in \mathcal{M}$, we find a special mass-redistribution $\mathbf{v} = \mathbf{b}^*\rho$ (see (3.1) for the definition of $\mathbf{b}^*$) of $\mathbf{u}$, that remains in $\mathcal{M}$, which has a lower total energy.

Next, we consider attractive spin-exchange interaction case($\tau_1 > 0$) and obtain the following

Theorem 1.2. *Assume that $\tau_1 > 0, \tau_2 = \tau_3 = 0$ and $M = 0$, then the ground state of* (1.3)-(1.4) *must be the form*

$$\left(-s, -t, r, \sqrt{1 - 2s^2 - 2t^2 - 2r^2}, -r, -t, s \right) W_{\tau_0,0,0},$$

where $(s, t, r) \in \mathbb{R}^3$ and $s^2 + t^2 + r^2 \le \frac{1}{2}$.

Remark 1.2. *Theorem 1.1 and Theorem 1.2 show that spin-3 BEC has independent characteristics in both attractive spin-exchange interaction case and attractive spin-exchange interaction cases, but also support the so-called single-mode approximation (SMA) in experimental observations. The requirement $M = 0$ in Theorem 1.2 is necessary. The ground state is unique for attractive spin-exchange interaction case, while are not unique for repulsive spin-exchange interaction case.*

Next, we give some vanishing phenomenon for ground states of (1.3)-(1.4) in nonzero magnetization case.

Theorem 1.3. *Assume that $\tau_1 > 0$, $\tau_2 = \tau_3 = 0$ and $M \neq 0$. Let $(u_3, u_2, u_1, u_0, u_{-1}, u_{-2}, u_{-3})$ be a ground state of (1.3)-(1.4), then*
(i) $u_0 \equiv 0$;
(ii) $u_i \equiv 0$, if $M \neq iN(i = \pm1, \pm2, \pm3)$;
(iii) $u_{\pm1} = u_{\pm i} \equiv 0$, if $M \neq \pm N$ and $M \neq iN(i = \pm2, \pm3)$;
(iv) $u_{\pm2} = u_{\pm3} \equiv 0$, if $M \neq \pm2N$ and $M \neq \pm3N$.

Remark 1.3. *Theorem 1.2 and Theorem 1.3 indicates that if $M = iN(i = 3, 2, 1, -1, -2, -3)$, then u_i is minimizer of following minimization problem*

$$\inf_{u\in\mathcal{N}}\left\{\frac{1}{2}\int_{\mathbb{R}^d}(|\nabla u_i|^2 + V(x)u_i^2 + \left(\frac{\tau_0}{4} + \frac{i^2\tau_1}{4}\right)u_i^4)dx\right\},$$

where

$$\mathcal{N} = \left\{u_i \in H^1(\mathbb{R}^d) \middle| \int_{\mathbb{R}^d} u_i^2 dx = N\right\}.$$

Remark 1.4. *Theorem 1.2 shows that for the attractive spin-exchange interaction case $\tau_1 > 0$, if $M = 0$, any ground state of (1.3)-(1.4) is nontrivial. While when $M \neq 0$, Theorem 1.3 shows that ground states of (1.3)-(1.4) must be semi-trivial. These results not only show that spin-3 BEC has independent characteristics in both $M \neq 0$ and the degenerate case $M = 0$, but also justify the so-called vanishing phenomenon in experimental observations. In this sense, the influence of the total magnetization on system (1.3) is important.*

For repulsive spin-singlet interaction($\tau_2 < 0$), we have

Theorem 1.4. *Assume that $\tau_1 = \tau_3 = 0, \tau_2 < 0$ and $M = 0$, then the ground state of (1.3)-(1.4) must be the form*

$$\left(-s, -t, r, \sqrt{1 - 2s^2 - 2t^2 - 2r^2}, -r, -t, s\right)W_{\tau_0,0,\tau_2}.$$

Theorem 1.5. *Assume that $\tau_1 = \tau_3 = 0, \tau_2 < 0$ and $M \neq 0$. Let $(u_3, u_2, u_1, u_0, u_{-1}, u_{-2}, u_{-3})$ be a ground state of (1.3)-(1.4), then $u_0 \equiv u_{\pm1} \equiv u_{\pm2} \equiv 0$.*

Remark 1.5. *For repulsive spin-singlet interaction case($\tau_2 < 0$), Theorem 1.4 and Theorem 1.5 indicate that ground states of (1.3)-(1.4) either have SMA phenomenon or exhibit vanishing phenomenon, which depends on whether the total magnetization M is zero.*

As an application of above classification results, we give the uniqueness and symmetry breaking for ground states of spinor Bose-Einstein condensates.

Recall the following nonlinear equation in $\mathbb{R}^d (d = 1, 2, 3)$:

$$-\Delta u + u = u^3, \quad u \in H^1(\mathbb{R}^d), \tag{1.8}$$

from [20], there exists a unique positive solution $Q(x)$ for (1.8). By the related Pohozaev identity, we get

$$a^* := \int_{\mathbb{R}^d} |Q|^2 dx = \frac{4-d}{d} \int_{\mathbb{R}^d} |\nabla Q|^2 dx = \frac{4-d}{4} \int_{\mathbb{R}^d} |Q|^4 dx. \tag{1.9}$$

Moreover, we obtain from [17] that $Q(x)$ satisfies

$$Q(x),\ |\nabla Q(x)| = O\big(|x|^{-\frac{d-1}{2}} e^{-|x|}\big), \quad \text{as } |x| \to \infty.$$

In the following, we always denote

$$N^* = -\frac{a^*}{\tau_0 + 9\tau_1}, \tag{1.10}$$

which is the critical number of atoms. It is natural to ask what would happen if $V(x)$ has infinitely many minima. Hence, we are interested in studying the GP functional with a trapping potential $V(x)$ with infinitely many minima and analyzing the detailed behavior of its minimizers as $N \nearrow N^*$. For this purpose, we focus on the following ring-shaped trapping potential

$$V(x) = (|x| - A)^2, \text{ where } A > 0, x \in \mathbb{R}^2. \tag{1.11}$$

and define the following constraint variational problem

$$m(N) := \inf_{\mathbf{u} \in \mathcal{M}} E(\mathbf{u}). \tag{1.12}$$

Our main results on uniqueness and symmetry breaking for ground states of (1.12) are the following.

Theorem 1.6. *Let $V(x)$ be given by (1.11), $\tau_2 > 0$ and $\tau_3 > 0$. Then there exist $N_* > 0$ and $N_{**} > 0$ satisfying $N_{**} \le N_* < N^*$ such that*

(i) $m(N)$ has a unique non-negative minimizer which is radially symmetric about the origin if $N \in (0, N_{**})$.

(ii) $m(N)$ has a unique non-negative minimizer up to rotation around the origin, which are not radially symmetric if $N \in [N_*, N^*)$.

Remark 1.6. *Noting that the trapping potential $V(x)$ of (1.11) is radially symmetric, it then follows from Theorem 1.6 that $m(N)$ has a unique non-negative minimizer which is also radially symmetric for small $N > 0$. On the other hand, Theorem 1.2 in [14] shows that any non-negative minimizer of (1.6) concentrates at a point on the ring $\{x \in \mathbb{R}^2 : |x| = A\}$ as $N \nearrow N^*$ and thus it cannot be radially symmetric. Based on the complete classification of ground states, we know that any non-negative minimizer of $m(N)$ cannot be radially symmetric as $N \nearrow N^*$. This implies that, as the strength of the interaction N increases from 0 to N^*, symmetry breaking occurs in the minimizers of $m(N)$.*

Remark 1.7. *The authors in [30] explored spontaneous symmetry breaking in ^{87}Rb spinor condensates, rapidly quenched across a quantum phase transition to a ferromagnetic state. They observed the formation of spin textures, ferromagnetic domains and domain walls, and demonstrate phase-sensitive in situ detection of spin vortices. The latter were topological defects resulting from the symmetry breaking, containing non-zero spin current but no net mass current. We show that rigorous mathematical justifications of these symmetry breaking conclusions are exactly what is expected in [30] and show the validity of SMA phenomenon in experimental observations [19].*

Remark 1.8. *The uniqueness result in Theorem 1.6 also holds for the one-dimensional (1D) case and three-dimensional (3D) case. Indeed, for spin-F (F=1,2,3) Bose-Einstein condensate, we show that each component of the ground state is a multiple of one single density function, which is independent of the dimension of whole space. This indicates that the ground states of spin-1 BEC obtained in Theorem 1 [8], Theorem 1.1 [22] and Theorem 1 [27] are unique.*

To prove the uniqueness of ground states for BEC, the authors in [15,16] studied carefully the limit structure of a suitable difference function, for which they needed to make full use of the non-degeneracy results for a corresponding limit system. In order to employed the non-degeneracy assumption to derive Pohozaev identities, some delicate estimates and new ideas are also needed to handle with the crossing terms in BEC systems. Although the authors in [15,16] developed an approach to establish the uniqueness of ground states for BEC, it does not work for our spin-F(F=1,2,3) BEC. In fact, it is not clear whether the solution for spin-F(F=1,2,3) BEC is non-degenerate. So, the method of Pohozaev identities does not work for our case. We provide a new way to establish the uniqueness of ground states based on the classification of ground states for spin-F(F=1,2,3) BEC, which mainly rely on a principle, that the mass-redistribution for n-tuple of real-valued functions will decrease the kinetic energy. Precisely, for any $\mathbf{u} \in \mathcal{M}$, we find a special mass-redistribution $\mathbf{v} = \mathbf{b}^*\rho$ (see (3.1) for the definition of $\mathbf{b}^*$) of $\mathbf{u}$, that remains in $\mathcal{M}$, which has a lower total energy.

Finally, under more general conditions on $\tau_0, \tau_1, \tau_2, \tau_3$ and $V(x)$ with

$$\lim_{|x|\to+\infty} V(x) = +\infty,\ V(x) > 0 \text{ and } V(\tau x) = \tau^p V(x), p > 0, \tag{1.13}$$

$$N^{**} = -\frac{7a^*}{7\tau_0 + 63\tau_1 + 7(\tau_3)^- + (\tau_2)^-}, \tag{1.14}$$

we consider the existence and concentration of ground states.

Theorem 1.7. *Let $\tau_1 < 0$.*
(i) $m(N)$ has at least one minimizer for $0 < N < N^{**}$, while $m(N)$ has no minimizer for $N > N^*$;
(ii) If $\tau_2 > 0$ and $\tau_3 > 0$, then $m(N)$ has at least one minimizer for $0 < N < N^*$, while $m(N)$ has no minimizer for $N \geq N^*$;
(iii) For any minimizer $\mathbf{u} = (u_3, u_2, u_1, u_0, u_{-1}, u_{-2}, u_{-3}) \in \mathcal{M}(N)$ of $m(N)$, there holds

$$\left\|\mathbf{u} - (l_{30}\Psi_0, l_{20}\Psi_0, l_{10}\Psi_0, l_{00}\Psi_0, l_{-10}\Psi_0, l_{-20}\Psi_0, l_{-30}\Psi_0)\right\|_\Lambda^2 = O(N), \quad \text{as } N \to 0^+, \tag{1.15}$$

where Ψ_0 is the unique normalized positive eigenvector of $-\Delta + V(x)$ and

$$l_{i0} = \int_{\mathbb{R}^3} u_i \Psi_0 dx,\ \text{for } i = 3, 2, 1, 0, -1, -2, -3;$$

(iv) Let $N_n \nearrow N^*$ as $n \to \infty$ and

$$\mathbf{u_n} = (u_{3n}, u_{2n}, u_{1n}, u_{0n}, u_{-1n}, u_{-2n}, u_{-3n}) \in \mathcal{M}(N_n)$$

be a minimizer of $m(N_n)$. We have

$$m(N_n) = \frac{p+2}{2}\left(\frac{1}{2}\right)^{\frac{2}{p+2}}\left(\frac{1}{p}\right)^{\frac{p}{p+2}}\left(\frac{N^*\int_{\mathbb{R}^2} V(x)Q^2(x)dx}{a^*}\right)^{\frac{2}{p+2}}(N^* - N_n)^{\frac{2}{p+2}},\ \text{as } n \to \infty. \tag{1.16}$$

In addition, $\mathbf{u_n}$ *satisfies*

$$\begin{cases} \lim_{n\to\infty} \varepsilon_n u_{\pm 3n}(\varepsilon_n x + \tilde{z}_{\pm 3n}) = \sqrt{\frac{N^*}{a^*}} \frac{(3N^* \pm M)^3}{216(N^*)^3} Q(x), \\ \lim_{n\to\infty} \varepsilon_n u_{\pm 2n}(\varepsilon_n x + \tilde{z}_{\pm 2n}) = \sqrt{\frac{N^*}{a^*}} \sqrt{6} \frac{(3N^* + M)^{\frac{3}{2}\pm 1}(3N^* - M)^{\frac{3}{2}\mp 1}}{216(N^*)^3} Q(x), \\ \lim_{n\to\infty} \varepsilon_n u_{\pm 1n}(\varepsilon_n x + \tilde{z}_{\pm 1n}) = \sqrt{\frac{N^*}{a^*}} \sqrt{15} \frac{(3N^* + M)^{\frac{3}{2}\pm \frac{1}{2}}(3N^* - M)^{\frac{3}{2}\mp \frac{1}{2}}}{216 N^3} Q(x), \\ \lim_{n\to\infty} \varepsilon_n u_{0n}(\varepsilon_n x + \tilde{z}_{0n}) = \sqrt{\frac{N^*}{a^*}} \sqrt{20} \frac{(9(N^*)^2 - M^2)^{\frac{3}{2}}}{216(N^*)^3} Q(x), \end{cases} \tag{1.17}$$

where $\tilde{z}_{in}$ $(i = 3, 2, 1, 0, -1, -2, -3)$ *is the unique maximum point of* u_{in} *with*

$$\lim_{n\to\infty} |\frac{\tilde{z}_{in} - \tilde{z}_{jn}}{\varepsilon_n}| = 0 \ (i, j = 3, 2, 1, 0, -1, -2, -3, i \neq j), \quad \lim_{n\to\infty} |\tilde{z}_{in}| = 0$$

and

$$\varepsilon_n = \frac{p+2}{2} \left(\frac{1}{2}\right)^{\frac{2}{p+2}} \left(\frac{1}{p}\right)^{\frac{p}{p+2}} \Big(\frac{N^* \int_{\mathbb{R}^2} V(x) Q^2(x) dx}{a^*}\Big)^{\frac{2}{p+2}} (N^* - N_n)^{\frac{2}{p+2}}. \tag{1.18}$$

Remark 1.9. *Theorem 1.7 shows that for attractive spin-singlet interaction and attractive spin-quintet interaction case, any minimizer* $\mathbf{u}$ *of* $m(N)$ *in the case of* $N \nearrow N^*$ *is nontrivial. It gives a complete classification of the existence and nonexistence of global minimizers and present the thresholds for the existence and nonexistence of ground state. In addition, asymptotic behavior of the ground state at the thresholds and atoms collapse behavior of the ground states are also analyzed.*

Remark 1.10. *For the general potential* $V(x)$*, in order to get a consistent upper and lower bound estimate of the energy more directly, we assume* $V(x)$ *is homogeneous of degree* p*. Indeed, for the ring-shaped trapping potential* (1.11)*, we also can obtain the detailed behavior of the minimizers for problem* (1.12) *as* $N \nearrow N^*$*. A more delicate estimate on the GP functional is required. As far as we know, it is usually not easy to derive directly the optimal energy estimates for the GP functional under general trapping potentials. Although the authors in [14] developed an approach to establish this kind of energy estimates for single equation with the trapping potentials, it does not work well for our problem. In fact, the spin-3 BEC with trapping potentials is more complicated and difficult. To get a uniformly energy estimate, by following the method in [14] we first get the following type of estimates*

$$C_1 (N^* - N)^{\frac{2}{3}} \leq m(N) \leq C_2 (N^* - N)^{\frac{1}{2}} \text{ as } N \nearrow N^*. \tag{1.19}$$

Then we provide some new ways to estimate precisely the GP energy under the potential (1.11)*, which may be used effectively to handle some general type potentials. Based on the estimates, we may improve the power* $\frac{2}{3}$ *at the left of* (1.19) *to be the same as that at the right, namely* $\frac{1}{2}$*.*

Notations. In the paper, we use the following notations. $L^p = L^p(\mathbb{R}^d)$ with norm $\|\cdot\|_{L^p(\mathbb{R}^d)} = \|\cdot\|_{L^p}$, $H^1(\mathbb{R}^d)$ is the usual Sobolev space and $H^1(\mathbb{R}^d, \mathbb{R}^7) = (H^1(\mathbb{R}^d))^7$ and $L^p(\mathbb{R}^d, \mathbb{R}^7) = (L^p(\mathbb{R}^d))^7$ are the vector-valued functions spaces.

The paper is organized as follows. In Section 2, we introduce some preliminary results. In Section 3, we prove Theorem 1.1. In section 4, we prove Theorem 1.2 and Theorem 1.3. In section 4, we Prove Theorem 1.4 and Theorem 1.5. In section 6, we prove Theorem 1.6. Finally, Theorem 1.7 will be proved in section 7.

