ON MINIMA OF DIFFERENCE OF THETA FUNCTIONS AND APPLICATION TO HEXAGONAL CRYSTALLIZATION

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Abstract. Let $z = x+iy \in \mathbb{H} := \{z = x+iy \in \mathbb{C} : y > 0\}$ and $\theta(\alpha; z) = \sum_{(m,n)\in\mathbb{Z}^2} e^{-\alpha \frac{\pi}{y}|mz+n|^2}$ be the theta function associated with the lattice $L = \mathbb{Z} \oplus z\mathbb{Z}$. In this paper we consider the following minimization problem of difference of two theta functions

$$\min_{\mathbb{H}} \Big(\theta(\alpha; z) - \beta\theta(2\alpha; z)\Big)$$

where $\alpha \geq 1$ and $\beta \in (-\infty, +\infty)$. We prove that there is a critical value $\beta_c = \sqrt{2}$ (independent of α) such that if $\beta \leq \beta_c$, the minimizer is $\frac{1}{2} + i\frac{\sqrt{3}}{2}$ (up to translation and rotation) which corresponds to the hexagonal lattice, and if $\beta > \beta_c$, the minimizer does not exist. Our result partially answers some questions raised in [7, 8, 10, 11] and gives a new proof in the crystallization of hexagonal lattice under Yukawa potential.

1. Introduction and Statement of Main Results

Let L be a two dimensional lattice. A large class of physical problems can be reduced to the following minimization problem:

$$\min_{L} E_f(\Lambda), \quad \text{where} \quad E_f(L) := \sum_{\mathbb{P}\in L\setminus\{0\}} f(|\mathbb{P}|^2). \tag{1.1}$$

The function f denotes the potential of the system and the summation ranges over all the lattice points except for the origin 0. The function $E_f(L)$ denotes the total energy of the system under the background potential f over a periodical lattice L, which arises in various physical problems ([6, 7, 8, 9, 11, 12, 32]). For example there is a clear connection of lattice sum and Abrikosov vortex lattices (see e.g. [1], [15], [35], [29], [30], [31], [21]). In the physical aspect, $E_f(L)$ refers to crystal lattice energy ([16, 33]) and Hamiltonian of crystals with long-ranged interaction ([3]). The function $E_f(L)$ has deep link with the partition function which is fundamental in equilibrium statistical physics. Locating the minimizer of such a total energy $E_f(L)$ over all the shapes of the lattices has important applications in physics ([12, 19, 20, 25, 24]), number theory (see e.g., [26], [28], [13]), adsorption on non-ideal surfaces [14], etc. The application of number theory to physics has many aspects and some of them reveal unexpected discovery. Significant examples of this direction went back to [17], [16], [34], [36] and the references therein. For other examples see the book [37].

Let $z \in \mathbb{H} := \{z = x+iy \in \mathbb{C} : y > 0\}$ and $L = \mathbb{Z}\oplus z\mathbb{Z}$ be the lattice in $\mathbb{R}^2$. When $f(\cdot) = e^{-\alpha|\cdot|^2}$ $(\alpha > 0)$ and the corresponding $E_f(L)$ becomes the the theta function

$$\theta(\alpha; z) := \sum_{\mathbb{P}\in\Lambda} e^{-\alpha|\mathbb{P}|^2} = \sum_{(m,n)\in\mathbb{Z}^2} e^{-\alpha\frac{\pi}{y}|mz+n|^2}, \tag{1.2}$$

a celebrated result of Montgomery ([27]) states that the hexagonal lattice attains the minimimizer in (1.1). By the classical Bernstein representation formula, Montgomery's result can be extended to any completely monotone functions, which are $C^\infty(0,\infty)$ satisfying

$$(-1)^j f^{(j)}(x) > 0, j = 0, 1, 2, \cdots \infty. \tag{1.3}$$

In [23] we have considered the following minimization problem of sum of two theta functions

$$\min_{z\in\mathbb{H}}\Big(\theta(\alpha;z)+\rho\theta(\alpha;\frac{z+1}{2})\Big) \tag{1.4}$$

and showed that the hexagonal-rhombic-square-rectangular transition appears as ρ goes from 0 to $+\infty$. This result can also be generalized to completely monotone functions.

However, in many physical models, the potential function f may not be completely monotone. A classical example is the Lennard-Jones potential $f(r)=\frac{1}{r^6}-\frac{2}{r^3}$ (see e.g. [4]) and its generalization $f(r)=\frac{a_1}{r^{t_1}}-\frac{a_2}{r^{t_2}}$, where $t_1>t_2>0, a_1, a_2>0$. In [8, 10, 11], Bétermin and his collaborators initiated a theoretical study of the (1.1) with the Lennard-Jones type potential. A numerical simulation suggests the hexagonal-rhombic-square-rectangular lattice phase transitions. Observe that the Lennard-Jones potential consists of two parts, each part being completely monotone. In addition, it is of one-well potential as introduced and defined in [8], i.e., there exists $a>0$ such that f is nonincreasing on $(0,a)$ and nondecreasing on $(a,+\infty)$. The following conjecture was made in [8]:

Conjecture 1.1 ([8], last page; open question 1.16 of [11]).
- *The behavior of the minimizers of (1.1) with respect to the lattice area A is qualitatively the same for all the Lennard-Jones type potentials (i.e., admits hexagonal-rhombic-square-rectangular lattice phase transitions);*
- *more generally, we can imagine that we should find the same result for any potential f written as $f:=f_1-f_2$, where f_1 and f_2 are both completely monotone and f is of one-well.*

For non-monotone potentials, there are other interesting open problems concerning the energy functional (1.1), which we list some of them here:

Conjecture 1.2 (Conjecture 2.7 of Bétermin, Faulhuber and Knüpfer [10]). *The existence of square lattice minimizes the lattice energy $E_f(L)$ when $f(r)=e^{-\beta\pi r}-e^{-\alpha\pi r}$ for $|\beta-\alpha|$ bigger than some small positive number.*

Open Quesion 1.1 (Bétermin-Petrache, [11]). *If $f(r^2)$ is not a positive superposition of Gaussians, can the triangular(hexagonal) lattice still be a minimizer of $E_f[L]$ among lattices at any fixed density?*

Open Quesion 1.2 (Question 1.8 of Bétermin-Petrache, [11]). *Is there any non-completely monotone f for which the minimizer of $E_f[L]$ is the triangular lattice for all $\lambda>0$, among periodic configurations C of unit density?*

Open Quesion 1.3 (Open Problem 1.9 of Bétermin-Petrache [11]). *[Stability of crystallization phenomena, with respect to perturbations of f] Study and classify natural distances between (or other measures of the size of perturbations of) interaction kernels f , with respect to which small perturbations of f can be ensured to preserve the crystallization properties of the kernels, such as the existence and shape of the global minimum amongst periodic configurations.*

In this paper, we study the existence and nonexistence of the minimizer for the potentials in the form of difference of two completely monotone functions. As a consequence, we give affirmative answers to Conjectures 1.1 and 1.2 and answer partially some other open questions listed as above.

Let $\theta(\alpha;z)$ be the theta function defined at (1.2). The following is the main result of this paper.

Theorem 1.1. *Let $\alpha\geq1$ and $\beta\in\mathbb{R}$. Consider the minimizing of difference of two theta functions with different frequencies, i.e.,*

$$\min_{\mathbb{H}}\Big(\theta(\alpha;z)-\beta\theta(2\alpha;z)\Big). \tag{1.5}$$

There is a critical value $\beta_c=\sqrt{2}$ (independent of α) such that

- *if $\beta\leq\beta_c$, the minimizer of the lattice energy functional is $\frac{1}{2}+i\frac{\sqrt{3}}{2}$ (up to translation and rotation), which corresponds to the hexagonal lattice;*
- *if $\beta>\beta_c$, the minimizer of the lattice energy functional does not exist.*

For many physical applications, we state an equivalent form of Theorem 1.1 in the following Theorem 1.2.

Theorem 1.2. *Let $\gamma \in (0,1]$ and $\beta \in \mathbb{R}$. Consider the minimizing problem*

$$\min_{\mathbb{H}} \Big(\theta(\gamma; z) - \beta\theta(\frac{1}{2}\gamma; z)\Big).$$

There is a critical value $\beta_s = \frac{\sqrt{2}}{2}$ (independent of α) such that if $\beta \leq \beta_s$, the minimizer of the lattice energy functional is $\frac{1}{2} + i\frac{\sqrt{3}}{2}$ (up to translation and rotation), and if $\beta > \beta_s$, the minimizer of the lattice energy functional does not exist.

In the following, we discuss applications to two special potentials: the exponential potential and Yukawa potential. For exponential potentials we have

Theorem 1.3. *Let $E_f[L]$ be defined as*

$$E_f(L) := \sum_{\mathbb{P}\in L\backslash\{0\}} f(|\mathbb{P}|^2)$$

with the area of two dimensional lattice L is normalized to 1. Consider the potential f has the following form

$$f_\alpha(r) := e^{-\pi\alpha\cdot r} - \beta e^{-2\pi\alpha\cdot r}, \;\; \alpha \geq 1,$$
$$g_\gamma(r) := e^{-\pi\gamma\cdot r} - \beta e^{-\frac{1}{2}\pi\gamma\cdot r}, \;\; \gamma \in (0,1).$$

- *There exists $\beta_c = \sqrt{2}$ (independent of α) such that*
  - *if $\beta \leq \beta_c$, the minimizer of $E_{f_\alpha}[L]$ exists and is always hexagonal lattice;*
  - *if $\beta > \beta_c$, the minimizer of $E_{f_\alpha}[L]$ does not exists.*
- *There exists $\beta_s = \frac{\sqrt{2}}{2}$ (independent of γ) such that*
  - *if $\beta \leq \beta_s$, the minimizer of $E_{g_\gamma}[L]$ exists and is always hexagonal lattice;*
  - *if $\beta > \beta_s$, the minimizer of $E_{g_\gamma}[L]$ does not exists.*

Remark 1.1 (An negative answer to Conjecture 1.1). *If one regards*

$$h_{\alpha_1,\beta}(r) := e^{-\pi\alpha_1\cdot r} - \beta e^{-\frac{1}{2}\pi\alpha_1\cdot r}, \beta > 0,$$
$$L_{\alpha_2} := \alpha_2 \cdot L, \alpha_2 > 0, \alpha := \alpha_1\cdot\alpha_2 < 1,$$

then $h_{\alpha_1,\beta}(r)$ is the difference of two completely monotone functions and is a one well potential and

$$E_{h_{\alpha_1,\beta}}[L_{\alpha_2}] = E_{g_\alpha}[L].$$

However $E_{h_{\alpha_1,\beta}}[L_\alpha]$ admits minimizer always at hexagonal lattice for any $\beta \in (0, \beta_s]$ as the lattice density α_2 changes in $(0, \frac{1}{\alpha_1})$(one can choose α_1 to close 0 then $\frac{1}{\alpha_1} \to \infty$). This disproves the hexagonal-rhombic-square-rectangular lattice phase transitions (see e.g.,[12, 10]) and gives an negative answer to Conjecture 1.1 and open question 1.16 of Bétermin-Petrache [11]. More general potentials of difference of two completely monotone type are shown in Theorem 1.4 via the Laplace transform.

Remark 1.2 (A negative answer to Conjecture 1.2 on dimension two). *As shown in the Theorem 1.3, there is no square lattice being minimizer for potential of such form.*

Remark 1.3 (A partial answer to Open Question 1.1). *Note that*

$$h_\beta(r^2) = e^{-\pi\cdot r^2} - \beta e^{-\frac{1}{2}\pi\cdot r^2}, \;\; \beta > 0,$$

is the difference of two Gaussians (hence not a positive superposition of Gaussians), the hexagonal lattice is always the minimizer for any fixed density $\alpha \geq 1$. This partially answers open question 1.1.

Remark 1.4 (Partial answer on Open Question 1.3). *We discuss two aspects: the stability and instability.*

- (**Instability under small perturbation: critical parameter**). *Let*

$$f_{0,\alpha}(r):=e^{-\pi\alpha\cdot r}-\sqrt{2}e^{-2\pi\alpha\cdot r},\ \ \alpha\geq 1,$$
$$g_{0,\gamma}(r):=e^{-\pi\gamma\cdot r}-\frac{\sqrt{2}}{2}e^{-\frac{1}{2}\pi\gamma\cdot r}\ \ \gamma\in(0,1).$$

A small perturbation of $f_{0,\alpha}(r)$ *and* $g_{0,\gamma}(r)$ *from left hand side by* $\varepsilon e^{-2c\cdot r}$ *will lead to the minimizer of* $E_f[L]$ *does not exist. Namely, let*

$$f_{0,\alpha,\varepsilon}(r):=e^{-\pi\alpha\cdot r}-\sqrt{2}e^{-2\pi\alpha\cdot r}-\varepsilon e^{-2c\cdot r},\ \ \alpha\geq 1$$
$$g_{0,\gamma,\varepsilon}(r):=e^{-\pi\gamma\cdot r}-\frac{\sqrt{2}}{2}e^{-\frac{1}{2}\pi\gamma\cdot r}-\varepsilon e^{-2c\cdot r}\ \ \gamma\in(0,1),$$

for $\forall c>0,\forall\varepsilon>0$ *be a small perturbation of* $f_{0,\alpha}(r),g_{0,\gamma}(r)$, *the minimizers of* $E_{f_{0,\alpha,\varepsilon}}[L]$ *and* $E_{g_{0,\gamma,\varepsilon}}[\Lambda]$ *do not exist and the minimizers of* $E_{f_{0,\alpha}}[L]$ *and* $E_{g_{0,\gamma}}[L]$ *are both hexagonal lattice. In this sense, the minimizers of* $E_{f_{0,\alpha}}[L]$ *are instable under small perturbation as above.*

- (**Stability under small perturbation: subcritical parameter**). *Assume* $\beta<\frac{\sqrt{2}}{2}$. *Let* $|\varepsilon|\leq\frac{\sqrt{2}}{2}-\beta$ *and*

$$g_{\alpha,\varepsilon}(r):=e^{-\pi\alpha\cdot r}-\beta e^{-2\pi\alpha\cdot r}\pm\varepsilon e^{-2\pi\alpha\cdot r},\ \ \alpha\geq 1,$$

be a small perturbation of $g_\alpha(r)$. *Then the minimizer of* $E_{g_{\alpha,\varepsilon}(r)}[L]$ *is still the hexagonal lattice, i.e., the minimizer of* $E_{g_\alpha(r)}[L]$ *is stable under small perturbation as above.*

Remark 1.5. *The numerical study of the potential*

$$f_\alpha(r):=e^{-\pi\alpha\cdot r}-\beta e^{-2\pi\alpha\cdot r}$$

is performed as an important cases in Bétermin-Faulhuber-Knüpfer [10], *see Figures 3, 8 and 10 of* [10].

Using the free parameter α of Theorem 1.3, we proceed Theorem 1.2 and 1.3 to a general form by the Laplace transform (inspired by Bétermin [7]). There is a difference between Theorem 1.4 and Theorems 1.1, 1.2, 1.3. In the former, one does not know the parameter is optimal or not, and in the latter, the parameters are optimal as stated in the Theorems.

