ON EXTREMUM OF QUOTIENT OF THETA AND ZETA FUNCTIONS AND THEIR APPLICATIONS

SENPING LUO AND JUNCHENG WEI

Abstract. We consider the minimization (or maximization) of quotients of two classical modular invariant functions. Let $z \in \mathbb{H} := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ and $\theta(\beta, z)$ and $\zeta(s, z)$ be the Theta and Epstein Zeta functions associated with a two-dimensional lattice:

$$\zeta(s,z) = \sum_{(m,n)\in\mathbb{Z}^2\backslash\{0\}} \frac{\mathrm{Im}(z)^s}{|mz+n|^{2s}},\ \theta(\alpha,z) = \sum_{(m,n)\in\mathbb{Z}^2} e^{-\pi\alpha\cdot\frac{|mz+n|^2}{\mathrm{Im}(z)}}.$$

We completely classify the optimal shape for

$$\min(\max)_{z\in\mathbb{H}} \frac{\theta(\beta,z)}{\theta^k(\alpha,z)},\ \ \min_{z\in\mathbb{H}} \frac{\zeta(s,z)}{\theta^k(\alpha,z)}\ \text{ for }\ \alpha,\beta,k>0, s>1.$$

These results have direct applications to conformal and Liouville filed theory, and string theory via the partition functions. Besides, these results yield extremum of differences of modular invariant functions, which have applications to mathematics of crystallization and interacting particle theory.

1. Introduction and main results

In this paper, we consider the minimum or maximum of quotients of modular invariant functions. A function $\mathcal{W}$ is called modular invariant if

$$\mathcal{W}(\gamma(z)) = \mathcal{W}(z),\ \ \forall\ \gamma \in \mathrm{SL}(2,\mathbb{Z}), \tag{1.1}$$

where the modular group $\mathrm{SL}(2,\mathbb{Z})$ is finitely generated by $z \mapsto -\frac{1}{z}$ and $z \mapsto z+1$.

In Subsection 1.1, we first introduce our main mathematical results. In Subsection 1.2, we introduce the physical background of quotients of modular invariant functions, and then state their applications in conformal field theory, string theory and physics. As one can see below, in many cases the minima are hexagonal patterns, which shadows light on why hexagonal shape prevails in nature.

1.1. **Brief introduction to our main results.** Let $z \in \mathbb{H} := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ and $\Lambda = \sqrt{\frac{1}{\mathrm{Im}(z)}}\Big(\mathbb{Z} \oplus z\mathbb{Z}\Big)$ be the lattice in $\mathbb{R}^2$ with area of unit cell and parameter z. The Epstein Zeta and Theta functions associated with the lattice Λ are defined as

$$\zeta(s,\Lambda) := \sum_{\mathbb{P}\in\Lambda\backslash\{0\}} \frac{1}{|\mathbb{P}|^{2s}}, \quad \theta(\alpha,\Lambda) := \sum_{\mathbb{P}\in\Lambda} e^{-\pi\alpha|\mathbb{P}|^2}.$$

Using the parametrization z of Λ, one has

$$\zeta(s,z) := \zeta(s,\Lambda) = \sum_{(m,n)\in\mathbb{Z}^2\backslash\{0\}} \frac{\mathrm{Im}(z)^s}{|mz+n|^{2s}},\ \theta(\alpha,z) := \theta(\alpha,\Lambda) = \sum_{(m,n)\in\mathbb{Z}^2} e^{-\pi\alpha\cdot\frac{|mz+n|^2}{\mathrm{Im}(z)}}. \tag{1.2}$$

In 1950s, in a series of work in number theory, Rankin [42], Cassels [10], Ennola [17], Diamond [16] established that

Theorem A (Rankin, Cassels, Ennola, Diamonda 1950-1960s). *For $s > 1$, up to rotations and translations,*

$$\mathrm{Minima}_{z\in\mathbb{H}}\zeta(s,z) = e^{i\frac{\pi}{3}}.$$

The higher dimensional Epstein Zeta becomes much more difficult. The first rigorous theorem appeared in 2006 by Sarnak-Strömbergsson [37].

Motivated by **Theorem A**, Montgomery [29] further proved that

Theorem B (Montgomery 1988)**.** *For $\alpha > 0$, up to rotations and translations*

$$\mathrm{Minima}_{z\in\mathbb{H}}\theta(\alpha,z) = e^{i\frac{\pi}{3}}.$$

Note that by Mellin transform, Zeta functions can be expressed by Theta functions, i.e., $\zeta(s,z) = \frac{\pi^s}{\Gamma(s)}\int_0^\infty \big(\theta(\alpha,z)-1\big)\alpha^{s-1}d\alpha$. Hence Theorem B implies Theorem A. Theorems A and B have become classic results in number theory, and have deep applications in other fields.

Theorems A and B laid down the foundations to the optimality of triangular(hexagonal) vortices in Ginzburg-Landau theory (Abrikosov [1], Sandier-Serfaty [36, 38]). Theorem B has direct applications to crystallization among lattices (Bétermin [6]), Ohta-Kawasaki models in di-block copolymers (Chen-Oshita [9], Goldman-Muratov-Serfaty [20]), Bose-Einstein condensates ([21]), crystallization of particle interactions (Blanc-Lewin [8], Luo-Wei [23]), minimal frame operator norms (Faulhuber [18]) and many others. Furthermore, Theta functions are deeply connected to string theory (Alvarez-Gaumé-Moore-Vafa [2]), Gauss core model (Cohn and Courcy-Ireland [11], Prestipino-Saija-Giaquinta [33]), sphere packings (Conway-Sloane [12], Viazovska [39]), the reverse Minkowski inequality (Regev and Stephens-Davidowitz [34], Regev [35]), and communications (Barreal-Damir-Freij-Hollanti [4]).

In this paper, we consider quotients of special modular invariant functions. The Epstein Zeta function and Theta function in Theorems A and B are two most important modular invariant functions. As Mumford [30] commented *"The theory of theta functions is far from a finished polished topic"*), the Theta and Zeta functions have rich inner structures and deep applications. Note that a power of modular invariant function is still modular invariant. Motivated by Theorems A and B, it is natural to consider the following problem:

Problem 1.1. *Assume that $\alpha,\beta,b,a>0, s>1$. Find the optimal lattice for*

$$\mathrm{Minima(Maxima)}_{z\in\mathbb{H}}\frac{\theta^b(\beta,z)}{\theta^a(\alpha,z)} \ \textit{and}\ \frac{\zeta^b(s,z)}{\theta^a(\alpha,z)}.$$

In this paper we give a complete answer to Problem 1.1 in Corollary 1.1 and Theorem 1.2. Our first theorem concerns the quotient of Theta functions.

Theorem 1.1 (Quotient of Theta functions)**.** *Assume that $\alpha>0,\beta>0$. Then, up to rotations and translations*

(a) *if $\beta>\alpha, \beta\alpha>1$ or $\beta<\alpha,\beta\alpha<1$, then*

$$\mathrm{Maxima}_{z\in\mathbb{H}}\frac{\theta(\beta,z)}{\theta(\alpha,z)} = e^{i\frac{\pi}{3}}, \tag{1.3}$$

and $\mathrm{Minima}_{z\in\mathbb{H}}\frac{\theta(\beta,z)}{\theta(\alpha,z)}$ *does not exist.*

(b) *if $\beta<\alpha,\beta\alpha>1$ or $\beta>\alpha,\beta\alpha<1$, then*

$$\mathrm{Minima}_{z\in\mathbb{H}}\frac{\theta(\beta,z)}{\theta(\alpha,z)} = e^{i\frac{\pi}{3}}, \tag{1.4}$$

and $\mathrm{Maxima}_{z\in\mathbb{H}}\frac{\theta(\beta,z)}{\theta(\alpha,z)}$ *does not exist.*

See the illustration of the regions in Figure 1.

Note that the four regions $\beta>\alpha,\beta\alpha>1$ or $\beta<\alpha,\beta\alpha<1$ or $\beta<\alpha,\beta\alpha>1$ or $\beta>\alpha,\beta\alpha<1$ are determined by a line $\beta=\alpha$ and a curve $\alpha\beta=1$ in the first quadrant, see Picture 1.