2 Preliminaries

In this section, we give some preliminaries which are useful for the rest of the paper. First, we give a compact embedding result.

Lemma 2.1. *(Pankov [29]) The embedding $\Lambda \hookrightarrow L^p(\mathbb{R}^d, \mathbb{R}^7)$ is compact for any $p \in [2, \frac{2d}{d-2})$, where Λ is defined in* (1.5).

For any $u \in H^1(\mathbb{R}^d)(d = 2, 3)$, by Lemma 2.4 in [5], u satisfies the classical Gagliardo-Nirenberg inequality

$$\int_{\mathbb{R}^2} u^4 dx \leq \frac{2}{a^*} \int_{\mathbb{R}^2} |\nabla u|^2 dx \cdot \int_{\mathbb{R}^2} u^2 dx \tag{2.1}$$

and

$$\int_{\mathbb{R}^3} u^4 dx \leq \frac{4\sqrt{3}}{9a^*} \Big(\int_{\mathbb{R}^3} |\nabla u|^2 dx \Big)^{\frac{3}{2}} \cdot \Big(\int_{\mathbb{R}^3} u^2 dx \Big)^{\frac{1}{2}}, \tag{2.2}$$

where a^* is defined in (1.9).

For any $(u_3, u_2, u_1, u_0, u_{-1}, u_{-2}, u_{-3}) \in H^1(\mathbb{R}^d, \mathbb{R}^7)$, there also holds the similar inequality.

Lemma 2.2. *For $\mathbf{u} \in H^1(\mathbb{R}^d, \mathbb{R}^7)$, there holds*

$$\int_{\mathbb{R}^2} (\sum_{j=-3}^{3} u_j^2)^2 dx \leq \frac{2}{a^*} \int_{\mathbb{R}^2} (\sum_{j=-3}^{3} |\nabla u_j|^2) dx \cdot \int_{\mathbb{R}^2} (\sum_{j=-3}^{3} u_j^2) dx \tag{2.3}$$

and

$$\int_{\mathbb{R}^3} (\sum_{j=-3}^{3} u_j^2)^2 dx \leq C_* \Big(\int_{\mathbb{R}^3} (\sum_{j=-3}^{3} |\nabla u_j|^2) dx \Big)^{\frac{3}{2}} \cdot \Big(\int_{\mathbb{R}^3} (\sum_{j=-3}^{3} u_j^2) dx \Big)^{\frac{1}{2}}, \tag{2.4}$$

where $C_* = \frac{4\sqrt{3}}{9a^*}$. Moreover, up to translations and suitable scalings, the equality (2.3) *holds only at*

$$\begin{cases} u_3(x) = Q(x) \cos\varphi_1, \\ u_2(x) = Q(x) \sin\varphi_1 \cos\varphi_2, \\ u_1(x) = Q(x) \sin\varphi_1 \sin\varphi_2 \cos\varphi_3, \\ u_0(x) = Q(x) \sin\varphi_1 \sin\varphi_2 \sin\varphi_3 \cos\varphi_4, \\ u_{-1}(x) = Q(x) \sin\varphi_1 \sin\varphi_2 \sin\varphi_3 \cos\varphi_4 \cos\varphi_5, \\ u_{-2}(x) = Q(x) \sin\varphi_1 \sin\varphi_2 \sin\varphi_3 \cos\varphi_4 \cos\varphi_5 \cos\varphi_6, \\ u_{-3}(x) = Q(x) \sin\varphi_1 \sin\varphi_2 \sin\varphi_3 \cos\varphi_4 \cos\varphi_5 \sin\varphi_6, \end{cases} \tag{2.5}$$

for $\varphi_j \in [0, \frac{\pi}{2})$ $(j = 6, 5, 4, 3, 2, 1)$ and $Q(x)$ is the unique positive solution to (1.8).

Proof. We only prove the 2D case, the proof of (2.4) are similar, we omit the details here. Consider the minimization problem:

$$k := \inf_{(0,0,0,0,0,0,0) \neq \mathbf{u} \in H^1(\mathbb{R}^2, \mathbb{R}^7)} K(\mathbf{u}), \tag{2.6}$$

where

$$K(\mathbf{u})=\frac{\int_{\mathbb{R}^2}\big(\sum\limits_{j=-3}^{3}|\nabla u_j|^2\big)dx\cdot\int_{\mathbb{R}^2}\big(\sum\limits_{j=-3}^{3}u_j^2\big)dx}{\int_{\mathbb{R}^2}\big(\sum\limits_{j=-3}^{3}u_j^2\big)^2dx}.$$

To obtain (2.4), it is sufficient to show $k=\frac{a^*}{2}$. Let $Q(x)$ be the unique positive solution to (1.8) and set

$$(u_3,u_2,u_1,u_0,u_{-1},u_{-2},u_{-3})=\Big(\frac{Q}{\sqrt{7}},\frac{Q}{\sqrt{7}},\frac{Q}{\sqrt{7}},\frac{Q}{\sqrt{7}},\frac{Q}{\sqrt{7}},\frac{Q}{\sqrt{7}},\frac{Q}{\sqrt{7}}\Big),$$

then by (1.9),

$$K(\mathbf{u})=\frac{\int_{\mathbb{R}^2}|\nabla Q|^2dx\cdot\int_{\mathbb{R}^2}Q^2dx}{\int_{\mathbb{R}^2}Q^4dx}=\frac{a^*}{2}.$$

By direct calculation, for arbitrary $(u_3,u_2,u_1,u_0,u_{-1},u_{-2},u_{-3})\in H^1(\mathbb{R}^2,\mathbb{R}^7)$, there holds

$$\Big|\nabla\sqrt{\sum_{j=-3}^{3}u_j^2}\Big|^2\le\sum_{j=-3}^{3}|\nabla u_j|^2,$$

therefore, by (2.2),

$$\begin{aligned}K(\mathbf{u})&\ge\frac{\int_{\mathbb{R}^2}(|\nabla\sqrt{\sum\limits_{j=-3}^{3}u_j^2}|^2)dx\cdot\int_{\mathbb{R}^2}\big(\sum\limits_{j=-3}^{3}u_j^2\big)dx}{\int_{\mathbb{R}^2}\big(\sum\limits_{j=-3}^{3}u_j^2\big)^2dx}\\&=\frac{\int_{\mathbb{R}^2}\big(|\nabla\sqrt{\sum\limits_{j=-3}^{3}u_j^2}|^2\big)dx\cdot\int_{\mathbb{R}^2}\big(\sqrt{\sum\limits_{j=-3}^{3}u_j^2}\big)^2dx}{\int_{\mathbb{R}^2}\big(\sqrt{\sum\limits_{j=-3}^{3}u_j^2}\big)^4dx}\ge\frac{a^*}{2}.\end{aligned}$$

Thus, $k=\frac{a^*}{2}$. Similar to [13], we conclude that to find the minimizer of (2.6) is equivalent to the ground state of the following system:

$$-\Delta u_i+u_i=u_i^3+\big(\sum_{j=-3}^{3}u_j^2\big)u_i,\, i=3,2,1,0,-1,-2,-3. \tag{2.7}$$

Moreover, we have

$$\int_{\mathbb{R}^2}\big(\sum_{j=-3}^{3}|\nabla u_j|^2\big)dx=\int_{\mathbb{R}^2}\big(\sum_{j=-3}^{3}u_j^2\big)dx=\frac{1}{2}\int_{\mathbb{R}^2}\big(\sum_{j=-3}^{3}u_j^2\big)^2dx.$$

Then similar to the arguments in [12] for three components system, the ground state of (2.7) is of the form (2.5), hence equality (2.3) holds only for the ground state of (2.7). □

Finally, we give the pure point spectrum and the associated eigenvectors for harmonic oscillator $-\Delta + V(x)$, which is useful for us to study the qualitative properties of solutions for $m(N)$.

Denote $L_V := -\Delta + V(x) : \hat{\Sigma}_V \to L^2(\mathbb{R}^d, \mathbb{R})$. We have the following spectral analysis for L_V:

Lemma 2.3. *Let V satisfy* (1.13)*. Then there hold:*

(i) Each eigenvalue of L_V is real.

(ii) If we repeat each eigenvalue $\lambda_{k,V}(k = 0, 1, 2, \cdots)$ according to its (finite) multiplicity, we have

$$0 < \lambda_{0,V} \leq \lambda_{1,V} \leq \lambda_{2,V} \leq \cdots$$

and

$$\lambda_{k,V} \to \infty \quad \text{as } k \to \infty.$$

(iii) Furthermore, there exists an orthonormal basis $\{\psi_{k,V}\}_{k=0}^{\infty}$ of $L^2(\mathbb{R}^d, \mathbb{R})$, where $\psi_{k,V} \in \Sigma_V$ is an eigenfunction corresponding to $\lambda_{k,V}$:

$$L_V \psi_{k,V} = \psi_{k,V} \lambda_{k,V},$$

for $k = 0, 1, 2, \ldots$.

(iv) Finally, the first eigenvalue $\lambda_{0,V}$ is simple.

We omit the proof since it is a standard argument like Section 6.5 in the classical book [11].

3 Proof of Theorem 1.1

The proofs of Theorem 1.1 mainly rely on a principle, that the mass-redistribution for n-tuple of real-valued functions will decrease the kinetic energy. We now introduce the definition and properties for the mass-redistribution.

Definition 3.1. *[24] Let $\mathbf{f} = (f_1, f_2, \cdots, f_n) \in H^1(\mathbb{R}^d, \mathbb{R}^n)$ be an n-tuple of real-valued functions and $\mathbf{g} = (g_1, g_2, \cdots, g_m)$ be an m-tuple of nonnegative functions. We say $\mathbf{g}$ is a mass-redistribution of $\mathbf{f}$, if $g_l^2 = \sum_{k=1}^{n} b_{lk} f_k^2$ for each $l = 1, 2, ...m$, where $b_{lk} \geq 0$ are constants and $\sum_{l=1}^{m} b_{lk} = 1$ for each $k = 1, 2, ..., n$.*

Proposition 3.1. *[24] For any mass-redistribution $\mathbf{g}$ of $\mathbf{f}$, we have*

(i) $|\mathbf{g}| = |\mathbf{f}|$;

(ii) $|\nabla \mathbf{g}| \leq |\nabla \mathbf{f}|$. Moreover, $|\nabla \mathbf{g}| = |\nabla \mathbf{f}|$ if and only if $f_j \nabla f_k = f_k \nabla f_j$ for each $k \neq j$ with $b_{lj} b_{lk} \neq 0$ for at least one l.

Suppose $b_j \geq 0 (j = 3, 2, 1, 0, -1, -2, -3)$ and $\mathbf{b} = (b_3, b_2, b_1, b_0, b_{-1}, b_{-2}, b_{-3})$, we consider the maximization problem

$$\max_{\mathbf{b} \in \mathrm{B}} Q(\mathbf{b})$$

where

$$Q(\mathbf{b}) = \left[\sqrt{6}(b_2 b_3 + b_{-2} b_{-3}) + \sqrt{10}(b_1 b_2 + b_{-1} b_{-2}) + 2\sqrt{3}(b_0 b_1 + b_0 b_{-1})\right]^2 + \left[\sum_{j=-3}^{3} (j b_j^2)\right]^2,$$

and

$$\mathrm{B} = \left\{\mathbf{b} \in \mathbb{R}^7 | b_j \geq 0 \text{ and } \mathbf{b} \text{ satisfies } \sum_{j=-3}^{3} b_j^2 = 1,\ \sum_{j=-3}^{3} (j b_j^2) = \frac{M}{N}\right\}.$$

Lemma 3.1. *Assume* $|M| \le 3N$, *then there exists a* $\mathbf{b}^* = (b_3^*, b_2^*, b_1^*, b_0^*, b_{-1}^*, b_{-2}^*, b_{-3}^*) \in \mathrm{B}$, *such that*

$$\max_{\mathbf{b}\in\mathrm{B}} Q(\mathbf{b}) = Q(\mathbf{b}^*) = 9,$$

where

$$\begin{aligned} b_3^* &= \frac{(3N+M)^3}{216N^3},\ b_{-3}^* = \frac{(3N-M)^3}{216N^3},\ b_0^* = \sqrt{20}\frac{(9N^2-M^2)^{\frac{3}{2}}}{216N^3}, \\ b_2^* &= \sqrt{6}\frac{(3N+M)^{\frac{5}{2}}(3N-M)^{\frac{1}{2}}}{216N^3}, \quad b_{-2}^* = \sqrt{6}\frac{(3N+M)^{\frac{1}{2}}(3N-M)^{\frac{5}{2}}}{216N^3}, \\ b_1^* &= \sqrt{15}\frac{(3N+M)^2(3N-M)}{216N^3}, \quad b_{-1}^* = \sqrt{15}\frac{(3N+M)(3N-M)^2}{216N^3}. \end{aligned} \tag{3.1}$$

Proof. By direct calculations, for any $\mathbf{b} \in \mathrm{B}$, we have

$$\begin{aligned} 9(\sum_{j=-3}^{3}(b_j^2))^2 - Q(\mathbf{b}) &= (3b_0^2 - 4b_1b_{-1})^2 + (2b_1b_{-1} - 5b_2b_{-2})^2 + (b_2b_{-2} - 6b_3b_{-3})^2 \\ &+ (\sqrt{5}b_2^2 - 2\sqrt{3}b_1b_3)^2 + (\sqrt{5}b_{-2}^2 - 2\sqrt{3}b_{-1}b_{-3})^2 \\ &+ (\sqrt{8}b_1^2 - \sqrt{15}b_0b_2)^2 + (\sqrt{8}b_{-1}^2 - \sqrt{15}b_0b_{-2})^2 \\ &+ 3(b_0b_2 - 2\sqrt{2}b_3b_{-1})^2 + 3(b_0b_{-2} - 2\sqrt{2}b_{-3}b_1)^2 \\ &+ (2b_1b_2 - 3\sqrt{2}b_0b_3)^2 + (2b_{-1}b_{-2} - 3\sqrt{2}b_0b_{-3})^2 \\ &+ (\sqrt{6}b_0b_1 - 2\sqrt{5}b_{-1}b_2)^2 + (\sqrt{6}b_0b_{-1} - 2\sqrt{5}b_1b_{-2})^2 \\ &+ 2(b_1b_{-2} - \sqrt{15}b_2b_{-3})^2 + 2(b_{-1}b_2 - \sqrt{15}b_{-2}b_3)^2 \ge 0. \end{aligned} \tag{3.2}$$

So, when $(b_3, b_2, b_1, b_0, b_{-1}, b_{-2}, b_{-3})$ satisfies following algebra system

$$\left\{ \begin{aligned} 3b_0^2 - 4b_1b_{-1} &= 0, \\ 2b_1b_{-1} - 5b_2b_{-2} &= 0, \\ b_2b_{-2} - 6b_3b_{-3} &= 0, \\ \sqrt{5}b_2^2 - 2\sqrt{3}b_1b_3 &= 0, \\ \sqrt{5}b_{-2}^2 - 2\sqrt{3}b_{-1}b_{-3} &= 0, \\ \sqrt{8}b_1^2 - \sqrt{15}b_0b_2 &= 0, \\ \sqrt{8}b_{-1}^2 - \sqrt{15}b_0b_{-2} &= 0, \\ b_0b_2 - 2\sqrt{2}b_3b_{-1} &= 0, \\ b_0b_{-2} - 2\sqrt{2}b_{-3}b_1 &= 0, \\ 2b_1b_2 - 3\sqrt{2}b_0b_3 &= 0, \\ 2b_{-1}b_{-2} - 3\sqrt{2}b_0b_{-3} &= 0, \\ \sqrt{6}b_0b_1 - 2\sqrt{5}b_{-1}b_2 &= 0, \\ \sqrt{6}b_0b_{-1} - 2\sqrt{5}b_1b_{-2} &= 0, \\ b_1b_{-2} - \sqrt{15}b_2b_{-3} &= 0, \\ b_{-1}b_2 - \sqrt{15}b_{-2}b_3 &= 0, \end{aligned} \right. \tag{3.3}$$

then

$$Q(\mathbf{b}) = 9(\sum_{j=-3}^{3}(b_j^2))^2 = 9.$$

By solving above algebraic system directly and using $\sum_{j=-3}^{3} b_j^2 = 1,\ \sum_{j=-3}^{3}(jb_j^2) = \frac{M}{N}$, we have

$$\max_{\mathbf{b}\in\mathrm{B}} Q(\mathbf{b}) = Q(\mathbf{b}^*) = 9,$$

where

$$b_3^* = \frac{(3N+M)^3}{216N^3},\ b_{-3}^* = \frac{(3N-M)^3}{216N^3},\ b_0^* = \sqrt{20}\frac{(9N^2-M^2)^{\frac{3}{2}}}{216N^3},$$
$$b_2^* = \sqrt{6}\frac{(3N+M)^{\frac{5}{2}}(3N-M)^{\frac{1}{2}}}{216N^3},\quad b_{-2}^* = \sqrt{6}\frac{(3N+M)^{\frac{1}{2}}(3N-M)^{\frac{5}{2}}}{216N^3},$$
$$b_1^* = \sqrt{15}\frac{(3N+M)^2(3N-M)}{216N^3},\quad b_{-1}^* = \sqrt{15}\frac{(3N+M)(3N-M)^2}{216N^3}.$$