Theorem 1.4. *Let* $E_f[L]$ *be defined as*

$$E_f(L):=\sum_{\mathbb{P}\in L\backslash\{0\}}f(|\mathbb{P}|^2)$$

with the area of two dimensional lattice L *is normalized to 1. Consider the potential* $f_{\alpha,P},g_{\alpha,P}$ *have the following form*

$$\begin{aligned}f_{\alpha,P}(r)&:=\int_1^\infty\Big(\big(e^{-\pi\alpha x\cdot r}-\beta e^{-2\pi\alpha x\cdot r}\big)\cdot P(x)\Big)dx,\ \ \alpha\geq 1,\\ g_{\gamma,P}(r)&:=\int_0^1\Big(\big(e^{-\pi\gamma x\cdot r}-\beta e^{-\frac{1}{2}\pi\gamma x\cdot r}\big)\cdot P(x)\Big)dx,\ \ \gamma\in(0,1),\end{aligned}\tag{1.6}$$

where the $P(x)$ *is any real function(not necessarily continuous) such that* $f_{\alpha,P}(r),g_{\alpha,P}(r)$ *are finite and*

$$P(x)\geq 0.$$

Then there exists $\beta_c=\sqrt{2},\beta_s=\frac{\sqrt{2}}{2}$ *and nonnegative function* P *such that*

- *if* $\beta\leq\beta_c$, *the minimizer of* $E_{f_{\alpha,P}}[\Lambda]$ *exists and is always hexagonal lattice.*
- *if* $\beta\leq\beta_s$, *the minimizer of* $E_{g_{\gamma,P}}[\Lambda]$ *exists and is always hexagonal lattice.*

Remark 1.6 (**Partial answer on open question 1.6 of Bétermin-Petrache** [11]). *Theorem 1.4 partially answers open question 1.6 of Bétermin-Petrache* [11] *on minimizers of difference of two Laplace transform of the potentials.*

Remark 1.7 (Connection to G-type potentials)**.** *The potential introduced by* (1.6) *are a subclass of G-type potentials* (*see Chapter 10 of monograph* [5]). *Here we show that under these potentials the minimizer of the lattice energy are hexagonal lattice under suitable competing strength* β.

Remark 1.8 (**Partial answer on Open Question 1.2**)**.** *Since here* $f_{\alpha,P}(r), (\alpha \geq 1)$ *is the difference of two completely monotone functions, and hence not completely monotone. The minimizer of* $E_{f_{\alpha,P}(r)}[\lambda \cdot L]$ *is hexagonal lattice for all* $\lambda \geq 1$. *This gives partial answer on Open question 1.2.*

Remark 1.9 (**More potentials to answer Conjecture 1.1**)**.** *Theorem 1.4 provides general potentials to a negative answer to Conjecture 1.1.*

A particular application of Theorem 1.4 is the classical Yukawa potential case.

Corollary 1.1 (**Yukawa potential**$\{\cong P(x) \equiv 1\}$ **of Theorem 1.4**)**.** *Let* $E_f[L]$ *be defined as*

$$E_f(L) := \sum_{\mathbb{P} \in L \backslash \{0\}} f(|\mathbb{P}|^2)$$

with the area of two dimensional lattice L *is normalized to 1. Consider the potential* $f_{1,\alpha}, g_{1,\alpha}$ *have the following form*

$$f_{1,\alpha}(r) := \frac{e^{-\pi\alpha r}}{r} - \beta \frac{e^{-2\pi\alpha r}}{2r}, \quad \alpha \geq 1.$$

Then there exists $\beta_c = \sqrt{2}$ *independent of parameter* α *such that*

- *if* $\beta \leq \beta_c$, *the minimizer of* $E_{f_{1,\alpha}}[L]$ *exists and is always hexagonal lattice.*

Remark 1.10. *The rigorous results on Yukawa potential of the minimizer of the crystal energy* $E_f[L]$, *as far as we know, is initiated by Bétermin* [7]. *Here Corollary 1.1 improves the result in* [7] *on some aspects. Note that we provide an effective way to prove the crystallization of hexagonal lattice under Yukawa potential.*

Remark 1.11. *In using the results of Corollary 1.1 and combining the method of Bétermin* [7], *one can obtain more general results on minimization results under Yukawa potential.*

Theorem 1.1 can be extended as follows by iteration.

Theorem 1.5. *Consider the minimizing problem of difference of two theta functions*

$$\min_{\mathbb{H}} \Big(\theta(\alpha; z) - \beta\theta(2^k\alpha; z)\Big), \quad \text{for any } \alpha \geq 1, k \geq 0, k \in \mathbb{Z}, \ \beta \in (-\infty, \infty).$$

There is a critical value $\beta_A = \sqrt{2^k}$ *independent of* α *such that*

- *if* $\beta \leq \beta_A$, *the minimizer of the lattice energy functional is* $\frac{1}{2} + i\frac{\sqrt{3}}{2}$ *up to translation and rotation, this minimizer corresponds to* Λ *is the hexagonal lattice;*
- *if* $\beta > \beta_A$, *the minimizer of the lattice energy functional does not exist.*

The paper is organized as follows: in Section 2, we state some basic preliminaries about the functional $\theta(\alpha; z) - \beta\theta(2\alpha; z)$. In Section 3, we prove that the minimization problem on the fundamental region (see (2.4) and figure 1) can be reduced to a vertical line (see figure 1). (See Theorem 3.1.) In Section 4, we prove that the minimization problem on the vertical line can be reduced to a particular point (hexagonal point (see figure 1)). We develop effective methods and delicate analysis to obtain the estimates, which can be generalized to solve related problems. As a consequence we prove Theorem 1.1. Finally Section 5 contains proofs of remaining Theorems.

2. Preliminaries

In this section we collect some simple symmetries of the theta function $\theta(\alpha; z)$ and the associated fundamental domain, and also the properties of Jacobi theta functions to be used in later sections.

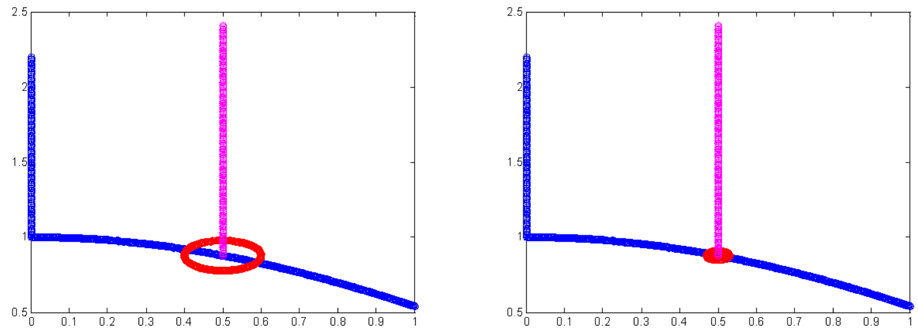

FIGURE 1. Location of the fundamental region and hexagonal point.

Let $\mathbb{H}$ denote the upper half plane and $\mathcal{S}$ denote the modular group

$$\mathcal{S} := SL_2(\mathbb{Z}) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix}, ad - bc = 1, a, b, c, d \in \mathbb{Z}\}. \tag{2.1}$$

We use the following definition of fundamental domain which is slightly different from the classical definition (see [27]):

Definition 1 (page 108, [18]). *The fundamental domain associated to group G is a connected domain $\mathcal{D}$ satisfies*

- *For any $z \in \mathbb{H}$, there exists an element $\pi \in G$ such that $\pi(z) \in \overline{\mathcal{D}}$;*
- *Suppose $z_1, z_2 \in \mathcal{D}$ and $\pi(z_1) = z_2$ for some $\pi \in G$, then $z_1 = z_2$ and $\pi = \pm Id$.*

By Definition 1, the fundamental domain associated to modular group $\mathcal{S}$ is

$$\mathcal{D}_{\mathcal{S}} := \{z \in \mathbb{H} : |z| > 1,\ -\frac{1}{2} < x < \frac{1}{2}\} \tag{2.2}$$

which is open. Note that the fundamental domain can be open. (See [page 30, [2]].)

Next we introduce another group related to the functionals $\theta(\alpha; z)$. The generators of the group are given by

$$\mathcal{G} : \text{the group generated by } \ \tau \mapsto -\frac{1}{\tau},\ \ \tau \mapsto \tau + 1,\ \ \tau \mapsto -\overline{\tau}. \tag{2.3}$$

It is easy to see that the fundamental domain associated to group $\mathcal{G}$ denoted by $\mathcal{D}_{\mathcal{G}}$ is

$$\mathcal{D}_{\mathcal{G}} := \{z \in \mathbb{H} : |z| > 1,\ 0 < x < \frac{1}{2}\}. \tag{2.4}$$

The following lemma characterizes the fundamental symmetries of the theta functions $\theta(s; z)$. The proof is easy so we omit it.

Lemma 2.1. *For any $s > 0$, any $\gamma \in \mathcal{G}$ and $z \in \mathbb{H}$, $\theta(s; \gamma(z)) = \theta(s; z)$.*

Let

$$\mathrm{W}_\beta(\alpha; z) := \theta(\alpha; z) - \beta\theta(2\alpha; z). \tag{2.5}$$

Lemma 2.2. *For any $\alpha > 0$, any $\gamma \in \mathcal{G}$ and $z \in \mathbb{H}$, $\mathrm{W}_\beta(\alpha; \gamma(z)) = \mathrm{W}_\beta(\alpha; z)$.*

We also need some delicate analysis of the Jacobi theta function.

We first recall the following well-known Jacobi triple product formula:

$$\prod_{m=1}^{\infty}(1 - x^{2m})(1 + x^{2m-1}y^2)(1 + \frac{x^{2m-1}}{y^2}) = \sum_{n=-\infty}^{\infty} x^{n^2} y^{2n} \tag{2.6}$$

for complex numbers x, y with $|x| < 1$, $y \neq 0$.

The Jacobi theta function is defined as

$$\vartheta_J(z;\tau) := \sum_{n=-\infty}^{\infty} e^{i\pi n^2\tau + 2\pi i n z},$$

and the classical one-dimensional theta function is given by

$$\vartheta(X;Y) := \vartheta_J(Y;iX) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 X} e^{2n\pi i Y}. \tag{2.7}$$

Hence by the Jacobi triple product formula (2.6), it holds

$$\vartheta(X;Y) = \prod_{n=1}^{\infty}(1-e^{-2\pi n X})(1+e^{-2(2n-1)\pi X} + 2e^{-(2n-1)\pi X}\cos(2\pi Y)). \tag{2.8}$$

The following Lemmas 2.3 and 2.4 are proved in [22].

Lemma 2.3. *Assume $X > \frac{1}{5}$. If $\sin(2\pi Y) > 0$, then*

$$-\overline{\vartheta}(X)\sin(2\pi Y) \leq \frac{\partial}{\partial Y}\vartheta(X;Y) \leq -\underline{\vartheta}(X)\sin(2\pi Y).$$

If $\sin(2\pi Y) < 0$, then

$$-\underline{\vartheta}(X)\sin(2\pi Y) \leq \frac{\partial}{\partial Y}\vartheta(X;Y) \leq -\overline{\vartheta}(X)\sin(2\pi Y).$$

Here

$$\underline{\vartheta}(X) := 4\pi e^{-\pi X}(1-\mu(X)),\ \ \overline{\vartheta}(X) := 4\pi e^{-\pi X}(1+\mu(X)),$$

and

$$\mu(X) := \sum_{n=2}^{\infty} n^2 e^{-\pi(n^2-1)X}. \tag{2.9}$$

Lemma 2.4. *Assume $X < \min\{\frac{\pi}{\pi+2}, \frac{\pi}{4\log\pi}\} = \frac{\pi}{\pi+2}$. If $\sin(2\pi Y) > 0$, then*

$$-\overline{\vartheta}(X)\sin(2\pi Y) \leq \frac{\partial}{\partial Y}\vartheta(X;Y) \leq -\underline{\vartheta}(X)\sin(2\pi Y).$$

If $\sin(2\pi Y) < 0$, then

$$-\underline{\vartheta}(X)\sin(2\pi Y) \leq \frac{\partial}{\partial Y}\vartheta(X;Y) \leq -\overline{\vartheta}(X)\sin(2\pi Y).$$

Here

$$\underline{\vartheta}(X) := \pi e^{-\frac{\pi}{4X}} X^{-\frac{3}{2}};\ \ \overline{\vartheta}(X) := X^{-\frac{3}{2}}.$$

3. The transversal monotonicity

Let $\mathcal{D}_{\mathcal{G}} := \{z \in \mathbb{H} : |z| > 1,\ 0 < x < \frac{1}{2}\}$ be the fundamental domain associated to the group $\mathcal{G}$. Define the vertical line

$$\Gamma := \{z \in \mathbb{H} : \mathrm{Re}(z) = \frac{1}{2},\ \mathrm{Im}(z) \geq \frac{\sqrt{3}}{2}\}. \tag{3.1}$$

By the group invariance (Lemma 2.2), one has

$$\min_{z\in\mathbb{H}}\Big(\theta(\alpha;z) - \beta\theta(2\alpha;z)\Big) = \min_{z\in\overline{\mathcal{D}_{\mathcal{G}}}}\Big(\theta(\alpha;z) - \beta\theta(2\alpha;z)\Big). \tag{3.2}$$

Let $\mu(X)$ be defined in (2.9) and

$$\beta_0 := \min\begin{cases} \frac{\pi e^{2\pi} - 4e^{-6\pi}}{2} \\ \frac{\sqrt{2}e^{\frac{\sqrt{3}\pi}{4}}(1-\mu(\frac{1}{2})) - 4e^{-\frac{5\sqrt{3}\pi}{4}}(1+\mu(\frac{1}{4}))}{1+\mu(\frac{1}{4})} \\ \frac{4\sqrt{2}\pi e^{\pi}\left(1-\mu(\frac{1}{4})\right) - 16\sqrt{2}\pi e^{-\frac{7\pi}{2}}\left(1+\mu(\frac{1}{4})\right)}{64}. \end{cases} \tag{3.3}$$

Numerically,

$$\beta_0 :\cong 3.801819\cdots.$$

As we shall see in the next Section, one only needs to require that $\beta_0 \geq \sqrt{2}$ to obtain all the main results.

In this section, we aim to establish that

Theorem 3.1. *Assume that $\alpha \geq 1$. Let β_0 be defined at (3.3). Then for $\beta < \beta_0$,*

$$\min_{z\in\mathbb{H}}\Big(\theta(\alpha;z)-\beta\theta(2\alpha;z)\Big)=\min_{z\in\overline{\mathcal{D}_{\mathcal{G}}}}\Big(\theta(\alpha;z)-\beta\theta(2\alpha;z)\Big)=\min_{z\in\Gamma}\Big(\theta(\alpha;z)-\beta\theta(2\alpha;z)\Big),$$

where Γ is a vertical line and defined at (3.1).

The proof of Theorem 3.1 follows from the following monotonicity result

Theorem 3.2. *Assume that $\alpha \geq 1$. Then for $\beta < \beta_0$*

$$\frac{\partial}{\partial x}\Big(\theta(\alpha;z)-\beta\theta(2\alpha;z)\Big)<0,\ \ \text{for}\ \ z\in\mathcal{D}_{\mathcal{G}}.$$

In the rest of this Section, we prove Theorem 3.2.

3.1. **The estimates.** We provide an exponential expansion of the theta function, which is useful in our estimates.

