A REMARK ON THE CASE-GURSKY-VÉTOIS IDENTITY AND ITS APPLICATIONS

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ABSTRACT. Based on the works of Gursky (CMP, 1997), Vétois (Potential Anal., 2023) and Case (Crelle's journal, 2024), we make use of an Obata type formula established in these works to obtain some Liouville type theorems on conformally Einstein manifolds. In particular, we solve Hang-Yang conjecture (IMRN, 2020) via an Obata-type argument and obtain optimal perturbation.

1. INTRODUCTION

Given a Riemannian manifold (M^n, g) with $n \geq 3$, it is well known that Branson's Q-curvature [2] is defined by

$$Q_g := -\frac{1}{2(n-1)}\Delta_g R_g - \frac{2}{(n-2)^2}|E_g|_g^2 + \frac{n^2-4}{8n(n-1)^2}R_g^2 \tag{1.1}$$

where R_g is the scalar curvature and E_g is the trace-free Ricci tensor defined by $E_g := Ric_g - \frac{R_g}{n}g$. Now, we give some notations and a brief overview of Q-curvature in conformal geometry. The Schoten tensor is given by

$$A_g = \frac{1}{n-2}\left(Ric_g - \frac{R_g}{2(n-1)}g\right)$$

and $\sigma_k(A_g)$ denote the k-th symmetric functions of the eigenvalues of A_g. Then we can rewrite Q-curvature in (1.1) as follows

$$Q_g = -\Delta_g\sigma_1(A_g) + 4\sigma_2(A_g) + \frac{n-4}{2}\sigma_1^2(A_g) \tag{1.2}$$

The remarkable Paneitz operator [17] is given by

$$P_g\varphi = \Delta_g^2\varphi + div_g\left\{(4A_g - (n-2)\sigma_1(A_g)g)(\nabla\varphi, \cdot)\right\} + \frac{n-4}{2}Q_g\varphi. \tag{1.3}$$

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For $n \geq 3$ and $n \neq 4$, the Q-curvature $Q_{\tilde{g}}$ of the conformal metric $\tilde{g} = \varphi^{\frac{4}{n-4}} g$ satisfies

$$P_g \varphi = \frac{n-4}{2} Q_{\tilde{g}} \varphi^{\frac{n+4}{n-4}}. \tag{1.4}$$

For $n = 4$, the Q-curvature $Q_{\tilde{g}}$ of the conformal metric $\tilde{g} = e^{2\varphi} g$ satisfies

$$P_g \varphi + Q_g = Q_{\tilde{g}} e^{4\varphi}. \tag{1.5}$$

Similar to the Yamabe problem, it is crucial to identify a conformal metric such that the Q-curvature is constant, which is equivalent to solving the fourth-order nonlinear equations (1.4) and (1.5) with $Q_{\tilde{g}} \equiv C$ for some constant C. For $n = 4$, significant advancements were made by Chang and Yang [5] and Djadli and Malchiodi [6]. In dimensions $n \geq 5$, Gursky and Malchiodi [10] and Hang and Yang [12] established existence results under suitable conditions related to scalar curvature. For $n = 3$, a similar result was achieved by Hang and Yang [13].

The well known Obata theorem ([15], [16]) states that for an Einstein manifold (M^n, g_0) with $n \geq 3$, if the scalar curvature of conformal metric $g = u^2 g_0$ is a constant, g must also be Einstein. Furthermore, if (M, g_0) is not conformally equivalent to round sphere, u must be a positive constant.

For the Einstein manifold (M^n, g_0) with scalar curvature R_0, the Paneitz operator can be succinctly expressed as follows (See [8] for more details)

$$P_{g_0} \varphi = \Delta_{g_0}^2 \varphi - \frac{n^2 - 2n - 4}{2n(n-1)} R_0 \Delta_{g_0} \varphi + \frac{n-4}{2} Q_0 \varphi \tag{1.6}$$

where the Q-curvature Q_0 is given by

$$Q_0 = \frac{n^2 - 4}{8n(n-1)^2} R_0^2. \tag{1.7}$$

It is natural to ask whether similar Obata type theorem holds for Q-curvature. Firstly, Vétois [18] made use of BochnerLichnerowiczWeitzenbock formula and a important Lemma 3.1 established by Gursky and Malchiodi [10] to deal with the equations (1.4) and (1.5). Then he establishes the following Obata type theorem.

Theorem 1.1. *(Vétois' theorem in* [18]*) Suppose that (M^n, g_0) where $n \geq 3$ is a compact Einstein manifold with non-negative scalar curvature R_0. Consider a conformal metric $g = u^2 g_0$ where $u > 0$. Suppose that the Q-curvature of conformal metric g is constant. Then g is Einstein. Furthermore, if (M, g_0) is not conformally equivalent to round sphere, u must be a positive constant.*

Remark 1.2. In original version of Vétois's theorem(Theorem 1.1 in [18]), he didn't cover the case $R_0 = 0$. In fact, by integrating (1.4) and (1.5) using the representation of (1.6), it is not hard to include this case.