□

Proof of Theorem 1.1. Let $\mathbf{u} \in \mathcal{M}$ be a minimizer of (1.12). We claim that $\mathbf{b}^*\rho$ is also a minimizer. Indeed, by direct calculations, we have

$$\mathbf{A}_{0,0}^2(\mathbf{b}^*\rho) = 0,\ \mathbf{A}_{2,0}^2(\mathbf{b}^*\rho) = 0,\ \mathbf{A}_{2,1}^2(\mathbf{b}^*\rho) = 0,\ \mathbf{A}_{2,-1}^2(\mathbf{b}^*\rho) = 0,$$

$$\mathbf{A}_{2,2}^2(\mathbf{b}^*\rho) = 0,\ \mathbf{A}_{2,-2}^2(\mathbf{b}^*\rho) = 0.$$

By (3.2), we have

$$\begin{aligned}
&\mathbf{F}^2(\mathbf{b}^*\rho) - \mathbf{F}^2(\mathbf{u}) = 9(\sum_{j=-3}^{3}(u_j^2))^2 - \mathbf{F}^2(\mathbf{u}) \\
&= (3u_0^2 - 4u_1u_{-1})^2 + (2u_1u_{-1} - 5u_2u_{-2})^2 + (u_2u_{-2} - 6u_3u_{-3})^2 \\
&+ (\sqrt{5}u_2^2 - 2\sqrt{3}u_1u_3)^2 + (\sqrt{5}u_{-2}^2 - 2\sqrt{3}u_{-1}u_{-3})^2 \\
&+ (\sqrt{8}u_1^2 - \sqrt{15}u_0u_2)^2 + (\sqrt{8}u_{-1}^2 - \sqrt{15}u_0u_{-2})^2 \\
&+ 3(u_0u_2 - 2\sqrt{2}u_3u_{-1})^2 + 3(u_0u_{-2} - 2\sqrt{2}u_{-3}u_1)^2 \\
&+ (2u_1u_2 - 3\sqrt{2}u_0u_3)^2 + (2u_{-1}u_{-2} - 3\sqrt{2}u_0u_{-3})^2 \\
&+ (\sqrt{6}u_0u_1 - 2\sqrt{5}u_{-1}u_2)^2 + (\sqrt{6}u_0u_{-1} - 2\sqrt{5}u_1u_{-2})^2 \\
&+ 2(u_1u_{-2} - \sqrt{15}u_2u_{-3})^2 + 2(u_{-1}u_2 - \sqrt{15}u_{-2}u_3)^2 \\
&\geq (3u_0^2 - 4u_1u_{-1})^2 + (2u_1u_{-1} - 5u_2u_{-2})^2 + (u_2u_{-2} - 6u_3u_{-3})^2 \\
&+ 2(\sqrt{5}u_2^2 - 2\sqrt{3}u_1u_3)(\sqrt{5}u_{-2}^2 - 2\sqrt{3}u_{-1}u_{-3}) + 2(\sqrt{8}u_1^2 - \sqrt{15}u_0u_2)(\sqrt{8}u_{-1}^2 - \sqrt{15}u_0u_{-2}) \\
&+ 6(u_0u_2 - 2\sqrt{2}u_3u_{-1})(u_0u_{-2} - 2\sqrt{2}u_{-3}u_1) - 2(2u_1u_2 - 3\sqrt{2}u_0u_3)(2u_{-1}u_{-2} - 3\sqrt{2}u_0u_{-3}) \\
&- 2(\sqrt{6}u_0u_1 - 2\sqrt{5}u_{-1}u_2)(\sqrt{6}u_0u_{-1} - 2\sqrt{5}u_1u_{-2}) - 4(u_1u_{-2} - \sqrt{15}u_2u_{-3})(u_{-1}u_2 - \sqrt{15}u_{-2}u_3) \\
&= 63\mathbf{A}_{0,0}^2(\mathbf{u}).
\end{aligned} \tag{3.4}$$

Thus, for $\tau_1 \le \frac{\tau_2}{63} \le 0, \tau_3 \ge 0$, we obtain

$$\frac{\tau_1}{4}(\mathbf{F}^2(\mathbf{b}^*\rho) - \mathbf{F}^2(\mathbf{u})) + \frac{\tau_2}{4}(\mathbf{A}_{0,0}^2(\mathbf{b}^*\rho) - \mathbf{A}_{0,0}^2(\mathbf{u})) + \frac{\tau_3}{4}\left(\sum_{j=-2}^{2}\mathbf{A}_{2,j}^2(\mathbf{b}^*\rho) - \sum_{j=-2}^{2}\mathbf{A}_{2,j}^2(\mathbf{u})\right) \tag{3.5}$$

$$\le \frac{1}{4}(63\tau_1 - \tau_2)\mathbf{A}_{0,0}^2(\mathbf{u}) \le 0.$$

For $\tau_1 \le 0, \tau_2 \ge 0,\ \tau_3 \ge 0$, we have

$$\frac{\tau_1}{4}(\mathbf{F}^2(\mathbf{b}^*\rho) - \mathbf{F}^2(\mathbf{u})) + \frac{\tau_2}{4}(\mathbf{A}_{0,0}^2(\mathbf{b}^*\rho) - \mathbf{A}_{0,0}^2(\mathbf{u})) + \frac{\tau_3}{4}(\sum_{j=-2}^{2}\mathbf{A}_{2,j}^2(\mathbf{b}^*\rho) - \sum_{j=-2}^{2}\mathbf{A}_{2,j}^2(\mathbf{u})) \le 0. \tag{3.6}$$

Therefore $E(\mathbf{b}^*\rho) \le E(\mathbf{u})$.Thus $\mathbf{b}^*\rho$ is also a minimizer. Consequently,

$$\big(\sum_{j=-3}^{3}|\nabla u_j|^2\big) - \big(\sum_{j=-3}^{3}|\nabla b_j\rho|^2\big) = 0,$$

$$\frac{\tau_1}{4}(\mathbf{F}^2(\mathbf{a}^*\rho) - \mathbf{F}^2(\mathbf{u})) + \frac{\tau_2}{4}(\mathbf{A}_{0,0}^2(\mathbf{b}^*\rho) - \mathbf{A}_{0,0}^2(\mathbf{u})) + \frac{\tau_3}{4}(\sum_{j=-2}^{2}\mathbf{A}_{2,j}^2(\mathbf{a}^*\rho) - \sum_{j=-2}^{2}\mathbf{A}_{2,j}^2(\mathbf{u})) = 0.$$

By (3.5), (3.6), $\mathbf{A}_{0,0}^2(\mathbf{b}^*\rho) = 0$, and $\sum_{j=-2}^{2}\mathbf{A}_{2,j}^2(\mathbf{b}^*\rho) = 0$, we get that

$$\mathbf{A}_{0,0}^2(\mathbf{u}) = 0,\ \sum_{j=-2}^{2}\mathbf{A}_{2,j}^2(\mathbf{u}) = 0,\ \mathbf{F}^2(\mathbf{b}^*\rho) = \mathbf{F}^2(\mathbf{u}).$$

Therefore $\mathbf{u} = \pm\mathbf{b}^*\rho$.

Next, we prove that the ground state must be positive if $\tau_1 < 0, \tau_2 = \tau_3 = 0$. Define

$$\mathcal{A} = \Big\{u \in \mathcal{M}|\ u_j \ge 0, j = 3, 2, 1, 0, -1, -2, -3\Big\}$$

and

$$\mathcal{G} = \Big\{\mathbf{u} \in \mathcal{A}|\ E(\mathbf{u}) = \inf_{\mathbf{v}\in\mathcal{A}} E(\mathbf{v}) = \inf_{\mathbf{v}\in\mathcal{M}} E(\mathbf{v})\Big\}.$$

If $\mathbf{u} \in \mathcal{M}$, then $\mathbf{b}^*\rho \in \mathcal{M}$ is a mass-redistribution of $\mathbf{u}$. By Proposition 3.1, we obtain

$$E(\mathbf{u}) \le \frac{1}{4}\int_{\mathbb{R}^d}(|\nabla\rho|^2 + V(x)\rho^2 + \frac{1}{4}(\tau_0 + 9\tau_1)\rho^4)dx = E(b^*\rho) \le E(\mathbf{u}). \tag{3.7}$$

Thus if ρ is a solution of (1.6) and (1.7), then $\mathbf{u}$ is a ground state solution of (1.3) and (1.4).

On the one hand, if $\mathbf{u}$ is a ground state of (1.3) and (1.4), then $|\mathbf{u}| \in \mathcal{G}$. We claim $|\mathbf{u}| = \mathbf{b}^*\rho$. Indeed,

$$E(\mathbf{u}) - E(\mathbf{b}^*\rho) = \frac{1}{2}\int_{\mathbb{R}^d}\left(\big(\sum_{j=-3}^{3}|\nabla u_j|^2\big) - \big(\sum_{j=-3}^{3}|\nabla b_j^*\rho|^2\big)\right)dx - \frac{\tau_1}{4}\int_{\mathbb{R}^d}(9\rho^4 - F_x^2 - F_z^2)dx \ge 0.$$

Since $\mathbf{u} \in \mathcal{G}$, we obtain that

$$\Big(\sum_{j=-3}^{3} |\nabla u_j|^2\Big) - \Big(\sum_{j=-3}^{3} |\nabla b_j \rho|^2\Big) = 0, \ 9\rho^4 - F_x^2 - F_z^2 = 0. \tag{3.8}$$

When $|M| \neq 3N$, by Proposition 3.1 and the first equality of (3.8), we get $u_j \nabla u_k = u_k \nabla u_j$ for $j \neq k$. Since $E(|\mathbf{u}|) \leq E(\mathbf{u})$, we may assume that $u_j \geq 0 (j = 3, 2, 1, 0, -1, -2, -3)$. Since $\int_{\mathbb{R}^d} \sum_{j=-3}^{3} u_j^2 = N$, at least one $u_j \not\equiv 0 (j = 3, 2, 1, 0, -1, -2, -3)$. Without loss of generality, we may assume $u_0 > 0$, so we have

$$\nabla(\frac{u_3}{u_0}) = \nabla(\frac{u_2}{u_0}) = \nabla(\frac{u_1}{u_0}) = \nabla(\frac{u_{-1}}{u_0}) = \nabla(\frac{u_{-2}}{u_0}) = \nabla(\frac{u_{-3}}{u_0}) = 0.$$

Then there are some $c_j \geq 0$ such that $u_j = c_j u_0$ for $j = 3, 2, 1, -1, -2, -3$. Together with the second equality of (3.8), we get $\mathbf{u} = \mathbf{b}^* \rho$. When $|M| = 3N$, the conclusion is obvious. So

$$\min_{\mathbf{u} \in \mathcal{M}} E(\mathbf{u}) \leq E(\mathbf{u}) \leq \frac{1}{2} \int_{\mathbb{R}^d} (|\nabla \rho|^2 + V(x)\rho^2 + \frac{1}{4}(\tau_0 + 9\tau_1)\rho^4) dx = E(\mathbf{b}^* \rho) = E(|\mathbf{u}|) \leq E(\mathbf{u}) = \min_{\mathbf{u} \in \mathbf{M}} E(\mathbf{u}).$$

Thus, it is easy to see that ρ is a solution to (1.6) and (1.7). Therefore, we complete the proof. $\square$

4 Proof of Theorem 1.2 and Theorem 1.3

Lemma 4.1. *Assume that $\tau_1 > 0$, $\tau_2 = \tau_3 = 0$ and $M = 0$, then* (1.12) *has infinitely many solutions if following minimizing problem*

$$\inf_{u \in \mathcal{N}} \Big\{ \frac{1}{2} \int_{\mathbb{R}^d} (|\nabla u|^2 + V(x)u^2 + \frac{\tau_0}{4} \int_{\mathbb{R}^d} u^4 dx \Big\}, \tag{4.1}$$

where

$$\mathcal{N} = \Big\{ u \in H^1(\mathbb{R}^d) | \int_{\mathbb{R}^d} u^2 dx = N \Big\}, \tag{4.2}$$

has a solution.

Proof. For any $(s, t, r) \in \mathbb{R}^3$ satisfying $s^2 + t^2 + r^2 \leq \frac{1}{2}$, let

$$\gamma = \left(-s, -t, r, \sqrt{1 - 2s^2 - 2t^2 - 2r^2}, -r, -t, s \right),$$

it is easy to see that when $M = 0$, for any $\mathbf{u} \in \mathcal{M}$, $\mathbf{v} = \gamma \rho \in \mathcal{M}$ is a mass-redistribution of $\mathbf{u}$. By Proposition 3.1, we have

$$E(\mathbf{u}) \geq E(\mathbf{v}) = \frac{1}{2} \int_{\mathbb{R}^d} (|\nabla \rho|^2) dx + \frac{1}{2} \int_{\mathbb{R}^d} V(x)\rho^2 dx + \frac{\tau_0}{4} \int_{\mathbb{R}^d} \rho^4 dx.$$

Thus, (1.12) has infinitely many solutions if (4.1)-(4.2) has a solution. $\square$

Proof of Theorem 1.2. Theorem 1.2 can be derived easily from Lemma 4.1.

$\square$

Next, we prove Theorem 1.3. Define a subset of $\mathcal{M}^*$ of $\mathcal{M}$ by

$$\mathcal{M}^* = \{\mathbf{u} \in \mathcal{M} : u_2 = u_1 = u_0 = u_{-1} = u_{-2} = 0\}.$$

For any $u \in \mathcal{M}$, we consider the following mass-redistribution $\mathbf{v}^* = (v_3^*, v_2^*, v_1^*, v_0^*, v_{-1}^*, v_{-2}^*, v_{-3}^*)$ of $\mathbf{u}$ with

$$\mathbf{v}^* = \left(\sqrt{u_3^2 + \frac{5}{6}u_2^2 + \frac{2}{3}u_1^2 + \frac{1}{2}u_0^2 + \frac{1}{3}u_{-1}^2 + \frac{1}{6}u_{-2}^2}, 0, 0, 0, 0, 0, \sqrt{u_{-3}^2 + \frac{5}{6}u_{-2}^2 + \frac{2}{3}u_{-1}^2 + \frac{1}{2}u_0^2 + \frac{1}{3}u_1^2 + \frac{1}{6}u_2^2}\right), \tag{4.3}$$

then we have

Lemma 4.2. *Assume that $\tau_1 > 0$ and $\tau_2 = \tau_3 = 0$, then a solution to the following minimizing problem*

$$\min_{\mathbf{u} \in \mathcal{M}^*} E(\mathbf{u})$$

is a solution to (1.12).