Lemma 3.1. *We have the following exponential expansion of theta function:*

$$\theta(\alpha;z)=2\sqrt{\frac{y}{\alpha}}\sum_{n=1}^{\infty}e^{-\alpha\pi yn^2}\vartheta(\frac{y}{\alpha};nx)+\sqrt{\frac{y}{\alpha}}\vartheta(\frac{y}{\alpha};0). \tag{3.4}$$

Proof. In view of (2.7), by Poisson Summation Formula, one has

$$\vartheta(X;Y)=X^{-\frac{1}{2}}\sum_{n\in\mathbb{Z}}e^{-\frac{\pi(n-Y)^2}{X}}. \tag{3.5}$$

Then

$$\begin{aligned}\theta(\alpha;z)&=\sum_{(m,n)\in\mathbb{Z}^2}e^{-\alpha\pi\frac{1}{y}|nz+m|^2}=\sum_{n\in\mathbb{Z}}e^{-\alpha\pi yn^2}\sum_{m\in\mathbb{Z}}e^{-\frac{\alpha\pi(nx+m)^2}{y}}\\&=\sqrt{\frac{y}{\alpha}}\sum_{n\in\mathbb{Z}}e^{-\alpha\pi yn^2}\vartheta(\frac{y}{\alpha};-nx)=\sqrt{\frac{y}{\alpha}}\sum_{n\in\mathbb{Z}}e^{-\alpha\pi yn^2}\vartheta(\frac{y}{\alpha};nx)\\&=2\sqrt{\frac{y}{\alpha}}\sum_{n=1}^{\infty}e^{-\alpha\pi yn^2}\vartheta(\frac{y}{\alpha};nx)+\sqrt{\frac{y}{\alpha}}\vartheta(\frac{y}{\alpha};0).\end{aligned}$$

□

Lemma 3.2. *We have the following identity for derivative of the theta function with respect to x*

$$-\frac{\partial}{\partial x}\big(\theta(\alpha;z)-\beta\theta(2\alpha;z)\big)=2\sqrt{\frac{y}{2\alpha}}e^{-\pi\alpha y}\mathcal{E}_{\alpha,\beta,x}(z).$$

Here

$$\mathcal{E}_{\alpha,\beta,x}(z):=\sqrt{2}\sum_{n=1}^{\infty}ne^{-\alpha\pi y(n^2-1)}\big(-\frac{\partial}{\partial Y}\vartheta(\frac{y}{\alpha};nx)\big)-\beta\sum_{n=1}^{\infty}ne^{-\alpha\pi y(2n^2-1)}\big(-\frac{\partial}{\partial Y}\vartheta(\frac{y}{2\alpha};nx)\big).$$

Proof. By Lemma 3.1,

$$
\begin{aligned}
-\frac{\partial}{\partial x}\big(\theta(\alpha;z)-\beta\theta(2\alpha;z)\big) &= -\frac{\partial}{\partial x}\Big(2\sqrt{\frac{y}{\alpha}}\sum_{n=1}^{\infty}e^{-\alpha\pi y n^2}\vartheta(\frac{y}{\alpha};nx)-2\beta\sqrt{\frac{y}{2\alpha}}\sum_{n=1}^{\infty}e^{-2\alpha\pi y n^2}\vartheta(\frac{y}{2\alpha};nx)\Big)\\
&=2\sqrt{\frac{y}{2\alpha}}e^{-\pi\alpha y}(-\frac{\partial}{\partial x})\Big(\sqrt{2}\sum_{n=1}^{\infty}e^{-\alpha\pi y(n^2-1)}\vartheta(\frac{y}{\alpha};nx)\\
&\quad-\beta\sum_{n=1}^{\infty}e^{-\alpha\pi y(2n^2-1)}\vartheta(\frac{y}{2\alpha};nx)\Big)\\
&=2\sqrt{\frac{y}{2\alpha}}e^{-\pi\alpha y}\Big(\sqrt{2}\sum_{n=1}^{\infty}ne^{-\alpha\pi y(n^2-1)}\big(-\frac{\partial}{\partial Y}\vartheta(\frac{y}{\alpha};nx)\big)\\
&\quad-\beta\sum_{n=1}^{\infty}ne^{-\alpha\pi y(2n^2-1)}\big(-\frac{\partial}{\partial Y}\vartheta(\frac{y}{2\alpha};nx)\big)\Big)\\
&=2\sqrt{\frac{y}{2\alpha}}e^{-\pi\alpha y}\mathcal{E}_{\alpha,\beta,x}(z).
\end{aligned}
$$

□

Lemma 3.3. *For $x\in[0,\frac{1}{3}]$, $z\in\overline{\mathcal{D}_{\mathcal{G}}}$,*

$$
\mathcal{E}_{\alpha,\beta,x}(z)\geq\sin(2\pi x)\Big(\sqrt{2}\underline{\vartheta}(\frac{y}{\alpha})-(\beta+\sigma_2)e^{-\alpha\pi y}\overline{\vartheta}(\frac{y}{2\alpha})-(4\sqrt{2}e^{-3\alpha\pi y}+\sigma_1)\overline{\vartheta}(\frac{y}{\alpha})\Big),
$$

where

$$
\sigma_1:=\sqrt{2}\sum_{n=2}^{\infty}n^2e^{-\alpha\pi y(n^2-1)},\sigma_2:=\beta\sum_{n=2}^{\infty}n^2e^{-\alpha\pi y(2n^2-1)}. \tag{3.6}
$$

Proof. We decompose

$$
\begin{aligned}
\mathcal{E}_{\alpha,\beta,x}(z):=&\sqrt{2}(-\frac{\partial}{\partial Y})\vartheta(\frac{y}{\alpha};x)-be^{-\pi\alpha y}(-\frac{\partial}{\partial Y})\vartheta(\frac{y}{2\alpha};x)+2\sqrt{2}e^{-3\pi\alpha y}(-\frac{\partial}{\partial Y})\vartheta(\frac{y}{\alpha};x)\\
&+\sqrt{2}\sum_{n=2}^{\infty}ne^{-\alpha\pi y(n^2-1)}\big(-\frac{\partial}{\partial Y}\vartheta(\frac{y}{\alpha};nx)\big)-\beta\sum_{n=2}^{\infty}ne^{-\alpha\pi y(2n^2-1)}\big(-\frac{\partial}{\partial Y}\vartheta(\frac{y}{2\alpha};nx)\big)\\
=&\mathcal{E}^{a,1}_{\alpha,\beta,x}(z)+\mathcal{E}^{b,1}_{\alpha,\beta,x}(z).
\end{aligned}
$$

where $\mathcal{E}^{a,1}_{\alpha,\beta,x}(z)$ and $\mathcal{E}^{b,1}_{\alpha,\beta,x}(z)$ are defined at the last equality.

For $\mathcal{E}^{b,1}_{\alpha,\beta,x}(z)$ we estimate as follows

$$
\begin{aligned}
|\mathcal{E}^{b,1}_{\alpha,\beta,x}(z)|&\leq\sqrt{2}\sum_{n=2}^{\infty}ne^{-\alpha\pi y(n^2-1)}\overline{\vartheta}(\frac{y}{\alpha})|\sin(2\pi nx)|+\beta\sum_{n=2}^{\infty}ne^{-\alpha\pi y(2n^2-1)}\overline{\vartheta}(\frac{y}{2\alpha})|\sin(2\pi nx)|\\
&\leq\sqrt{2}\sum_{n=2}^{\infty}n^2e^{-\alpha\pi y(n^2-1)}\overline{\vartheta}(\frac{y}{\alpha})\sin(2\pi x)+\beta\sum_{n=2}^{\infty}n^2e^{-\alpha\pi y(2n^2-1)}\overline{\vartheta}(\frac{y}{2\alpha})\sin(2\pi x)\\
&\leq\sigma_1\cdot\overline{\vartheta}(\frac{y}{\alpha})\sin(2\pi x)+\sigma_2\cdot\overline{\vartheta}(\frac{y}{2\alpha})\sin(2\pi x),
\end{aligned}
$$

where σ_1 and σ_2 are defined at (3.6).

For $\mathcal{E}^{b,1}_{\alpha,\beta,x}(z)$ we have

$$
\begin{aligned}
\mathcal{E}^{a,1}_{\alpha,\beta,x}(z)&\geq\sqrt{2}\underline{\vartheta}(\frac{y}{\alpha})\sin(2\pi x)-\beta e^{-\pi\alpha y}\overline{\vartheta}(\frac{y}{2\alpha})\sin(2\pi x)+2\sqrt{2}e^{-3\pi\alpha y}\underline{\vartheta}(\frac{y}{\alpha})\sin(4\pi x)\\
&\geq\sin(2\pi x)\Big(\sqrt{2}\underline{\vartheta}(\frac{y}{\alpha})-\beta e^{-\pi\alpha y}\overline{\vartheta}(\frac{y}{2\alpha})\Big).
\end{aligned}
$$

Therefore,

$$\begin{aligned}\mathcal{E}_{\alpha,\beta,x}(z) =&\mathcal{E}^{a,1}_{\alpha,\beta,x}(z)+\mathcal{E}^{b,1}_{\alpha,\beta,x}(z)\\ &\geq \sin(2\pi x)\Big(\sqrt{2}\underline{\vartheta}(\frac{y}{\alpha})-(\beta+\sigma_2)e^{-\pi\alpha y}\overline{\vartheta}(\frac{y}{2\alpha})-\sigma_1\overline{\vartheta}(\frac{y}{\alpha})\Big).\end{aligned}$$

□

Lemma 3.4. *For* $x\in[\frac{1}{3},\frac{1}{2}]$, $z\in\overline{\mathcal{D}_{\mathcal{G}}}$,

$$\mathcal{E}_{\alpha,\beta,x}(z)\geq \sin(2\pi x)\Big(\sqrt{2}\underline{\vartheta}(\frac{y}{\alpha})-(\beta+\sigma_4)e^{-\alpha\pi y}\overline{\vartheta}(\frac{y}{2\alpha})-(4\sqrt{2}e^{-3\alpha\pi y}+\sigma_3)\overline{\vartheta}(\frac{y}{\alpha})\Big).$$

Here

$$\sigma_3:=\sqrt{2}\sum_{n=4}^{\infty}n^2e^{-\alpha\pi y(n^2-1)},\sigma_4:=\beta\sum_{n=3}^{\infty}n^2e^{-\alpha\pi y(2n^2-1)}. \tag{3.7}$$

Proof. In this case, $\big(-\frac{\partial}{\partial Y}\vartheta(y;nx)\big)\geq 0$ is positive for $n=1,3$, $y>0$ and negative for $n=2$, $y>0$. We then decompose

$$\begin{aligned}\mathcal{E}_{\alpha,\beta,x}(z):=&\sqrt{2}\sum_{n=1}^{\infty}ne^{-\alpha\pi y(n^2-1)}\big(-\frac{\partial}{\partial Y}\vartheta(\frac{y}{\alpha};nx)\big)-\beta\sum_{n=1}^{\infty}ne^{-\alpha\pi y(2n^2-1)}\big(-\frac{\partial}{\partial Y}\vartheta(\frac{y}{2\alpha};nx)\big)\\ =&\sqrt{2}e^{-\alpha\pi y}\big(-\frac{\partial}{\partial Y}\vartheta(\frac{y}{\alpha};x)\big)+2\sqrt{2}e^{-3\alpha\pi y}\big(-\frac{\partial}{\partial Y}\vartheta(\frac{y}{\alpha};2x)\big)+3\sqrt{2}e^{-8\alpha\pi y}\big(-\frac{\partial}{\partial Y}\vartheta(\frac{y}{\alpha};3x)\big)\\ &-\beta e^{-\alpha\pi y}\big(-\frac{\partial}{\partial Y}\vartheta(\frac{y}{2\alpha};x)\big)-2\beta e^{-7\alpha\pi y}\big(-\frac{\partial}{\partial Y}\vartheta(\frac{y}{2\alpha};2x)\big)\\ &+\sqrt{2}\sum_{n=4}^{\infty}ne^{-\alpha\pi y(n^2-1)}\big(-\frac{\partial}{\partial Y}\vartheta(\frac{y}{\alpha};nx)\big)-\beta\sum_{n=3}^{\infty}ne^{-\alpha\pi y(2n^2-1)}\big(-\frac{\partial}{\partial Y}\vartheta(\frac{y}{2\alpha};nx)\big)\\ &:=\mathcal{E}^{a,2}_{\alpha,\beta,x}(z)+\mathcal{E}^{b,2}_{\alpha,\beta,x}(z).\end{aligned}$$

where $\mathcal{E}^{a,2}_{\alpha,\beta,x}(z)$ and $\mathcal{E}^{b,2}_{\alpha,\beta,x}(z)$ are defined at the last equality.

For $\mathcal{E}^{a,2}_{\alpha,\beta,x}(z)$ we have

$$\begin{aligned}\mathcal{E}^{a,2}_{\alpha,\beta,x}(z):=&\sqrt{2}\big(-\frac{\partial}{\partial Y}\vartheta(\frac{y}{\alpha};x)\big)+2\sqrt{2}e^{-3\alpha\pi y}\big(-\frac{\partial}{\partial Y}\vartheta(\frac{y}{\alpha};2x)\big)+3\sqrt{2}e^{-8\alpha\pi y}\big(-\frac{\partial}{\partial Y}\vartheta(\frac{y}{\alpha};3x)\big)\\ &-\beta e^{-\alpha\pi y}\big(-\frac{\partial}{\partial Y}\vartheta(\frac{y}{2\alpha};x)\big)-2\beta e^{-7\alpha\pi y}\big(-\frac{\partial}{\partial Y}\vartheta(\frac{y}{2\alpha};2x)\big).\end{aligned}$$

For $\mathcal{E}^{b,2}_{\alpha,\beta,x}(z)$,

$$\mathcal{E}^{b,2}_{\alpha,\beta,x}(z):=\sqrt{2}\sum_{n=4}^{\infty}ne^{-\alpha\pi y(n^2-1)}\big(-\frac{\partial}{\partial Y}\vartheta(\frac{y}{\alpha};nx)\big)-\beta\sum_{n=3}^{\infty}ne^{-\alpha\pi y(2n^2-1)}\big(-\frac{\partial}{\partial Y}\vartheta(\frac{y}{2\alpha};nx)\big).$$

A lower bound of $\mathcal{E}^{a,2}_{\alpha,\beta,x}(z)$ yields

$$\begin{aligned}\mathcal{E}^{a,2}_{\alpha,\beta,x}(z):=&\sqrt{2}\big(-\frac{\partial}{\partial Y}\vartheta(\frac{y}{\alpha};x)\big)+2\sqrt{2}e^{-3\alpha\pi y}\big(-\frac{\partial}{\partial Y}\vartheta(\frac{y}{\alpha};2x)\big)+3\sqrt{2}e^{-8\alpha\pi y}\big(-\frac{\partial}{\partial Y}\vartheta(\frac{y}{\alpha};3x)\big)\\ &-\beta e^{-\alpha\pi y}\big(-\frac{\partial}{\partial Y}\vartheta(\frac{y}{2\alpha};x)\big)-2\beta e^{-7\alpha\pi y}\big(-\frac{\partial}{\partial Y}\vartheta(\frac{y}{2\alpha};2x)\big)\\ \geq&\sqrt{2}\big(-\frac{\partial}{\partial Y}\vartheta(\frac{y}{\alpha};x)\big)+2\sqrt{2}e^{-3\alpha\pi y}\big(-\frac{\partial}{\partial Y}\vartheta(\frac{y}{\alpha};2x)\big)-\beta e^{-\alpha\pi y}\big(-\frac{\partial}{\partial Y}\vartheta(\frac{y}{2\alpha};x)\big)\\ \geq&\sqrt{2}\underline{\vartheta}(\frac{y}{\alpha})\sin(2\pi x)+2\sqrt{2}e^{-3\alpha\pi y}\overline{\vartheta}(\frac{y}{\alpha})\sin(4\pi x)-\beta e^{-\alpha\pi y}\overline{\vartheta}(\frac{y}{2\alpha})\sin(2\pi x)\\ \geq&\sin(2\pi x)\Big(\sqrt{2}\underline{\vartheta}(\frac{y}{\alpha})-\beta e^{-\alpha\pi y}\overline{\vartheta}(\frac{y}{2\alpha})-4\sqrt{2}e^{-3\alpha\pi y}\overline{\vartheta}(\frac{y}{\alpha})\Big).\end{aligned}$$

A upper bound of $\mathcal{E}^{b,2}_{\alpha,\beta,x}(z)$ is given by

$$
\begin{aligned}
|\mathcal{E}^{b,2}_{\alpha,\beta,x}(z)| &\leq \sqrt{2}\sum_{n=4}^{\infty} n e^{-\alpha\pi y(n^2-1)}\overline{\vartheta}(\frac{y}{\alpha})|\sin(2n\pi x)| + \beta\sum_{n=3}^{\infty} n e^{-\alpha\pi y(2n^2-1)}\overline{\vartheta}(\frac{y}{2\alpha})|\sin(2n\pi x)|\\
&\leq \sin(2\pi x)\Big(\sqrt{2}\sum_{n=4}^{\infty} n^2 e^{-\alpha\pi y(n^2-1)}\overline{\vartheta}(\frac{y}{\alpha}) + \beta\sum_{n=3}^{\infty} n^2 e^{-\alpha\pi y(2n^2-1)}\overline{\vartheta}(\frac{y}{2\alpha})\Big)\\
&\leq \sin(2\pi x)\Big(\sigma_3\cdot\overline{\vartheta}(\frac{y}{\alpha}) + \sigma_4\cdot\overline{\vartheta}(\frac{y}{2\alpha})\Big).
\end{aligned}
$$

Here σ_3 and σ_4 are defined at (3.7).