Recently, Case [4] consider a more general Obata-Vétois type theorem. For a compact manifold (M^n, g), he introduced the following mixed curvature

$$I_a(g) = Q_g + a\sigma_2(A_g) \tag{1.8}$$

where Q_g and A_g are the Q-curvature and Schouten tensor respectively given before.

Theorem 1.3. *(Case's theorem in [4]) Suppose that (M^n, g_0) is an Einstein manifolds with the scalar curvature $R_0 \geq 0$ and $n \geq 3$. Consider the conformal metric $g = u^2 g_0$ where $u > 0$ satisfying $R_g \geq 0$ and $I_a(g)$ is a constant. If the constant a satisfies*

$$\frac{(n-2)(n-4) - n\sqrt{n^2+4n}}{2(n-1)} \leq a \leq \frac{(n-2)(n-4) + n\sqrt{n^2+4n}}{2(n-1)}, \tag{1.9}$$

then g is Einstein.

Remark 1.4. In original version of Case's theorem, he considered the following interval

$$B_1 := [\frac{n^2 - 7n + 8 - \sqrt{n^4 + 2n^3 - 3n^2}}{2(n-1)}, \frac{n^2 - 7n + 8 + \sqrt{n^4 + 2n^3 - 3n^2}}{2(n-1)}].$$

It is easy to check that

$$B_1 \subset [\frac{(n-2)(n-4) - n\sqrt{n^2+4n}}{2(n-1)}, \frac{(n-2)(n-4) + n\sqrt{n^2+4n}}{2(n-1)}].$$

In fact, with the help of the sharp inequality Lemma 2.3, we slightly extend the range of the constant a. However, we still do not know how to get the optimal range which is a very interesting question.

To prove Theorem 1.3, Case established a remarkable identity (See Lemma 3.2 in [4]) as follows

$$\begin{aligned} 0 =& \frac{1}{2}\int_M u|\nabla R_g|^2 \mathrm{d}v_g + \frac{n-(n-1)(a+4)}{n-2}\int_M E_g(\nabla R_g, \nabla u)\mathrm{d}v_g \\ &+ \frac{n(n-1)^2(a+4)}{2(n-2)^2}\int_M |E_g|^2 u^{-1}|\nabla u|^2 \mathrm{d}v_g \\ &+ \frac{(n-1)(a+4)+2n^2-4n}{2(n-2)^2}\int_M u|E_g|^2 R_g \mathrm{d}v_g \\ &+ \frac{(n-1)(a+4)R_0}{2(n-2)^2}\int_M |E_g|^2 u^{-1}\mathrm{d}v_g \end{aligned} \tag{1.10}$$

under the assumption that $I_a(g)$ is a constant. Such formula is also obtained by Gursky(See the equation (1.13) in [9]) on $\mathbb{S}^4$ and Vétois (See equation (1.4) in [18]) for $a = 0$.

In [18], Vétois established a more general Liouville-type theorem by considering the following equations

$$P_{g_0}\varphi=\varphi^p,\quad n\neq 4,\ p\leq\frac{n+4}{n-4}, \tag{1.11}$$

$$P_{g_0}\varphi=e^{p\varphi},\quad n=4,\ p\leq 4, \tag{1.12}$$

on compact Einstein manifolds. He showed that the positive solutions to (1.11) and smooth solutions to (1.12) must be constant if (M^n,g) is not conformally equivalent to the round sphere.

Based on Case's formula (1.10) and Gursky's work [9], we observe that we can add the term $I_g(g)$ into the formula (1.10) without assuming $I_a(g)$ is a constant. Such formula will play an important role in the proof of Liouville type theorems. On the whole, the Case-Gursky-Vétois formula can be written as follows

$$\begin{aligned}
&2(n-1)^2\int_M\langle\nabla I_a(g),\nabla u\rangle_g\mathrm{d}v_g\\
=&\frac{1}{2}\int_M u|\nabla R_g|^2\mathrm{d}v_g+\frac{n-(n-1)(a+4)}{n-2}\int_M E_g(\nabla R_g,\nabla u)\mathrm{d}v_g\\
&+\frac{n(n-1)^2(a+4)}{2(n-2)^2}\int_M|E_g|^2u^{-1}|\nabla u|^2\mathrm{d}v_g\\
&+\frac{(n-1)(a+4)+2n^2-4n}{2(n-2)^2}\int_M u|E_g|^2R_g\mathrm{d}v_g\\
&+\frac{(n-1)(a+4)R_0}{2(n-2)^2}\int_M|E_g|^2u^{-1}\mathrm{d}v_g.
\end{aligned} \tag{1.13}$$

With help of such formula, we are able to streamline the proof of the Liouville-type theorems of Vétois(See Theorem 2.1 and Theorem 2.2 in [18]). Besides, we can generalize the Liouville-type results from [1], [3], and [7] related to second-order nonlinear equations to fourth-order cases by considering the perturbation of linear term of Paneitz operator. Since the situations differ slightly for $n\geq 5, n=4$ and $n=3$, we establish the results for each case separately.