Note that neither Theorem B implies Theorem 1.1 nor Theorem 1.1 implies Theorem B, while Theorem 1.1 reveals the inner structure of Theorem B. In fact, through the simple deformation $\theta(\alpha,z) = \frac{\theta(\alpha,z)}{\theta(\beta,z)}\cdot\theta(\beta,z)$ and Theorem 1.1, one infers that if $\mathrm{Minima}_{z\in\mathbb{H}}\theta(\beta,z) = e^{i\frac{\pi}{3}}$ for some

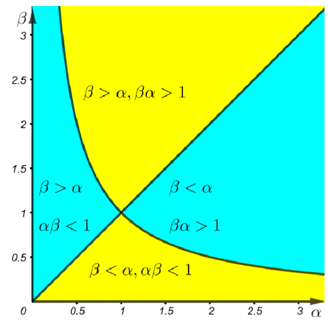

FIGURE 1. (α, β) plane for extremes of $\frac{\theta(\beta,z)}{\theta(\alpha,z)}$.

$\beta > 1$, then $\text{Minima}_{z\in\mathbb{H}}\theta(\alpha, z) = e^{i\frac{\pi}{3}}$ for any $1 \leq \alpha < \beta$; roughly speaking, the minima of $\beta > 1$ in $\theta(\beta, z)$ implies the same minima of smaller β.

Next we consider general quotients of two powers of Theta functions. Since $\frac{\theta^b(\beta,z)}{\theta^a(\alpha,z)} = (\frac{\theta(\beta,z)}{\theta^{a/b}(\alpha,z)})^b$, it is enough to consider $\text{Minima}(\text{Maxima})_{z\in\mathbb{H}}\frac{\theta(\beta,z)}{\theta^k(\alpha,z)}$ for $k > 0$. By Theorem 1.1 we have the following two corollaries.

Corollary 1.1 (Quotient of Theta functions with different powers)**.** *Assume that* $\beta > \alpha \geq 1$. *Then, up to rotations and translations*

(1) *for all* $k \in [1, \infty)$, $\text{Maxima}_{z\in\mathbb{H}}\frac{\theta(\beta,z)}{\theta^k(\alpha,z)} = e^{i\frac{\pi}{3}}$;
(2) *for* $k \in (0, 1)$, $\text{Maxima}_{z\in\mathbb{H}}\frac{\theta(\beta,z)}{\theta^k(\alpha,z)}$ *does not exists.*

Corollary 1.1 provides a complete new perspective on quotient of Theta functions with different powers. Corollary 1.1 shows that the form in Theorem 1.1 is critical.

Theorem 1.1 can be generalized to sums of Theta functions.

Corollary 1.2 (Quotient of sum of Theta functions)**.** *Let* $\min_{1\leq j\leq K}\beta_j \geq \max_{1\leq j\leq K}\alpha_j \geq 1$ *and any* $a_j, b_j \geq 0$, *where* $i, j = 1\cdots K$ *and* $K \geq 2$ *is arbitrary. Then, up to rotations and translations*

$$\text{Maxima}_{z\in\mathbb{H}}\frac{\sum_{j=1}^K b_j\theta(\beta_j, z)}{\sum_{j=1}^K a_j\theta(\alpha_j, z)} = e^{i\frac{\pi}{3}}.$$

Next, we shall consider the quotient of Zeta and Theta functions, namely,

$$\frac{\zeta(s, z)}{\theta(\alpha, z)}. \tag{1.5}$$

In general, when we consider the quotient form (1.5) with a parameter k, i.e., $\frac{\zeta(s,z)}{\theta^k(\alpha,z)}$, we find that there is a threshold $k = 2s$ in the classification of the minimum. Precisely,

Theorem 1.2 (Quotient of Zeta and Theta functions)**.** *Assume that* $s > 1, \alpha \geq 3s$. *Then up to rotations and translations*

(a) *if* $k \leq 2s$(*independent of* α), *then*

$$\text{Minima}_{z\in\mathbb{H}}\frac{\zeta(s, z)}{\theta^k(\alpha, z)} = e^{i\frac{\pi}{3}}. \tag{1.6}$$

(b) *if* $k > 2s$(*independent of* α), *then* $\text{Minima}_{z\in\mathbb{H}}\frac{\zeta(s,z)}{\theta^k(\alpha,z)}$ *does not exist.*

With some extra lower bound of the minimum value of the quotient form, we could deduce that the minimizer of the quotient form implies the minimum of difference form. Precisely, we have the following

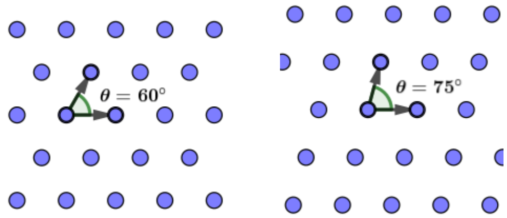

FIGURE 2. Hexagonal and rhombic lattices.

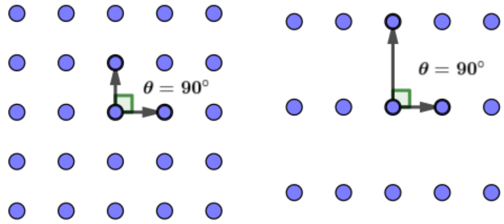

FIGURE 3. Square and rectangular lattices.

Corollary 1.3 (Differences of Zeta and Theta functions with different powers)**.** *Assume that* $s \in (1, 12], \ \ \alpha \geq 3s$. *Then up to rotations and translations*

(a) *if* $k \leq 2s$(*independent of* α), *then*

$$\mathrm{Minima}_{z\in\mathbb{H}}\big(\zeta(s,z)-\theta^k(\alpha,z)\big)=e^{i\frac{\pi}{3}}. \tag{1.7}$$

(b) *if* $k > 2s$(*independent of* α), *then* $\mathrm{Minima}_{z\in\mathbb{H}}\big(\zeta(s,z)-\theta^k(\alpha,z)\big)$ *does not exist.*

The minimum of differences of Zeta and Theta functions with different powers given by Corollary 1.3 have many applications in the mathematics of crystallization and lattice minimization problems. In particular, for $k = 1$, it yields that $\mathrm{Minima}_{z\in\mathbb{H}}\big(\zeta(s,z)-\theta(\alpha,z)\big)=e^{i\frac{\pi}{3}}$ whenever $s \in (1, 12]$ and $\alpha \geq 3s$. This implies hexagonal crystallization under one-well potentials.

1.2. **Quotients of modular invariant functions: background and applications.** Let L be a $n-$dimensional lattice spanned by the basis $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$. Denote that $\sum_{\mathbb{P}\in L}\mathcal{F}(\beta\cdot|\mathbb{P}|^2)$ and $\sum_{\mathbb{P}\in L}\mathcal{H}(\beta\cdot|\mathbb{P}|^2)$ be the summations on the lattice L with respect to the background potentials(functions) $\mathcal{F}$ and $\mathcal{H}$ respectively, here usually $\alpha, \beta > 0$ are the free parameters representing some physical quantity(like temperature or its variant). In this paper, we propose and consider the min(max) problem of the quotient of summations on the lattice as follows

$$\min_L \frac{\sum_{\mathbb{P}\in L}\mathcal{F}(\beta\cdot|\mathbb{P}|^2)}{\sum_{\mathbb{P}\in L}\mathcal{H}(\alpha\cdot|\mathbb{P}|^2)} \ \text{ and } \ \max_L \frac{\sum_{\mathbb{P}\in L}\mathcal{F}(\beta\cdot|\mathbb{P}|^2)}{\sum_{\mathbb{P}\in L}\mathcal{H}(\alpha\cdot|\mathbb{P}|^2)}. \tag{1.8}$$

As we can see below problem (1.8) arises naturally in physical and number theoretical problems. To the best of our knowledge, this paper is the first time to consider to the min(max) problem of quotient of summations on the lattice(1.8). Denote that $\rho = |L| := \det\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ be the density of the lattice L, it is straightforward to show that between different densities, one has the following relation

$$\sum_{\mathbb{P}\in L,\ |L|=\rho}\mathcal{F}(\beta\cdot|\mathbb{P}|^2)=\sum_{\mathbb{P}\in L,\ |L|=1}\mathcal{F}(\beta\rho^2\cdot|\mathbb{P}|^2). \tag{1.9}$$

Therefore, one usually normalizes the density of the lattice to be 1. The summation on the lattice enjoys another important property: it is invariant under special linear group $\mathrm{SL}(n,\mathbb{Z})$ of degree n