Combining all the estimates we get

$$
\begin{aligned}
\mathcal{E}_{\alpha,\beta,x}(z) &= \mathcal{E}^{a,2}_{\alpha,\beta,x}(z) + \mathcal{E}^{b,2}_{\alpha,\beta,x}(z)\\
&\geq \sin(2\pi x)\Big(\sqrt{2}\underline{\vartheta}(\frac{y}{\alpha}) - (\beta+\sigma_4)e^{-\alpha\pi y}\overline{\vartheta}(\frac{y}{2\alpha}) - (4\sqrt{2}e^{-3\alpha\pi y}+\sigma_3)\overline{\vartheta}(\frac{y}{\alpha})\Big).
\end{aligned}
$$

□

3.2. **The estimates of the lower bound of a useful function and completing of the proof.** In view of Lemma 3.3 and 3.4. We define

$$
\mathcal{R}_{\alpha,\beta}(y) := \Big(\sqrt{2}\underline{\vartheta}(\frac{y}{\alpha}) - \beta e^{-\alpha\pi y}\overline{\vartheta}(\frac{y}{2\alpha}) - 4\sqrt{2}e^{-3\alpha\pi y}\overline{\vartheta}(\frac{y}{\alpha})\Big).
$$

Lemma 3.5. *For $\beta < \beta_0$, and $\forall \alpha \geq 1, y \geq \frac{\sqrt{3}}{2}$,*

$$
\mathcal{R}_{\alpha,\beta}(y) > 0.
$$

Proof. We divide its proof to three cases.

Case A: $\frac{y}{\alpha}, \frac{y}{2\alpha} \in (0, \frac{1}{4}]$**.** In this case, $\alpha \geq 4y \geq 2\sqrt{3}$.

$$
\begin{aligned}
\mathcal{R}_{\alpha,\beta}(y) &= \Big(\sqrt{2}\underline{\vartheta}(\frac{y}{\alpha}) - \beta e^{-\alpha\pi y}\overline{\vartheta}(\frac{y}{2\alpha}) - 4\sqrt{2}e^{-3\alpha\pi y}\overline{\vartheta}(\frac{y}{\alpha})\Big)\\
&\geq \sqrt{2}\pi e^{-\frac{\pi\alpha}{4y}}(\frac{y}{\alpha})^{-\frac{3}{2}} - \beta e^{-\pi\alpha y}(\frac{y}{2\alpha})^{-\frac{3}{2}} - 4\sqrt{2}e^{-3\pi\alpha y}(\frac{y}{\alpha})^{-\frac{3}{2}}\\
&= e^{-\frac{\pi\alpha}{4y}}(\frac{y}{\alpha})^{-\frac{3}{2}}\Big(\sqrt{2}\pi - 2\sqrt{2}\beta e^{-\pi\alpha(y-\frac{1}{4y})} - 4\sqrt{2}e^{-\pi\alpha(3y-\frac{1}{4y})}\Big)
\end{aligned}
$$

Trivially $y - \frac{1}{4y} \geq \frac{\sqrt{3}}{3}$ and $3y - \frac{1}{4y} \geq \frac{4\sqrt{3}}{3}$ since $y \geq \frac{\sqrt{3}}{2}$. Hence

$$
\begin{aligned}
\mathcal{R}_{\alpha,\beta}(y) &\geq e^{-\frac{\pi\alpha}{4y}}(\frac{y}{\alpha})^{-\frac{3}{2}}\Big(\sqrt{2}\pi - 2\sqrt{2}\beta e^{-\pi\alpha(y-\frac{1}{4y})} - 4\sqrt{2}e^{-\pi\alpha(3y-\frac{1}{4y})}\Big)\\
&\geq e^{-\frac{\pi\alpha}{4y}}(\frac{y}{\alpha})^{-\frac{3}{2}}\Big(\sqrt{2}\pi - 2\sqrt{2}\beta e^{-\pi\alpha\frac{\sqrt{3}}{3}} - 4\sqrt{2}e^{-\pi\alpha\frac{4\sqrt{3}}{3}}\Big)\\
&\geq e^{-\frac{\pi\alpha}{4y}}(\frac{y}{\alpha})^{-\frac{3}{2}}\Big(\sqrt{2}\pi - 2\sqrt{2}\beta e^{-2\pi} - 4\sqrt{2}e^{-8\pi}\Big)\\
&> 0 \ \text{ if } \ \beta < \frac{\pi e^{2\pi} - 4e^{-6\pi}}{2}.
\end{aligned}
$$

Case B: $\frac{y}{\alpha}, \frac{y}{2\alpha} \in [\frac{1}{4}, \infty)$**.** In this case, $\frac{y}{\alpha} \geq \frac{1}{2}$ and there holds

$$
\begin{aligned}
\mathcal{R}_{\alpha,\beta}(y) &= \Big(\sqrt{2}\underline{\vartheta}(\frac{y}{\alpha}) - \beta e^{-\alpha\pi y}\overline{\vartheta}(\frac{y}{2\alpha}) - 4\sqrt{2}e^{-3\alpha\pi y}\overline{\vartheta}(\frac{y}{\alpha})\Big)\\
&\geq \Big(4\sqrt{2}\pi e^{-\pi\frac{y}{\alpha}}\big(1-\mu(\frac{y}{\alpha})\big) - 4\pi\beta e^{-\pi\alpha y}e^{-\pi\frac{y}{2\alpha}}\big(1+\mu(\frac{y}{2\alpha})\big)\\
&\qquad - 16\sqrt{2}\pi\beta e^{-3\pi\alpha y}e^{-\pi\frac{y}{\alpha}}\big(1+\mu(\frac{y}{\alpha})\big)\Big)\\
&= 4\pi e^{-\frac{y}{\alpha}}\Big(\sqrt{2}\big(1-\mu(\frac{y}{\alpha})\big) - \beta e^{-\pi y(\alpha-\frac{1}{2\alpha})}\big(1+\mu(\frac{y}{2\alpha})\big) - 4\sqrt{2}e^{-3\pi\alpha y}\big(1+\mu(\frac{y}{\alpha})\big)\Big)
\end{aligned}
$$

Trivially $e^{-\pi y(\alpha-\frac{1}{2\alpha})}\leq e^{-\frac{\sqrt{3}\pi}{4}}$, $\mu(\frac{y}{\alpha})\leq\mu(\frac{1}{2})$ and $\mu(\frac{y}{2\alpha})\leq\mu(\frac{1}{4})$. Hence we have

$$\begin{aligned}\mathcal{R}_{\alpha,\beta}(y)&\geq 4\pi e^{-\frac{y}{\alpha}}\Big(\sqrt{2}\big(1-\mu(\frac{y}{\alpha})\big)-\beta e^{-\pi y(\alpha-\frac{1}{2\alpha})}\big(1+\mu(\frac{y}{2\alpha})\big)-4\sqrt{2}e^{-3\pi\alpha y}\big(1+\mu(\frac{y}{\alpha})\big)\Big)\\&\geq 4\pi e^{-\frac{y}{\alpha}}\Big(\sqrt{2}\big(1-\mu(\frac{1}{2})\big)-\beta e^{-\frac{\sqrt{3}\pi}{4}}\big(1+\mu(\frac{1}{4})\big)-4\sqrt{2}e^{-\frac{3\sqrt{3}\pi}{2}}\big(1+\mu(\frac{1}{2})\big)\Big)\\&>0\ \text{ if }\beta<\frac{\sqrt{2}e^{\frac{\sqrt{3}\pi}{4}}(1-\mu(\frac{1}{2}))-4e^{-\frac{5\sqrt{3}\pi}{4}}(1+\mu(\frac{1}{4}))}{1+\mu(\frac{1}{4})}.\end{aligned}$$

Case C: $\frac{y}{\alpha}\in[\frac{1}{4},\infty)$, $\frac{y}{2\alpha}\in(0,\frac{1}{4}]$**.** In this case, $2y\leq\alpha\leq 4y$ and we have

$$\begin{aligned}\mathcal{R}_{\alpha,\beta}(y)&=\Big(\sqrt{2}\underline{\vartheta}(\frac{y}{\alpha})-\beta e^{-\alpha\pi y}\overline{\vartheta}(\frac{y}{2\alpha})-4\sqrt{2}e^{-3\alpha\pi y}\overline{\vartheta}(\frac{y}{\alpha})\Big)\\&\geq 4\sqrt{2}\pi e^{-\pi\frac{y}{\alpha}}\big(1-\mu(\frac{y}{\alpha})\big)-\beta e^{-\pi\alpha y}(\frac{y}{2\alpha})^{-\frac{3}{2}}-16\sqrt{2}\pi e^{-3\pi\alpha y}e^{-\pi\frac{y}{\alpha}}\big(1+\mu(\frac{y}{\alpha})\big)\\&=e^{-\pi\frac{y}{\alpha}}\Big(4\sqrt{2}\pi\big(1-\mu(\frac{y}{\alpha})\big)-\beta e^{-\pi y(\alpha-\frac{1}{\alpha})}(\frac{y}{2\alpha})^{-\frac{3}{2}}-16\sqrt{2}\pi e^{-3\pi\alpha y}\big(1+\mu(\frac{y}{\alpha})\big)\Big).\end{aligned}$$

Trivially $y(\alpha-\frac{1}{\alpha})\geq 1$ and $(\frac{y}{2\alpha})^{-\frac{3}{2}}\leq 64$, then

$$\begin{aligned}\mathcal{R}_{\alpha,\beta}(y)&\geq e^{-\pi\frac{y}{\alpha}}\Big(4\sqrt{2}\pi\big(1-\mu(\frac{y}{\alpha})\big)-\beta e^{-\pi y(\alpha-\frac{1}{\alpha})}(\frac{y}{2\alpha})^{-\frac{3}{2}}-16\sqrt{2}\pi e^{-3\pi\alpha y}\big(1+\mu(\frac{y}{\alpha})\big)\Big)\\&\geq e^{-\pi\frac{y}{\alpha}}\Big(4\sqrt{2}\pi\big(1-\mu(\frac{1}{4})\big)-64e^{-\pi}\beta-16\sqrt{2}\pi e^{-\frac{9\pi}{2}}\big(1+\mu(\frac{1}{4})\big)\Big)\\&>0\ \text{ if }\ \beta<\frac{4\sqrt{2}\pi e^{\pi}\big(1-\mu(\frac{1}{4})\big)-16\sqrt{2}\pi e^{-\frac{7\pi}{2}}\big(1+\mu(\frac{1}{4})\big)}{64}.\end{aligned}$$

□

Finally we complete the proof of Theorem 3.2.

Proof. Combining Lemma 3.5 with Lemmas 3.3 and 3.4, yield that

$$\mathcal{E}_{\alpha,\beta,x}(z)\begin{cases}>0, & \text{for }\ x\in(0,\frac{1}{3}),\alpha\geq1,y\geq\frac{\sqrt{3}}{2},\\>0, & \text{for }\ x\in[\frac{1}{3},\frac{1}{2}),\alpha\geq1,y\geq\frac{\sqrt{3}}{2},\end{cases}\tag{3.8}$$

where $\mathcal{E}_{\alpha,\beta,x}(z)$ is defined in Lemma 3.2.

Since trivially

$$\mathcal{D}_{\mathcal{G}}\subseteq\Big(\{z|:x\in[0,\frac{1}{3}],y\geq\frac{\sqrt{3}}{2}\}\cup\{z|:x\in[\frac{1}{3},\frac{1}{2}],y\geq\frac{\sqrt{3}}{2}\}\Big).\tag{3.9}$$

(3.8) and (3.9) yield

$$\mathcal{E}_{\alpha,\beta,x}(z)>0,\ \text{ for }\ \alpha\geq1\ \text{ and }\ z\in\mathcal{D}_{\mathcal{G}}.\tag{3.10}$$

Therefore, (3.10) and Lemma 3.2 complete the proof of Theorem 3.2.

□

4. The monotonicity on the vertical line $y=\frac{1}{2}$ and proof of Theorem 1.1

In Theorem 3.1, we have established that for $\alpha\geq1,\beta<\beta_0\cong3.8$,

$$\min_{z\in\mathbb{H}}\Big(\theta(\alpha;z)-\beta\theta(2\alpha;z)\Big)=\min_{z\in\Gamma}\Big(\theta(\alpha;z)-\beta\theta(2\alpha;z)\Big),\tag{4.1}$$

where Γ is a vertical line and defined at (3.1).

The following Lemma, which proves the non-existence part of Theorem 1.1, shows that one only needs to consider the minimum of $\Big(\theta(\alpha;z)-\beta\theta(2\alpha;z)\Big)$ for $\beta\leq\sqrt{2}$.

Lemma 4.1. *The minimum of* $\Big(\theta(\alpha;z)-\beta\theta(2\alpha;z)\Big)$ *does not exist if* $\beta>\sqrt{2}$.

We postpone the proof of Lemma 4.1 to the late section. Combining (4.1) of Theorem 3.1 and Lemma 4.1, it suffices to consider minimization problem on a vertical line Γ for $\beta \leq \sqrt{2}$. For this, we establish the following, which proves the first part of Theorem 1.1.

Theorem 4.1. *Assume that $\alpha \geq 1$. For $\beta \leq \sqrt{2}$,*

$$\min_{z\in\Gamma}\Big(\theta(\alpha;z)-\beta\theta(2\alpha;z)\Big) \text{ is achieved uniquely at } \frac{1}{2}+i\frac{\sqrt{3}}{2}.$$

Since $\frac{\partial}{\partial y}\theta(\alpha;\frac{1}{2}+iy)\geq 0$ for $y\geq\frac{\sqrt{3}}{2}$ with equality attained at $\frac{\sqrt{3}}{2}$ ([27]), it suffices to prove the critical case of Theorem 4.1, namely,

Theorem 4.2. *Assume that $\alpha \geq 1$ and $\beta = \sqrt{2}$. Then*

$$\min_{z\in\Gamma}\Big(\theta(\alpha;z)-\sqrt{2}\theta(2\alpha;z)\Big) \text{ is achieved uniquely at } \frac{1}{2}+i\frac{\sqrt{3}}{2}.$$

In the rest of this section, we aim to prove Theorem 4.2 (which is a consequence of Lemma 4.4 and 4.6). Due to its difficulty and complexity, we shall divide its proof into two cases.