Theorem 1.5. *Suppose that (M^n,g_0) where $n\geq 5$ is a compact Einstein manifold with positive scalar curvature R_0. Consider the positive solution φ to the equation*

$$P_{g_0}\varphi-\varepsilon\varphi=\varphi^p \tag{1.14}$$

where $0\leq\varepsilon<\frac{n-4}{2}Q_0$ and $p\leq\frac{n+4}{n-4}$. If (M^n,g_0) is conformally equivalent to round sphere, we additionally assume that $\varepsilon+\frac{n+4}{n-4}-p>0$. Then φ must be a constant.

Remark 1.6. By integrating the equation (1.14) over (M,g_0), it is easy to see that the condition $\varepsilon<\frac{n-4}{2}Q_0$ is necessary for $\varphi>0$.

Theorem 1.7. *Suppose that (M^4, g_0) is a compact Einstein manifold with positive scalar curvature R_0. Consider the solution φ to the equation*

$$P_{g_0}\varphi + Q_0 - \varepsilon = e^{p\varphi} \tag{1.15}$$

where $0 \le \varepsilon < Q_0$ and $p \le 4$. If (M^4, g_0) is conformally equivalent to round sphere, we additionally assume that $\varepsilon + 4 - p > 0$. Then φ must be a constant.

Theorem 1.8. *Suppose that (M^3, g_0) is a compact Einstein manifold with positive scalar curvature R_0. Consider the positive solution φ to the equation*

$$P_{g_0}\varphi + \varepsilon\varphi = -\varphi^p \tag{1.16}$$

where $0 \le \varepsilon < \frac{Q_0}{2}$ and $p \ge -7$. If (M^3, g_0) is conformally equivalent to round sphere, we additionally assume that $\varepsilon + 7 + p > 0$. Then φ must be a constant.

This result recovers Conjecture 1.1 of Hang and Yang [11], which addresses the standard sphere $\mathbb{S}^3$ via an Obata-type proof. Firstly, Zhang [19] solved this conjecture by transforming the equation on the sphere into Euclidean space and applying the moving plane method. Later, Hyder and Ng [14] generalized this theorem to higher order cases by using similar moving plane method.

We should point out that their approaches can only handle small values of ε since their proofs need a compactness theorem. When $p = -7$ and (M^3, g_0) is the round sphere $\mathbb{S}^3$, Theorem 1.8 establishes this Liouville-type theorem under the optimal range $0 < \varepsilon < \frac{Q_0}{2}$ via an Obata type argument.

In fact, with the help of the strong maximum principle established by Gursky and Malchiodi (Theorem 2.2 in [10]) and Hang-Yang (Proposition 2.1 in [13]), we are able to establish a more general result as below.

Theorem 1.9. *Suppose that (M^n, g_0) where $n \ge 3$ and $n \ne 4$ is a compact Einstein manifold with positive scalar curvature R_0. Consider the solution $\varphi \in C^4(M^n, g_0)$ to the following equation*

$$P_{g_0}\varphi = \frac{n-4}{2} f(\varphi)$$

where $f(t) \ge 0$ for all $t \in \mathbb{R}$ is a smooth function satisfying

$$\frac{n-4}{2}\partial_t\left(t^{-\frac{n+4}{n-4}} f(t)\right) \le 0, \quad \forall\, t > 0. \tag{1.17}$$

If (M^n, g_0) is conformally equivalent to the round sphere, we additionally assume that the inequality in (1.17) is strict. Then φ must be a constant.

For $n = 4$, a similar result holds.

Theorem 1.10. *Suppose that (M^4, g_0) is a four-dimensional compact Einstein manifold with positive scalar curvature R_0. Consider the solution $\varphi \in C^4(M^4, g_0)$ to the following equation*

$$P_{g_0}\varphi + Q_0 = f(e^{\varphi})$$

where $f(t)\geq 0$ for all $t>0$ is a smooth function satisfying

$$\partial_t\left(t^{-4}f(t)\right)\leq 0,\quad \forall\, t>0. \tag{1.18}$$

If (M^4,g_0) is conformally equivalent to the round sphere, we additionally assume that the inequality in (1.18) is strict. Then φ must be a constant.

This paper is organized as follows. In Section 2, we prove the Case-Gursky-Vétois formula by following the argument of Gursky [9] and Case [4]. With the help of such identity, we establish two Case-Gursky-Vétois rigidity inequalities. Finally, in Section 3, with the help of rigidity inequalities, we give the proofs of Theorem 1.5, Theorem 1.7, Theorem 1.8, Theorem 1.9 and Theorem 1.10.