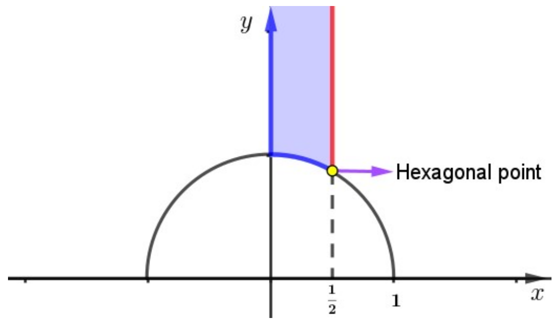

FIGURE 4. Fundamental domain and hexagonal point.

over a field $\mathbb{Z}$ which is the set of $n\times n$ matrices with determinant 1. It is ready to check that

$$\sum_{\mathbb{P}\in L,\ |L|=1}\mathcal{F}(\beta\cdot|\mathbb{P}|^2)=\sum_{\mathbb{P}\in AL,\ |L|=1}\mathcal{F}(\beta\cdot|\mathbb{P}|^2),\ \ \text{where}\ \ A\in\mathrm{SL}(n,\mathbb{Z}).$$

In this paper, we are interested in dimension $n=2$, since *the two-dimensional theories capture many essential features of higher dimensions, without sharing the complexities of higher dimensions* (Alvarez-Gaumé-Moore-Vafa [2]). In this case, the lattice L spanned by $\{\mathbf{v}_1,\mathbf{v}_2\}$(see Figures 2 and 3) can be determined by a complex variable z defined as follows

$$z=\frac{\mathbf{v}_2}{\mathbf{v}_1},\ \text{or}\ \ \mathbf{v}_2=z\mathbf{v}_1.$$

In this way, a lattice with density 1 can be parameterized by $L=\sqrt{\frac{1}{\mathrm{Im}(z)}}\Big(\mathbb{Z}\oplus z\mathbb{Z}\Big)$, where z belongs to the upper half plane $\mathbb{H}:=\{z=x+iy\in\mathbb{C}:y>0\}$. In this way, we define

$$S_{\mathcal{F}}(\beta,z)=S_{\mathcal{F}}(\beta,L):=\sum_{\mathbb{P}\in L,\ |L|=1}\mathcal{F}(\beta\cdot|\mathbb{P}|^2).$$

Then as in (1.9) or check directly, $S_{\mathcal{F}}(\beta,z)$ is modular invariant, i.e.,

$$S_{\mathcal{F}}(\beta,\gamma(z))=S_{\mathcal{F}}(\beta,z),\ \ \forall\gamma\in\mathrm{SL}(2,\mathbb{Z}).\tag{1.10}$$

The fundamental domain associated to the group $\mathrm{SL}(2,\mathbb{Z})$ is

$$\mathcal{D}_0:=\{z\in\mathbb{H}:|z|>1,\ |\,\mathrm{Re}(z)|<\frac{1}{2}\}.\tag{1.11}$$

We also note that $S_{\mathcal{F}}(\beta,z)$ is symmetric about the $y-$axis, i.e.,

$$S_{\mathcal{F}}(\beta,-\overline{z})=S_{\mathcal{F}}(\beta,z).\tag{1.12}$$

Then the fundamental domain associated to the symmetries of $S_{\mathcal{F}}(\beta,z)$ in (1.10) and (1.12) is

$$\mathcal{D}_{\mathcal{G}}:=\{z\in\mathbb{H}:|z|>1,\ 0<\mathrm{Re}(z)<\frac{1}{2}\},\tag{1.13}$$

see an illustration in Figure 4.

When the background potential $\mathcal{F}$ takes the forms of Gaussian(exponential) and Riesz (inverse power) respectively, one gets the Theta and Zeta functions (1.2) respectively.

The Theta, Zeta functions and their variants play a fundamental role in number theory and statistical physics. They still provide many insights to many more abstract theories and have many new applications to newly developed physical problems/theories(see detailed description in the previous Subsection). In many references and textbooks (see e.g. Cohen [15]), Zeta functions defined above are also called as real analytic Eisenstein series and denoted by $E(s,\tau)$. There is a third type of basic modular invariant functions (besides Theta and Zeta functions) that appeared as

$$\sqrt{\operatorname{Im}(\tau)}|\eta(\tau)|^2. \tag{1.14}$$

Here the Dedekind eta function $\eta(\tau)$ is defined by

$$\eta(\tau)=q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n),\quad q=\exp(2\pi i\tau).$$

The modular invariant function (1.14) has applications to minimal frame operator norms and Ohta-Kawasaki models in di-block copolymers(Chen-Oshita [9]), and Ginzburg-Landau theory (Sandier-Serfaty [36]). In fact, $\sqrt{\operatorname{Im}(\tau)}|\eta(\tau)|^2$ is a variant of Zeta function (or real analytical Eisenstein series). Indeed, their relation are included in Kronecker first limit formula, which states that

$$\zeta(s,\tau)=\frac{\pi}{s-1}+2\pi(\gamma-\log(2)-\log(\sqrt{\operatorname{Im}(\tau)}|\eta(\tau)|^2))+O(s-1),\quad s\to 1^+, \tag{1.15}$$

here γ is the Euler-Mascheroni constant. On the other hand, the Dedekind eta functions still can be expressed by Theta functions, it can be found in the book by Nakayama [31](page 233). We state it as follows

Proposition 1.1 (Dedekind η functions in terms of Theta functions: Nakayama [31])**.** *For $\tau\in\mathbb{H}$, it holds that*

$$\sqrt{\operatorname{Im}(\tau)}|\eta(\tau)|^2=-\frac{\sqrt{6}}{4}\Big(\theta(\frac{3}{2},\tau)-2\theta(6,\tau)\Big).$$

On locating the extremals of determinants of Laplacians, in 1988, Osgood-Phillips-Sarnak found that

Proposition 1.2 (Maximum of Dedekind eta function, Osgood-Phillips-Sarnak [32], page 206)**.**

$$\operatorname{Maxima}_{z\in\mathbb{H}}\sqrt{\operatorname{Im}(\tau)}|\eta(\tau)|^2=e^{i\frac{\pi}{3}}, \tag{1.16}$$

Due the deep applications in various fields, Chen-Oshita [9]) and Sandier-Serfaty [36] encountered Dedekind eta function in their problems, provide completely different proofs of Proposition 1.2 in their papers. An alternatively proof can also be obtained by Proposition 1.1 and Theorem 1 in our paper (Luo-Wei [26]).

The min(max) problem of a single form of Theta, Zeta, eta functions and their variants are well studied(Theorems A, B and Proposition 1.2). In this paper, we study the min(max) problem of quotients of Theta, and Zeta functions and their variants. Namely, by studying them serving as model cases, we get a better understanding of quotients of modular invariant functions or the general problem (1.8). This looks like a purely mathematical problem, while it turns out that quotients of Theta, Zeta functions, and their variants appear naturally in conformal field theory, statistical field theory, and string theory, as we shall see below.

In the classical book of conformal field theory (Francesco-Mathieu-Sénéchal [40]), the free-boson partition function(without zero-mode) is

$$\mathbf{Z}_{\text{bos}}(\tau)=\frac{1}{\sqrt{\operatorname{Im}(\tau)}|\eta(\tau)|^2}, \tag{1.17}$$

see Section 10.2 in [40].

When the free bosonic theory compactified in a circle with radius R, the corresponding partition function on the torus is

$$\mathbf{Z}(R,\tau)=\frac{R}{\sqrt{2}}\mathbf{Z}_{\text{bos}}(\tau)\sum_{m,m'}\exp(-\frac{\pi R^2|m\tau-m'|^2}{2\operatorname{Im}(\tau)}). \tag{1.18}$$

See Section 10.4 in Francesco-Mathieu-Sénéchal [40], Bershadsky-Klebanov [7] and Alvarez-Gaum-Moore-Vafa [2](page 28). By the definition of Theta function given by (1.2), the widely used partition function (1.18) can be rewritten as

$$\mathbf{Z}(R,\tau)=\frac{R}{\sqrt{2}}\frac{\theta(\frac{R^2}{2},\tau)}{\sqrt{\mathrm{Im}(\tau)}|\eta(\tau)|^2}, \tag{1.19}$$

which is quotient form of Theta and eta functions (up to a power and scaling). See (1.17) and (1.19) also in the classical book of statistical field theory (Mussardo [41], chapter 12, pages 404-408) for the free energy.