Proof. For any $\mathbf{u} \in \mathcal{M}$, by (4.3), we have that $\mathbf{v}^* \in \mathcal{M}^*$, by Proposition 3.1 and the assumptions $\tau_1 > 0$, $\tau_2 = \tau_3 = 0$, we have

$$\begin{aligned}
E(\mathbf{v}^*) - E(\mathbf{u}) = & \frac{1}{2}\int_{\mathbb{R}^d} \left(|\nabla v_3^*|^2 + |\nabla v_{-3}^*|^2\right) dx + \frac{1}{2}\int_{\mathbb{R}^d} \left(V(x)((v_3^*)^2 + (v_{-3}^*)^2)\right) dx + \frac{\tau_0}{4}\int_{\mathbb{R}^d} \left((v_3^*)^2 + (v_{-3}^*)^2\right)^2 \\
& + \frac{\tau_1}{4}\int_{\mathbb{R}^d} \left(3(v_3^*)^2 - 3(v_{-3}^*)^2\right)^2 dx \\
& - \frac{1}{2}\int_{\mathbb{R}^d} \left(\sum_{j=-3}^{3} |\nabla u_j|^2\right) dx - \frac{1}{2}\int_{\mathbb{R}^d} \left(V(x) \sum_{j=-3}^{3} u_j^2\right) dx - \frac{\tau_0}{4}\int_{\mathbb{R}^d} \left(\sum_{j=-3}^{3} u_j^2\right)^2 \\
& - \frac{\tau_1}{4}\int_{\mathbb{R}^d} (\sqrt{6}(u_2u_3 + u_{-2}u_{-3}) + \sqrt{10}(u_1u_2 + u_{-1}u_{-2}) + 2\sqrt{3}(u_1u_0 + u_{-1}u_0))^2 dx \\
& - \frac{\tau_1}{4}\int_{\mathbb{R}^d} \left(\sum_{j=-3}^{3} (ju_j^2)\right)^2 dx \\
\leq & -\frac{\tau_1}{4}\int_{\mathbb{R}^d} (\sqrt{6}(u_2u_3 + u_{-2}u_{-3}) + \sqrt{10}(u_1u_2 + u_{-1}u_{-2}) + 2\sqrt{3}(u_1u_0 + u_{-1}u_0))^2 dx \leq 0,
\end{aligned}$$

therefore a minimizer of $\min_{\mathbf{u} \in \mathcal{M}^*} E(\mathbf{u})$ is a minimizer of $\min_{\mathbf{u} \in \mathcal{M}} E(\mathbf{u})$. □

Lemma 4.3. *Assume that $\tau_1 > 0$, $\tau_2 = \tau_3 = 0$ and $M \neq 0$. If $\mathbf{u}$ is a minimizer of* (1.12)*, then $u_0 \equiv 0$.*

Proof. When $\tau_1 > 0$, $\tau_2 = \tau_3 = 0$, if $\mathbf{u}$ is a minimizer of (1.12), then $\mathbf{v}^*$ in (4.3) is also a minimizer. By lemma 4.2, we have that

$$F_x(\mathbf{u}) = F_x(\mathbf{v}^*) = 0,$$

thus $\mathbf{u}$ satisfies that

$$\begin{cases}
-\Delta u_{\pm 3} + V(x)u_{\pm 3} + (\lambda \pm 3\mu)u_{\pm 3} + \tau\rho^2 u_{\pm 3} \pm 3\tau_1 F_z u_{\pm 3} = 0, \\
-\Delta u_{\pm 2} + V(x)u_{\pm 2} + (\lambda \pm 2\mu)u_{\pm 2} + \tau\rho^2 u_{\pm 2} \pm 2\tau_1 F_z u_{\pm 2} = 0, \\
-\Delta u_{\pm 1} + V(x)u_{\pm 1} + (\lambda \pm \mu)u_{\pm 1} + \tau\rho^2 u_{\pm 1} \pm \tau_1 F_z u_{\pm 1} = 0, \\
-\Delta u_0 + V(x)u_0 + \lambda u_0 + \tau\rho^2 u_0 = 0.
\end{cases} \tag{4.4}$$

By regularity theorem, it is clear that $\mathbf{u} \in (C^\infty(\mathbb{R}^n))^7$. Suppose there exists a point $x_0 \in \mathbb{R}^n$ with $u_0(x_0) \neq 0$, then there exists an open connected set $I \subset \mathbb{R}^n$, such that $u_0(x) \neq 0$ in I and $u_0(x) = 0$ in ∂I. By lemma 4.2 and Proposition 3.1, we obtain

$$\nabla u_j u_0 = \nabla u_0 u_j,\ \forall x \in I, j = 3, 2, 1, -1, -2, -3,$$

which implies that there exists $b_j \in \mathbb{R}$, such that

$$u_j = a_j u_0,\ \forall x \in I, j = 3, 2, 1, -1, -2, -3,$$

and

$$\sqrt{6}(a_2a_3 + a_{-2}a_{-3}) + \sqrt{10}(a_1a_2 + a_{-1}a_{-2}) + 2\sqrt{3}(a_1 + a_{-1}) = 0. \tag{4.5}$$

Denote

$$\mathbb{R}^n = (\cup_i I^i_+) \cup I_0 \cup (\cup_i I^i_-),$$

where i, j are some index sets and

$$\begin{aligned} &I^i_+ \text{ are connected subsets in } I^+ = \{x \in \mathbb{R}^n | u_0(x) > 0\},\\ &I^j_- \text{ are connected subsets in } I^- = \{x \in \mathbb{R}^n | u_0(x) < 0\}, \end{aligned}$$

and

$$I_0 = \{x \in \mathbb{R}^n | u_0(x) = 0\}.$$

We define $\mathbf{v}$ by

$$\mathbf{v} = \mathbf{a}|u_0| = \begin{cases} (a^i_3, a^i_2, a^i_1, 1, a^i_{-1}, a^i_{-2}, a^i_{-3})u_0, & x \in I^i_+,\\ (a^i_3, a^i_2, a^i_1, 1, a^i_{-1}, a^i_{-2}, a^i_{-3})|u_0|, & x \in I^i_-,\\ \mathbf{u}, & x \in I_0. \end{cases}$$

It is easy to see that $\mathbf{v} \in \mathcal{M}$. By (4.5), we have that $F_x(\mathbf{v}) \equiv 0$, therefore $E(\mathbf{v}) \leq E(\mathbf{u})$. Hence $\mathbf{v}$ is also a minimizer which satisfies system (4.4) and $v_0 \geq 0$. By maximum principle, we get that $v_0 > 0$ in $\mathbb{R}^n$. Thus, we may assume that $u_0 > 0$ in $\mathbb{R}^n$. Since $\mathbf{u}$ satisfies (4.4), we get that

$$\begin{cases} a_{\pm 3}\left(-\Delta u_0 + V(x)u_0 + (\lambda \pm 3\mu)u_0 + (\tau \mathbf{a}^2 \pm 3\tau_1 M_0)u_0^3\right) = 0,\\ a_{\pm 2}\left(-\Delta u_0 + V(x)u_0 + (\lambda \pm 2\mu)u_0 + (\tau \mathbf{a}^2 \pm 2\tau_1 M_0)u_0^3\right) = 0,\\ a_{\pm 1}\left(-\Delta u_0 + V(x)u_0 + (\lambda \pm \mu)u_0 + (\tau \mathbf{a}^2 \pm \tau_1 M_0)u_0^3\right) = 0,\\ -\Delta u_0 + V(x)u_0 + \lambda u_0 + \tau \mathbf{a}^2 u_0^3 = 0, \end{cases} \tag{4.6}$$

where $M_0 = \sum_{j=-3}^{3}(j a_j^2)$ satisfies that

$$M_0 \int_{\mathbb{R}^n} u_0^2 dx = M \neq 0. \tag{4.7}$$

If one of a_j is nonzero for $j = 3, 2, 1, -1, -2, -3$, by (4.6), we obtain that

$$\tau_1 M_0 u_0^3 + \mu u_0 = 0, \forall\, x \in \mathbb{R}^n,$$

which contradict to $M \neq 0,\ \tau_1 > 0$ and $0 \not\equiv u_0 \in H^1_0(\mathbb{R}^n)$. If $a_j = 0$, for $j = 3, 2, 1, -1, -2, -3$, then $M = 0$, which contracts to (4.7). Therefore, $u_0 \equiv 0$. □

Lemma 4.4. *Assume that $\tau_1 > 0$, $\tau_2 = \tau_3 = 0$ and $M \neq 0$, $M \neq \pm N$. If $\mathbf{u}$ is a minimizer of* (1.12), *then $u_{\pm 1} \equiv 0$.*

Proof. By Lemma 4.3, we have $u_0 \equiv 0$. We may assume there exists a point $x_0 \in \mathbb{R}^n$ with $u_1(x_0) \neq 0$, then there exists an open connected set $I \subset \mathbb{R}^n$, such that $u_1(x) \neq 0$ in I and $u_1(x) = 0$ in ∂I. Similar to the proof of Lemma 4.3, we have

$$u_j = b_j u_1, \ \forall x \in I, j = 3, 2, 0, -1, -2, -3,$$

with

$$b_0 = 0 \text{ and } \sqrt{6}(b_2 b_3 + b_{-2} b_{-3}) + \sqrt{10}(b_2 + b_{-1} b_{-2}) = 0.$$

Moreover, we may assume that $u_1 > 0$ and $\mathbf{u} = (b_3, b_2, 1, 0, b_{-1}, b_{-2}, b_{-3}) u_1$. Since $\mathbf{u}$ satisfies (4.4), we get that

$$\begin{cases} b_{\pm 3}\left(-\Delta u_1 + V(x)u_1 + (\lambda \pm 3\mu)u_1 + (\tau \mathbf{b}^2 \pm 3\tau_1 M_1)u_1^3\right) = 0, \\ b_{\pm 2}\left(-\Delta u_1 + V(x)u_1 + (\lambda \pm 2\mu)u_1 + (\tau \mathbf{b}^2 \pm 2\tau_1 M_1)u_1^3\right) = 0, \\ b_{-1}\left(-\Delta u_1 + V(x)u_1 + (\lambda - \mu)u_{-1} + (\tau \mathbf{b}^2 - \tau_1 M_1)u_1^3\right) = 0, \\ -\Delta u_1 + V(x)u_1 + (\lambda + \mu)u_1 + (\tau \mathbf{b}^2 + \tau_1 M_1)u_1^3 = 0, \end{cases} \tag{4.8}$$

where $M_1 = \sum_{j=-3}^{3}(j b_j^2)$ satisfies that

$$M_1 \int_{\mathbb{R}^n} u_1^2 dx = M \neq 0 \text{ and } M_1 \int_{\mathbb{R}^n} u_1^2 dx = M \neq \pm N. \tag{4.9}$$

If one of b_j is nonzero for $j = 3, 2, -1, -2, -3$, by (4.10), we obtain that

$$\tau_1 M_1 u_1^3 + \mu u_1 = 0, \forall \ x \in \mathbb{R}^n,$$

which contradict to $M \neq 0$, $\tau_1 > 0$ and $0 \not\equiv u_1 \in H_0^1(\mathbb{R}^n)$. If $b_j = 0$, for $j = 3, 2, -1, -2, -3$, then we have $M = N$, which contracts to $M \neq N$. Therefore, $u_1 \equiv 0$. Similarly, we can deduce that $u_{-1} = 0$. ☐

Lemma 4.5. *Assume that $\tau_1 > 0$, $\tau_2 = \tau_3 = 0$ and $M \neq 0$, $M \neq \pm 2N$. If $\mathbf{u}$ is a minimizer of* (1.12), *then $u_{\pm 2} \equiv 0$.*

Proof. By Lemma 4.3, we have $u_0 \equiv 0$. We may assume there exists a point $x_0 \in \mathbb{R}^n$ with $u_2(x_0) \neq 0$, then there exists an open connected set $I \subset \mathbb{R}^n$, such that $u_2(x) \neq 0$ in I and $u_2(x) = 0$ in ∂I. Similar to the proof of Lemma 4.3, we have

$$u_j = c_j u_2, \ \forall x \in I, j = 3, 1, 0, -1, -2, -3,$$

with

$$c_0 = 0 \text{ and } \sqrt{6}(c_3 + c_{-2} c_{-3}) + \sqrt{10}(c_1 + c_{-1} c_{-2}) = 0.$$

Moreover, we may assume that $u_2 > 0$ and $\mathbf{u} = (c_3, 1, c_1, 0, c_{-1}, c_{-2}, c_{-3}) u_2$. Since $\mathbf{u}$ satisfies (4.4), we get that

$$\begin{cases} c_{\pm 3}\left(-\Delta u_2 + V(x)u_2 + (\lambda \pm 3\mu)u_2 + (\tau \mathbf{c}^2 \pm 3\tau_1 M_2)u_2^3\right) = 0, \\ c_{\pm 1}\left(-\Delta u_2 + V(x)u_2 + (\lambda \pm \mu)u_2 + (\tau \mathbf{c}^2 \pm \tau_1 M_2)u_2^3\right) = 0, \\ c_{-2}\left(-\Delta u_2 + V(x)u_2 + (\lambda - 2\mu)u_2 + (\tau \mathbf{c}^2 - 2\tau_1 M_2)u_2^3\right) = 0, \\ -\Delta u_2 + V(x)u_2 + (\lambda + 2\mu)u_2 + (\tau \mathbf{c}^2 + 2\tau_1 M_2)u_2^3 = 0, \end{cases} \tag{4.10}$$

where $M_2 = \sum_{j=-3}^{3}(jc_j^2)$ satisfies that

$$M_2 \int_{\mathbb{R}^n} u_2^2 dx = M \neq 0 \text{ and } M_2 \int_{\mathbb{R}^n} u_2^2 dx = M \neq \pm 2N. \tag{4.11}$$

If one of c_j is nonzero for $j = 3, 1, -1, -2, -3$, by (4.10), we obtain that

$$\tau_1 M_2 u_2^3 + \mu u_2 = 0, \forall\, x \in \mathbb{R}^n,$$

which contradict to $M \neq 0$, $\tau_1 > 0$ and $0 \not\equiv u_2 \in H_0^1(\mathbb{R}^n)$. If $c_j = 0$, for $j = 3, 1, -1, -2, -3$, then we have $M = 2N$, which contracts to $M \neq 2N$. Therefore, $u_2 \equiv 0$. Similarly, we can deduce that $u_{-2} = 0$. □

Lemma 4.6. *Assume that $\tau_1 > 0$, $\tau_2 = \tau_3 = 0$ and $M \neq 0$, $M \neq \pm 3N$. If $\mathbf{u}$ is a minimizer of* (1.12), *then $u_{\pm 3} \equiv 0$.*

Proof. By Lemma 4.3, we have $u_0 \equiv 0$. We may assume there exists a point $x_0 \in \mathbb{R}^n$ with $u_3(x_0) \neq 0$, then there exists an open connected set $I \subset \mathbb{R}^n$, such that $u_3(x) \neq 0$ in I and $u_3(x) = 0$ in ∂I. Similar to the proof of Lemma 4.3, we have

$$u_j = d_j u_3,\ \forall x \in I, j = 2, 1, 0, -1, -2, -3,$$

with

$$d_0 = 0 \text{ and } \sqrt{6}(d_2 + d_{-2}d_{-3}) + \sqrt{10}(d_1 d_2 + d_{-1}d_{-2}) = 0.$$

Moreover, we may assume that $u_3 > 0$ and $\mathbf{u} = (1, d_2, d_1, 0, d_{-1}, d_{-2}, d_{-3})u_3$. Since $\mathbf{u}$ satisfies (4.4), we get that

$$\begin{cases} d_{\pm 2}\left(-\Delta u_3 + V(x)u_3 + (\lambda \pm 2\mu)u_3 + (\tau \mathbf{d}^2 \pm 2\tau_1 M_3)u_3^3\right) = 0, \\ d_{\pm 1}\left(-\Delta u_0 + V(x)u_3 + (\lambda \pm \mu)u_3 + (\tau \mathbf{d}^2 \pm \tau_1 M_3)u_3^3\right) = 0, \\ d_{-3}\left(-\Delta u_0 + V(x)u_3 + (\lambda - 3\mu)u_3 + (\tau \mathbf{d}^2 - 3\tau_1 M_3)u_3^3\right) = 0, \\ -\Delta u_3 + V(x)u_3 + (\lambda + 3\mu)u_3 + (\tau \mathbf{d}^2 + 3\tau_1 M_3)u_3^3 = 0, \end{cases} \tag{4.12}$$

where $M_3 = \sum_{j=-3}^{3}(jd_j^2)$ satisfies that

$$M_3 \int_{\mathbb{R}^n} u_3^2 dx = M \neq 0 \text{ and } M_3 \int_{\mathbb{R}^n} u_3^2 dx = M \neq \pm 3N. \tag{4.13}$$

If one of d_j is nonzero for $j = 2, 1, -1, -2, -3$, by (4.12), we obtain that

$$\tau_1 M_3 u_3^3 + \mu u_3 = 0, \forall\, x \in \mathbb{R}^n,$$

which contradict to $M \neq 0$, $\tau_1 > 0$ and $0 \not\equiv u_3 \in H_0^1(\mathbb{R}^n)$. If $d_j = 0$, for $j = 2, 1, -1, -2, -3$, then we have $M = 3N$, which contracts to $M \neq 3N$. Therefore, $u_3 \equiv 0$. Similarly, we can deduce that $u_{-3} = 0$. □

Proof of Theorem 1.3. Theorem 1.3 can be obtained from Lemma 4.3-Lemma 4.6. □

5 Proof of Theorem 1.4 and Theorem 1.5

Lemma 5.1. *Assume that $\tau_1 = 0, \tau_3 = 0$ and $\tau_2 < 0$, for any $\mathbf{u} \in \mathcal{M}$, let $\mathbf{v}^*$ be defined by (4.3), then*

$$E(\mathbf{v}^*) \leq E(\mathbf{u}).$$

Proof. Since

$$\begin{aligned}E(\mathbf{u}) = &\frac{1}{2}\int_{\mathbb{R}^d}\left(\sum_{j=-3}^{3}|\nabla u_j|^2\right)dx + \frac{1}{2}\int_{\mathbb{R}^d}\left(V(x)\sum_{j=-3}^{3}u_j^2\right)dx + \frac{\tau_0}{4}\int_{\mathbb{R}^d}\left(\sum_{j=-3}^{3}u_j^2\right)^2 \\ &+ \frac{\tau_2}{4}\int_{\mathbb{R}^d}\mathbf{A}_{0,0}(\mathbf{u})^2dx.\end{aligned}$$

For all $\mathbf{u} \in \mathcal{M}$. By direct calculation, we get that

$$\begin{aligned}&7(\mathbf{A}_{0,0}(\mathbf{v}^*))^2 - 7(\mathbf{A}_{0,0}(\mathbf{u}))^2 = 4(v_3^*)^2(v_{-3}^*)^2 - (2u_3u_{-3} - 2u_2u_{-2} + 2u_1u_{-1} - u_0^2)^2 \\ &= 4\left(u_3^2 + \frac{5}{6}u_2^2 + \frac{2}{3}u_1^2 + \frac{1}{2}u_0^2 + \frac{1}{3}u_{-1}^2 + \frac{1}{6}u_{-2}^2\right)\left(u_{-3}^2 + \frac{5}{6}u_{-2}^2 + \frac{2}{3}u_{-1}^2 + \frac{1}{2}u_0^2 + \frac{1}{3}u_1^2 + \frac{1}{6}u_2^2\right) \\ &\quad - \left[(2u_3u_{-3} - 2u_2u_{-2})^2 + 8(u_3u_{-3} - u_2u_{-2})(u_1u_{-1} - u_0^2) + (2u_1u_{-1} - u_0^2)\right] \\ &= \frac{5}{9}(u_2^2 - u_{-2}^2)^2 + \frac{8}{9}(u_1^2 - u_{-1}^2)^2 + \frac{2}{3}(u_2u_3 + u_{-2}u_{-3})^2 + \frac{10}{3}(u_2u_{-3} + u_{-2}u_3)^2 \\ &\quad + \frac{4}{3}(u_1u_3 - u_{-1}u_{-3})^2 + \frac{8}{3}(u_3u_{-1} - u_1u_{-3})^2 + \frac{14}{9}(u_2u_1 + u_{-1}u_{-2})^2 + \frac{22}{9}(u_2u_{-1} + u_{-2}u_1)^2 \\ &\quad + 2u_0^2(u_3 + u_{-3})^2 + 2u_0^2(u_2 - u_{-2})^2 + 2u_0^2(u_1 + u_{-1})^2 \geq 0.\end{aligned} \tag{5.1}$$

Thus,

$$(\mathbf{A}_{0,0}(\mathbf{v}^*))^2 \geq (\mathbf{A}_{0,0}(\mathbf{u}))^2.$$

By the definition of $E(\mathbf{u})$, it is easy to see that

$$E(\mathbf{v}^*) \leq E(\mathbf{u}).$$

□

Lemma 5.2. *Assume that $\tau_1 = \tau_3 = 0, \tau_2 < 0$ and $M = 0$. Then (1.12) has infinitely many solutions if following minimizing problem*

$$\inf_{u\in\mathcal{N}}\left\{\frac{1}{2}\int_{\mathbb{R}^d}(|\nabla u|^2 + V(x)u^2 + \left(\frac{\tau_0}{4} + \frac{\tau_2}{28}\right)\int_{\mathbb{R}^d}u^4dx\right\}, \tag{5.2}$$

where

$$\mathcal{N} = \left\{u \in H^1(\mathbb{R}^d) \Big| \int_{\mathbb{R}^d}u^2dx = N\right\}, \tag{5.3}$$

has a solution.