2. Case-Gursky-Vétois identity on conformal Einstein manifolds

Before doing so, we introduce some notations for later use. Notice that

$$\sigma_2(A_g)=-\frac{|E_g|_g^2}{2(n-2)^2}+\frac{1}{8(n-1)n}R_g^2.$$

Then, with help of (1.1) and (1.8), one has

$$2(n-1)I_a(g)=-\Delta_g R_g-\alpha_1|E_g|_g^2+\alpha_2R_g^2 \tag{2.1}$$

where there constants α_1 and α_2 are defined as follows

$$\alpha_1=\frac{(n-1)(4+a)}{(n-2)^2},\quad \alpha_2=\frac{(n-1)a+n^2-4}{4(n-1)n}. \tag{2.2}$$

Now, we are going to give give the proof of Case-Gursky-Vétois identity (1.13). In the proof of this formula, we basically follow Case [4], using idea of Gursky [9]. The key observation during Gursky's proof (1997, CMP, Page 660) is that integrating $u\langle E_g,\nabla^2R_g\rangle_g$ over (M,g), then insert the representations of $\Delta_g u$ and $\Delta_g R_g$. Here, we needt assume that $I_a(g)$ is a constant. Then we obtain an integral identity (Theorem 2.1). We should point out that this identity can also be obtained by inserting the tensor T defined in Page 4 of [4] into the equation (1.6) without assuming $I_a(g)$ is a constant.

Theorem 2.1. *(Case-Gursky-Vétois identity) Suppose that (M^n,g_0) is an Einstein manifold with constant scalar curvature R_0 and $n\geq 3$. Consider a conformal metric $g=u^2g_0$ where $u>0$. Then there holds*

$$\begin{aligned}&2(n-1)^2\int_M\langle\nabla I_a(g),\nabla u\rangle_g \mathrm{d}v_g\\ =&\frac{1}{2}\int_M u|\nabla R_g|^2\mathrm{d}v_g+\frac{n-(n-1)(a+4)}{n-2}\int_M E_g(\nabla R_g,\nabla u)\mathrm{d}v_g\\ &+\frac{n(n-1)^2(a+4)}{2(n-2)^2}\int_M|E_g|^2u^{-1}|\nabla u|^2\mathrm{d}v_g\end{aligned}$$

$$+\frac{(n-1)(a+4)+2n^2-4n}{2(n-2)^2}\int_M u|E_g|^2R_g\mathrm{d}v_g$$
$$+\frac{(n-1)(a+4)R_0}{2(n-2)^2}\int_M |E_g|^2u^{-1}\mathrm{d}v_g.$$

Proof. Since g_0 is an Einstein metric, a direct computation yields that

$$u(E_g)_{ij}=-(n-2)\left(\nabla^2_{ij}u-\frac{\Delta_g u}{n}g_{ij}\right) \tag{2.3}$$

and

$$\Delta_g u=\frac{n}{2}u^{-1}|\nabla_g u|^2-\frac{R_g u}{2(n-1)}+\frac{u^{-1}R_0}{2(n-1)}. \tag{2.4}$$

Multiplying $\nabla^2_{ij}R_g$ on both sides of the equations (2.3) and integrating it over (M,g), then there holds

$$\int_M u\langle E_g,\nabla^2R_g\rangle_g\mathrm{d}v_g=-(n-2)\int_M\langle\nabla^2u,\nabla^2R_g\rangle_g\mathrm{d}v_g+\frac{n-2}{n}\int_M\Delta_g u\cdot\Delta_g R_g\mathrm{d}v_g \tag{2.5}$$

We are going to deal with these three terms one by one.

Firstly, with the help of second Bianchi identity $\nabla_i E_{ij}=\frac{n-2}{2n}\nabla_j R$ and integration by parts, one has

$$\int_M u\langle E_g,\nabla^2R_g\rangle_g\mathrm{d}v_g=-\frac{n-2}{2n}\int_M u|\nabla R_g|^2_g\mathrm{d}v_g-\int_M E_g(\nabla u,\nabla R_g)\mathrm{d}v_g. \tag{2.6}$$

Secondly, integrating by parts, one has

$$\begin{aligned}&\int_M\langle\nabla^2u,\nabla^2R_g\rangle_g\mathrm{d}v_g\\ =&-\int_M\langle\nabla R_g,\nabla\Delta_g u\rangle_g\mathrm{d}v_g-\int_M Ric_g(\nabla u,\nabla R_g)\mathrm{d}v_g\\ =&\int_M\Delta_g R_g\cdot\Delta_g u\mathrm{d}v_g-\int_M E_g(\nabla u,\nabla R_g)\mathrm{d}v_g-\frac{1}{n}\int_M R_g\langle\nabla R_g,\nabla u\rangle_g\mathrm{d}v_g.\end{aligned}$$

Combing these estimates, we can rewrite (2.5) as follows

$$\begin{aligned}0=&-\frac{1}{2n}\int_M u|\nabla R_g|^2\mathrm{d}v_g-\frac{n-1}{n-2}\int_M E_g(\nabla u,\nabla R_g)\mathrm{d}v_g\\ &+\frac{n-1}{n}\int_M\Delta_g R_g\cdot\Delta_g u\mathrm{d}v_g-\frac{1}{n}\int_M R_g\langle\nabla R_g,\nabla u\rangle_g\mathrm{d}v_g\end{aligned} \tag{2.7}$$