When $R=\sqrt{\frac{2p'}{p}}$, with $p'>p$ two coprime integers, the partition function for the O(n) model of the $A_{p'-1},A_{p-1}$ minimal theory is (Francesco-Mathieu-Sénéchal [40])

$$\mathbf{Z}_{A_{p'-1},A_{p-1}}(\tau)=\frac{1}{2}\Big(\mathbf{Z}(\sqrt{2pp'},\tau)-\mathbf{Z}(\sqrt{\frac{2p'}{p}},\tau)\Big). \tag{1.20}$$

In the ADE classification, one of the partition functions is

$$\mathbf{Z}_{A_{p'-1},D_{p/2+1}}(\tau)=\frac{1}{2}\Big(\mathbf{Z}(\sqrt{\frac{8p'}{p}},\tau)-\mathbf{Z}(\sqrt{\frac{2p'}{p}},\tau)-\mathbf{Z}(\sqrt{\frac{p'p}{2}},\tau)+\mathbf{Z}(\sqrt{2pp'},\tau)\Big).$$

The others have the familiar forms and we omit them here. See more in pages 406-407 in [40]. They are still the quotient form of Theta, eta and their variants. For example, by (1.19), the partition function in (1.20) can be deformed as

$$\mathbf{Z}_{A_{p'-1},A_{p-1}}(\tau)=\frac{\sqrt{2p'/p}}{2}\frac{p\theta(pp',\tau)-\theta(\frac{p'}{p},\tau)}{\sqrt{\mathrm{Im}(\tau)}|\eta(\tau)|^2}.$$

Besides, the quotients of modular invariant functions play a key role in two-dimensional conformal field theory. The holographic correspondence between topological gravity coupled to an average of Narain's family of massless free bosons in two dimensions, discovered by Maloney-Witten [28] and by Afhkami-Jeddi et al. [22], involves the quotients of Epstein Zeta and eta functions.

The real analytic Eisenstein series $E(s,\tau)$ is defined as

$$E(s,\tau):=\sum_{m,m'\in\mathbb{Z}^2\setminus\{0\}}\frac{(\mathrm{Im}(\tau))^s}{\mid m\tau+m'\mid^{2s}}.$$

The average of partition functions of c massless free bosons in two dimensions over Narain moduli space and a $U(1)^c\times U(1)^c$ Chern-Simons gauge in three dimensions coupled to topological gravity. The three quantities are the same (as summarized by Benjamin-Keller-Ooguri-Zadeh [5]):

1: The average partition function of c free bosons(Afhkami-Jeddi et al. [22]) or the average of the genus 1 partition function over the Narain moduli space(Maloney-Witten [28]).

$$\mathbf{Z}_{\mathcal{M}}(\tau)=\frac{\int_{\mathcal{M}}d\mu Z(\mu)}{\int_{\mathcal{M}}d\mu}=\frac{E(\frac{c}{2},\tau)}{(\sqrt{\mathrm{Im}(\tau)}|\eta(\tau)|^2)^c},\ \mathcal{M}=O(c,c;\mathbb{Z})\setminus O(c,c)/O(c)\times O(c). \tag{1.21}$$

2. The Poincaré sum of a $U(1)^c$ vacuum character.

$$\mathbf{Z}_{T^c}(\tau)=\sum_{\gamma\in\Gamma_\infty\setminus SL(2,\mathbb{Z})}\mid\chi^{vac}(\gamma\tau)\mid^2=\frac{E(\frac{c}{2},\tau)}{(\sqrt{\mathrm{Im}(\tau)}|\eta(\tau)|^2)^c},\ \ \chi^{vac}(\tau)=\frac{1}{\eta(\tau)^c} \tag{1.22}$$

3: An exotic 3d gravity computation of a sum over geometries of a $U(1)^c\times U(1)^c$ abelian Chern-Simons theory:

$$\mathbf{Z}_{T^c}(\tau)=\sum_{\textit{3-manifold geometries}}e^{-S_{CS}}=\frac{E(\frac{c}{2},\tau)}{(\sqrt{\mathrm{Im}(\tau)}|\eta(\tau)|^2)^c}. \tag{1.23}$$

In the purely mathematical side, the explicit expression $\mathbf{Z}_{T^c}(\tau)$ appeared items 1,2,3 above is a reformulation of an argument originally by Siegel, and is known as the Siegel-Weil formula (Benjamin-Keller-Ooguri-Zadeh [5]).

In the physical and applied side, the explicit expressions $\mathbf{Z}_{T^c}(\tau)$ and $\mathbf{Z}(R,\tau)$ (given by (1.18) or (1.19)) and many others have appeared as partition functions in physical systems. Partition function plays fundamental role in statistical physics: the total energy, free energy, entropy, and pressure, can all be expressed in terms of the partition function or its derivatives. In particular, the Helmholtz free energy(F) and the partition function($\mathbf{Z}$) have the following relation

$$F=-k_BT\log(\mathbf{Z}).$$

Here k_B is Boltzmann's constant and T is the temperature. Therefore, at a given temperature, locating the min(max) of the partition functions is equivalent to finding the max(min) of the Helmholtz free energy. In this line, since the partition function determines many basic physical quantities, we are led to the following problem

Problem A (Torus geometry and max(min) partition functions)**.** *How does the geometry of the torus affect the value of partition functions in various physical models? In particular, what kind of geometry of torus such that the partition functions achieve the extreme values?*

Regarding $\mathbf{Z}_{T^c}(\tau)$(given by (1.21),(1.22), or (1.23)) and $\mathbf{Z}(R,\tau)$(given by (1.18) or (1.19)), we have the following corollary which follows from Theorems A, B and Proposition 1.2.

Corollary 1.4. *Assume that $c>2, \alpha>0$. Then*

(a) $\mathrm{Minima}_{\tau\in\mathbb{H}}\mathbf{Z}_{T^c}(\tau)=e^{i\frac{\pi}{3}}$,
(b) $\mathrm{Minima}_{\tau\in\mathbb{H}}\mathbf{Z}(R,\tau)=e^{i\frac{\pi}{3}}$.

Note that $\mathbf{Z}(R,\tau)$ denotes the partition function of free bosonic theory compactified in a circle with radius R. Considering the effect of the value of the circle radius to partition functions, one would ask the following question

Problem 1.2. *Assume that $R_1, R_2>0$. Classify*

$$\min(\max)_{\tau\in\mathbb{H}}\frac{\mathbf{Z}(R_2,\tau)}{\mathbf{Z}(R_1,\tau)}.$$

By (1.18) or (1.19), one has

$$\frac{\mathbf{Z}(R_2,\tau)}{\mathbf{Z}(R_1,\tau)}=\frac{\theta(\frac{R_2^2}{2},\tau)}{\theta(\frac{R_1^2}{2},\tau)}.$$

As a result, Problem 1.2 is completely solved in Theorem 1.1. A similar problem to Problem 1.2 is the following

Problem 1.3. *Assume that $R>0, c>2$. Classify*

$$\min_{\tau\in\mathbb{H}}\frac{\mathbf{Z}_{T^c}(\tau)}{(\mathbf{Z}(R,\tau))^c}.$$

By (1.22) or (1.23) and (1.18) or (1.19), one has

$$\frac{\mathbf{Z}_{T^c}(\tau)}{(\mathbf{Z}(R,\tau))^c}=(\frac{\sqrt{2}}{R})^c\frac{\zeta(\frac{c}{2},\tau)}{\theta^c(\frac{R^2}{2},\tau)}$$

Problem 1.3 is solved by Theorem 1.2.

The paper is organized as follows: In Section 2, we give the proof of Theorem 1.1 and Corollaries 1.1 and 1.2. In Section 3, we establish a minimum principle for modular invariant functions and collect some summation formulas for Zeta functions. Finally, we give the proof of Theorem 1.2 and Corollary 1.3 in Section 4.