Proof. For any $(s,t,r)\in\mathbb{R}^3$ satisfying $s^2+t^2+r^2\le\frac{1}{2}$, let

$$\gamma=\left(-s,-t,r,\sqrt{1-2s^2-2t^2-2r^2},-r,-t,s\right),$$

it is easy to see that when $M=0$, for any $\mathbf{u}\in\mathcal{M}$, $\mathbf{v}=\gamma\rho\in\mathcal{M}$ is a mass-redistribution of $\mathbf{u}$. By Proposition 3.1, we have

$$E(\mathbf{u})\ge E(\mathbf{v})=\frac{1}{2}\int_{\mathbb{R}^d}(|\nabla\rho|^2)dx+\frac{1}{2}\int_{\mathbb{R}^d}V(x)\rho^2dx+\left(\frac{\tau_0}{4}+\frac{\tau_2}{28}\right)\int_{\mathbb{R}^d}\rho^4dx.$$

Thus, (1.12) has infinitely many solutions if (5.2)-(5.3) has a solution. □

Lemma 5.3. *Assume that $\tau_1=\tau_3=0,\tau_2<0$ and $M\neq0$. If $\mathbf{u}$ is a minimizer of* (1.12)*, then $u_0\equiv0$ and $u_{\pm2}\equiv0$.*

Proof. When $\tau_1=0,\tau_3=0,\tau_2<0$ and $\mathbf{u}$ is a minimizer of (1.12), by direct calculations, we have

$$E(|u_3|,|u_2|,|u_1|,|u_0|,-|u_{-1}|,|u_{-2}|,-|u_{-3}|)\le E(u_3,u_2,u_1,u_0,u_{-1},u_{-2},u_{-3}).$$

Since $\mathbf{u}$ satisfies Let $\mathbf{v}^*$ be defined by (4.3), by Lemma 5.1 and the fact that $\mathbf{u}$ is a minimizer of (1.12), we get $E(\mathbf{v}^*)=E(\mathbf{u})$. If $u_0\not\equiv0$, then by (5.1), we get

$$u_3=-u_{-3},u_2=u_{-2}\text{ and }\ u_1=-u_{-1},$$

which contracts to $M\neq0$. Hence $u_0\equiv0$. Next, we prove that $u_{\pm2}=0$. If $u_2\not\equiv0$ in $\mathbb{R}^d$, since $\mathbf{u}$ is a minimizer of (1.12), we get $E(\mathbf{v}^*)=E(\mathbf{u})$. By (5.1), we get

$$u_3=-u_{-3},u_2=u_{-2}\text{ and }\ u_1=-u_{-1},$$

which contracts to $M\neq0$. Hence $u_2\equiv0$. Similarly, we can prove that $u_{-2}\equiv0$ and $u_{\pm1}\equiv0$. □

Proof of Theorem 1.4 and Theorem 1.5. Theorem 1.4 and Theorem 1.5 can be obtained by Lemma 5.2 and Lemma 5.3.

□

6 Proof of Theorem 1.6

Proof of Theorem 1.6 . From Theorem 1.1 in [14] and Theorem 1.1 in [28], we know that (1.6)-(1.7) has a unique non-negative minimizer in H when $N\in(0,N^*)$ suitable small and (1.6)-(1.7) has a unique non-negative minimizer in H up to rotation around the origin when $N\nearrow N^*$. By Theorem 1.1, when $\tau_2>0$, $\tau_3>0$ and $N\in(0,N^*)$ suitable small or $N\nearrow N^*$ up to rotation around the origin, we obtain that (1.3)-(1.4) has a unique non-negative minimizer.

Noting that the trapping potential $V(x)$ of (1.11) is radially symmetric, it then follows from (ii) of Theorem 1.1 in [14] that (1.6)-(1.7) has a unique non-negative minimizer which is also radially symmetric for small $N>0$. On the other hand, Theorem 1.2 in [14] shows that any non-negative minimizer of (1.6)-(1.7) concentrates at a point on the ring $\{x\in\mathbb{R}^2:|x|=A\}$ as $N\nearrow N^*$ and thus it cannot be radially symmetric. This implies that, as the strength of the interaction N increases from 0 to N^*, symmetry breaking occurs in the minimizers of (1.6)-(1.7). Thus, we complete the proof of (i) and (ii) of Theorem 1.6. □

7 Proof of Theorem 1.7

For any $\mathbf{u} \in \mathcal{M}$, by the definition of N^{**}, we obtain that

$$E(\mathbf{u}) \geq \frac{1}{2}\int_{\mathbb{R}^2}(\sum_{j=-3}^{3}|\nabla u_j|^2)dx + \frac{1}{2}\int_{\mathbb{R}^2}V(x)(\sum_{j=-3}^{3}u_j^2)dx + \left(\frac{\tau_0+9\tau_1+\tau_3}{4}+\frac{\tau_2}{28}\right)\int_{\mathbb{R}^2}(\sum_{j=-3}^{3}u_j^2)^2dx \tag{7.1}$$

$$\geq \frac{1}{2}\int_{\mathbb{R}^2}(\sum_{j=-3}^{3}|\nabla u_j|^2)dx - \frac{a^*}{4N^{**}}\int_{\mathbb{R}^2}(\sum_{j=-3}^{3}u_j^2)^2dx$$

$$\geq \frac{1}{2}\int_{\mathbb{R}^2}(\sum_{j=-3}^{3}|\nabla u_j|^2)dx - \frac{a^*}{4N^{**}}\cdot\frac{2N}{a^*}\int_{\mathbb{R}^2}(\sum_{j=-3}^{3}|\nabla u_j|^2)dx$$

$$= \frac{1}{2N^{**}}(N^{**}-N)\int_{\mathbb{R}^2}(\sum_{j=-3}^{3}|\nabla u_j|^2)dx.$$

When $\tau_0 < 0,\ \tau_1 < 0,\ \tau_2 > 0,\ \tau_3 > 0$, for any $\mathbf{u} \in \mathcal{M}$, by the definition of $N^* := -\frac{a^*}{\tau_0+9\tau_1}$, we obtain that

$$E(\mathbf{u}) \geq \frac{1}{2}\int_{\mathbb{R}^2}(\sum_{j=-3}^{3}|\nabla u_j|^2)dx + \frac{1}{2}\int_{\mathbb{R}^2}V(x)(\sum_{j=-3}^{3}u_j^2)dx + \left(\frac{\tau_0+9\tau_1}{4}\right)\int_{\mathbb{R}^2}(\sum_{j=-2}^{2}u_j^2)^2dx \tag{7.2}$$

$$\geq \frac{1}{2}\int_{\mathbb{R}^2}(\sum_{j=-3}^{3}|\nabla u_j|^2)dx - \frac{a^*}{4N^{*}}\int_{\mathbb{R}^2}(\sum_{j=-3}^{3}u_j^2)^2dx$$

$$\geq \frac{1}{2}\int_{\mathbb{R}^2}(\sum_{j=-3}^{3}|\nabla u_j|^2)dx - \frac{a^*}{4N^{*}}\cdot\frac{2N}{a^*}\int_{\mathbb{R}^2}(\sum_{j=-3}^{3}|\nabla u_j|^2)dx$$

$$= \frac{1}{2N^{*}}(N^{*}-N)\int_{\mathbb{R}^2}(\sum_{j=-3}^{3}|\nabla u_j|^2)dx.$$

Proof of Theorem 1.7. (i) When $\tau_0 < 0,\ \tau_1 \leq 0,\ \tau_2 < 0,\ \tau_3 < 0$. Let $\{\mathbf{u_n}\} \subset \mathcal{M}$ be a minimizing sequence of $m(N)$, then by (7.1), $\{\mathbf{u_n}\}$ is bounded in Λ if $0 < N < N^{**}$. Applying Lemma 2.1, there exists $\mathbf{w} = (w_3, w_2, w_1, w_0, w_{-1}, w_{-2}, w_{-3}) \in \Lambda$, such that up to a subsequence, as $n \to +\infty$,

$$\begin{cases} \mathbf{u_n} \rightharpoonup \mathbf{w}, & \text{in } \Lambda. \\ \mathbf{u_n} \to \mathbf{w}, & \text{in } L^t(\mathbb{R}^2, \mathbb{R}^7),\ \forall t \in [2, +\infty). \\ \mathbf{u_n} \to \mathbf{w}, & a.e. \text{ in } \mathbb{R}^2. \end{cases}$$

Then $\mathbf{w} \in \mathcal{M}$. Further, by the lower semi-continuity of the norm, there holds

$$m(N) \leq I(\mathbf{w}) \leq \lim_{n\to\infty} I(\mathbf{u_n}) = m(N).$$

It yields $I(\mathbf{w}) = m(N)$, that is, $\mathbf{w} \in \mathcal{M}$ is a minimizer of $m(N)$ for any $N \in (0, N^{**})$.

(ii) When $\tau_0 < 0$, $\tau_1 < 0$, $\tau_2 > 0$, $\tau_3 > 0$. Let $\{\mathbf{u_n}\} \subset \mathcal{M}$ be a minimizing sequence of $m(N)$, then by (7.2), $\{\mathbf{u_n}\}$ is bounded in Λ if $0 < N < N^*$. Applying Lemma 2.1, there exists $\mathbf{w} = (w_3, w_2, w_1, w_0, w_{-1}, w_{-2}, w_{-3}) \in \Lambda$, such that up to a subsequence, as $n \to +\infty$,

$$\begin{cases} \mathbf{u_n} \rightharpoonup \mathbf{w}, & \text{in } \Lambda. \\ \mathbf{u_n} \to \mathbf{w}, & \text{in } L^t(\mathbb{R}^2, \mathbb{R}^7),\ \forall t \in [2, +\infty). \\ \mathbf{u_n} \to \mathbf{w}, & a.e. \text{ in } \mathbb{R}^2. \end{cases}$$

Then $\mathbf{w} \in \mathcal{M}$. Further, by the lower semi-continuity of the norm, there holds

$$m(N) \le I(\mathbf{w}) \le \varliminf_{n\to\infty} I(\mathbf{u_n}) = m(N).$$

It yields $I(\mathbf{w}) = m(N)$, that is, $\mathbf{w} \in \mathcal{M}$ is a minimizer of $m(N)$ for any $N \in (0, N^*)$.

Next, we show that there has no minimizer for $m(N)$ when $N > N^*$ by carefully and skilfully choosing some proper test functions. For $\sigma > 0$, we define $\mathbf{\Phi} = (\Phi_3, \Phi_2, \Phi_1, \Phi_0, \Phi_{-1}, \Phi_{-2}, \Phi_{-3}) \in \mathcal{M}$ as

$$\begin{aligned} &\Phi_3(x) = \frac{(3N+M)^3}{216N^{\frac{5}{2}}\sqrt{a^*}}\sigma Q(\sigma x),\ \Phi_{-3}(x) = \frac{(3N-M)^3}{216N^{\frac{5}{2}}\sqrt{a^*}}\sigma Q(\sigma x),\ \Phi_0(x) = \sqrt{20}\frac{(9N^2-M^2)^{\frac{3}{2}}}{216N^{\frac{5}{2}}\sqrt{a^*}}, \\ &\Phi_2(x) = \sqrt{6}\frac{(3N+M)^{\frac{5}{2}}(3N-M)^{\frac{1}{2}}}{216N^{\frac{5}{2}}\sqrt{a^*}}\sigma Q(\sigma x), \quad \Phi_{-2}(x) = \sqrt{6}\frac{(3N+M)^{\frac{1}{2}}(3N-M)^{\frac{5}{2}}}{216N^{\frac{5}{2}}\sqrt{a^*}}\sigma Q(\sigma x), \\ &\Phi_1(x) = \sqrt{15}\frac{(3N+M)^2(3N-M)}{216N^{\frac{5}{2}}\sqrt{a^*}}\sigma Q(\sigma x), \quad \Phi_{-1}(x) = \sqrt{15}\frac{(3N+M)(3N-M)^2}{216N^{\frac{5}{2}}\sqrt{a^*}}\sigma Q(\sigma x), \end{aligned} \tag{7.3}$$

where $Q(x)$ is the unique positive solution of equation (1.8) . By direct calculations, we get

$$\frac{1}{2}\int_{\mathbb{R}^2}(\sum_{j=-3}^{3}|\nabla\Phi_j|^2)dx - \frac{a^*}{4N}\int_{\mathbb{R}^2}(\sum_{j=-3}^{3}\Phi_j^2)^2dx = \frac{1}{2}\cdot N\sigma^2 - \frac{a^*}{4N}\cdot\frac{2N^2\sigma^2}{a^*} = 0,$$

$$\int_{\mathbb{R}^2}V(x)(\sum_{j=-3}^{3}|\Phi_j|^2)dx = \frac{N}{a^*}\int_{\mathbb{R}^2}V(x)\sigma^2Q^2(\sigma x)dx = \frac{N\sigma^{-p}}{a^*}\int_{\mathbb{R}^2}V(x)Q^2(x)dx,$$

$$\begin{aligned} &\Big(\frac{a^*}{4N} + \frac{\tau_0+9\tau_1}{4}\Big)\int_{\mathbb{R}^2}(\sum_{j=-3}^{3}\Phi_j^2)^2dx = \Big(\frac{a^*}{4N} + \frac{\tau_0+9\tau_1}{4}\Big)\cdot\frac{2N^2\sigma^2}{a^*} \\ &= \sigma^2\cdot\Big(\frac{a^*}{4N} + \frac{\tau_0+9\tau_1}{4}\Big)\frac{2N^2}{a^*}, \end{aligned}$$

$$\begin{aligned} &\frac{\tau_1}{4}\int_{\mathbb{R}^2}(\sqrt{6}(\Phi_2\Phi_3 + \Phi_{-2}\Phi_{-3}) + \sqrt{10}(\Phi_1\Phi_2 + \Phi_{-1}\Phi_{-2}) + 2\sqrt{3}(\Phi_1\Phi_0 + \Phi_{-1}\Phi_0))^2dx \\ &+ \frac{\tau_1}{4}\int_{\mathbb{R}^2}\left(\sum_{j=-3}^{3}(j\Phi_j^2)\right)^2dx = \frac{9\tau_1}{4}\int_{\mathbb{R}^2}\left(\sum_{j=-3}^{3}\Phi_j^2\right)^2, \end{aligned}$$

$$\frac{\tau_2}{28}\int_{\mathbb{R}^2}\mathbf{A}_{00}^2dx = \frac{\tau_2}{28}\int_{\mathbb{R}^2}(2\Phi_3\Phi_{-3} - 2\Phi_2\Phi_{-2} + 2\Phi_1\Phi_{-1} - \Phi_0^2)^2dx = 0,$$

and

$$\frac{\tau_3}{4}\int_{\mathbb{R}^2}(\sum_{j=-2}^{2}\mathbf{A}_{2j}^2)dx=0.$$

Denote

$$K:=\Big(\frac{a^*}{4N}+\frac{\tau_0+9\tau_1}{4}\Big)\frac{2N^2}{a^*},$$

then it follows that

$$\begin{aligned}E(\mathbf{\Phi}):=&\frac{1}{2}\int_{\mathbb{R}^2}\Big(\sum_{j=-3}^{3}|\nabla\Phi_j|^2\Big)dx-\frac{a^*}{4N}\int_{\mathbb{R}^2}(\sum_{j=-3}^{3}\Phi_j^2)^2dx-\frac{9\tau_1}{4}\int_{\mathbb{R}^2}\Bigg(\sum_{j=-3}^{3}\Phi_j^2\Bigg)^2\\&+\frac{1}{2}\int_{\mathbb{R}^2}\Bigg(V(x)\sum_{j=-3}^{3}\Phi_j^2\Bigg)dx+\Big(\frac{a^*}{4N}+\frac{\tau_0+9\tau_1}{4}\Big)\int_{\mathbb{R}^2}\Bigg(\sum_{j=-3}^{3}\Phi_j^2\Bigg)^2\\&+\frac{\tau_1}{4}\int_{\mathbb{R}^2}(\sqrt{6}(\Phi_2\Phi_3+\Phi_{-2}\Phi_{-3})+\sqrt{10}(\Phi_1\Phi_2+\Phi_{-1}\Phi_{-2})+2\sqrt{3}(\Phi_1\Phi_0+\Phi_{-1}\Phi_0))^2dx\\&+\frac{\tau_1}{4}\int_{\mathbb{R}^2}\Bigg(\sum_{j=-3}^{3}(j\Phi_j^2)\Bigg)^2dx+\frac{\tau_2}{28}\int_{\mathbb{R}^2}(2\Phi_3\Phi_{-3}-2\Phi_2\Phi_{-2}+2\Phi_1\Phi_{-1}-\Phi_0^2)^2dx\\&+\frac{\tau_3}{4}\int_{\mathbb{R}^2}(\sum_{j=-2}^{2}\mathbf{A}_{2j}^2)dx=\frac{N\sigma^{-p}}{2a^*}\int_{\mathbb{R}^2}V(x)Q^2(x)dx+K\sigma^2.\end{aligned}\tag{7.4}$$