With the help of the identities (2.1) and (2.4), one has

$$\begin{aligned}&\int_M\Delta_g R_g\cdot\Delta_g u\mathrm{d}v_g\\ =&\int_M\left(-2(n-1)I_a-\alpha_1|E_g|^2+\alpha_2R_g^2\right)\cdot\Delta_g u\mathrm{d}v_g\\ =&2(n-1)\int_M\langle\nabla I_a,\nabla u\rangle_g\mathrm{d}v_g-\alpha_1\int_M|E_g|^2\left(\frac{n}{2}u^{-1}|\nabla u|^2-\frac{R_g u}{2(n-1)}+\frac{R_0u^{-1}}{2(n-1)}\right)\mathrm{d}v_g\end{aligned}$$

$$- 2\alpha_2 \int_M R_g \langle \nabla R_g, \nabla u \rangle_g \mathrm{d}v_g.$$

Inserting the above formula into (2.7), we obtain that

$$\begin{aligned}
&2(n-1)^2 \int_M \langle \nabla I_a, \nabla u \rangle_g \mathrm{d}v_g \\
=&\frac{1}{2} \int_M u|\nabla R_g|^2 \mathrm{d}v_g + \frac{n(n-1)}{n-2} \int_M E_g(\nabla R_g, \nabla u) \mathrm{d}v_g \\
&+ \frac{n(n-1)\alpha_1}{2} \int_M |E_g|^2 u^{-1} |\nabla u|^2 \mathrm{d}v_g - \frac{\alpha_1}{2} \int_M |E_g|^2 R_g u \mathrm{d}v_g \\
&+ \frac{\alpha_1 R_0}{2} \int_M |E_g|^2 u^{-1} \mathrm{d}v_g \\
&+ (2(n-1)\alpha_2 + 1) \int_M R_g \langle \nabla R_g, \nabla u \rangle_g \mathrm{d}v_g
\end{aligned}$$

Multiplying $R_g E_{ij}$ on both sides of (2.3) and integrating by parts, we obtain that

$$\begin{aligned}
&\int_M u R_g |E_g|^2 \mathrm{d}v_g \\
=&-(n-2) \int_M R_g \langle E_g, \nabla^2 u \rangle_g \mathrm{d}v_g \\
=&\frac{(n-2)^2}{2n} \int_M R_g \langle \nabla R_g, \nabla u \rangle_g \mathrm{d}v_g + (n-2) \int_M E_g(\nabla R_g, \nabla u) \mathrm{d}v_g
\end{aligned}$$

which is equivalent to

(2.8)

$$\int_M R_g \langle \nabla R_g, \nabla u \rangle_g \mathrm{d}v_g = \frac{2n}{(n-2)^2} \int_M u R_g |E_g|^2 \mathrm{d}v_g - \frac{2n}{n-2} \int_M E_g(\nabla R_g, \nabla u) \mathrm{d}v_g$$

Inserting the identity (2.8) and the notations (2.2) into (2.7), there holds

$$\begin{aligned}
&2(n-1)^2 \int_M \langle \nabla I_a, \nabla u \rangle_g \mathrm{d}v_g \\
=&\frac{1}{2} \int_M u|\nabla R_g|^2 \mathrm{d}v_g + \frac{n-(n-1)(a+4)}{n-2} \int_M E_g(\nabla R_g, \nabla u) \mathrm{d}v_g \\
&+ \frac{n(n-1)^2(a+4)}{2(n-2)^2} \int_M |E_g|^2 u^{-1} |\nabla u|^2 \mathrm{d}v_g \\
&+ \frac{(n-1)(a+4) + 2n^2 - 4n}{2(n-2)^2} \int_M u |E_g|^2 R_g \mathrm{d}v_g \\
&+ \frac{(n-1)(a+4)R_0}{2(n-2)^2} \int_M |E_g|^2 u^{-1} \mathrm{d}v_g.
\end{aligned}$$

Thus we finish our proof.

□

With the help of above identity, we are able to establish the following Case-Gursky-Vétois inequalities.

Corollary 2.2. *Suppose that (M^n, g_0) where $n \geq 3$ is a compact Einstein manifold with constant scalar curvature R_0. Consider a conformal metric $g = u^2 g_0$ where $u > 0$. Then there holds*

$$(n-2)^2(n-1)\int_M \langle \nabla Q_g, \nabla u\rangle_g \mathrm{d}v_g \geq \frac{n^2-2}{2(n-1)}\int_M u|E_g|_g^2 R_g \mathrm{d}v_g + R_0\int_M |E_g|_g^2 u^{-1}\mathrm{d}v_g$$

with the equality holds if and only if g is Einstein.