2. Proof of Theorem 1.1 and its corollaries

Recall that the fundamental domain associated to the group $\mathcal{G}$ is given by

$$\mathcal{D}_{\mathcal{G}} := \{z \in \mathbb{H} : |z| > 1,\ 0 < \mathrm{Re}(z) < \frac{1}{2}\}.$$

For convenience, we define

$$\Gamma := \{z \in \mathbb{H} : \mathrm{Re}(z) = \frac{1}{2},\ \mathrm{Im}(z) \geq \frac{\sqrt{3}}{2}\}. \tag{2.1}$$

First we show that by deformation Theorem 1.1 follows from the following

Theorem 2.1. *Assume that $\beta > \alpha \geq 1$. Then*

(1) $\mathrm{Maxima}_{z\in\mathbb{H}} \frac{\theta(\beta,z)}{\theta(\alpha,z)} = e^{i\frac{\pi}{3}}$.
(2) $\mathrm{Minima}_{z\in\mathbb{H}} \frac{\theta(\beta,z)}{\theta(\alpha,z)}$ *does not exist.*

In fact, in the case (a) of Theorem 1.1: $\beta > \alpha, \beta\alpha > 1$, we consider two subcases, (a1): $\beta > \alpha \geq 1$ and (a2): $\beta > \frac{1}{\alpha} \geq 1$. The subcases (a1) are exactly proved in Theorem 2.1. For subcases (a2), we use the deformation $\frac{\theta(\beta,z)}{\theta(\alpha,z)} = \alpha \frac{\theta(\beta,z)}{\theta(\frac{1}{\alpha},z)}$, then it reduces to Theorem 2.1. The case (b): $\beta < \alpha, \beta\alpha < 1$ in Theorem 1.1 contains two subcases, (b1): $\frac{1}{\beta} > \frac{1}{\alpha} \geq 1$ and (b2): $\frac{1}{\beta} > \alpha \geq 1$. In subcases (b1) and (b2), one uses the deformations $\frac{\theta(\beta,z)}{\theta(\alpha,z)} = \frac{\alpha}{\beta} \frac{\theta(\frac{1}{\beta},z)}{\theta(\frac{1}{\alpha},z)}$ and $\frac{\theta(\beta,z)}{\theta(\alpha,z)} = \frac{1}{\beta} \frac{\theta(\frac{1}{\beta},z)}{\theta(\alpha,z)}$ respectively, then they are reduced to Theorem 2.1. The case (c): $\beta < \alpha, \beta\alpha > 1$ in Theorem 1.1 contains two subcases, (c1): $\alpha > \beta \geq 1$ and (c2): $\alpha > \frac{1}{\beta} \geq 1$. In subcases (c1) and (c2), one uses the deformations $\frac{\theta(\beta,z)}{\theta(\alpha,z)} = \frac{1}{\frac{\theta(\alpha,z)}{\theta(\beta,z)}}$ and $\frac{\theta(\beta,z)}{\theta(\alpha,z)} = \frac{1}{\beta} \frac{1}{\frac{\theta(\alpha,z)}{\theta(\frac{1}{\beta},z)}}$ respectively, then they are reduced to Theorem 2.1. Similar analysis applied to case (d): $\beta > \alpha, \beta\alpha < 1$ in Theorem 1.1, we omit the details here.

We now prove Theorem 2.1. The proof consists of two main steps.

In Step One we show that the maximizer can be reduced to the vertical line Γ. We shall prove that

$$\max_{z\in\mathbb{H}} \frac{\theta(\beta,z)}{\theta(\alpha,z)} = \max_{z\in\overline{\mathcal{D}_{\mathcal{G}}}} \frac{\theta(\beta,z)}{\theta(\alpha,z)} = \max_{z\in\Gamma} \frac{\theta(\beta,z)}{\theta(\alpha,z)} \ \text{ for } \ \beta > \alpha \geq 1. \tag{2.2}$$

This is a consequence of Proposition 2.1. Proposition 2.1 also implies that assuming the existence of the minimizers, one has

$$\min_{z\in\mathbb{H}} \frac{\theta(\beta,z)}{\theta(\alpha,z)} = \min_{z\in\overline{\mathcal{D}_{\mathcal{G}}}} \frac{\theta(\beta,z)}{\theta(\alpha,z)} = \min_{z=iy,y\geq1} \frac{\theta(\beta,z)}{\theta(\alpha,z)} \ \text{ for } \ \beta > \alpha \geq 1. \tag{2.3}$$

In Step Two, we show that the maximizer is located on $z = \frac{1}{2} + i\frac{\sqrt{3}}{2} = e^{i\frac{\pi}{3}}$. We shall prove that

$$\max_{z\in\Gamma} \frac{\theta(\beta,z)}{\theta(\alpha,z)} \ \text{ is achieved at } \ \frac{1}{2} + i\frac{\sqrt{3}}{2} \ \text{ for } \ \beta > \alpha \geq 1. \tag{2.4}$$

This follows from Proposition 2.2. By Proposition 2.3 we have

$$\min_{z=iy,y\geq1} \frac{\theta(\beta,z)}{\theta(\alpha,z)} \ \text{ does not exist for } \ \beta > \alpha \geq 1. \tag{2.5}$$

Combining Step One and Two we complete the proof of Theorem 2.1.

In the remaining part we prove these Propositions.

2.1. Transversal monotonicity.

In this subsection, we aim to prove a transversal monotonicity on quotient of theta functions. It is stated as follows

Proposition 2.1. *Assume that $\beta > \alpha \geq 1$. Then*

$$\frac{\partial}{\partial x} \frac{\theta(\beta,z)}{\theta(\alpha,z)} \geq 0 \ \textit{ for } \ z \in \mathcal{D}_{\mathcal{G}}.$$

The proof of Proposition 2.1 will be given at the end of this subsection. Before that we shall prove some preliminary lemma first.

In terms of one dimensional Theta function, one has an alternative expression of Theta functions.

Lemma 2.1. *Assume that* $z \in \mathbb{H}$ *and* $\alpha > 0$. *Then*

$$\sum_{n\in\mathbb{Z}} e^{-\pi\alpha y n^2}\vartheta(\frac{y}{\alpha}; nx) = \sqrt{\frac{\alpha}{y}}\cdot\theta(\alpha, z). \tag{2.6}$$

Here the classical one-dimensional theta function is given by

$$\vartheta(X;Y) := \sum_{n\in\mathbb{Z}} e^{-\pi n^2 X} e^{2n\pi i Y},\ X>0,\ Y\in\mathbb{R}. \tag{2.7}$$

Recall that

Lemma 2.2 (Montgomery's first Lemma [29])**.** *Assume that* $\alpha \geq 1$. *Then*

$$\frac{\partial}{\partial x}\theta(\alpha, z) \leq 0 \ \ for \ \ z \in \mathcal{D}_{\mathcal{G}}.$$

Or equivalently,

$$\frac{\partial}{\partial x}\sum_{n\in\mathbb{Z}} e^{-\pi\alpha y n^2}\vartheta(\frac{y}{\alpha}; nx) \leq 0 \ \ for \ \ z \in \mathcal{D}_{\mathcal{G}}.$$

In our previous work [24], we have established that

Lemma 2.3 (Corollary of Theorem 3.4 in [24])**.** *Assume that* $s \geq 1$. *Then*

$$\frac{\partial}{\partial x}\frac{\partial}{\partial s}(\sqrt{s}\theta(s, z)) \geq 0 \ \ for \ \ z \in \mathcal{D}_{\mathcal{G}}.$$

Using Lemma 2.3 and fundamental theorem of calculus, one has

$$\sqrt{\beta}\theta(\beta, z) - \sqrt{\alpha}\theta(\alpha, z) = \int_\alpha^\beta \frac{\partial}{\partial s}(\sqrt{s}\theta(s, z))ds.$$

Then

$$\frac{\partial}{\partial x}\Big(\sqrt{\beta}\theta(\beta, z) - \sqrt{\alpha}\theta(\alpha, z)\Big) = \int_\alpha^\beta \frac{\partial}{\partial x}\frac{\partial}{\partial s}(\sqrt{s}\theta(s, z))ds. \tag{2.8}$$

Therefore, by Lemma 2.3 and (2.8), it holds that

Lemma 2.4. *Assume that* $\beta > \alpha \geq 1$. *Then*

$$\frac{\partial}{\partial x}\Big(\sqrt{\beta}\theta(\beta, z) - \sqrt{\alpha}\theta(\alpha, z)\Big) \geq 0 \ \ for \ \ z \in \mathcal{D}_{\mathcal{G}}. \tag{2.9}$$

Or equivalently,

$$\frac{\partial}{\partial x}\Big(\sum_{n\in\mathbb{Z}} e^{-\pi\beta y n^2}\vartheta(\frac{y}{\beta}; nx) - \sum_{n\in\mathbb{Z}} e^{-\pi\alpha y n^2}\vartheta(\frac{y}{\alpha}; nx)\Big) \geq 0 \ \ for \ \ z \in \mathcal{D}_{\mathcal{G}}. \tag{2.10}$$

We shall also prove that

Lemma 2.5. *Assume that* $\beta > \alpha \geq 1$. *Then*

$$\sqrt{\beta}\theta(\beta, z) \geq \sqrt{\alpha}\theta(\alpha, z) \ \ for \ \ z \in \mathcal{D}_{\mathcal{G}}.$$

Or equivalently,

$$\sum_{n\in\mathbb{Z}} e^{-\pi\beta y n^2}\vartheta(\frac{y}{\beta}; nx) \geq \sum_{n\in\mathbb{Z}} e^{-\pi\alpha y n^2}\vartheta(\frac{y}{\alpha}; nx) \ \ for \ \ z \in \mathcal{D}_{\mathcal{G}}. \tag{2.11}$$

By Lemma 2.4, to prove Lemma 2.5, it suffices to prove that $\sqrt{\beta}\theta(\beta, z) \geq \sqrt{\alpha}\theta(\alpha, z)$ on the left boundary of half fundamental domain $\mathcal{D}_{\mathcal{G}}$. These are done in Lemmas 2.6 and 2.7.