We conclude that for any $\sigma>0$,

$$\begin{aligned}m(N)&=\frac{N\sigma^{-p}}{2a^*}\int_{\mathbb{R}^2}V(x)Q^2(x)dx+\Big(\frac{N}{2}+\Big(\frac{\tau_0+9\tau_1}{4}\Big)\frac{2N^2}{a^*}\Big)\sigma^2\\&=\frac{N\sigma^{-p}}{2a^*}\int_{\mathbb{R}^2}V(x)Q^2(x)dx+\frac{N}{2N^*}(N^*-N)\sigma^2.\end{aligned}\tag{7.5}$$

If $N>N^*$, let $\sigma\to\infty$ in (7.5), then $m(N)\to-\infty$. Thus, there has no minimizer for $m(N)$.

When $\tau_2>0$ and $\tau_3>0$, we take $\sigma=\Big(\frac{p}{2}\frac{N^*\int_{\mathbb{R}^2}V(x)Q^2(x)dx}{a^*(N^*-N)}\Big)^{\frac{1}{p+2}}$, we get

$$m(N)\le\frac{p+2}{2p}\frac{N}{N^*}\Big(\frac{p}{2}\frac{N^*\int_{\mathbb{R}^2}V(x)Q^2(x)dx}{a^*}\Big)^{\frac{2}{p+2}}(N^*-N)^{\frac{p}{p+2}}\to0,\quad\text{as }N\nearrow N^*,\tag{7.6}$$

that is, $\lim\limits_{N\nearrow N^*}m(N)\le0$. On the other hand, when $N\in(0,N^*)$, we obtain from (7.2) that $E(\boldsymbol{u})\ge0$ for any $\boldsymbol{u}=(u_3,u_2,u_1,u_0,u_{-1},u_{-2},u_{-3})\in\mathcal{M}$, which implies $\lim\limits_{N\nearrow N^*}m(N)\ge0$. Thus

$$\lim_{N\nearrow N^*}m(N)=0.$$

Next, we show that there has no minimizer for $m(N)$ if $N=N^*$. We argue by contradiction to show that there has no minimizer for $m(N^*)$. Suppose $\boldsymbol{u^*}=(u_3^*,u_2^*,u_1^*,u_0^*,u_{-1}^*,u_{-2}^*,u_{-3}^*)$ is a minimizer of

$m(N^*)$. From the proof of (7.4), we have

$$E(\boldsymbol{u^*}) \geq \frac{1}{2}\int_{\mathbb{R}^2}(\sum_{j=-3}^{3}|\nabla u_j^*|^2)dx - \frac{a^*}{4N}\int_{\mathbb{R}^2}(\sum_{j=-3}^{3}u_j^*)^2dx$$
$$+\frac{1}{2}\int_{\mathbb{R}^2}V(x)(\sum_{j=-3}^{3}|u_j^*|^2)dx \geq 0.$$

Together with (7.6), we get $m(N^*) = 0$. As a consequence,

$$\frac{1}{2}\int_{\mathbb{R}^2}(\sum_{j=-3}^{3}|\nabla u_j^*|^2)dx = \frac{a^*}{4N}\int_{\mathbb{R}^2}(\sum_{j=-3}^{3}u_j^*)^2dx \tag{7.7}$$

and

$$\int_{\mathbb{R}^2}V(x)(\sum_{j=-3}^{3}|u_j^*|^2)dx = 0. \tag{7.8}$$

From (7.7), $\boldsymbol{u^*}$ is an optimal function of the Gagliardo-Nirenberg inequality for $d = 2$. By Lemma $\boldsymbol{u^*}$ can be formed as a scaling of $Q(x)$. However, this contradicts to (7.8). Therefore, there has no minimizer for $m(N^*)$ and we complete the proof. □

Next, we prove (iii) of Theorem 1.7. Before that, we give an estimate for the least energy $m(N)$.

Lemma 7.1. *Suppose $\tau_0 < 0$, $\tau_1 \leq 0$, $\tau_2 < 0$ and $\tau_3 < 0$ (resp. $\tau_2 \geq$ and $\tau_3 \geq 0$), then there holds $m(N) < \frac{\lambda_{0,V}N}{2}$, for $N \in (0, N^{**})$ (resp. $N \in (0, N^*)$).*

Proof. If $\tau_2 < 0$,$\tau_3 < 0$, since $\Big(0, 0, \sqrt{\frac{N+M}{2}}\Psi_0, 0, \sqrt{\frac{N-M}{2}}\Psi_0, 0, 0\Big) \in \mathcal{M}$, we get from Lemma (2.3),

$$m(N) = \inf_{\mathbf{u}\in\mathcal{M}} E(\mathbf{u}) \leq E\Big(0, 0, \sqrt{\frac{N+M}{2}}\Psi_0, 0, \sqrt{\frac{N-M}{2}}\Psi_0, 0, 0\Big)$$
$$< \frac{N}{2}\int_{\mathbb{R}^2}\Big(|\nabla\Psi_0|^2 + V(x){\Psi_0}^2\Big)dx = \frac{\lambda_{0,V}N}{2}.$$

If $\tau_2 \geq 0$, $\tau_3 \geq 0$, let

$$\widetilde{\Phi}_3(x) = \frac{(3N+M)^3}{216N^{\frac{5}{2}}}\Psi_0,\ \widetilde{\Phi}_{-3}(x) = \frac{(3N-M)^3}{216N^{\frac{5}{2}}}\Psi_0,\ \widetilde{\Phi}_0(x) = \sqrt{20}\frac{(9N^2-M^2)^{\frac{3}{2}}}{216N^{\frac{5}{2}}}\Psi_0,$$
$$\widetilde{\Phi}_2(x) = \sqrt{6}\frac{(3N+M)^{\frac{5}{2}}(3N-M)^{\frac{1}{2}}}{216N^{\frac{5}{2}}}\Psi_0,\quad \widetilde{\Phi}_{-2}(x) = \sqrt{6}\frac{(3N+M)^{\frac{1}{2}}(3N-M)^{\frac{5}{2}}}{216N^{\frac{5}{2}}}\Psi_0,$$
$$\widetilde{\Phi}_1(x) = \sqrt{15}\frac{(3N+M)^2(3N-M)}{216N^{\frac{5}{2}}}\Psi_0,\quad \widetilde{\Phi}_{-1}(x) = \sqrt{15}\frac{(3N+M)(3N-M)^2}{216N^{\frac{5}{2}}}\Psi_0,$$

then $\Big(\widetilde{\Phi}_3(x), \widetilde{\Phi}_2(x), \widetilde{\Phi}_1(x), \widetilde{\Phi}_0(x), \widetilde{\Phi}_{-1}(x), \widetilde{\Phi}_{-2}(x), \widetilde{\Phi}_{-3}(x)\Big) \in \mathcal{M}$. Further, we get from Lemma (2.3) that

$$m(N) = \inf_{\mathbf{u}\in\mathcal{M}} E(\mathbf{u}) \leq E\Big(\widetilde{\Phi}_3(x), \widetilde{\Phi}_2(x), \widetilde{\Phi}_1(x), \widetilde{\Phi}_0(x), \widetilde{\Phi}_{-1}(x), \widetilde{\Phi}_{-2}(x), \widetilde{\Phi}_{-3}(x)\Big)$$
$$< \frac{N}{2}\int_{\mathbb{R}^2}\Big(|\nabla\Psi_0|^2 + V(x){\Psi_0}^2\Big)dx = \frac{\lambda_{0,V}N}{2}.$$

□

Proof of Theorem 1.7. (iii) Set $l_{ik} = \int_{\mathbb{R}^2} u_i \Psi_k dx$ for $i = 3, 2, 1, 0, -1, -2, -3$, then

$$\mathbf{u} = \Big(\sum_{k=0}^{\infty} l_{3k}\Psi_k, \sum_{k=0}^{\infty} l_{2k}\Psi_k, \sum_{k=0}^{\infty} l_{1k}\Psi_k, \sum_{k=0}^{\infty} l_{0k}\Psi_k, \sum_{k=0}^{\infty} l_{-1k}\Psi_k, \sum_{k=0}^{\infty} l_{-2k}\Psi_k, \sum_{k=0}^{\infty} l_{-3k}\Psi_k \Big).$$

Moreover, we conclude

$$\begin{aligned} N &= \|(u_3, u_2, u_1, u_0, u_{-1}, u_{-2}, u_{-3})\|_{L^2}^2 = \sum_{k=0}^{\infty}(l_{3k}^2 + l_{2k}^2 + l_{1k}^2 + l_{0k}^2 + l_{-1k}^2 + l_{-2k}^2 + l_{-3k}^2)\|\Psi_k\|_{L^2}^2 \\ &= \sum_{k=0}^{\infty}(l_{3k}^2 + l_{2k}^2 + l_{1k}^2 + l_{0k}^2 + l_{-1k}^2 + l_{-2k}^2 + l_{-3k}^2) \end{aligned} \tag{7.9}$$

and

$$\begin{aligned} \|\mathbf{u}\|_\Lambda^2 &= \sum_{k=0}^{\infty}(l_{3k}^2 + l_{2k}^2 + l_{1k}^2 + l_{0k}^2 + l_{-1k}^2 + l_{-2k}^2 + l_{-3k}^2)\|\Psi_k\|_\Lambda^2 \\ &= \sum_{k=0}^{\infty} \lambda_{k,V}(l_{3k}^2 + l_{2k}^2 + l_{1k}^2 + l_{0k}^2 + l_{-1k}^2 + l_{-2k}^2 + l_{-3k}^2). \end{aligned}$$

Denote $M_0 := \frac{1}{2N^*}(N^* - N) \in \big(0, \frac{1}{2}\big)$, then we get

$$\begin{aligned} m(N) = E(\mathbf{u}) &\geq M_0 \int_{\mathbb{R}^2} (\sum_{j=-3}^{3} |\nabla u_j|^2) dx + \frac{1}{2}\int_{\mathbb{R}^2} V(x)(\sum_{j=-3}^{3} |u_j|^2) dx \geq M_0 \|\mathbf{u}\|_\Lambda^2 \\ &= M_0 \cdot \sum_{k=0}^{\infty} \lambda_{k,V}(l_{3k}^2 + l_{2k}^2 + l_{1k}^2 + l_{0k}^2 + l_{-1k}^2 + l_{-2k}^2 + l_{-3k}^2) \\ &= M_0 \cdot \sum_{k=0}^{\infty}(\lambda_{k,V} - \lambda_{0,V})(l_{3k}^2 + l_{2k}^2 + l_{1k}^2 + l_{0k}^2 + l_{-1k}^2 + l_{-2k}^2 + l_{-3k}^2) \\ &\quad + M_0 \cdot \sum_{k=0}^{\infty} \lambda_{0,V}(l_{3k}^2 + l_{2k}^2 + l_{1k}^2 + l_{0k}^2 + l_{-1k}^2 + l_{-2k}^2 + l_{-3k}^2). \end{aligned}$$

By Lemma 7.1 and (7.9), we have

$$\begin{aligned} &(\lambda_{1,V} - \lambda_{0,V}) \sum_{k=1}^{\infty}(l_{3k}^2 + l_{2k}^2 + l_{1k}^2 + l_{0k}^2 + l_{-1k}^2 + l_{-2k}^2 + l_{-3k}^2) \\ &\leq \sum_{k=1}^{\infty}(\lambda_{k,V} - \lambda_{0,V})(l_{3k}^2 + l_{2k}^2 + l_{1k}^2 + l_{0k}^2 + l_{-1k}^2 + l_{-2k}^2 + l_{-3k}^2) \\ &\leq \frac{m(N)}{M_0} - \sum_{k=0}^{\infty} \lambda_{0,V}(l_{3k}^2 + l_{2k}^2 + l_{1k}^2 + l_{0k}^2 + l_{-1k}^2 + l_{-2k}^2 + l_{-3k}^2) \leq \Big(\frac{\lambda_{0,V}}{M_0} - \lambda_{0,V}\Big)N, \end{aligned}$$

then

$$\sum_{k=1}^{\infty}(l_{3k}^2 + l_{2k}^2 + l_{1k}^2 + l_{0k}^2 + l_{-1k}^2 + l_{-2k}^2 + l_{-3k}^2) \leq \Big(\frac{\lambda_{0,V}}{M_0} - \lambda_{0,V}\Big) \cdot \frac{N}{\xi_1 - \xi_0}.$$

Thus

$$
\begin{aligned}
&\sum_{k=1}^{\infty}\lambda_{k,V}(l_{3k}^2+l_{2k}^2+l_{1k}^2+l_{0k}^2+l_{-1k}^2+l_{-2k}^2+l_{-3k}^2)\\
&=\sum_{k=1}^{\infty}(\lambda_{k,V}-\lambda_{0,V})(l_{3k}^2+l_{2k}^2+l_{1k}^2+l_{0k}^2+l_{-1k}^2+l_{-2k}^2+l_{-3k}^2)\\
&+\lambda_{0,V}\sum_{k=1}^{\infty}(l_{3k}^2+l_{2k}^2+l_{1k}^2+l_{0k}^2+l_{-1k}^2+l_{-2k}^2+l_{-3k}^2)\\
&\le\Big(\frac{\lambda_{0,V}}{M_0}-\lambda_{0,V}\Big)N+\lambda_{0,V}\Big(\frac{\lambda_{0,V}}{M_0}-\lambda_{0,V}\Big)\cdot\frac{N}{\lambda_{l,V}-\lambda_{0,V}}=\frac{\lambda_{l,V}}{\lambda_{l,V}-\lambda_{0,V}}\cdot\Big(\frac{\lambda_{0,V}}{M_0}-\lambda_{0,V}\Big)N.
\end{aligned}
$$