Proof. Choosing $a = 0$ in Theorem 2.1, one has

$$\begin{aligned}
(2.9)\qquad & 2(n-1)^2\int_M \langle \nabla Q_g, \nabla u\rangle_g \mathrm{d}v_g \\
=& \frac{1}{2}\int_M u|\nabla R_g|^2 \mathrm{d}v_g + \frac{4-3n}{n-2}\int_M E_g(\nabla R_g, \nabla u)\mathrm{d}v_g \\
&+ \frac{2n(n-1)^2}{(n-2)^2}\int_M |E_g|^2 u^{-1}|\nabla u|^2 \mathrm{d}v_g \\
&+ \frac{n^2-2}{(n-2)^2}\int_M u|E_g|^2 R_g \mathrm{d}v_g + \frac{2(n-1)R_0}{(n-2)^2}\int_M |E_g|^2 u^{-1}\mathrm{d}v_g.
\end{aligned}$$

With help of Cauchy inequality and Young's inequality, one has

$$\frac{4-3n}{n-2}E_g(\nabla R_g, \nabla u) \geq -\frac{2n(n-1)^2}{(n-2)^2}|E_g|^2 u^{-1}|\nabla u|^2 - \frac{(3n-4)^2}{8n(n-1)^2}u|\nabla R_g|^2.$$

Inserting it into the above identity (2.9), one has

$$\begin{aligned}
(2.10)\qquad & 2(n-1)^2\int_M \langle \nabla Q_g, \nabla u\rangle_g \mathrm{d}v_g \\
\geq& \frac{4(n-4)(n-1)^2 + 7n^2 - 8n}{8n(n-1)^2}\int_M u|\nabla R_g|^2 \mathrm{d}v_g \\
&+ \frac{n^2-2}{(n-2)^2}\int_M u|E_g|^2 R_g \mathrm{d}v_g + \frac{2(n-1)R_0}{(n-2)^2}\int_M |E_g|^2 u^{-1}\mathrm{d}v_g.
\end{aligned}$$

Immediately, one has

(2.11)

$$(n-2)^2(n-1)\int_M \langle \nabla Q_g, \nabla u\rangle_g \mathrm{d}v_g \geq \frac{n^2-2}{2(n-1)}\int_M u|E_g|_g^2 R_g \mathrm{d}v_g + R_0\int_M |E_g|_g^2 u^{-1}\mathrm{d}v_g.$$

On one hand, if the equality holds in (2.11), with the help of the inequality (2.10), one must have $|\nabla R_g| = 0$ which means that the scalar curvature R_g is a constant. Then Obata theorem shows that g is Einstein. On the other hand, if g is Einstein, it is not hard to check that the equality is achieved by (1.7).

Thus we finish our proof.

□

To slightly extend Case's Theorem 1.3, we need the following well known lemma. For reader's convenience, we sketch the proof here.

Lemma 2.3. *Let A be a $n\times n$ symmetric matrix satisfying trace$(A)=0$. Let x and y be two $n\times 1$ vector. There holds*

$$|x^TAy|\leq\sqrt{\frac{n-1}{n}}|A|\cdot|x|\cdot|y|.$$

Proof. With the help of orthogonality, one may assume that A is a diagonal matrix and let λ_i be the diagonal elements where $1\leq i\leq n$ and $|\lambda_i|\leq|\lambda_1|$ for all $1\leq i\leq n$. Notice that

$$|A|^2=\sum_i^n\lambda_i\geq\lambda_1^2+\frac{1}{n-1}\left(\sum_{i=2}^n\lambda_i\right)^2=\frac{n}{n-1}\lambda_1^2. \tag{2.12}$$

With the help of (2.12) and Cauchy inequality, one has

$$|x^TAy|=|\sum_i\lambda_ix_iy_i|\leq|\lambda_1|\cdot|x|\cdot|y|\leq\sqrt{\frac{n-1}{n}}|A|\cdot|x|\cdot|y|.$$

□

Corollary 2.4. *Suppose that (M^n,g_0) where $n\geq3$ is a compact Einstein manifolds with constant scalar curvature R_0 and $n\geq3$. Consider the conformal metric $g=u^2g_0$ where $u>0$. If the constant a satisfies*

$$\frac{(n-2)(n-4)-n\sqrt{n^2+4n}}{2(n-1)}\leq a\leq\frac{(n-2)(n-4)+n\sqrt{n^2+4n}}{2(n-1)}, \tag{2.13}$$

then there holds

$$\int_M\langle\nabla I_a(g),\nabla u\rangle_g\mathrm{d}v_g\geq a_1\int_M u|E_g|^2R_g\mathrm{d}v_g+a_2R_0\int_M|E_g|^2u^{-1}\mathrm{d}v_g.$$

where $a_1(n)=\frac{(n-1)(a+4)+2n^2-4n}{4(n-2)^2(n-1)^2}$ and $a_2=\frac{a+4}{4(n-2)^2(n-1)}$.

Proof. With the help of Lemma 2.3 and Cauchy-Schwarz inequality as well as Young's inequality, there holds

$$\begin{aligned}&\frac{n-(n-1)(a+4)}{n-2}\int_M E_g(\nabla R_g,\nabla u)\mathrm{d}v_g\\ \geq&-|\frac{n-(n-1)(a+4)}{n-2}|\sqrt{\frac{n-1}{n}}\int_M|E_g|\cdot|\nabla R_g|\cdot|\nabla u|\mathrm{d}v_g\\ \geq&-\frac{1}{2}\int_M u|\nabla R_g|^2\mathrm{d}v_g-\frac{\left(n-(n-1)(a+4)\right)^2(n-1)}{2(n-2)^2n}\int_M|E_g|^2u^{-1}|\nabla u|^2.\end{aligned}$$

Using the condition (2.13) and Theorem 2.1, we obtain our desired result.