Lemma 2.6. *Assume that* $\beta > \alpha \geq 1$. *Then*

$$\sqrt{\beta}\theta(\beta, z)\mid_{\mathrm{Re}(z)=0} \geq \sqrt{\alpha}\theta(\alpha, z)\mid_{\mathrm{Re}(z)=0} \quad \textit{for} \ \ \mathrm{Im}(z) \geq 1.$$

Lemma 2.7. *Assume that* $\beta > \alpha \geq 1$. *Then*

$$\sqrt{\beta}\theta(\beta, z)\mid_{|z|=1, 0\leq \mathrm{Re}(z)\leq \frac{1}{2}} \geq \sqrt{\alpha}\theta(\alpha, z)\mid_{|z|=1, 0\leq \mathrm{Re}(z)\leq \frac{1}{2}}.$$

By Lemma 2.1,

$$\sum_{n\in\mathbb{Z}} e^{-\pi\alpha y n^2}\vartheta(\frac{y}{\alpha}; nx)\mid_{x=0} = \vartheta_3(\alpha y)\vartheta_3(\frac{y}{\alpha}).$$

Then we have

Lemma 2.8 (Evaluation of theta function on $x-$axis)**.**

$$\begin{aligned}\theta(\alpha, iy) &= \sqrt{\frac{y}{\alpha}}\vartheta_3(\alpha y)\vartheta_3(\frac{y}{\alpha}) \ \textit{for} \ \frac{y}{\alpha} \ \textit{has a positive lower bound},\\ &= \vartheta_3(\alpha y)\vartheta_3(\frac{\alpha}{y}) \ \textit{for} \ \frac{\alpha}{y} \ \textit{has a positive lower bound}.\end{aligned}$$

Here ϑ_3 *is the Jacobi theta function of third type and defined as*

$$\vartheta_3(x) := \sum_{n\in\mathbb{Z}} e^{-\pi n^2 x}.$$

By Lemma 2.8, Lemma 2.6 is equivalent to

Lemma 2.9 (=Lemma 2.6)**.** *Assume that* $\beta > \alpha \geq 1$. *Then*

$$\vartheta_3(\beta y)\vartheta_3(\frac{y}{\beta}) \geq \vartheta_3(\alpha y)\vartheta_3(\frac{y}{\alpha}) \ \ \textit{for} \ \ y \geq 1.$$

To prove Lemma 2.9, it suffices to prove that

Lemma 2.10. *Assume that* $\alpha \geq 1$. *Then*

$$\frac{\partial}{\partial\alpha}\Big(\vartheta_3(\alpha y)\vartheta_3(\frac{y}{\alpha})\Big) \geq 0 \ \ \textit{for} \ \ y \geq 1.$$

By symmetry, Lemma 2.10 is equivalent to

Lemma 2.11. *Assume that* $\alpha \geq 1$. *Then*

$$\frac{\partial}{\partial y}\Big(\vartheta_3(\alpha y)\vartheta_3(\frac{\alpha}{y})\Big) \geq 0 \ \ \textit{for} \ \ y \geq 1.$$

By Lemma 2.8, Lemma 2.11 is equivalent to following Montegomery's Lemma [29].

Lemma 2.12 (Montegomery's second Lemma [29])**.** *Assume that* $\alpha \geq 1$. *Then*

$$\frac{\partial}{\partial y}\theta(\alpha, z) \geq 0 \ \ \textit{for} \ \ z \in \mathcal{D}_{\mathcal{G}}.$$

Therefore, Lemma 2.6 is proved. It remains to prove Lemma 2.7. By the group invariance($z \mapsto \frac{1}{1-z}$), one has

Lemma 2.13 (From arc to $\frac{1}{2}-$vertical line)**.** *Assume that* $\alpha, \beta > 0$, *it holds that*

$$\sqrt{\beta}\theta(\beta, z) - \sqrt{\alpha}\theta(\alpha, z)\mid_{|z|=1, \mathrm{Re}(z)\in[0,\frac{1}{2}]} = \sqrt{\beta}\theta(\beta, \frac{1}{2} + iy') - \sqrt{\alpha}\theta(\alpha, \frac{1}{2} + iy'), \ y' \in [\frac{1}{2}, \frac{\sqrt{3}}{2}],$$

explicitly, $y' = \frac{1}{2}\sqrt{\frac{1+\mathrm{Re}(z)}{1-\mathrm{Re}(z)}}$. *In particular,*

$$\sqrt{\beta}\theta(\beta, i) - \sqrt{\alpha}\theta(\alpha, i) = \sqrt{\beta}\theta(\beta, \frac{1}{2} + i\frac{1}{2}) - \sqrt{\alpha}\theta(\alpha, \frac{1}{2} + i\frac{1}{2}). \tag{2.12}$$

In fact, one has

Lemma 2.14. *Assume that* $\alpha, \beta > 0$*, it holds that*

$$\frac{\partial}{\partial y}\Big(\sqrt{\beta}\theta(\beta, \frac{1}{2}+iy) - \sqrt{\alpha}\theta(\alpha, \frac{1}{2}+iy)\Big) \geq 0 \ \ for \ \ y \in [\frac{1}{2}, \frac{\sqrt{3}}{2}].$$

Lemma 2.14 is proved by fundamental theorem of calculus

$$\sqrt{\beta}\theta(\beta, \frac{1}{2}+iy) - \sqrt{\alpha}\theta(\alpha, \frac{1}{2}+iy) = \int_\alpha^\beta \frac{\partial}{\partial s}\Big(\sqrt{s}\theta(s, \frac{1}{2}+iy)\Big)ds$$

and Lemma 2.15 as follows

Lemma 2.15 ((2) of Lemma 2.17)**.** *For* $s \geq 1$*,*

$$\frac{\partial}{\partial y}\frac{\partial}{\partial s}\Big(\sqrt{s}\theta(s, \frac{1}{2}+iy)\Big) \geq 0 \ \ for \ \ y \in [\frac{1}{2}, \frac{\sqrt{3}}{2}].$$

On the other hand, by Lemma 2.6,

$$\sqrt{\beta}\theta(\beta, i) - \sqrt{\alpha}\theta(\alpha, i) \geq 0 \ \ \text{for} \ \ \beta > \alpha \geq 1. \tag{2.13}$$

This and (2.12) in Lemma 2.13 implies that

$$\sqrt{\beta}\theta(\beta, \frac{1}{2}+i\frac{1}{2}) - \sqrt{\alpha}\theta(\alpha, \frac{1}{2}+i\frac{1}{2}) \geq 0 \ \ \text{for} \ \ \beta > \alpha \geq 1. \tag{2.14}$$

Therefore, (2.14) and Lemmas 2.13, 2.14 yield Lemma 2.7.

We are in a position to prove the main result(Proposition 2.1) in this subsection.