For $N\to 0^+$, we can see that

$$
\begin{aligned}
&\|\boldsymbol{u}-(l_3\Psi_0,l_2\Psi_0,l_1\Psi_0,l_0\Psi_0,l_{-1}\Psi_0,l_{-2}\Psi_0,l_{-3}\Psi_0)\|_{\dot\Lambda}^2\\
&=\Big\|\Big(\sum_{k=1}^{\infty}l_{3k}\Psi_k,\sum_{k=1}^{\infty}l_{2k}\Psi_k,\sum_{k=1}^{\infty}l_{1k}\Psi_k,\sum_{k=1}^{\infty}l_{0k}\Psi_k,\sum_{k=1}^{\infty}l_{-1k}\Psi_k,\sum_{k=1}^{\infty}l_{-2k}\Psi_k,\sum_{k=1}^{\infty}l_{-3k}\Psi_k\Big)\Big\|_{\dot\Lambda}^2\\
&=\sum_{k=1}^{\infty}\lambda_{k,V}(l_{3k}^2+l_{2k}^2+l_{1k}^2+l_{0k}^2+l_{-1k}^2+l_{-2k}^2+l_{-3k}^2)=O(N)
\end{aligned}
$$

and

$$
\begin{aligned}
&\|\boldsymbol{u}-(l_3\Psi_0,l_2\Psi_0,l_1\Psi_0,l_0\Psi_0,l_{-1}\Psi_0,l_{-2}\Psi_0,l_{-3}\Psi_0)\|_{L^2}^2\\
&=\Big\|\Big(\sum_{k=1}^{\infty}l_{3k}\Psi_k,\sum_{k=1}^{\infty}l_{2k}\Psi_k,\sum_{k=1}^{\infty}l_{1k}\Psi_k,\sum_{k=1}^{\infty}l_{0k}\Psi_k,\sum_{k=1}^{\infty}l_{-1k}\Psi_k,\sum_{k=1}^{\infty}l_{-2k}\Psi_k,\sum_{k=1}^{\infty}l_{-3k}\Psi_k\Big)\Big\|_{L^2}^2\\
&=\sum_{k=1}^{\infty}(l_{3k}^2+l_{2k}^2+l_{1k}^2+l_{0k}^2+l_{-1k}^2+l_{-2k}^2+l_{-3k}^2)=O(N).
\end{aligned}
$$

Therefore, it is obvious the conclusion holds and we complete the proof. □

Assume $\tau<0$, $\tau_1<0$,$\tau_2>0$, $\tau_3>0$ and $N_n\nearrow N^*$ as $n\to\infty$, let

$$\mathbf{u_n}=(u_{3n},u_{2n},u_{1n},u_{0n},u_{-1n},u_{-2n},u_{-3n})\in\mathcal{M}(N_n)$$

be a minimizer for $m(N_n)$. Then $\mathbf{u_n}$ satisfies system (1.3) where λ_n and μ_n are the corresponding

Lagrange multipliers. By (7.2), we have

$$
\begin{aligned}
E(\mathbf{u_n}) :=& \frac{1}{2}\int_{\mathbb{R}^2}\Big(\sum_{j=-3}^{3}|\nabla u_{jn}|^2\Big)dx - \frac{a^*}{4N^*}\int_{\mathbb{R}^2}(\sum_{j=-3}^{3}u_{jn}^2)^2dx - \frac{9\tau_1}{4}\int_{\mathbb{R}^2}\left(\sum_{j=-3}^{3}u_{jn}^2\right)^2 \\
&+ \frac{1}{2}\int_{\mathbb{R}^2}\left(V(x)\sum_{j=-3}^{3}u_{jn}^2\right)dx + \Big(\frac{a^*}{4N^*}+\frac{\tau_0+9\tau_1}{4}\Big)\int_{\mathbb{R}^2}\left(\sum_{j=-3}^{3}u_{jn}^2\right)^2 \\
&+ \frac{\tau_1}{4}\int_{\mathbb{R}^2}(\sqrt{6}(u_{2n}u_{3n}+u_{-2n}u_{-3n})+\sqrt{10}(u_{1n}u_{2n}+u_{-1n}u_{-2n})+2\sqrt{3}(u_{1n}u_{0n}+u_{-1n}u_{0n}))^2dx \\
&+ \frac{\tau_1}{4}\int_{\mathbb{R}^2}\left(\sum_{j=-3}^{3}(ju_{jn}^2)\right)^2dx + \frac{\tau_2}{28}\int_{\mathbb{R}^2}(2u_{3n}u_{-3n}-2u_{2n}u_{-2n}+2u_{1n}u_{-1n}-u_{0n}^2)^2dx \\
&+ \frac{\tau_3}{4}\int_{\mathbb{R}^2}(\sum_{j=-2}^{2}\mathbf{A}_{2j}^2)dx \\
&\geq \frac{1}{2N^*}(N^*-N_n)\int_{\mathbb{R}^2}\Big(\sum_{j=-3}^{3}|\nabla u_{jn}|^2\Big)dx + \frac{1}{2}\int_{\mathbb{R}^2}V(x)\Big(\sum_{j=-3}^{3}u_{jn}^2\Big)dx \geq 0.
\end{aligned}
\tag{7.10}
$$

Combining with the fact that $\lim\limits_{N\nearrow N^*} m(N) = 0$, we can see that

$$
\lim_{n\to\infty}\frac{\tau_2}{28}\int_{\mathbb{R}^2}(2u_{3n}u_{-3n}-2u_{2n}u_{-2n}+2u_{1n}u_{-1n}-u_{0n}^2)^2dx = 0, \tag{7.11}
$$

$$
\begin{aligned}
&\lim_{n\to\infty}\frac{\tau_1}{4}\int_{\mathbb{R}^2}(\sqrt{6}(u_{2n}u_{3n}+u_{-2n}u_{-3n})+\sqrt{10}(u_{1n}u_{2n}+u_{-1n}u_{-2n})+2\sqrt{3}(u_{1n}u_{0n}+u_{-1n}u_{0n}))^2dx \\
&+\lim_{n\to\infty}\frac{\tau_1}{4}\int_{\mathbb{R}^2}\Big(\sum_{j=-3}^{3}(ju_{jn}^2)\Big)^2dx - \lim_{n\to\infty}\frac{9\tau_1}{4}\int_{\mathbb{R}^2}\Big(\sum_{j=-3}^{3}u_{jn}^2\Big)^2dx = 0
\end{aligned}
\tag{7.12}
$$

and

$$
\lim_{n\to\infty}\frac{\int_{\mathbb{R}^2}(\sum\limits_{j=-3}^{3}|\nabla u_{jn}|^2)dx}{\int_{\mathbb{R}^2}(\sum\limits_{j=-3}^{3}u_{jn}^2)^2dx} = \frac{a^*}{2N^*}. \tag{7.13}
$$

We claim that

$$
\lim_{n\to\infty}\int_{\mathbb{R}^2}(\sum_{j=-3}^{3}|\nabla u_{jn}|^2)dx = +\infty.
$$

We argue by contradiction. Suppose there exists a positive constant C, such that $\int_{\mathbb{R}^2}(\sum_{j=-3}^{3}|\nabla u_{jn}|^2)dx \leq C$ for large n. Then $\{\mathbf{u_n}\}$ is a bounded sequence in Λ, which implies that there exist a subsequence (still denoted by $\{\mathbf{u_n}\}$) and $\mathbf{u}^* := (u_3^*, u_2^*, u_1^*, u_0^*, u_{-1}^*, u_{-2}^*, u_{-3}^*)$, such that as $n\to\infty$,

$$
\mathbf{u_n} \to \mathbf{u}^* \text{ in } L^t(\mathbb{R}^2,\mathbb{R}^7),\ \forall t\in[2,+\infty).
$$

Hence, by the weak lower semi-continuity of the norm, we get

$$0 = \lim_{n\to\infty} E(\mathbf{u_n}) \geq E(\mathbf{u}^*) \geq m(N^*) = 0.$$

It shows $\mathbf{u}^*$ is a minimizer of $m(N^*)$, which contradicts to Theorem 1.7. Thus, we obtain the claim.
Now, defining

$$\varepsilon_n := \sqrt{N^*}\Big(\int_{\mathbb{R}^2}(\sum_{j=-3}^{3}|\nabla u_{jn}|^2)dx\Big)^{-\frac{1}{2}}, \tag{7.14}$$

then it is easy to see that $\varepsilon_n \to 0$ as $n \to \infty$.

Proof (iv) of Theorem 1.7. On the one hand, by (7.5), we get for any $\sigma > 0$,

$$m(N_n) \leq \frac{N\sigma^{-p}}{2a^*}\int_{\mathbb{R}^2} V(x)Q^2(x)dx + \frac{N_n}{2N^*}(N^* - N_n)\sigma^2.$$

By (7.6), it follows that

$$\lim_{n\to\infty}\frac{m(N_n)}{(N^*-N_n)^{\frac{p}{p+2}}} \leq \frac{p+2}{2}\left(\frac{1}{2}\right)^{\frac{2}{p+2}}\left(\frac{1}{p}\right)^{\frac{p}{p+2}}\Big(\frac{N^*\int_{\mathbb{R}^2}V(x)Q^2(x)dx}{a^*}\Big)^{\frac{2}{p+2}}. \tag{7.15}$$

On the other hand, let $\tilde{\mathbf{w}}_\mathbf{n} := (\tilde{w}_{3n}, \tilde{w}_{2n}, \tilde{w}_{1n}, \tilde{w}_{0n}, \tilde{w}_{-1n}, \tilde{w}_{-2n}, \tilde{w}_{-3n})$ with $\tilde{w}_{jn}(x) := \varepsilon_n u_{jn}(\varepsilon_n x)$ ($j = 3, 2, 1, 0, -1, -2, -3$), then

$$\int_{\mathbb{R}^2}(\sum_{j=-3}^{3}|\nabla\tilde{w}_{jn}|^2)dx = \varepsilon_n^2\int_{\mathbb{R}^2}(\sum_{j=-3}^{3}|\nabla u_{jn}|^2)dx = N^*.$$

Moreover, from (7.13), we have

$$\lim_{n\to\infty}\frac{\int_{\mathbb{R}^2}(\sum_{j=-3}^{3}|\nabla\tilde{w}_{jn}|^2)dx}{\int_{\mathbb{R}^2}(\sum_{j=-3}^{3}\tilde{w}_{jn}^2)^2dx} = \lim_{n\to\infty}\frac{\int_{\mathbb{R}^2}(\sum_{j=-3}^{3}|\nabla u_{jn}|^2)dx}{\int_{\mathbb{R}^2}(\sum_{j=-3}^{3}u_{jn}^2)^2dx} = \frac{a^*}{2N^*}, \tag{7.16}$$

which yields that

$$\lim_{n\to\infty}\int_{\mathbb{R}^2}(\sum_{j=-3}^{3}\tilde{w}_{jn}^2)^2dx = \frac{2(N^*)^2}{a^*}. \tag{7.17}$$

We claim that there exist $\{y_n\} \subset \mathbb{R}^2$ and $R_0,\ \eta > 0$, such that at least one $j \in \{3, 2, 1, 0, -1, -2, -3\}$ satisfies $\liminf\limits_{n\to\infty}\int_{B_{R_0}(y_n)}\tilde{w}_{jn}^2dx \geq \eta > 0$. Otherwise, suppose for any $R > 0$, there has a subsequence $\{\tilde{w}_{jn_k}\}$ ($j = 3, 2, 1, 0, -1, -2, -3$), such that $\lim\limits_{k\to\infty}\sup\limits_{x\in\mathbb{R}^2}\int_{B_R(y)}\tilde{w}_{jn_k}^2dx = 0$. Then by Lion's vanishing Lemma, we conclude that $\tilde{w}_{jn_k} \to 0$ ($j = 3, 2, 1, 0, -1, -2, -3$) in $L^t(\mathbb{R}^2)$ for $t \in (2, \infty)$, which contradicts to (7.17). Hence, we obtain the claim. Now we define $\mathbf{w_n} := (w_{3n}, w_{2n}, w_{1n}, w_{0n}, w_{-1n}, w_{-2n}, w_{-3n})$ with

$$w_{jn}(x) := \tilde{w}_{jn}(x + y_n) = \varepsilon_n u_{jn}(\varepsilon_n x + \varepsilon_n y_n), \quad j = 3, 2, 1, 0, -1, -2, -3 \tag{7.18}$$

then

$$\lim_{n\to\infty}\int_{\mathbb{R}^2}(\sum_{j=-3}^{3}|\nabla w_{jn}|^2)dx=\lim_{n\to\infty}\int_{\mathbb{R}^2}(\sum_{j=-3}^{3}w_{jn}^2)dx=N^*$$

and

$$\lim_{n\to\infty}\int_{\mathbb{R}^2}(\sum_{j=-3}^{3}w_{jn}^2)^2dx=\frac{2(N^*)^2}{a^*}.$$

Moreover, there exists some $j\in\{3,2,1,0,-1,-2,-3\}$, such that

$$\liminf_{n\to\infty}\int_{B_{R_0}(0)}|w_{jn}|^2dx\geq\eta>0. \tag{7.19}$$

It follows that

$$\lim_{n\to\infty}\frac{\int_{\mathbb{R}^2}(\sum\limits_{j=-3}^{3}|\nabla w_{jn}|^2)dx\int_{\mathbb{R}^2}(\sum\limits_{j=-3}^{3}w_{jn}^2)dx}{\int_{\mathbb{R}^2}(\sum\limits_{j=-3}^{3}w_{jn}^2)^2dx}=\frac{a^*}{2}. \tag{7.20}$$

By Lemma 2.2, $\{\mathbf{w_n}\}$ is a minimizing sequence for the following minimization problem:

$$k:=\inf_{(0,0,0,0,0,0,0)\neq\mathbf{u}\in H}K(\mathbf{u}),$$

where

$$K(\mathbf{u}):=\frac{\int_{\mathbb{R}^2}(\sum\limits_{j=-3}^{3}|\nabla u_{jn}|^2)dx\int_{\mathbb{R}^2}(\sum\limits_{j=-3}^{3}u_{jn}^2)dx}{\int_{\mathbb{R}^2}(\sum\limits_{j=-3}^{3}u_{jn}^2)^2dx}.$$

From Lemma 2.2, the minimizer $\mathbf{w}=(w_3,w_2,w_1,w_0,w_{-1},w_{-2},w_{-3})$ must be in form

$$\begin{cases}
w_3(x)=\sqrt{\frac{N^*}{a^*}}Q(x)\cos\varphi_1,\\
w_2(x)=\sqrt{\frac{N^*}{a^*}}Q(x)\sin\varphi_1\cos\varphi_2,\\
w_1(x)=\sqrt{\frac{N^*}{a^*}}Q(x)\sin\varphi_1\sin\varphi_2\cos\varphi_3,\\
w_0(x)=\sqrt{\frac{N^*}{a^*}}Q(x)\sin\varphi_1\sin\varphi_2\sin\varphi_3\cos\varphi_4,\\
w_{-1}(x)=\sqrt{\frac{N^*}{a^*}}Q(x)\sin\varphi_1\sin\varphi_2\sin\varphi_3\cos\varphi_4\cos\varphi_5,\\
w_{-2}(x)=\sqrt{\frac{N^*}{a^*}}Q(x)\sin\varphi_1\sin\varphi_2\sin\varphi_3\cos\varphi_4\cos\varphi_5\cos\varphi_6\\
w_{-3}(x)=\sqrt{\frac{N^*}{a^*}}Q(x)\sin\varphi_1\sin\varphi_2\sin\varphi_3\cos\varphi_4\cos\varphi_5\sin\varphi_6,
\end{cases}$$

for $\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5, \varphi_6 \in [0, \frac{\pi}{2})$. Since $\int_{\mathbb{R}^2} (\sum_{j=-3}^{3} w_j^2) dx = N^*$, we get $\mathbf{w_n} \to \mathbf{w}$ in $L^2(\mathbb{R}^2, \mathbb{R}^7)$. Further, using the interpolation inequality, there holds $\mathbf{w_n} \to \mathbf{w}$ in $L^4(\mathbb{R}^2, \mathbb{R}^7)$. From (7.20), we obtain

$$
\begin{aligned}
&\frac{a^*}{2} \int_{\mathbb{R}^2} (\sum_{j=-3}^{3} w_j^2)^2 dx = N^* \int_{\mathbb{R}^2} (\sum_{j=-3}^{3} |\nabla w_j|^2) dx \leq \lim_{n\to\infty} N_n \int_{\mathbb{R}^2} (\sum_{j=-3}^{3} |\nabla w_{jn}|^2) dx \\
&= \frac{a^*}{2} \lim_{n\to\infty} \int_{\mathbb{R}^2} (\sum_{j=-3}^{3} w_{jn}^2)^2 dx = \frac{a^*}{2} \int_{\mathbb{R}^2} (\sum_{j=-3}^{3} w_j^2)^2 dx,
\end{aligned}
$$

which gives that $\lim_{n\to\infty} \int_{\mathbb{R}^2} (\sum_{j=-3}^{3} |\nabla w_{jn}|^2) dx = \int_{\mathbb{R}^2} (\sum_{j=-3}^{3} |\nabla w_j|^2) dx$, that is, $\mathbf{w_n} \to \mathbf{w}$ in $H^1(\mathbb{R}^2, \mathbb{R}^7)$ as $n \to \infty$. Therefore, there exists some $x_1 \in \mathbb{R}^2$, such that

$$
\begin{cases}
\lim_{n\to\infty} w_{3n}(x) = \sqrt{\frac{N^*}{a^*}} Q(x - x_1) \cos \varphi_1, \\
\lim_{n\to\infty} w_{2n}(x) = \sqrt{\frac{N^*}{a^*}} Q(x - x_1) \sin \varphi_1 \cos \varphi_2, \\
\lim_{n\to\infty} w_{1n}(x) = \sqrt{\frac{N^*}{a^*}} Q(x - x_1) \sin \varphi_1 \sin \varphi_2 \cos \varphi_3, \\
\lim_{n\to\infty} w_{0n}(x) = \sqrt{\frac{N^*}{a^*}} Q(x - x_1) \sin \varphi_1 \sin \varphi_2 \sin \varphi_3 \cos \varphi_4, \\
\lim_{n\to\infty} w_{-1n}(x) = \sqrt{\frac{N^*}{a^*}} Q(x - x_1) \sin \varphi_1 \sin \varphi_2 \sin \varphi_3 \cos \varphi_4 \cos \varphi_5, \\
\lim_{n\to\infty} w_{-2n}(x) = \sqrt{\frac{N^*}{a^*}} Q(x - x_1) \sin \varphi_1 \sin \varphi_2 \sin \varphi_3 \cos \varphi_4 \cos \varphi_5 \cos \varphi_6, \\
\lim_{n\to\infty} w_{-3n}(x) = \sqrt{\frac{N^*}{a^*}} Q(x - x_1) \sin \varphi_1 \sin \varphi_2 \sin \varphi_3 \cos \varphi_4 \cos \varphi_5 \sin \varphi_6,
\end{cases}
\tag{7.21}
$$

for $\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5, \varphi_6 \in [0, \frac{\pi}{2})$.