□

For reader's convenience, we repeat the proof of Vétois' Theorem 1.1 and Case's Theorem 1.3 via the Case-Gursky-Vétois identity.

Proof of Theorem 1.1:

When the scalar curvature $R_0 = 0$, then Q-curvature of g_0 also vanishes by (1.7). For $n \geq 3$ and $n \neq 4$, if the Q-curvature of conformal metric of $g = \varphi^{\frac{4}{n-4}} g_0$ is constant, the equation (1.6) yields that

$$\Delta_{g_0}^2 \varphi = C\varphi^{\frac{n+4}{n-4}} \tag{2.14}$$

where C is a constant. By integrating (2.14) over (M, g_0), it is easy to say that the constant C must be zero. Immediately, $\Delta_{g_0}\varphi$ must be a constant and such constant must be zero by integrating it over (M, g_0) again. Then φ must be a constant. For $n = 4$, consider $g = e^{2\varphi} g_0$ and the same argument yields that φ must be a constant.

When $R_0 > 0$, with the help of (1.1), one has

$$Q_{g_0} = \frac{n^2 - 4}{8n(n-1)^2} R_0^2 > 0.$$

If the Q-curvature Q_g of the conformal metric $g = u^2 g_0$ is constant. By integrating the equations (1.4) and (1.5) over (M, g_0), it is easy to see that the constant $Q_g > 0$. Apply Lemma 3.1, the scalar curvature R_g is positive. Making use of Corollary 2.2, one has

$$0 \geq \frac{n^2 - 2}{2(n-1)} \int_M u|E_g|_g^2 R_g \mathrm{d}v_g + R_0 \int_M |E_g|_g^2 u^{-1} \mathrm{d}v_g.$$

Since u, R_g, R_0 are all positive, one must have $E_g \equiv 0$ which means that g is Einstein. If Furthermore, if (M, g_0) is not conformally equivalent to round sphere, Obata theorem yields that u must be a positive constant.

Proof of Theorem 1.3:

If $R_0 = 0$, one has $-\frac{4(n-1)}{n-2}\Delta_0 u^{\frac{n-2}{2}} = R_g u^{\frac{n+2}{2}} \geq 0$. Integrating it over (M^n, g_0), the left side vanishes and then we must have $R_g \equiv 0$. Immediately, u must be a constant. Otherwise, $R_0 > 0$, Corollary 2.4 yields that $E_g = 0$ which means that g is Einstein.

3. Applications

With the help of a continuity method and the maximum principle, Gursky and Malchiodi [10] establish an important theorem related to the positivity of Q-curvature and scalar curvature in conformal classes of the metrics.

Lemma 3.1. *(See Theorem 2.2 in* [10]*, Proposition 2.1 in* [13]*, Theorem 2.3 in* [18]*) Given a compact manifold (M^n, g) with $n \geq 3$ with positive scalar* *curvature and non-negative Q-curvature. Consider the conformal metric $\tilde{g} = u^2 g$. If the Q-curvature $Q_{\tilde{g}} \geq 0$, then the scalar curvature $R_{\tilde{g}}$ is positive.*

Proof of Theorem 1.5:

For $n \geq 5$, consider the conformal metric $g = \varphi^{\frac{4}{n-4}} g_0$ and then

$$Q_g = \frac{2}{n-4}\left(\epsilon \varphi^{-\frac{8}{n-4}} + \varphi^{p-\frac{n+4}{n-4}}\right) > 0.$$

Lemma 3.1 yields that $R_g > 0$.

Choosing $u = \varphi^{\frac{2}{n-4}}$ in Corollary 2.2, one has

$$\begin{aligned}\langle \nabla Q_g, \nabla u\rangle_g =& -\frac{32\varepsilon}{(n-4)^3}\varphi^{-\frac{6}{n-4}-2}|\nabla \varphi|_g^2 - \frac{4}{(n-4)^2}(\frac{n+4}{n-4} - p)\varphi^{p-\frac{n+2}{n-4}-2}|\nabla \varphi|_g^2 \\ =& C(\varphi, \varepsilon, p, n)|\nabla \varphi|_g^2\end{aligned}$$

where

$$C(\varphi, \varepsilon, p, n) = -\frac{32\varepsilon}{(n-4)^3}\varphi^{-\frac{6}{n-4}-2} - \frac{4}{(n-4)^2}(\frac{n+4}{n-4} - p)\varphi^{p-\frac{n+2}{n-4}-2}.$$

By using Corollary 2.2 and the facts the scalar curvatures R_g and R_0 are both positive, we know that

$$\int_M C(\varphi, \varepsilon, p, n)|\nabla \varphi|_g^2 \mathrm{d}v_g \geq 0. \tag{3.1}$$