Proof. **Proof of Proposition 2.1.** The key is to use a new but equivalent quotient form,

$$\frac{\sqrt{\beta}\theta(\beta, z)}{\sqrt{\alpha}\theta(\alpha, z)}.$$

It suffices to prove that

$$\frac{\partial}{\partial x}\frac{\sqrt{\beta}\theta(\beta, z)}{\sqrt{\alpha}\theta(\alpha, z)} \geq 0 \ \ \text{for} \ \ z \in \mathcal{D}_{\mathcal{G}}.$$

A direct calculation shows that

$$\frac{\partial}{\partial x}\frac{\sqrt{\beta}\theta(\beta, z)}{\sqrt{\alpha}\theta(\alpha, z)} = \frac{\frac{\partial}{\partial x}(\sqrt{\beta}\theta(\beta, z))\sqrt{\alpha}\theta(\alpha, z) - \frac{\partial}{\partial x}(\sqrt{\alpha}\theta(\alpha, z))\sqrt{\beta}\theta(\beta, z)}{\alpha\theta^2(\alpha, z)}.$$

Then it is also equivalent to proving that

$$\frac{\partial}{\partial x}(\sqrt{\beta}\theta(\beta, z))\sqrt{\alpha}\theta(\alpha, z) - \frac{\partial}{\partial x}(\sqrt{\alpha}\theta(\alpha, z))\sqrt{\beta}\theta(\beta, z) \geq 0 \ \ \text{for} \ \ z \in \mathcal{D}_{\mathcal{G}}. \tag{2.15}$$

Regrouping the terms, we get that

$$\begin{aligned}
&\frac{\partial}{\partial x}(\sqrt{\beta}\theta(\beta, z))\sqrt{\alpha}\theta(\alpha, z) - \frac{\partial}{\partial x}(\sqrt{\alpha}\theta(\alpha, z))\sqrt{\beta}\theta(\beta, z)\\
=&\frac{\partial}{\partial x}(\sqrt{\beta}\theta(\beta, z))\sqrt{\alpha}\theta(\alpha, z) - \frac{\partial}{\partial x}(\sqrt{\beta}\theta(\beta, z))\sqrt{\beta}\theta(\beta, z)\\
&+\frac{\partial}{\partial x}(\sqrt{\beta}\theta(\beta, z))\sqrt{\beta}\theta(\beta, z) - \frac{\partial}{\partial x}(\sqrt{\alpha}\theta(\alpha, z))\sqrt{\beta}\theta(\beta, z)
\end{aligned}$$

Then it holds

$$\begin{aligned}
&\frac{\partial}{\partial x}(\sqrt{\beta}\theta(\beta, z))\sqrt{\alpha}\theta(\alpha, z) - \frac{\partial}{\partial x}(\sqrt{\alpha}\theta(\alpha, z))\sqrt{\beta}\theta(\beta, z)\\
=&\sqrt{\beta}\frac{\partial}{\partial x}(\theta(\beta, z))\Big(\sqrt{\alpha}\theta(\alpha, z) - \sqrt{\beta}\theta(\beta, z)\Big) + \sqrt{\beta}\theta(\beta, z)\frac{\partial}{\partial x}\Big(\sqrt{\alpha}\theta(\alpha, z) - \sqrt{\beta}\theta(\beta, z)\Big).
\end{aligned}$$

To simplify the expression, let

$$\mathcal{B}_a(\alpha,\beta,z):=\sqrt{\beta}\frac{\partial}{\partial x}(\theta(\beta,z))\Big(\sqrt{\alpha}\theta(\alpha,z)-\sqrt{\beta}\theta(\beta,z)\Big)$$
$$\mathcal{B}_b(\alpha,\beta,z):=\sqrt{\beta}\theta(\beta,z)\frac{\partial}{\partial x}\Big(\sqrt{\alpha}\theta(\alpha,z)-\sqrt{\beta}\theta(\beta,z)\Big).$$

Then

$$\frac{\partial}{\partial x}(\sqrt{\beta}\theta(\beta,z))\sqrt{\alpha}\theta(\alpha,z)-\frac{\partial}{\partial x}(\sqrt{\alpha}\theta(\alpha,z))\sqrt{\beta}\theta(\beta,z)=\mathcal{B}_a(\alpha,\beta,z)+\mathcal{B}_b(\alpha,\beta,z). \tag{2.16}$$

On the other hand, by Lemmas 2.2 and 2.5

$$\mathcal{B}_a(\alpha,\beta,z)\geq 0. \tag{2.17}$$

And similarly by Lemma 2.4

$$\mathcal{B}_b(\alpha,\beta,z)\geq 0. \tag{2.18}$$

(2.16), (2.17) and (2.18) yield (2.15). These complete the proof.

□

2.2. **Monotonicity on the $\frac{1}{2}-$Vertical line.** In this subsection, we aim to prove that

Proposition 2.2. *Assume that $\beta>\alpha\geq 1$. Then*

$$\frac{\partial}{\partial y}\frac{\theta(\beta,z)}{\theta(\alpha,z)}\mid_{\mathrm{Re}(z)=\frac{1}{2}}\leq 0 \;\; for \;\; \mathrm{Im}(z)\geq\frac{\sqrt{3}}{2}.$$

To prove Proposition 2.2, we establish one more auxiliary lemma except those in previous subsection.

Lemma 2.16. *Assume that $\beta>\alpha\geq 1$. Then*

$$\frac{\partial}{\partial y}\Big(\sqrt{\beta}\theta(\beta,z)-\sqrt{\alpha}\theta(\alpha,z)\Big)\mid_{\mathrm{Re}(z)=\frac{1}{2}}\leq 0 \;\; for \;\; \mathrm{Im}(z)\geq\frac{\sqrt{3}}{2}.$$

Via the deformation,

$$\begin{aligned}\frac{\partial}{\partial y}\Big(\sqrt{\beta}\theta(\beta,z)-\sqrt{\alpha}\theta(\alpha,z)\Big)&=\frac{\partial}{\partial y}\int_\alpha^\beta\frac{\partial}{\partial s}(\sqrt{s}\theta(s,z))ds\\&=\int_\alpha^\beta\frac{\partial^2}{\partial y\partial s}(\sqrt{s}\theta(s,z))ds,\end{aligned}$$

Lemma 2.16 is deduced by item (1) in Lemma 2.17, which is proved by our previous paper [24]. In fact, item (1) in Lemma 2.17 is followed by Proposition 4.1 in [24], the proof of items (2), (3) is similar, hence we omit the detail here.

Lemma 2.17 ([24])**.** *Assume that $s\geq 1$. Then*

(1) $\frac{\partial^2}{\partial y\partial s}\Big(\sqrt{s}\theta(s,z)\Big)\mid_{\mathrm{Re}(z)=\frac{1}{2}}\leq 0$ *for* $\mathrm{Im}(z)\geq\frac{\sqrt{3}}{2}$.
(2) $\frac{\partial^2}{\partial y\partial s}\Big(\sqrt{s}\theta(s,z)\Big)\mid_{\mathrm{Re}(z)=\frac{1}{2}}\geq 0$ *for* $\mathrm{Im}(z)\in[\frac{1}{2},\frac{\sqrt{3}}{2}]$.
(3) $\frac{\partial^2}{\partial y\partial s}\Big(\sqrt{s}\theta(s,z)\Big)\mid_{\mathrm{Re}(z)=0}\leq 0$ *for* $\mathrm{Im}(z)\geq 1$.

Proof. **Proof of Proposition 2.2.** Using the deformation,

$$\sqrt{\frac{\beta}{\alpha}}\cdot\frac{\partial}{\partial y}\frac{\theta(\beta,z)}{\theta(\alpha,z)}=\frac{\partial}{\partial y}\frac{\sqrt{\beta}\theta(\beta,z)}{\sqrt{\alpha}\theta(\alpha,z)}$$

A direct calculation and deformation show that

$$
\begin{aligned}
&(\sqrt{\alpha}\theta(\alpha,z))^2\cdot\frac{\partial}{\partial y}\frac{\sqrt{\beta}\theta(\beta,z)}{\sqrt{\alpha}\theta(\alpha,z)}\\
=&\frac{\partial}{\partial y}(\sqrt{\beta}\theta(\beta,z))\cdot(\sqrt{\alpha}\theta(\alpha,z))-\frac{\partial}{\partial y}(\sqrt{\alpha}\theta(\alpha,z))\cdot(\sqrt{\beta}\theta(\beta,z))\\
=&\Big(\frac{\partial}{\partial y}(\sqrt{\beta}\theta(\beta,z))\cdot(\sqrt{\alpha}\theta(\alpha,z))-\frac{\partial}{\partial y}(\sqrt{\alpha}\theta(\alpha,z))\cdot(\sqrt{\alpha}\theta(\alpha,z))\Big)\\
+&\Big(\frac{\partial}{\partial y}(\sqrt{\alpha}\theta(\alpha,z))\cdot(\sqrt{\alpha}\theta(\alpha,z))-\frac{\partial}{\partial y}(\sqrt{\alpha}\theta(\alpha,z))\cdot(\sqrt{\beta}\theta(\beta,z))\Big)\\
=&\sqrt{\alpha}\theta(\alpha,z)\cdot\frac{\partial}{\partial y}\Big(\sqrt{\beta}\theta(\beta,z)-\sqrt{\alpha}\theta(\alpha,z)\Big)+\frac{\partial}{\partial y}(\sqrt{\alpha}\theta(\alpha,z))\cdot\Big(\sqrt{\alpha}\theta(\alpha,z)-\sqrt{\beta}\theta(\beta,z)\Big).
\end{aligned}
$$