We first establish the following

Lemma 4.2. *Assume that $\alpha \geq 1$ and $y \geq \frac{\sqrt{3}}{2}$. Then we have the following lower bound estimate*

$$(\frac{\partial^2}{\partial y^2}+\frac{2}{y}\frac{\partial}{\partial y})\Big(\theta(\alpha;\frac{1}{2}+iy)-\sqrt{2}\theta(2\alpha;\frac{1}{2}+iy)\Big)\geq\frac{2(\pi\alpha)^2}{y^4}e^{-\pi\frac{\alpha}{y}}\cdot\mathcal{W}(y;\alpha),$$

where

$$\mathcal{W}(y;\alpha):=1+(2(y^2-\frac{1}{4})^2-\frac{4}{\pi\alpha}y^3\cdot(1+\epsilon_a))\cdot e^{-\pi\alpha(y-\frac{3}{4y})}-4\sqrt{2}(1+\epsilon_b)\cdot e^{-\pi\frac{\alpha}{y}}. \tag{4.2}$$

Here ϵ_a is small and can be explicitly controlled by

$$\epsilon_a:=\epsilon_{a,1}+\epsilon_{a,2}+\epsilon_{a,3}+\epsilon_{a,4}.$$

and

$$\epsilon_a\rightarrow 0 \;\; as \;\; y\mapsto\infty.$$

Here each $\epsilon_{a,j}(j=1,2,3,4)$ is small and expressed by

$$\begin{aligned}
\epsilon_{a,1}&:=\sum_{n=2}^{\infty}(2n-1)^2e^{-\pi\alpha y((2n-1)^2-1)}\\
\epsilon_{a,2}&:=\sum_{n=2}^{\infty}e^{-\frac{\pi\alpha}{4y}((2n-1)^2-1)}\\
\epsilon_{a,3}&:=\epsilon_{a,1}\cdot\epsilon_{a,2}\\
\epsilon_{a,4}&:=2e^{-\pi\alpha(3y-\frac{1}{4y})}\big(1+\sum_{n=2}^{\infty}n^2e^{-4\pi\alpha y(n^2-1)}\big)\cdot\vartheta_3(\frac{\alpha}{y}).
\end{aligned}$$

And ϵ_b is small and consist of four smaller parts

$$\epsilon_b:=\epsilon_{b,1}+\epsilon_{b,2}+\epsilon_{b,3}+\epsilon_{b,4},$$

and

$$\epsilon_b\rightarrow 0 \;\; as \;\; y\mapsto\infty.$$

Here

$$\begin{aligned}
\epsilon_{b,1} :&= 2y^4 e^{-2\pi\alpha y}\cdot(1+\sum_{n=2}^{\infty} e^{-\frac{2\pi\alpha}{y}((2n-1)^2-1)})\cdot(1+\sum_{n=2}^{\infty}(2n-1)^4 e^{-2\pi\alpha y((2n-1)^2-1)})\\
\epsilon_{b,2} :&= \frac{1}{8}e^{-2\pi\alpha y}\cdot(1+\sum_{n=2}^{\infty}(2n-1)^4 e^{-\frac{2\pi\alpha}{y}((2n-1)^2-1)})\cdot(1+\sum_{n=2}^{\infty} e^{-2\pi\alpha y((2n-1)^2-1)})\\
\epsilon_{b,3} :&= 16y^4 e^{-\pi\alpha(8y-\frac{2}{y})}\cdot(1+\sum_{n=2}^{\infty} n^4 e^{-8\pi\alpha y(n^2-1)})\cdot(1+2\sum_{n=1}^{\infty} e^{-2\pi\frac{\alpha}{y}n^2})\\
\epsilon_{b,4} :&= y^4 e^{-\pi\alpha(8y-\frac{2}{y})}\cdot(1+\sum_{n=2}^{\infty} e^{-8\pi\alpha y(n^2-1)})\cdot(1+2\sum_{n=1}^{\infty}\frac{n^4}{y^4} e^{-2\pi\frac{\alpha}{y}n^2}).
\end{aligned}$$

An elementary estimate of $\mathcal{W}(y;\alpha)$ in Lemma 4.2, we obtain

Lemma 4.3. *Assume that $\alpha\geq 1$. If $y\in[\frac{\sqrt{3}}{2},1.8\alpha]$,*

$$(\frac{\partial^2}{\partial y^2}+\frac{2}{y}\frac{\partial}{\partial y})\Big(\theta(\alpha;\frac{1}{2}+iy)-\sqrt{2}\theta(2\alpha;\frac{1}{2}+iy)\Big)>0.$$

In view of Lemma 4.3, one has

Lemma 4.4. *Assume that $\alpha\geq 1$. If $y\in[\frac{\sqrt{3}}{2},1.8\alpha]$,*

$$\frac{\partial}{\partial y}\Big(\theta(\alpha;\frac{1}{2}+iy)-\sqrt{\theta}(2\alpha;\frac{1}{2}+iy)\Big)>0.$$

Proof. The proof follows from Lemma 4.3. In fact it suffices to notice that

$$\begin{aligned}
&\frac{\partial}{\partial y}\Big(y^{-2}\Big(\frac{\partial}{\partial y}\big(\theta(\alpha;\frac{1}{2}+iy)-\sqrt{\theta}(2\alpha;\frac{1}{2}+iy)\big)\Big)\Big)\\
=&(\frac{\partial^2}{\partial y^2}+\frac{2}{y}\frac{\partial}{\partial y})\Big(\theta(\alpha;\frac{1}{2}+iy)-\sqrt{2}\theta(2\alpha;\frac{1}{2}+iy)\Big)
\end{aligned}$$

and

$$\Big(\frac{\partial}{\partial y}\big(\theta(\alpha;\frac{1}{2}+iy)-\sqrt{\theta}(2\alpha;\frac{1}{2}+iy)\big)\Big)\mid_{y=\frac{\sqrt{3}}{2}}=0.$$

□

Lemma 4.5. *The following estimates hold*

$$\frac{\partial}{\partial y}\Big(\theta(\alpha;\frac{1}{2}+iy)-\sqrt{2}\theta(2\alpha;\frac{1}{2}+iy)\Big)\geq\frac{2}{\sqrt{y\alpha}}e^{-\pi\frac{y}{2\alpha}}\cdot\mathcal{Q}(y;\alpha),$$

where

$$\mathcal{Q}(y;\alpha):=\frac{\pi y}{2\alpha}-\frac{1}{2}-(\frac{\pi}{\alpha}-\frac{1}{2})e^{-\pi\frac{y}{2\alpha}}-ye^{-\pi y(\alpha-\frac{1}{2\alpha})}\mathcal{P}(y;\alpha)-ye^{-\pi y(2\alpha-\frac{1}{2\alpha})}\mathcal{P}(y;2\alpha).$$

$$\mathcal{P}(y;\alpha):=\sigma_5+\sigma_6+\sigma_7.$$

Here

$$\begin{aligned}
\sigma_5 :&= \frac{1}{2y}\big(1+2e^{-\pi\frac{y}{\alpha}}(1+\nu(\frac{y}{\alpha}))\big)(1+\nu(y\alpha))\\
\sigma_6 :&= \alpha\pi(1+\mu(y\alpha))(1+2e^{-\pi\frac{y}{\alpha}}(1+\nu(\frac{y}{\alpha})))\\
\sigma_7 :&= \frac{2\pi}{\alpha}e^{-\pi\frac{y}{\alpha}}(1+\nu(y\alpha))(1+\mu(\frac{y}{\alpha})).
\end{aligned}$$

$$\mu(y) := \sum_{n=2}^{\infty} n^2 e^{-\pi y(n^2-1)}$$
$$\nu(y) := \sum_{n=2}^{\infty} e^{-\pi y(n^2-1)}$$

Based on Lemma 4.5, an elementary analysis of $\mathcal{Q}(y;\alpha)$ yields

Lemma 4.6. *Assume that $\alpha \geq 1$. If $y \in [1.15\alpha, \infty]$,*

$$\frac{\partial}{\partial y}\Big(\theta(\alpha; \frac{1}{2}+iy) - \sqrt{2}\theta(2\alpha; \frac{1}{2}+iy)\Big) > 0.$$

Trivially, for $\alpha \geq 1$,

$$[\frac{\sqrt{3}}{2}, \infty) \subseteq [\frac{\sqrt{3}}{2}, 1.8\alpha] \cup [1.15\alpha, \infty)$$

then Lemmas 4.6 and 4.4 complete the proof of Theorem 4.2.

In the rest of this Section, we provide the proof of Lemmas 4.6 and 4.4.

4.1. **Some basic identities.**

Lemma 4.7. *The following identity for $\big(\frac{\partial^2}{\partial y^2} + \frac{2}{y}\frac{\partial}{\partial y}\big)\big(\theta(\alpha;z) - \sqrt{2}\theta(2\alpha;z)\big)$ holds*

$$\begin{aligned}
\big(\frac{\partial^2}{\partial y^2} + \frac{2}{y}\frac{\partial}{\partial y}\big)\big(\theta(\alpha;z) - \sqrt{2}\theta(2\alpha;z)\big) &= (\pi\alpha)^2 \sum_{n,m} (n^2 - \frac{(m+nx)^2}{y^2})^2 e^{-\pi\alpha(yn^2+\frac{(m+nx)^2}{y})} \\
&+ \frac{4\sqrt{2}\pi\alpha}{y} \sum_{n,m} n^2 e^{-2\pi\alpha(yn^2+\frac{(m+nx)^2}{y})} - \frac{2\pi\alpha}{y} \sum_{n,m} n^2 e^{-\pi\alpha(yn^2+\frac{(m+nx)^2}{y})} \\
&- 4\sqrt{2}(\pi\alpha)^2 \sum_{n,m} (n^2 - \frac{(m+nx)^2}{y^2})^2 e^{-2\pi\alpha(yn^2+\frac{(m+nx)^2}{y})}
\end{aligned}$$

Proof. By definition of the theta function (1.2), we have

$$\frac{\partial}{\partial y}\theta(\alpha;z) = \pi\alpha \sum_{n,m} n^2 e^{-\pi\alpha(yn^2+\frac{(m+nx)^2}{y})} - \pi\alpha \sum_{n,m} \frac{(m+nx)^2}{y^2} e^{-\pi\alpha(yn^2+\frac{(m+nx)^2}{y})}$$

and

$$\begin{aligned}
\big(\frac{\partial^2}{\partial y^2} + \frac{2}{y}\frac{\partial}{\partial y}\big)\theta(\alpha;z) =& (\pi\alpha)^2 \sum_{n,m} (n^2 - \frac{(m+nx)^2}{y^2})^2 e^{-\pi\alpha(yn^2+\frac{(m+nx)^2}{y})} \\
&- \frac{2\pi\alpha}{y} \sum_{n,m} n^2 e^{-\pi\alpha(yn^2+\frac{(m+nx)^2}{y})}.
\end{aligned} \tag{4.3}$$

The identity follows by (4.3).

□

Similar to the proof of Lemma 4.7 (the details are omit here), one has

Lemma 4.8. *It holds*

$$\begin{aligned}
\frac{\partial}{\partial y}\big(\theta(\alpha;z) - \sqrt{2}\theta(2\alpha;z)\big) &= \pi\alpha \sum_{n,m} \frac{(m+nx)^2}{y^2} e^{-\pi\alpha(yn^2+\frac{(m+nx)^2}{y})} + 2\sqrt{2}\pi\alpha \sum_{n,m} n^2 e^{-2\pi\alpha(yn^2+\frac{(m+nx)^2}{y})} \\
&- \pi\alpha \sum_{n,m} n^2 e^{-\pi\alpha(yn^2+\frac{(m+nx)^2}{y})} - 2\sqrt{2}\pi\alpha \sum_{n,m} \frac{(m+nx)^2}{y^2} e^{-2\pi\alpha(yn^2+\frac{(m+nx)^2}{y})}
\end{aligned}$$

4.2. **The analysis of $\frac{\partial}{\partial y}\Big(\theta(\alpha;z)-\sqrt{2}\theta(2\alpha;z)\Big)$.** We use the following expression of theta function, which is a variant of Lemma 3.1.

Lemma 4.9. *A variant expression of $\theta(\alpha;z)$ is the following*

$$\begin{aligned}\theta(\alpha;z)&=2\sqrt{\frac{y}{\alpha}}\sum_{n=1}^{\infty}e^{-\alpha\pi yn^2}\vartheta(\frac{y}{\alpha};nx)+\sqrt{\frac{y}{\alpha}}\vartheta_3(\frac{y}{\alpha})\\&=\sqrt{\frac{y}{\alpha}}\big(1+2\sum_{n=1}^{\infty}e^{-n^2\pi\frac{y}{\alpha}}\big)+2\sqrt{\frac{y}{\alpha}}\sum_{n=1}^{\infty}e^{-\alpha\pi yn^2}\vartheta(\frac{y}{\alpha};nx).\end{aligned}\tag{4.4}$$

Now we give the proof of Lemma 4.1:

Proof. In view of Lemma 4.9, one has

$$\theta(\alpha;z)=\sqrt{\frac{y}{\alpha}}\cdot\big(1+2e^{-\pi\frac{y}{\alpha}}+2e^{-\alpha\pi y}+o(e^{-\pi\frac{y}{\alpha}})+o(e^{-\alpha\pi y})\big).$$

Then

$$\begin{aligned}\theta(\alpha;z)-\beta\theta(2\alpha;z)&=\sqrt{\frac{y}{2\alpha}}\cdot\Big(\sqrt{2}-\beta+2\sqrt{2}e^{-\pi\alpha y}-2e^{-\pi\frac{y}{2\alpha}}+o(e^{-\pi\alpha y})+o(e^{-\pi\frac{y}{2\alpha}})\Big),\\&=\sqrt{\frac{y}{2\alpha}}\cdot\Big(\sqrt{2}-\beta+o(1)\Big)\end{aligned}$$

Therefore, for $\forall\alpha>0$,

$$\begin{aligned}\theta(\alpha;z)-\beta\theta(2\alpha;z)&=\sqrt{\frac{y}{2\alpha}}\cdot\Big(\sqrt{2}-\beta+o(1)\Big)\\&\mapsto-\infty,\quad\text{if}\quad\beta>\sqrt{2},\end{aligned}$$

proves the nonexistence result.

□

In the next two lemmas(Lemmas 4.10 and 4.11), we analyze the two parts of $\theta(\alpha;z)$ in Lemma 4.9.

Lemma 4.10 (Analysis of second part arising from Lemma 4.9)**.** *Assume that $\alpha\geq1$. If $\frac{y}{\alpha}\geq\frac{4}{5}$, then*

$$\frac{\partial}{\partial y}\Big(\sqrt{y}\Big(\vartheta_3(\frac{y}{\alpha})-\vartheta_3(\frac{y}{2\alpha})\Big)\Big)>0.$$

Proof. By a straightforward computation, we have

$$\begin{aligned}&\frac{\partial}{\partial y}\Big(\sqrt{y}\Big(\vartheta_3(\frac{y}{\alpha})-\vartheta_3(\frac{y}{2\alpha})\Big)\Big)\\=&2\frac{\partial}{\partial y}\Big(\sqrt{y}\sum_{n=1}^{\infty}\big(e^{-n^2\pi\frac{y}{\alpha}}-e^{-n^2\pi\frac{y}{2\alpha}}\big)\Big)\\=&\frac{2}{\sqrt{y}}\sum_{n=1}^{\infty}e^{-n^2\pi\frac{y}{\alpha}}\Big((\frac{n^2\pi y}{2\alpha}-\frac{1}{2})e^{n^2\pi\frac{y}{2\alpha}}-(\frac{n^2\pi}{\alpha}-\frac{1}{2})\Big)\end{aligned}\tag{4.5}$$

Since $\alpha\geq1$, $\frac{y}{\alpha}\geq\frac{4}{5}$, then

$$\begin{aligned}(\frac{\pi y}{2\alpha}-\frac{1}{2})e^{\pi\frac{y}{2\alpha}}-(\frac{\pi}{\alpha}-\frac{1}{2})&\geq(\frac{\pi y}{2\alpha}-\frac{1}{2})e^{\pi\frac{y}{2\alpha}}-(\pi-\frac{1}{2})\\&>0.\end{aligned}$$

Therefore, since each term in the sum of (4.5) is positive, the result then follows. □

To control the error terms, we recall

$$\mu(y) := \sum_{n=2}^{\infty} n^2 e^{-\pi y(n^2-1)}$$
$$\nu(y) := \sum_{n=2}^{\infty} e^{-\pi y(n^2-1)}$$

Lemma 4.11 (The estimate of first part in Lemma 4.9)**.**

$$\frac{\partial}{\partial y}\Big(\sqrt{y}\sum_{n=1}^{\infty} e^{-\alpha\pi y n^2}\vartheta(\frac{y}{\alpha};nx)\Big) \leq \sqrt{y}e^{-\pi y\alpha}\cdot\mathcal{P}(y;\alpha)$$

where the $\mathcal{P}(y;\alpha)$ can be controlled by some constant and is expressed by

$$\mathcal{P}(y;\alpha) := \sigma_5+\sigma_6+\sigma_7,$$

and

$$\mathcal{P}(y;\alpha) < \alpha\pi, \text{ as } y\rightarrow\infty.$$

Here

$$\sigma_5 := \frac{1}{2y}\big(1+2e^{-\pi\frac{y}{\alpha}}(1+\nu(\frac{y}{\alpha}))\big)(1+\nu(y\alpha))$$
$$\sigma_6 := \alpha\pi(1+\mu(y\alpha))(1+2e^{-\pi\frac{y}{\alpha}}(1+\nu(\frac{y}{\alpha})))$$
$$\sigma_7 := \frac{2\pi}{\alpha}e^{-\pi\frac{y}{\alpha}}(1+\nu(y\alpha))(1+\mu(\frac{y}{\alpha})).$$

Remark 4.1. *Lemma 4.11 shows that*

$$\frac{\partial}{\partial y}\Big(\sqrt{y}\sum_{n=1}^{\infty} e^{-\alpha\pi y n^2}\vartheta(\frac{y}{\alpha};nx)\Big)$$

is small in related estimates.