If $\epsilon + \frac{n+4}{n-4} - p > 0$, it is easy to check that $C(\varphi, \varepsilon, p, n) < 0$ based on our assumptions. Immediately, by using (3.1), one has $|\nabla \varphi|_g = 0$ which means that φ must be a constant. If $\epsilon = 0$ and $p = \frac{n+4}{n-4}$, due to our assumption, (M^n, g_0) is not conformally equivalent to a round sphere. In this case, one has

$$C(\varphi, \varepsilon, p, n) = 0,$$

Corollary 2.2 yields that

$$0 \geq \int_M u|E_g|_g^2 R_g \mathrm{d}v_g + R_0 \int_M |E_g|_g^2 u^{-1} \mathrm{d}v_g.$$

Noticing that $R_g > 0$ and $R_0 > 0$, one has $E_g \equiv 0$ which yields that g has constant scalar curvature. Immediately, Obata theorem deuces that φ must be a constant.

Proof of Theorem 1.7: For $n = 4$, consider the conformal metric $g = e^{2\varphi} g_0$ and Q-curvature satisfies

$$Q_g = \varepsilon e^{-4\varphi} + e^{(p-4)\varphi} > 0.$$

Then Lemma 3.1 yields that $R_g > 0$. Then choosing $u = e^{\varphi}$, one may easily check that

$$\langle \nabla Q_g, \nabla u\rangle_g = -4\varepsilon e^{-3\varphi}|\nabla \varphi|_g^2 - (4-p)e^{(p-3)\varphi}|\nabla \varphi|_g^2 \leq 0.$$

Similarly as the proof of Theorem 1.5, φ must be a constant.

Proof of Theorem 1.8:

For $n=3$, consider the conformal metric $g=\varphi^{-7}g_0$ and Q-curvature satisfies

$$Q_g = 2\varepsilon\varphi^8 + 2\varphi^{p+7} > 0$$

which yields that $R_g > 0$ by Lemma 3.1. Choose $u=\varphi^{-2}$ and then a direct computation yields that

$$\langle \nabla Q_g, \nabla u\rangle_g = -32\varepsilon\varphi^4|\nabla\varphi|_g^2 - 4(p+7)\varphi^{p+3}|\nabla\varphi|_g^2 \leq 0.$$

Similarly as the proof of Theorem 1.5, φ must be a constant.

Proof of Theorem 1.9:

For $n \geq 5$, our assumption $f(t) \geq 0$ yields that

$$P_{g_0}\varphi \geq 0.$$

Then the strong maximum principle of Theorem 2.2 in [10] shows that $\varphi \equiv 0$ or $\varphi > 0$. Thus we only need to deal with $\varphi > 0$. In this situation, consider the conformal metric $g = \varphi^{\frac{4}{n-4}}g_0$. Then the Q-curvature satisfies

$$Q_g = \varphi^{-\frac{n+4}{n-4}}f(\varphi) \geq 0.$$

Making use of this fact, Lemma 3.1 shows that $R_g > 0$. Then choosing $u = \varphi^{\frac{2}{n-4}}$, one has

$$\langle \nabla Q_g, \nabla u\rangle_g = \frac{4}{n-4}\varphi^{\frac{4}{n-4}-1}F'(\varphi)|\nabla\varphi|_g^2 \leq 0 \tag{3.2}$$

where $F(t) = t^{-\frac{n+4}{n-4}}f(t)$. Corollary 2.2 and the inequality yield that

$$0 \geq \frac{n^2-2}{2(n-1)}\int_M u|E_g|_g^2 R_g \mathrm{d}v_g + R_0\int_M |E_g|_g^2 u^{-1}\mathrm{d}v_g.$$

Since $R_g > 0$ and $R_0 > 0$, one must have $E_g = 0$ which means that R_g is a constant. On one hand, if(M^n, g_0) is not conformally equivalent to the round sphere, Obata theorem implies that φ must be a constant. On the other hand, if(M^n, g_0) is conformally equivalent to the round sphere, Corollary 2.2 and the (3.2) show that

$$\frac{4}{n-4}\int_M \varphi^{\frac{4}{n-4}-1}F'(\varphi)|\nabla\varphi|_g^2\mathrm{d}v_g \geq 0.$$

Since $F'(t) < 0$ for all $t > 0$, one must have $|\nabla\varphi|_g \equiv 0$ which means that φ is a constant.

For $n = 3$, one has

$$P_{g_0}\varphi \leq 0.$$

Proposition 2.1 in [13] shows that $\varphi \equiv 0$ or $\varphi > 0$. Since the left of the proof is the same as before, we omit the details.

Thus we finish our proof.

Proof of Theorem 1.10:

Firstly, consider the conformal metric $g = e^{2\varphi} g_0$ and then

$$Q_g = e^{-4\varphi} f(e^{\varphi}) \geq 0.$$

Immediately, Lemma 3.1 yields that $R_g > 0$. The remain is the same as the proof of Theorem 1.9 and we omit it.

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