By direct calculations, we obtain from (7.18) that

$$
\begin{aligned}
&\int_{\mathbb{R}^2} V(x) \mathbf{u_n}^2 dx = \sum_{i=-3}^{3} \int_{\mathbb{R}^2} V(x) \cdot \frac{1}{\varepsilon_n^2} \Big(w_{in} \Big(\frac{x - \varepsilon_n y_n}{\varepsilon_n} \Big) \Big)^2 dx \\
&= \sum_{i=-3}^{3} \int_{\mathbb{R}^2} V(\varepsilon_n x + \varepsilon_n y_n) w_{in}^2(x) dx = \sum_{i=-3}^{3} \varepsilon_n^p \int_{\mathbb{R}^2} V(x + y_n + x_1) w_{in}^2(x + x_1) dx.
\end{aligned}
\tag{7.22}
$$

We now claim $\lim_{n\to\infty} |y_n| \leq C$ for some positive constant C. Otherwise, suppose that $\lim_{n\to\infty} \big| y_n + x_1 \big| = +\infty$, then it follows from (7.22) that for arbitrary $C_1 > 0$, there holds $\int_{\mathbb{R}^2} V(x) |\mathbf{u_n}|^2 dx \geq C_1 \varepsilon_n^P$, as $n \to \infty$.

By (7.16) and (7.17), we have

$$
\begin{aligned}
E(\mathbf{u_n}) \geq & \frac{1}{2}\int_{\mathbb{R}^2}(\sum_{j=-3}^{3}|\nabla u_{jn}|^2)dx - \frac{a^*}{4N_n}\int_{\mathbb{R}^2}(\sum_{j=-3}^{3}u_{jn}^2)^2dx + \frac{1}{2}\int_{\mathbb{R}^2}V(x)(\sum_{j=-3}^{3}u_{jn}^2)dx \\
& + \frac{a^*}{4N_n}\int_{\mathbb{R}^2}(\sum_{j=-3}^{3}u_{jn}^2)^2dx + \Big(\frac{\tau_0+9\tau_1}{4}\Big)\int_{\mathbb{R}^2}(\sum_{j=-3}^{3}u_{jn}^2)^2dx \\
\geq & \frac{1}{2}\int_{\mathbb{R}^2}V(x)(\sum_{j=-3}^{3}u_{jn}^2)dx + \Big(\frac{a^*}{4N_n}+\frac{\tau_0+9\tau_1}{4}\Big)\int_{\mathbb{R}^2}(\sum_{j=-3}^{3}u_{jn}^2)^2dx \\
\geq & \frac{1}{2}C_1\varepsilon_n^p + \Big(\frac{a^*}{2N_n}+\frac{\tau_0+9\tau_1}{2}\Big)\cdot\frac{(N^{**})^2\varepsilon_n^{-2}}{a^*}+o_n(1) \\
= & \frac{1}{2}C_1\varepsilon_n^p + \frac{N^*}{2N_n}(N^*-N_n)\varepsilon_n^{-2}+o_n(1),
\end{aligned}
$$

where $o_n(1)\to 0$ as $n\to\infty$. Taking the infimum with respect to $\varepsilon_n>0$, then we conclude

$$
\lim_{n\to\infty}\frac{m(N_n)}{(N^*-N_n)^{\frac{p}{p+2}}}\geq C_1^{\frac{2}{p+2}}.
$$

However, it contradicts to (7.15). Thus, there exists $x_2\in\mathbb{R}^2$, such that

$$
\lim_{n\to\infty}\big(y_n+x_1\big)=x_2, \tag{7.23}
$$

which yields $\lim\limits_{n\to\infty}|y_n|\leq C$. Therefore, by (7.16), (7.22) and Fatou's Lemma, we have

$$
\begin{aligned}
E(\mathbf{u_n}) \geq & \frac{1}{2}\int_{\mathbb{R}^2}V(x)(\sum_{j=-3}^{3}u_{jn}^2)dx + \Big(\frac{a^*}{4N_n}+\frac{\tau_0+9\tau_1}{4}\Big)\int_{\mathbb{R}^2}(\sum_{j=-3}^{3}u_{jn}^2)^2dx \\
\geq & \frac{1}{2}\frac{N^*\varepsilon_n^p}{a^*}\cdot\int_{\mathbb{R}^2}V(x)Q^2(x)dx+\frac{N^*}{2N_n}(N^*-N_n)\varepsilon_n^{-2}+o_n(1).
\end{aligned} \tag{7.24}
$$

Then taking

$$
\varepsilon_n=\Big(\frac{2a^*(N^*-N_n)}{pN_n\int_{\mathbb{R}^2}V(x)Q^2(x)dx}\Big)^{\frac{1}{p+2}},
$$

we get

$$
\lim_{n\to\infty}\frac{m(N_n)}{(N^*-N_n)^{\frac{p}{p+2}}}\geq\frac{p+2}{2}\Big(\frac{1}{2}\Big)^{\frac{2}{p+2}}\Big(\frac{1}{p}\Big)^{\frac{p}{p+2}}\Big(\frac{N^*\int_{\mathbb{R}^2}V(x)Q^2(x)dx}{a^*}\Big)^{\frac{2}{p+2}}.
$$

Combining with (7.15), we conclude

$$
m(N_n)=\frac{p+2}{2}\Big(\frac{1}{2}\Big)^{\frac{2}{p+2}}\Big(\frac{1}{p}\Big)^{\frac{p}{p+2}}\Big(\frac{N^*\int_{\mathbb{R}^2}V(x)Q^2(x)dx}{a^*}\Big)^{\frac{2}{p+2}}N^*-N_n)^{\frac{2}{p+2}},\quad \text{as } n\to\infty.
$$

Now, we are ready to prove the limit behavior of $\{\mathbf{u_n}\}$ as $n\to\infty$. By (7.10)-(7.13) and the fact that

$$
\lim_{N\nearrow N^*}m(N)=0,
$$

we get

$$\frac{1}{\varepsilon_n^2}\Big\{\lim_{n\to\infty}\frac{\tau_1}{4}\int_{\mathbb{R}^d}(\sqrt{6}(u_{2n}u_{3n}+u_{-2n}u_{-3n})+\sqrt{10}(u_{1n}u_{2n}+u_{-1n}u_{-2n})+2\sqrt{3}(u_{1n}u_{0n}+u_{-1n}u_{0n}))^2dx$$
$$+\lim_{n\to\infty}\frac{\tau_1}{4}\int_{\mathbb{R}^2}\Big(\sum_{j=-3}^{3}(ju_{jn}^2)\Big)^2dx-\lim_{n\to\infty}\frac{9\tau_1}{4}\int_{\mathbb{R}^2}\Big(\sum_{j=-3}^{3}u_{jn}^2\Big)^2dx\Big\}=0.$$

Since $\varepsilon_n\to 0$ as $n\to\infty$, we deduce

$$\begin{aligned}&\frac{\tau_1}{4}\int_{\mathbb{R}^d}(\sqrt{6}(w_2w_3+w_{-2}w_{-3})+\sqrt{10}(w_1w_2+w_{-1}w_{-2})+2\sqrt{3}(w_1w_0+w_{-1}w_0))^2dx\\&+\frac{\tau_1}{4}\int_{\mathbb{R}^2}\Big(\sum_{j=-3}^{3}(jw_j^2)\Big)^2dx-\frac{9\tau_1}{4}\int_{\mathbb{R}^2}\Big(\sum_{j=-3}^{3}w_j^2\Big)^2dx=0.\end{aligned}\tag{7.25}$$

Denoting

$$w_j(x):=c_jQ(x-x_1),\quad j=3,2,1,0,-1,-2,-3,$$

we get from (7.21) that

$$\sum_{j=-3}^{3}c_j^2=\frac{N^*}{a^*},\quad \sum_{j=-3}^{3}(jc_j^2)=\frac{M}{a^*}.$$

Thus $b_j:=\sqrt{\frac{a^*}{N^*}}c_j\in\mathcal{B}$. By (7.25) and Lemma 3.1, we obtain $b_j=b_j^*$ with $N=N^*$ in (3.1) for $j=3,2,1,0,-1,-2,-3$. Hence, we conclude

$$\begin{aligned}&b_3^*=\frac{(3N+M)^3}{216N^3},\ b_{-3}^*=\frac{(3N-M)^3}{216N^3},\ b_0^*=\sqrt{20}\frac{(9N^2-M^2)^{\frac32}}{216N^3},\\&b_2^*=\sqrt{6}\frac{(3N+M)^{\frac52}(3N-M)^{\frac12}}{216N^3},\quad b_{-2}^*=\sqrt{6}\frac{(3N+M)^{\frac12}(3N-M)^{\frac52}}{216N^3},\\&b_1^*=\sqrt{15}\frac{(3N+M)^2(3N-M)}{216N^3},\quad b_{-1}^*=\sqrt{15}\frac{(3N+M)(3N-M)^2}{216N^3}.\end{aligned}\tag{7.26}$$

$$\begin{cases}\lim_{n\to\infty}w_{3n}(x)=\sqrt{\frac{N^*}{a^*}}\frac{(3N^*+M)^3}{216(N^*)^3}Q(x-x_1),\\ \lim_{n\to\infty}w_{2n}(x)=\sqrt{\frac{N^*}{a^*}}\sqrt{6}\frac{(3N^*+M)^{\frac52}(3N^*-M)^{\frac12}}{216(N^*)^3}Q(x-x_1),\\ \lim_{n\to\infty}w_{1n}(x)=\sqrt{\frac{N^*}{a^*}}\sqrt{15}\frac{(3N^*+M)^2(3N^*-M)}{216N^3}Q(x-x_1),\\ \lim_{n\to\infty}w_{0n}(x)=\sqrt{\frac{N^*}{a^*}}\sqrt{20}\frac{(9(N^*)^2-M^2)^{\frac32}}{216(N^*)^3}Q(x-x_1),\\ \lim_{n\to\infty}w_{-1n}(x)=\sqrt{\frac{N^*}{a^*}}\sqrt{15}\frac{(3N^*+M)(3N^*-M)^2}{216(N^*)^3}Q(x-x_1),\\ \lim_{n\to\infty}w_{-2n}(x)=\sqrt{\frac{N^*}{a^*}}\sqrt{6}\frac{(3N^*+M)^{\frac12}(3N^*-M)^{\frac52}}{216(N^*)^3}Q(x-x_1),\\ \lim_{n\to\infty}w_{-3n}(x)=\sqrt{\frac{N^*}{a^*}}\frac{(3N^*-M)^3}{216(N^*)^3}Q(x-x_1).\end{cases}\tag{7.27}$$

Noting that $\mathbf{u_n}$ satisfies the Euler-Lagrange system (1.3), then

$$-(\lambda_n N_n + \mu_n M) = 4E(\mathbf{u_n}) - \int_{\mathbb{R}^2}\Big(\sum_{j=-3}^{3}|\nabla u_{jn}|^2\Big)dx - \int_{\mathbb{R}^2} V(x)(\sum_{j=-3}^{3} u_{jn}^2)dx.$$

By (1.3) and (7.18), $\mathbf{w_n}$ satisfies the following system

$$\begin{cases} -\Delta w_{\pm 3n} + \varepsilon_n^p V(x+y_n)w_{\pm 3n} + \varepsilon_n^2(\lambda \pm 3\mu)w_{\pm 3n} + \tau\rho^2 w_{\pm 3n} + \tau_1\left(\frac{\sqrt{6}}{2}F_x w_{\pm 2n} \pm 3F_z w_{\pm 3n}\right) \\ \quad + \frac{\tau_2}{\sqrt{7}}\mathbf{A}_{0,0}(\mathbf{w}_n)w_{\mp 3n} + \frac{5\tau_3}{2\sqrt{21}}\mathbf{A}_{2,0}(\mathbf{w}_n)w_{\mp 3n} + \frac{5\tau_3}{2\sqrt{21}}\mathbf{A}_{2,\pm 1}(\mathbf{w}_n)w_{\mp 2n} + \frac{\sqrt{10}\tau_3}{2\sqrt{21}}\mathbf{A}_{2,\pm 2}(\mathbf{w}_n)w_{\mp 1n} = 0, \\ -\Delta w_{\pm 2n} + \varepsilon_n^p V(x+y_n)w_{\pm 2n} + \varepsilon_n^2(\lambda \pm 2\mu)w_{\pm 2n} + \tau\rho^2 w_{\pm 2n} + \tau_1\left(\frac{\sqrt{10}}{2}F_x w_{\pm 1n} + \frac{\sqrt{6}}{2}F_x w_{\pm 3n} \pm 2F_z w_{\pm 2n}\right) \\ \quad - \frac{\tau_2}{\sqrt{7}}\mathbf{A}_{0,0}(\mathbf{w}_n)w_{\mp 2n} - \frac{\sqrt{20}\tau_3}{2\sqrt{21}}\mathbf{A}_{2,\pm 2}(\mathbf{w}_n)w_{0n} = 0, \\ -\Delta w_{\pm 1n} + \varepsilon_n^p V(x+y_n)w_{\pm 1n} + \varepsilon_n^2(\lambda \pm \mu)w_{\pm 1n} + \tau\rho^2 w_{\pm 1n} \\ \quad + \tau_1\left(\frac{\sqrt{6}}{2}F_x w_{0n} + \frac{\sqrt{10}}{2}F_x w_{\pm 2n} + \sqrt{3}F_x w_{0n} \pm F_z w_{\pm 1n}\right) \\ \quad + \frac{\tau_2}{\sqrt{7}}\mathbf{A}_{0,0}(\mathbf{w}_n)w_{\mp 1n} - \frac{3\tau_3}{2\sqrt{21}}\mathbf{A}_{2,0}(\mathbf{w}_n)w_{\mp 1n} - \frac{\sqrt{15}\tau_3}{2\sqrt{21}}\mathbf{A}_{2,\mp 1}(\mathbf{w}_n)w_{\mp 2n} \\ \quad + \frac{\sqrt{2}\tau_3}{\sqrt{21}}\mathbf{A}_{2,\pm 1}(\mathbf{w}_n)w_{0n} + \frac{\sqrt{6}\tau_3}{2\sqrt{21}}\mathbf{A}_{2,\pm 2}(\mathbf{w}_n)w_{\pm 1n} = 0, \\ -\Delta w_{0n} + \varepsilon_n^p V(x+y_n)w_{0n} + \varepsilon_n^2\lambda w_{0n} + \tau\rho^2 w_{0n} + \sqrt{3}\tau_1\left(F_x w_{-1n} + F_x w_{1n}\right) \\ \quad - \frac{\tau_2}{\sqrt{7}}\mathbf{A}_{0,0}(\mathbf{w}_n)w_{0n} + \frac{2\tau_3}{\sqrt{21}}\mathbf{A}_{2,0}(\mathbf{w}_n)w_{0n} = 0, \end{cases} \tag{7.28}$$

If we let $\lim_{n\to\infty}\varepsilon_n^2\lambda_n N_n = N_1, \lim_{n\to\infty}\varepsilon_n^2\mu_n M = N_2$, using (7.27) and taking limit on both sides of the first equation and the seventh equation in (7.28), we can deduce that $\lim\limits_{n\to\infty}\mu_n\varepsilon_n^2 = 0$. Therefore

$$\lim_{n\to\infty}\lambda_n\varepsilon_n^2 = 1.$$

The following proof details are similar to the proof of Theorem 2 in [27], we omit it here. □

Acknowledgments

This paper was completed when M.D. Zhen was visiting the Chinese University of Hong Kong. He is grateful to the members in the department of Mathematics at Chinese University of Hong Kong for their invitation and hospitality. The research of J.C. Wei is partially supported by National R&D Program of China (No. 2022YFA1005602), and Hong Kong General Research Fund "New frontiers in singular limits of nonlinear partial differential equation". M.D. Zhen is supported by the National Natural Science Foundation of China (No. 12201167) and the Fundamental Research Funds for the Central Universities (No. JZ2023HGTB0218).

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