Proof. A direct calculation shows that

$$\frac{\partial}{\partial y}\Big(\sqrt{y}\sum_{n=1}^{\infty} e^{-\alpha\pi y n^2}\vartheta(\frac{y}{\alpha};nx)\Big) =\frac{1}{2\sqrt{y}}\sum_{n=1}^{\infty} e^{-\alpha\pi y n^2}\vartheta(\frac{y}{\alpha};nx)+\sqrt{y}\sum_{n=1}^{\infty}(-\alpha\pi y)e^{-\alpha\pi y n^2}\vartheta(\frac{y}{\alpha};nx)$$
$$+\sqrt{y}\sum_{n=1}^{\infty}(-\alpha\pi y)e^{-\alpha\pi y n^2}\frac{1}{\alpha}\frac{\partial}{\partial X}\vartheta(\frac{y}{\alpha};nx).$$

For convenience, we denote that

$$I_1 :=\frac{1}{2\sqrt{y}}\sum_{n=1}^{\infty} e^{-\alpha\pi y n^2}\vartheta(\frac{y}{\alpha};nx)$$
$$I_2 :=\sqrt{y}\sum_{n=1}^{\infty}(-\alpha\pi n^2)e^{-\alpha\pi y n^2}\vartheta(\frac{y}{\alpha};nx)$$
$$I_3 :=\sqrt{y}\sum_{n=1}^{\infty} e^{-\alpha\pi y n^2}\frac{1}{\alpha}\frac{\partial}{\partial X}\vartheta(\frac{y}{\alpha};nx).$$

Then

$$\frac{\partial}{\partial y}\Big(\sqrt{y}\sum_{n=1}^{\infty} e^{-\alpha\pi y n^2}\vartheta(\frac{y}{\alpha};nx)\Big) = I_1+I_2+I_3. \tag{4.6}$$

Next, we estimate $I_j, j=1,2,3$ in order. For I_1,

$$
\begin{aligned}
|I_1| &= |\frac{1}{2\sqrt{y}}\sum_{n=1}^{\infty} e^{-\alpha\pi y n^2}\vartheta(\frac{y}{\alpha};nx)| \\
&\leq \frac{1}{2\sqrt{y}}\sum_{n=1}^{\infty} e^{-\alpha\pi y n^2}|\vartheta(\frac{y}{\alpha};nx)| \\
&\leq \frac{1}{2\sqrt{y}}\sum_{n=1}^{\infty} e^{-\alpha\pi y n^2}(1+2\sum_{n=1}^{\infty} e^{-n^2\pi\frac{y}{\alpha}}) \\
&= \frac{1}{2\sqrt{y}}e^{-\alpha\pi y}\big(1+\sum_{n=2}^{\infty} e^{-\alpha\pi y(n^2-1)}\big)\cdot\big(1+2e^{-\pi\frac{y}{\alpha}}(1+\sum_{n=2}^{\infty} e^{-\pi\frac{y}{\alpha}(n^2-1)})\big) \\
&= \frac{1}{2\sqrt{y}}e^{-\alpha\pi y}\big(1+\nu(y\alpha)\big)\big(1+2e^{-\pi\frac{y}{\alpha}}(1+\nu(\frac{y}{\alpha}))\big).
\end{aligned}
\tag{4.7}
$$

And I_2,

$$
\begin{aligned}
|I_2| &= |\sqrt{y}\sum_{n=1}^{\infty}(-\alpha\pi n^2)e^{-\alpha\pi y n^2}\vartheta(\frac{y}{\alpha};nx)| \\
&\leq \alpha\pi\sqrt{y}\sum_{n=1}^{\infty} n^2 e^{-\alpha\pi y n^2}|\vartheta(\frac{y}{\alpha};nx)| \\
&\leq \alpha\pi\sqrt{y}\sum_{n=1}^{\infty} n^2 e^{-\alpha\pi y n^2}\vartheta_3(\frac{y}{\alpha}) \\
&= \alpha\pi\sqrt{y}\sum_{n=1}^{\infty} n^2 e^{-\alpha\pi y n^2}\big(1+2\sum_{n=1}^{\infty} e^{-\pi n^2\frac{y}{\alpha}}\big) \\
&= \alpha\pi\sqrt{y}e^{-\alpha\pi y}\big(1+\sum_{n=2}^{\infty} e^{-\alpha\pi y(n^2-1)}\big)\cdot\big(1+2e^{-\pi\frac{y}{\alpha}}(1+\sum_{n=2}^{\infty} e^{-\pi(n^2-1)\frac{y}{\alpha}})\big) \\
&= \alpha\pi\sqrt{y}e^{-\alpha\pi y}\big(1+\mu(y\alpha)\big)\cdot\big(1+2e^{-\pi\frac{y}{\alpha}}(1+\nu(\frac{y}{\alpha}))\big).
\end{aligned}
\tag{4.8}
$$

The I_3 is estimated by

$$
\begin{aligned}
|I_3| =& |\sqrt{y}\sum_{n=1}^{\infty} e^{-\alpha\pi y n^2}\frac{1}{\alpha}\frac{\partial}{\partial X}\vartheta(\frac{y}{\alpha};nx)| \\
\leq& 2\pi\sqrt{y}\frac{1}{\alpha}\sum_{n=1}^{\infty} e^{-\alpha\pi y n^2}\cdot\sum_{n=1}^{\infty} n^2 e^{-\pi n^2\frac{y}{\alpha}} \\
=& \frac{2}{\alpha}\pi\sqrt{y}e^{-\alpha\pi y}e^{-\pi\frac{y}{\alpha}}\big(1+\sum_{n=2}^{\infty} e^{-\alpha\pi y(n^2-1)}\big)\cdot\sum_{n=2}^{\infty} n^2 e^{-\pi(n^2-1)\frac{y}{\alpha}} \\
=& \frac{2}{\alpha}\pi\sqrt{y}e^{-\pi y(\alpha+\frac{1}{\alpha})}\cdot(1+\nu(y\alpha))\cdot(1+\mu(\frac{y}{\alpha})).
\end{aligned}
\tag{4.9}
$$

The (4.6) togethers with (4.7), (4.8) and (4.9) yield the result.

□

The following lemma is a variant of Lemma 4.5.

Lemma 4.12. *Assume that $\alpha \geq 1$. If $\frac{y}{\alpha} \geq \frac{4}{5}$, then it holds*

$$
\frac{\partial}{\partial y}\Big(\theta(\alpha;z)-\sqrt{2}\theta(2\alpha;z)\Big) \geq \frac{2}{\sqrt{y\alpha}}e^{-\pi\frac{y}{2\alpha}}\cdot\mathcal{Q}(y;\alpha),
$$

where

$$\mathcal{Q}(y;\alpha):=\frac{\pi y}{2\alpha}-\frac{1}{2}-(\frac{\pi}{\alpha}-\frac{1}{2})e^{-\pi\frac{y}{2\alpha}}-ye^{-\pi y(\alpha-\frac{1}{2\alpha})}\mathcal{P}(y;\alpha)-ye^{-\pi y(2\alpha-\frac{1}{2\alpha})}\mathcal{P}(y;2\alpha),$$

and $\mathcal{P}(y;\alpha)$ *is introduced in Lemma 4.11.*

Proof. By Lemmas 4.9, 4.10 and 4.11,

$$\begin{aligned}\frac{\sqrt{\alpha}}{2}\frac{\partial}{\partial y}\Big(\theta(\alpha;z)-\sqrt{2}\theta(2\alpha;z)\Big)&\geq\frac{1}{\sqrt{y}}\sum_{n=1}^{\infty}e^{-n^2\pi\frac{y}{\alpha}}\Big((\frac{n^2\pi y}{2\alpha}-\frac{1}{2})e^{n^2\pi\frac{y}{2\alpha}}-(\frac{n^2\pi}{\alpha}-\frac{1}{2})\Big)\\&-\sqrt{y}e^{-\pi y\alpha}\mathcal{P}(y;\alpha)-\sqrt{y}e^{-2\pi y\alpha}\mathcal{P}(y;2\alpha)\\&=\frac{1}{\sqrt{y}}e^{-\pi\frac{y}{\alpha}}\Big((\frac{\pi y}{2\alpha}-\frac{1}{2})e^{\pi\frac{y}{2\alpha}}-(\frac{\pi}{\alpha}-\frac{1}{2})\Big)\\&-\sqrt{y}e^{-\pi y\alpha}\mathcal{P}(y;\alpha)-\sqrt{y}e^{-2\pi y\alpha}\mathcal{P}(y;2\alpha)\\&+\frac{1}{\sqrt{y}}\sum_{n=2}^{\infty}e^{-n^2\pi\frac{y}{\alpha}}\Big((\frac{n^2\pi y}{2\alpha}-\frac{1}{2})e^{n^2\pi\frac{y}{2\alpha}}-(\frac{n^2\pi}{\alpha}-\frac{1}{2})\Big)\\&=\frac{1}{\sqrt{y}}e^{-\pi\frac{y}{2\alpha}}\cdot\mathcal{Q}(y;\alpha)+\mathcal{R}_0(y;\alpha).\end{aligned}\tag{4.10}$$

Here

$$\mathcal{R}_0(y;\alpha):=\frac{1}{\sqrt{y}}\sum_{n=2}^{\infty}e^{-n^2\pi\frac{y}{\alpha}}\Big((\frac{n^2\pi y}{2\alpha}-\frac{1}{2})e^{n^2\pi\frac{y}{2\alpha}}-(\frac{n^2\pi}{\alpha}-\frac{1}{2})\Big).$$

Since $\alpha\geq1$, $\frac{y}{\alpha}\geq\frac{4}{5}$,

$$\begin{aligned}(\frac{\pi y}{2\alpha}-\frac{1}{2})e^{\pi\frac{y}{2\alpha}}-(\frac{\pi}{\alpha}-\frac{1}{2})&\geq(\frac{\pi y}{2\alpha}-\frac{1}{2})e^{\pi\frac{y}{2\alpha}}-(\pi-\frac{1}{2})\\&>0.\end{aligned}$$

Then trivially,

$$\mathcal{R}_0(y;\alpha)>0.$$

Then the estimate follows by (4.10).

∎

Lemma 4.13 (The upper bounds of $y\cdot\mathcal{P}(y;\alpha)$ and $y\cdot\mathcal{P}(y;2\alpha)$)**.** *Assume that* $\alpha\geq1, y\geq\frac{\sqrt{3}}{2}$*. If* $\frac{y}{\alpha}\geq1$*, then*

$$\begin{aligned}y\cdot\mathcal{P}(y;\alpha)&\leq4.232412\cdots,\\y\cdot\mathcal{P}(y;2\alpha)&\leq10.268696\cdots.\end{aligned}$$

Proof. In view of the expression $y\cdot\mathcal{P}(y;\alpha), y\cdot\mathcal{P}(y;2\alpha)$. The only technical part is to control

$$\alpha ye^{-\pi y(\alpha-\frac{1}{2\alpha})},\alpha ye^{-\pi y(2\alpha-\frac{1}{2\alpha})}.$$

These two terms are similar. We estimate by

$$\begin{aligned}\alpha ye^{-\pi y(\alpha-\frac{1}{2\alpha})}&=\alpha^2\cdot\frac{y}{\alpha}e^{-\pi\alpha^2\cdot\frac{y}{\alpha}\cdot(1-\frac{1}{2\alpha^2})}\\&\geq\frac{y}{\alpha}e^{-\pi\frac{y}{\alpha}\cdot(1-\frac{1}{2\alpha^2})},\end{aligned}\tag{4.11}$$

where we shall control the growth of α by the monotonically decreasing of $xe^{-A\cdot x}$ as $x\geq\frac{1}{A}$. Similar to (4.11)

$$\alpha ye^{-\pi y(2\alpha-\frac{1}{2\alpha})}\geq\frac{y}{\alpha}e^{-\pi\frac{y}{\alpha}\cdot(2-\frac{1}{2\alpha^2})}.\tag{4.12}$$

Then we can view $\frac{y}{\alpha}$ as an variable in estimates. The rest of estimate using the decreasing of μ, ν. Namely,

$$\mu(y\alpha) \le \mu(\frac{\sqrt{3}}{2}), \mu(y\alpha) \le \mu(1);$$
$$\nu(y\alpha) \le \nu(\frac{\sqrt{3}}{2}), \nu(y\alpha) \le \nu(1).$$

Here $\alpha \ge 1, y \ge \frac{\sqrt{3}}{2}$ and $\frac{y}{\alpha} \ge 1$ used. □

Lemma 4.14. *Assume that $\alpha \ge 1$, if $\frac{y}{\alpha} \ge 1.15$, then*

$$\mathcal{Q}(y;\alpha) > 0,$$

where

$$\mathcal{Q}(y;\alpha) = \frac{\pi y}{2\alpha} - \frac{1}{2} - (\frac{\pi}{\alpha} - \frac{1}{2})e^{-\pi\frac{y}{2\alpha}} - ye^{-\pi y(\alpha - \frac{1}{2\alpha})}\mathcal{P}(y;\alpha) - ye^{-\pi y(2\alpha - \frac{1}{2\alpha})}\mathcal{P}(y;2\alpha)$$

is defined in Lemma 4.12.

Proof. The proof follows from Lemma 4.13:

$$\begin{aligned}
\mathcal{Q}(y;\alpha) &= \frac{\pi y}{2\alpha} - \frac{1}{2} - (\frac{\pi}{\alpha} - \frac{1}{2})e^{-\pi\frac{y}{2\alpha}} - ye^{-\pi y(\alpha - \frac{1}{2\alpha})}\mathcal{P}(y;\alpha) - ye^{-\pi y(2\alpha - \frac{1}{2\alpha})}\mathcal{P}(y;2\alpha)\\
&\ge \frac{\pi y}{2\alpha} - \frac{1}{2} - (\pi - \frac{1}{2})e^{-\pi\frac{y}{2\alpha}} - 4.5e^{-\pi y(\alpha - \frac{1}{2\alpha})} - 10.5e^{-\pi y(2\alpha - \frac{1}{2\alpha})}\\
&= \frac{\pi}{2}\frac{y}{\alpha} - \frac{1}{2} - (\pi - \frac{1}{2})e^{-\frac{\pi}{2}\frac{y}{\alpha}} - 4.5e^{-\alpha^2\cdot\pi\frac{y}{\alpha}(1 - \frac{1}{2\alpha^2})} - 10.5e^{-\alpha^2\cdot\pi\frac{y}{\alpha}(2 - \frac{1}{2\alpha^2})}\\
&\ge \frac{\pi}{2}\frac{y}{\alpha} - \frac{1}{2} - (\pi + 4)e^{-\frac{\pi}{2}\frac{y}{\alpha}} - 10.5e^{-\frac{3\pi}{2}\frac{y}{\alpha}},
\end{aligned} \tag{4.13}$$

where $\alpha \ge 1$ is used. Next, a simple calculation shows that

$$\frac{\pi}{2}\cdot x - \frac{1}{2} - (\pi + 4)e^{-\frac{\pi}{2}\cdot x} - 10.5e^{-\frac{3\pi}{2}\cdot x} > 0 \Leftrightarrow x > 1.126371\cdots.$$

Then by (4.14),

$$\begin{aligned}
\mathcal{Q}(y;\alpha) &\ge \frac{\pi}{2}\frac{y}{\alpha} - \frac{1}{2} - (\pi + 4)e^{-\frac{\pi}{2}\frac{y}{\alpha}} - 10.5e^{-\frac{3\pi}{2}\frac{y}{\alpha}}\\
&> 0 \quad \text{if} \quad \frac{y}{\alpha} > 1.126371\cdots,
\end{aligned} \tag{4.14}$$

yields the result. □

4.3. **The estimates of $(\frac{\partial^2}{\partial y^2} + \frac{2}{y}\frac{\partial}{\partial y})\Big(\theta(\alpha; \frac{1}{2} + iy) - \sqrt{2}\theta(\alpha; \frac{1}{2} + iy)\Big)$.** The following Lemma is a particular case of Lemma 4.7, our analysis relies on this expression.

Lemma 4.15. *The identity for $\big(\frac{\partial^2}{\partial y^2} + \frac{2}{y}\frac{\partial}{\partial y}\big)\big(\theta(\alpha; \frac{1}{2} + iy) - \sqrt{2}\theta(2\alpha; \frac{1}{2} + iy)\big)$ holds*

$$\begin{aligned}
(\frac{\partial^2}{\partial y^2} + \frac{2}{y}\frac{\partial}{\partial y})\big(\theta(\alpha; \frac{1}{2} + iy) - \sqrt{2}\theta(2\alpha; \frac{1}{2} + iy)\big) &= (\pi\alpha)^2\sum_{n,m}(n^2 - \frac{(m + \frac{n}{2})^2}{y^2})^2 e^{-\pi\alpha(yn^2 + \frac{(m+\frac{n}{2})^2}{y})}\\
&+ \frac{4\sqrt{2}\pi\alpha}{y}\sum_{n,m}n^2 e^{-2\pi\alpha(yn^2 + \frac{(m+\frac{n}{2})^2}{y})} - \frac{2\pi\alpha}{y}\sum_{n,m}n^2 e^{-\pi\alpha(yn^2 + \frac{(m+\frac{n}{2})^2}{y})}\\
&- 4\sqrt{2}(\pi\alpha)^2\sum_{n,m}(n^2 - \frac{(m + \frac{n}{2})^2}{y^2})^2 e^{-2\pi\alpha(yn^2 + \frac{(m+\frac{n}{2})^2}{y})}.
\end{aligned}$$

There are two types of double sums appeared in Lemma 4.15(with slightly different frequencies), as follows

$$\begin{aligned}\text{double sum A} &:= \sum_{n,m} n^2 e^{-\pi\alpha(yn^2+\frac{(m+\frac{n}{2})^2}{y})},\\ \text{double sum B} &:= \sum_{n,m} (n^2-\frac{(m+\frac{n}{2})^2}{y^2})^2 e^{-2\pi\alpha(yn^2+\frac{(m+\frac{n}{2})^2}{y})}.\end{aligned} \tag{4.15}$$

We shall estimate these two double sums of (4.15) in Lemmas 4.16 and 4.17.

Lemma 4.16. *We have the following upper bound function of* $\sum_{n,m} n^2 e^{-\pi\alpha(yn^2+\frac{(m+\frac{n}{2})^2}{y})}$*:*

$$\sum_{n,m} n^2 e^{-\pi\alpha(yn^2+\frac{(m+\frac{n}{2})^2}{y})} \leq 4e^{-\pi\alpha(y+\frac{1}{4y})}\cdot\big(1+\epsilon_a\big),$$

where ϵ_a *is small and can be explicitly controlled by*

$$\epsilon_a := \epsilon_{a,1}+\epsilon_{a,2}+\epsilon_{a,3}+\epsilon_{a,4}$$

and

$$\epsilon_a \to 0 \ \ as \ \ y \mapsto \infty.$$

Here each $\epsilon_{a,j}(j=1,2,3,4)$ *is small and expressed by*

$$\begin{aligned}\epsilon_{a,1} &:= \sum_{n=2}^{\infty}(2n-1)^2 e^{-\pi\alpha y((2n-1)^2-1)}\\ \epsilon_{a,2} &:= \sum_{n=2}^{\infty} e^{-\frac{\pi\alpha}{4y}((2n-1)^2-1)}\\ \epsilon_{a,3} &:= \epsilon_{a,1}\cdot\epsilon_{a,2}\\ \epsilon_{a,4} &:= 2e^{-\pi\alpha(3y-\frac{1}{4y})}\big(1+\sum_{n=2}^{\infty}n^2 e^{-4\pi\alpha y(n^2-1)}\big)\cdot\vartheta_3(\frac{\alpha}{y}).\end{aligned}$$

Proof. We shall divide the sum into two parts,

$$\begin{aligned}\sum_{n,m} n^2 e^{-\pi\alpha(yn^2+\frac{(m+\frac{n}{2})^2}{y})} &= \sum_{p,q,p\equiv q(mod2)} p^2 e^{-\pi\alpha(yp^2+\frac{q^2}{4y})}\\ &= \sum_{p\equiv q\equiv 0(mod2)} p^2 e^{-\pi\alpha(yp^2+\frac{q^2}{4y})} + \sum_{p\equiv q\equiv 1(mod2)} p^2 e^{-\pi\alpha(yp^2+\frac{q^2}{4y})}.\end{aligned}$$

For convenience, we denote that

$$\begin{aligned}J_1 &:= \sum_{p\equiv q\equiv 0(mod2)} p^2 e^{-\pi\alpha(yp^2+\frac{q^2}{4y})},\\ J_2 &:= \sum_{p\equiv q\equiv 1(mod2)} p^2 e^{-\pi\alpha(yp^2+\frac{q^2}{4y})}.\end{aligned}$$

Then

$$\sum_{n,m} n^2 e^{-\pi\alpha(yn^2+\frac{(m+\frac{n}{2})^2}{y})} = J_1+J_2. \tag{4.16}$$

We now estimate J_1 and J_2 respectively. First J_1 can be rewritten as

$$\begin{aligned} J_1 =& \sum_{p\equiv q\equiv 0(mod2)} p^2 e^{-\pi\alpha(yp^2+\frac{q^2}{4y})} = \sum_{p=2n,q=2m} p^2 e^{-\pi\alpha(yp^2+\frac{q^2}{4y})} \\ =&4\sum_n n^2 e^{-4\pi\alpha yn^2}\sum_m e^{-\pi\frac{\alpha}{y}m^2} = 8\sum_{n=1}^{\infty} n^2 e^{-4\pi\alpha yn^2}\cdot(1+2\sum_{m=1}^{\infty} e^{-\pi\frac{\alpha}{y}m^2}) \\ =&8e^{-4\pi\alpha y}(1+\sum_{n=2}^{\infty} n^2 e^{-4\pi\alpha y(n^2-1)})\cdot(1+2\sum_{m=1}^{\infty} e^{-\pi\frac{\alpha}{y}m^2}) \\ =&4e^{-\pi\alpha(y+\frac{1}{4y})}\cdot\epsilon_{a,4}, \end{aligned} \tag{4.17}$$

as we can see later, J_1 is the remainder terms.

Next J_2 can be deformed as

$$\begin{aligned} J_2 =& \sum_{p\equiv q\equiv 1(mod2)} p^2 e^{-\pi\alpha(yp^2+\frac{q^2}{4y})} = \sum_{p=2n-1,q=2m-1} p^2 e^{-\pi\alpha(yp^2+\frac{q^2}{4y})} \\ =&4\sum_{n=1}^{\infty}(2n-1)^2 e^{-\pi\alpha y(2n-1)^2}\cdot\sum_{m=1}^{\infty} e^{-\pi\alpha\frac{(2m-1)^2}{4y}} \\ =&4e^{-\pi\alpha(y+\frac{1}{4y})}\cdot(1+\sum_{n=2}^{\infty}(2n-1)^2 e^{-\pi\alpha y((2n-1)^2-1)})\cdot(1+\sum_{m=2}^{\infty} e^{-\frac{\pi\alpha}{4y}((2m-1)^2-1)}) \\ =&4e^{-\pi\alpha(y+\frac{1}{4y})}\cdot(1+\epsilon_{a,1}+\epsilon_{a,2}+\epsilon_{a,1}\cdot\epsilon_{a,2}) \\ =&4e^{-\pi\alpha(y+\frac{1}{4y})}\cdot(1+\epsilon_{a,1}+\epsilon_{a,2}+\epsilon_{a,3}). \end{aligned} \tag{4.18}$$

The result follows by (4.16), (4.17) and (4.18).

□

Lemma 4.17. *We have the following upper bound*

$$\sum_{n,m}(n^2-\frac{(m+\frac{n}{2})^2}{y^2})^2 e^{-2\pi\alpha(yn^2+\frac{(m+\frac{n}{2})^2}{y})} \leq \frac{2}{y^4}e^{-2\pi\frac{\alpha}{y}}\cdot(1+\epsilon_b),$$

where ϵ_b is small and consist of four smaller parts

$$\epsilon_b := \epsilon_{b,1}+\epsilon_{b,2}+\epsilon_{b,3}+\epsilon_{b,4},$$

and

$$\epsilon_b \to 0 \;\; as \;\; y\mapsto\infty.$$

Here

$$\begin{aligned} \epsilon_{b,1} :=& 2y^4 e^{-2\pi\alpha y}\cdot(1+\sum_{n=2}^{\infty} e^{-\frac{2\pi\alpha}{y}((2n-1)^2-1)})\cdot(1+\sum_{n=2}^{\infty}(2n-1)^4 e^{-2\pi\alpha y((2n-1)^2-1)}) \\ \epsilon_{b,2} :=& \frac{1}{8}e^{-2\pi\alpha y}\cdot(1+\sum_{n=2}^{\infty}(2n-1)^4 e^{-\frac{2\pi\alpha}{y}((2n-1)^2-1)})\cdot(1+\sum_{n=2}^{\infty} e^{-2\pi\alpha y((2n-1)^2-1)}) \\ \epsilon_{b,3} :=& 16y^4 e^{-\pi\alpha(8y-\frac{2}{y})}\cdot(1+\sum_{n=2}^{\infty} n^4 e^{-8\pi\alpha y(n^2-1)})\cdot(1+2\sum_{n=1}^{\infty} e^{-2\pi\frac{\alpha}{y}n^2}) \\ \epsilon_{b,4} :=& y^4 e^{-\pi\alpha(8y-\frac{2}{y})}\cdot(1+\sum_{n=2}^{\infty} e^{-8\pi\alpha y(n^2-1)})\cdot(1+2\sum_{n=1}^{\infty}\frac{n^4}{y^4}e^{-2\pi\frac{\alpha}{y}n^2}). \end{aligned}$$

Proof. We shall divide the sum into two different parts as follows

$$\begin{aligned}
&\sum_{n,m}(n^2-\frac{(m+\frac{n}{2})^2}{y^2})^2e^{-2\pi\alpha(yn^2+\frac{(m+\frac{n}{2})^2}{y})}\\
&=\sum_{p,q,p\equiv q(mod 2)}(p^2-\frac{q^2}{y^2})^2e^{-2\pi\alpha(yp^2+\frac{q^2}{y})}\\
&=\sum_{p,q,p\equiv q\equiv 0(mod 2)}(p^2-\frac{q^2}{y^2})^2e^{-2\pi\alpha(yp^2+\frac{q^2}{y})}\\
&+\sum_{p,q,p\equiv q\equiv 1(mod 2)}(p^2-\frac{q^2}{y^2})^2e^{-2\pi\alpha(yp^2+\frac{q^2}{y})}.
\end{aligned}$$

For convenience, one denotes that

$$\begin{aligned}
\mathcal{K}_a&:=\sum_{p,q,p\equiv q\equiv 0(mod 2)}(p^2-\frac{q^2}{y^2})^2e^{-2\pi\alpha(yp^2+\frac{q^2}{y})},\\
\mathcal{K}_b&:=\sum_{p,q,p\equiv q\equiv 1(mod 2)}(p^2-\frac{q^2}{y^2})^2e^{-2\pi\alpha(yp^2+\frac{q^2}{y})}.
\end{aligned}$$

Hence we have

$$\sum_{n,m}(n^2-\frac{(m+\frac{n}{2})^2}{y^2})^2e^{-2\pi\alpha(yn^2+\frac{(m+\frac{n}{2})^2}{y})}=\mathcal{K}_a+\mathcal{K}_b.$$

One deforms $\mathcal{K}_a$ and $\mathcal{K}_b$ respectively:

$$\begin{aligned}
\mathcal{K}_a&=\sum_{p,q,p\equiv q\equiv 0(mod 2)}(p^2-\frac{q^2}{y^2})^2e^{-2\pi\alpha(yp^2+\frac{q^2}{y})}\\
&=\sum_{n,m}(4n^2-\frac{m^2}{y^2})^2e^{-2\pi\alpha(4n^2y+\frac{m^2}{y})}
\end{aligned}$$

To leading order, we single out the major terms by regrouping the terms as follows

$$\begin{aligned}
\mathcal{K}_a&=\sum_{n,m}(4n^2-\frac{m^2}{y^2})^2e^{-2\pi\alpha(4n^2y+\frac{m^2}{y})}\\
&=\frac{2}{y^4}e^{-2\pi\frac{\alpha}{y}}+\sum_{n\notin\{0\},m}(4n^2-\frac{m^2}{y^2})^2e^{-2\pi\alpha(4n^2y+\frac{m^2}{y})}\\
&\quad+\sum_{m\notin\{-1,1\},n}(4n^2-\frac{m^2}{y^2})^2e^{-2\pi\alpha(4n^2y+\frac{m^2}{y})}.
\end{aligned}$$

To further simplify the structure, one denotes that

$$\begin{aligned}
\mathcal{K}_{a,1}&:=\sum_{n\notin\{0\},m}(4n^2-\frac{m^2}{y^2})^2e^{-2\pi\alpha(4n^2y+\frac{m^2}{y})},\\
\mathcal{K}_{a,2}&:=\sum_{m\notin\{-1,1\},n}(4n^2-\frac{m^2}{y^2})^2e^{-2\pi\alpha(4n^2y+\frac{m^2}{y})}.
\end{aligned}$$

Then,

$$\mathcal{K}_a=\frac{2}{y^4}e^{-2\pi\frac{\alpha}{y}}+\mathcal{K}_{a,1}+\mathcal{K}_{a,2}\tag{4.19}$$

and

$$\sum_{n,m}(n^2-\frac{(m+\frac{n}{2})^2}{y^2})^2 e^{-2\pi\alpha(yn^2+\frac{(m+\frac{n}{2})^2}{y})}=\frac{2}{y^4}e^{-2\pi\frac{\alpha}{y}}+\mathcal{K}_{a,1}+\mathcal{K}_{a,2}+\mathcal{K}_b. \tag{4.20}$$

To control $\mathcal{K}_{a,j}, j=1,2$, we use a basic mean value inequality and

$$\begin{aligned}
\mathcal{K}_{a,1} &= \sum_{n\notin\{0\},m}(4n^2-\frac{m^2}{y^2})^2 e^{-2\pi\alpha(4n^2y+\frac{m^2}{y})}\\
&\leq \sum_{n\notin\{0\},m} 16n^4 e^{-2\pi\alpha(4n^2y+\frac{m^2}{y})}+\sum_{n\notin\{0\},m}\frac{m^4}{y^4}e^{-2\pi\alpha(4n^2y+\frac{m^2}{y})}\\
&= 32\sum_{n=1}^{\infty}n^4 e^{-8\pi\alpha yn^2}\sum_m e^{-2\pi\frac{\alpha}{y}m^2}+2\sum_{n=1}^{\infty}e^{-8\pi\alpha yn^2}\sum_m\frac{m^4}{y^4}e^{-2\pi\frac{\alpha}{y}m^2}\\
&= \frac{2}{y^4}e^{-2\pi\frac{\alpha}{y}}\cdot\sigma_{\mathcal{K}_1},
\end{aligned}$$

where we single out the small remainder terms denoted by $\sigma_{\mathcal{K}}$ as follows

$$\begin{aligned}
\sigma_{\mathcal{K}_1}:&=16y^4 e^{-\pi\alpha(8y-\frac{2}{y})}\cdot(1+\sum_{n=2}^{\infty}n^4 e^{-8\pi\alpha y(n^2-1)})\cdot(1+2\sum_{n=1}^{\infty}e^{-2\pi\frac{\alpha}{y}n^2})\\
&+e^{-\pi\alpha(8y-\frac{2}{y})}\cdot(1+\sum_{n=2}^{\infty}e^{-8\pi\alpha y(n^2-1)})\cdot(1+2\sum_{n=1}^{\infty}n^4 e^{-2\pi\frac{\alpha}{y}n^2})
\end{aligned} \tag{4.21}$$

Similar to $\mathcal{K}_{a,2}$,

$$\begin{aligned}
\mathcal{K}_{a,2} &= \sum_{m\notin\{-1,1\},n}(4n^2-\frac{m^2}{y^2})^2 e^{-2\pi\alpha(4n^2y+\frac{m^2}{y})}\\
&\leq \sum_{m\notin\{-1,1\},n}16n^4 e^{-2\pi\alpha(4n^2y+\frac{m^2}{y})}+\sum_{m\notin\{-1,1\},n}\frac{m^4}{y^4}e^{-2\pi\alpha(4n^2y+\frac{m^2}{y})}\\
&\leq \sum_{m,n}16n^4 e^{-2\pi\alpha(4n^2y+\frac{m^2}{y})}+\sum_{m\notin\{-1,1\},n}\frac{m^4}{y^4}e^{-2\pi\alpha(4n^2y+\frac{m^2}{y})}\\
&\leq 32\sum_{n=1}^{\infty}n^4 e^{-8\pi\alpha yn^2}\sum_m e^{-2\pi\frac{\alpha}{y}m^2}+2\sum_{m=2}^{\infty}\frac{m^4}{y^4}e^{-2\pi\frac{\alpha}{y}m^2}\sum_n e^{-8\pi\alpha yn^2}\\
&= \frac{2}{y^4}e^{-2\pi\frac{\alpha}{y}}\cdot\sigma_{\mathcal{K}_2},
\end{aligned}$$

where

$$\begin{aligned}
\sigma_{\mathcal{K}_2}&=16e^{-\pi\alpha(8y-\frac{2}{y})}\cdot(1+\sum_{n=2}^{\infty}n^4 e^{-8\pi\alpha y(n^2-1)})\cdot(1+2\sum_{m=1}^{\infty}e^{-2\pi\frac{\alpha}{y}m^2})\\
&+\sum_{m=2}^{\infty}m^4 e^{-2\pi\frac{\alpha}{y}(m^2-1)}\cdot(1+2\sum_{m=1}^{\infty}e^{-8\pi\alpha yn^2}).
\end{aligned} \tag{4.22}$$

Recall in (4.19), one has

$$\mathcal{K}_a=\frac{2}{y^4}e^{-2\pi\frac{\alpha}{y}}+\mathcal{K}_{a,1}+\mathcal{K}_{a,2}\quad=\frac{2}{y^4}e^{-2\pi\frac{\alpha}{y}}\cdot(1+\sigma_{\mathcal{K}_1}+\sigma_{\mathcal{K}_2}), \tag{4.23}$$

where $\sigma_{\mathcal{K}_1}$ and $\sigma_{\mathcal{K}_2}$ are defined in (4.21) and (4.22) respectively.

Next, we estimate $\mathcal{K}_b$.

$$
\begin{aligned}
\mathcal{K}_b =& \sum_{p,q,p\equiv q\equiv 1(mod 2)} (p^2-\frac{q^2}{y^2})^2 e^{-2\pi\alpha(yp^2+\frac{q^2}{y})}\\
=& \sum_{n,m}((2n-1)^2-\frac{(2m-1)^2}{4y^2})^2 e^{-2\pi\alpha(y(2n-1)^2+\frac{(2m-1)^2}{y})}\\
\leq& \sum_{n,m}(2n-1)^4 e^{-2\pi\alpha(y(2n-1)^2+\frac{(2m-1)^2}{y})}+\sum_{n,m}\frac{(2m-1)^4}{16y^4}e^{-2\pi\alpha(y(2n-1)^2+\frac{(2m-1)^2}{y})}\\
=& \sum_{m} e^{-2\pi\frac{\alpha}{y}(2m-1)^2}\cdot\sum_{n}(2n-1)^4 e^{-2\pi\alpha y(2n-1)^2}\\
&+\sum_{m}\frac{(2m-1)^4}{16y^4}e^{-2\pi\frac{\alpha}{y}(2m-1)^2}\cdot\sum_{n} e^{-2\pi\alpha y(2n-1)^2}\\
=& \frac{2}{y^4}e^{-2\pi\frac{\alpha}{y}}\cdot\sigma_{\mathcal{K}_3},
\end{aligned}
\tag{4.24}
$$

where

$$
\begin{aligned}
\sigma_{\mathcal{K}_3} :=& 2y^4 e^{-2\pi\alpha y}\cdot(1+\sum_{n=2}^{\infty}e^{-2\pi\frac{\alpha}{y}((2n-1)^2-1)})\cdot(1+\sum_{n=2}^{\infty}(2n-1)^4 e^{-2\pi\alpha y((2n-1)^2-1)})\\
&+\frac{1}{8}e^{-2\pi\alpha y}\cdot(1+\sum_{n=2}^{\infty}(2n-1)^4 e^{-2\pi\frac{\alpha}{y}((2n-1)^2-1)})\cdot(1+\sum_{n=2}^{\infty}e^{-2\pi\alpha y((2n-1)^2-1)}).
\end{aligned}
\tag{4.25}
$$

Combining (4.20) with (4.23) and (4.24), one deduces that

$$
\begin{aligned}
\sum_{n,m}(n^2-\frac{(m+\frac{n}{2})^2}{y^2})^2 e^{-2\pi\alpha(yn^2+\frac{(m+\frac{n}{2})^2}{y})} =& \frac{2}{y^4}e^{-2\pi\frac{\alpha}{y}}+\mathcal{K}_{a,1}+\mathcal{K}_{a,2}+\mathcal{K}_b\\
\leq& \frac{2}{y^4}e^{-2\pi\frac{\alpha}{y}}\cdot(1+\sigma_{\mathcal{K}_1}+\sigma_{\mathcal{K}_2}+\sigma_{\mathcal{K}_3}),
\end{aligned}
\tag{4.26}
$$

where $\sigma_{\mathcal{K}_1}$, $\sigma_{\mathcal{K}_2}$ and $\sigma_{\mathcal{K}_3}$ are defined in (4.21), (4.22) and (4.25) respectively. The inequality (4.26) yields the result.

□

The next two Lemmas provide the lower bound functions of the double sums in Lemma 4.15, where the positiveness is used effectively.

Lemma 4.18. *A lower bound function of the double sum* $\sum_{n,m}(n^2-\frac{(m+\frac{n}{2})^2}{y^2})^2 e^{-\pi\alpha(yn^2+\frac{(m+\frac{n}{2})^2}{y})}$ *is as follows*

$$
\sum_{n,m}(n^2-\frac{(m+\frac{n}{2})^2}{y^2})^2 e^{-\pi\alpha(yn^2+\frac{(m+\frac{n}{2})^2}{y})}\geq \frac{2}{y^4}e^{-\pi\frac{\alpha}{y}}+4(1-\frac{1}{4y^2})^2 e^{-\pi\alpha(y+\frac{1}{4y})}.
$$

Remark 4.2. *In the proof of Lemma 4.18(and Lemma 4.19 below), we have used the positive structure of the double sum.*

Proof. The double sum evaluates at

$$
(m,n)=\{(1,0),(-1,0)\}\ \text{contributing}\ \frac{1}{y^4}e^{-\pi\frac{\alpha}{y}}\ \text{each}
$$

and

$$
(m,n)=\{(0,1),(0,-1),(1,-1),(-1,1)\}\ \text{contributing}\ (1-\frac{1}{4y^2})^2 e^{-\pi\alpha(y+\frac{1}{4y})}\ \text{each}.
$$

The rest of other terms in the double sum all are positive and hence the result follows. □

Lemma 4.19. *A lower bound function of* $\sum_{n,m} n^2 e^{-2\pi\alpha(yn^2+\frac{(m+\frac{n}{2})^2}{y})}$ *is*

$$\sum_{n,m} n^2 e^{-2\pi\alpha(yn^2+\frac{(m+\frac{n}{2})^2}{y})} \geq 4e^{-2\pi\alpha(y+\frac{1}{4y})}.$$

Proof. The double sum can be evaluated at

$$(m,n) = \{(0,1),(0,-1),(1,-1),(-1,1)\} \text{ contributing } e^{-2\pi\alpha(y+\frac{1}{4y})} \text{ each.}$$

The rest of other terms in the double sum all are positive and hence the result follows. □

Lemma 4.20. *We have the following lower bound estimate*

$$(\frac{\partial^2}{\partial y^2}+\frac{2}{y}\frac{\partial}{\partial y})(\theta(\alpha;\frac{1}{2}+iy)-\sqrt{2}\theta(2\alpha;\frac{1}{2}+iy)) \geq \frac{2(\pi\alpha)^2}{y^4}e^{-\pi\frac{\alpha}{y}}\cdot\mathcal{W}(y;\alpha),$$

where

$$\mathcal{W}(y;\alpha) := 1+(2(y^2-\frac{1}{4})^2-\frac{4}{\pi\alpha}y^3\cdot(1+\epsilon_a))\cdot e^{-\pi\alpha(y-\frac{3}{4y})}-4\sqrt{2}(1+\epsilon_b)\cdot e^{-\pi\frac{\alpha}{y}}. \tag{4.27}$$

Here ϵ_a and ϵ_b are defined in Lemmas 4.16 and 4.17 respectively.

Proof. In view of Lemma 4.15, combining with the bound functions in Lemmas 4.16-4.20, we have

$$\begin{aligned}
&(\frac{\partial^2}{\partial y^2}+\frac{2}{y}\frac{\partial}{\partial y})\big(\theta(\alpha;\frac{1}{2}+iy)-\sqrt{2}\theta(2\alpha;\frac{1}{2}+iy)\big) = (\pi\alpha)^2\sum_{n,m}(n^2-\frac{(m+\frac{n}{2})^2}{y^2})^2 e^{-\pi\alpha(yn^2+\frac{(m+\frac{n}{2})^2}{y})}\\
&\quad+\frac{4\sqrt{2}\pi\alpha}{y}\sum_{n,m}n^2e^{-2\pi\alpha(yn^2+\frac{(m+\frac{n}{2})^2}{y})}-\frac{2\pi\alpha}{y}\sum_{n,m}n^2e^{-\pi\alpha(yn^2+\frac{(m+\frac{n}{2})^2}{y})}\\
&\quad-4\sqrt{2}(\pi\alpha)^2\sum_{n,m}(n^2-\frac{(m+\frac{n}{2})^2}{y^2})^2e^{-2\pi\alpha(yn^2+\frac{(m+\frac{n}{2})^2}{y})}\\
&\geq(\pi\alpha)^2\frac{2}{y^4}e^{-\pi\frac{\alpha}{y}}+4(\pi\alpha)^2(1-\frac{1}{4y^2})^2e^{-\pi\alpha(y+\frac{1}{4y})}+\frac{16\sqrt{2}\pi\alpha}{y}e^{-2\pi\alpha(y+\frac{1}{4y})}\\
&\quad-\frac{8\pi\alpha}{y}(1+\epsilon_a)e^{-\pi\alpha(y+\frac{1}{4y})}-\frac{8\sqrt{2}(\pi\alpha)^2}{y^4}(1+\epsilon_b)e^{-2\pi\frac{\alpha}{y}}\\
&\geq\frac{2(\pi\alpha)^2}{y^4}e^{-\pi\frac{\alpha}{y}}\cdot\mathcal{W}(y;\alpha).
\end{aligned}$$

□

Lemma 4.21. *Assume that $\alpha\geq1, y\geq\frac{\sqrt{3}}{2}$. If $\frac{y}{\alpha}\leq3$, then*

$$\begin{aligned}\epsilon_a &= 0.1264717\cdots<0.15,\\ \epsilon_b &= 0.0054169\cdots<0.006.\end{aligned}$$

Proof. The terms of ϵ_a,ϵ_b are exponentially decaying. One needs to use the fact that $\alpha\geq1, y\geq\frac{\sqrt{3}}{2}$ and $\frac{\alpha}{y}\geq\frac{1}{3}$. Note that the positive lower bounds of $\frac{\alpha}{y}$ and αy are used effectively to control the summation. □

Lemma 4.22. *A refined lower bound of $\mathcal{W}(y;\alpha)$, which is defined in (4.27), is the following*

$$\mathcal{W}(y;\alpha)\geq\begin{cases}1-(\frac{4(1+\epsilon_a)}{\pi}-\frac{9}{8})-4\sqrt{2}(1+\epsilon_b)e^{-\pi}, & \text{if } y\in[\frac{\sqrt{3}}{2},1];\\ 1-(\frac{4(1+\epsilon_a)}{\pi}y^3-2(y^2-\frac{1}{4})^2)\cdot e^{-\frac{\pi}{4}}-4\sqrt{2}(1+\epsilon_b)e^{-\frac{1}{y_\epsilon}\pi}, & \text{if } y\in[1,y_\epsilon],\\ 1-4\sqrt{2}(1+\epsilon_b)e^{-\pi\frac{\alpha}{y}}, & \text{if } y\in[y_\epsilon,\infty).\end{cases}$$

Here y_ϵ is the unique root of

$$2(y^2-\frac{1}{4})^2-\frac{4}{\pi}y^3\cdot(1+\epsilon_a)=0$$

on $[\frac{\sqrt{3}}{2},\infty)$*. Numerically,* $y_\epsilon\cong 1.130998\cdots$.

Proof. The proof is based on the explicit expression of $\mathcal{W}(y;\alpha)$ in Lemma 4.20. Each part of $\mathcal{W}(y;\alpha)$ is analyzed separately.

□

Lemma 4.23. *Assume that* $\alpha\geq 1$*. If* $y\in[\frac{\sqrt{3}}{2},1.8\alpha]$*, then*

$$\mathcal{W}(y;\alpha)>0.$$

Proof. The proof follows from Lemma 4.22. Note that

$$1-(\frac{4(1+\epsilon_a)}{\pi}-\frac{9}{8})-4\sqrt{2}(1+\epsilon_b)e^{-\pi}>0.414852\cdots>0$$

$$1-(\frac{4(1+\epsilon_a)}{\pi}y^3-2(y^2-\frac{1}{4})^2)\cdot e^{-\frac{\pi}{4}}-4\sqrt{2}(1+\epsilon_b)e^{-\frac{1}{y_\epsilon}\pi}>0.491478\cdots,\ \text{if } y\in[1,y_\epsilon],$$

and that

$$1-4\sqrt{2}(1+\epsilon_b)e^{-\pi\frac{\alpha}{y}}>0\ \text{ if }\ \frac{\alpha}{y}>0.553493\cdots.$$

Next, a simple computation shows that

$$\frac{\alpha}{y}>0.553493\cdots\Leftrightarrow y\leq(1.806707\cdots)\alpha.$$

In view of Lemma 4.22, the result then follows.

□

5. Proofs of Theorems 1.2-1.5

Proof of Theorem 1.2: By Fourier transform, we have

$$\theta(\frac{1}{\alpha};z)=\alpha\cdot\theta(\alpha;z),\quad \alpha>0.$$

Theorem 1.2 is equivalent to Theorem 1.1.

Proof of Theorem 1.3-1.4: These two theorems are easy consequences of Theorems 1.1-1.2.

Proof of Theorem 1.5: The proof is based on an effective iteration scheme.
Case A: $\beta\leq(\sqrt{2})^k$**.** We use the scheme

$$\begin{aligned}&\Big(\theta(\alpha;z)-\beta\theta(2^k\alpha;z)\Big)\\=&\big((\sqrt{2})^k-\beta\big)\theta(2^k\alpha;z)+\sum_{n=0}^{k-1}(\sqrt{2})^n\Big(\theta(2^n\alpha;z)-\sqrt{2}\theta(2^{n+1}\alpha;z)\Big).\end{aligned}\tag{5.1}$$

Note that all the coefficients in (5.1) are nonnegative. We apply Theorem 1.1 on each term of (5.1) to arrive that the minimizer of $\Big(\theta(\alpha;z)-\beta\theta(2^k\alpha;z)\Big)$ is $\frac{1}{2}+i\frac{\sqrt{3}}{2}$ again in this case.

Case B: $\beta>(\sqrt{2})^k$**.** The proof of nonexistence of the minimizer is similar to that of Lemma 4.1.

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