For convenience, we denote that

$$
\begin{aligned}
\mathcal{H}_a(\alpha,\beta,z):=&\sqrt{\alpha}\theta(\alpha,z)\cdot\frac{\partial}{\partial y}\Big(\sqrt{\beta}\theta(\beta,z)-\sqrt{\alpha}\theta(\alpha,z)\Big),\\
\mathcal{H}_b(\alpha,\beta,z):=&\frac{\partial}{\partial y}(\sqrt{\alpha}\theta(\alpha,z))\cdot\Big(\sqrt{\alpha}\theta(\alpha,z)-\sqrt{\beta}\theta(\beta,z)\Big).
\end{aligned}
$$

Then

$$
(\sqrt{\alpha}\theta(\alpha,z))^2\cdot\frac{\partial}{\partial y}\frac{\sqrt{\beta}\theta(\beta,z)}{\sqrt{\alpha}\theta(\alpha,z)}=\mathcal{H}_a(\alpha,\beta,z)+\mathcal{H}_b(\alpha,\beta,z).
$$

By Lemma item (1) of Lemma 2.17, if $\beta>\alpha\geq1$, then

$$
\mathcal{H}_a(\alpha,\beta,z)\mid_{\mathrm{Re}(z)=\frac{1}{2}}\leq0\ \ \text{for}\ \ \mathrm{Im}(z)\geq\frac{\sqrt{3}}{2}.
$$

By Lemmas 2.5 and 2.12, if $\beta>\alpha\geq1$, then

$$
\mathcal{H}_b(\alpha,\beta,z)\mid_{\mathrm{Re}(z)=\frac{1}{2}}\leq0\ \ \text{for}\ \ \mathrm{Im}(z)\geq\frac{\sqrt{3}}{2}.
$$

□

Similar to the proof of Proposition 2.2, using Lemmas 2.5, 2.12 and 2.17(item (3)), we have

Proposition 2.3. *Assume that $\beta>\alpha\geq1$. Then*

$$
\frac{\partial}{\partial y}\frac{\theta(\beta,z)}{\theta(\alpha,z)}\mid_{z=iy,\ y\geq1}\leq0.
$$

2.3. **Proof of Corollaries 1.1 and 1.2.** Proof of Corollary 1.1. For $k\geq1$, we use the deformation as follows

$$
\frac{\theta(\beta,z)}{\theta^k(\alpha,z)}=\frac{\theta(\beta,z)}{\theta(\alpha,z)}\cdot\frac{1}{\theta^{k-1}(\alpha,z)}.
$$

Here $k-1\geq0$, the desired result follows from Theorem 1.1 and Montgomery's Theorem A.

For $k<1$, by Lemmas 2.1 and 2.8, we have the asymptotic

$$
\frac{\theta(\beta,z)}{\theta^k(\alpha,z)}\rightarrow\frac{\sqrt{\frac{y}{\beta}}}{(\sqrt{\frac{y}{\alpha}})^k}=\frac{(\sqrt{\alpha})^k}{\sqrt{\beta}}(\sqrt{y})^{1-k}\rightarrow+\infty,\ \ \text{as}\ \ y\rightarrow+\infty.
$$

This proves the nonexistence of the maximum.

For Corollary 1.2, we follow the steps of proof of Theorem 1.1. Namely, Corollary 1.2 yielded by Propositions 2.4 and 2.5 in the following.

Proposition 2.4. *If $\min_{1\leq j\leq K}\beta_j\geq\max_{1\leq j\leq K}\alpha_j\geq1$ and any $a_j,b_j\geq0$, where $i,j=1\cdots K$ and $K\geq2$ is arbitrary. Then*

$$
\frac{\partial}{\partial x}\frac{\sum_{j=1}^K b_j\theta(\beta_j,z)}{\sum_{j=1}^K a_j\theta(\alpha_j,z)}\geq0\ \ \textit{for}\ \ z\in\mathcal{D}_{\mathcal{G}}.
$$

Proof. It is shown that the derivative of quotient of sum of Theta functions can be decomposed into sum of derivative of quotient of Theta functions. In fact, a direct calculation shows that

$$\begin{aligned}\partial_x \frac{\sum_j b_j \theta(\beta_j, z)}{\sum_i a_i \theta(\alpha_i, z)} &= \sum_{i,j} a_i b_j \frac{\theta_x(\beta_j, z)\theta(\alpha_i, z) - \theta(\beta_j, z)\theta_x(\alpha_i, z)}{(\sum_k a_k \theta(\alpha_k, z))^2} \\ &= \sum_{i,j} a_i b_j \frac{\theta^2(\alpha_i, z)}{(\sum_k a_k \theta(\alpha_k, z))^2} \cdot \partial_x \frac{\theta(\beta_j, z)}{\theta(\alpha_i, z)}.\end{aligned}$$

That is, there exists non-negative functions c_{ij} such that

$$\partial_x \frac{\sum_j b_j \theta(\beta_j, z)}{\sum_i a_i \theta(\alpha_i, z)} = \sum_{i,j} c_{ij} \cdot \partial_x \frac{\theta(\beta_j, z)}{\theta(\alpha_i, z)}. \tag{2.19}$$

(2.19) and Proposition 2.1 yield the result.

□

Proposition 2.5. *If* $\min_{1 \le j \le K} \beta_j \ge \max_{1 \le j \le K} \alpha_j \ge 1$ *and any* $a_j, b_j \ge 0$*, where* $i, j = 1 \cdots K$ *and* $K \ge 2$ *is arbitrary. Then*

$$\frac{\partial}{\partial y} \frac{\sum_{j=1}^K b_j \theta(\beta_j, z)}{\sum_{j=1}^K a_j \theta(\alpha_j, z)} \mid_{\mathrm{Re}(z)=\frac{1}{2}} \le 0 \ \ for \ \ \mathrm{Im}(z) \ge \frac{\sqrt{3}}{2}.$$

Proof. The idea of the proof is similar to that of Proposition 2.4. We compute that

$$\begin{aligned}\partial_y \frac{\sum_j b_j \theta(\beta_j, z)}{\sum_i a_i \theta(\alpha_i, z)} &= \sum_{i,j} a_i b_j \frac{\theta_y(\beta_j, z)\theta(\alpha_i, z) - \theta(\beta_j, z)\theta_y(\alpha_i, z)}{(\sum_k a_k \theta(\alpha_k, z))^2} \\ &= \sum_{i,j} a_i b_j \frac{\theta^2(\alpha_i, z)}{(\sum_k a_k \theta(\alpha_k, z))^2} \cdot \partial_y \frac{\theta(\beta_j, z)}{\theta(\alpha_i, z)}.\end{aligned}$$

Then there exists non-negative functions c_{ij} such that it holds the following kind of linear relation

$$\partial_y \frac{\sum_j b_j \theta(\beta_j, z)}{\sum_i a_i \theta(\alpha_i, z)} = \sum_{i,j} c_{ij} \cdot \partial_y \frac{\theta(\beta_j, z)}{\theta(\alpha_i, z)}.$$

One then restricts the relation on the $\frac{1}{2}-$vertical line,

$$\partial_y \frac{\sum_j b_j \theta(\beta_j, z)}{\sum_i a_i \theta(\alpha_i, z)} \mid_{\mathrm{Re}(z)=\frac{1}{2}} = \sum_{i,j} c_{ij} \cdot \partial_y \frac{\theta(\beta_j, z)}{\theta(\alpha_i, z)} \mid_{\mathrm{Re}(z)=\frac{1}{2}}. \tag{2.20}$$

The sign of $\partial_y \frac{\theta(\beta_j, z)}{\theta(\alpha_i, z)} \mid_{\mathrm{Re}(z)=\frac{1}{2}}$ is non-positive by Proposition 2.2. Then the result follows by (2.20).

□

3. Minimum principles and summation formulas

There are some nice structures in $\frac{\theta(\beta,z)}{\theta(\alpha,z)}$, as shown in the proof of Theorem 1.1. While to prove Theorem 1.2, we need some minimum principles. In the latter part of this section, we collect some summation formulas and lower, upper-bounds estimates of one-dimensional Theta functions.

3.1. Minimum princples.

The first minimum principle(inspired by Rankin [42]) is a baby version of the general ones. It concludes that for any symmetric modular invariant functions satisfying two monotonicity conditions admit the minimum at hexagonal point($e^{i\frac{\pi}{3}}$).

Proposition 3.1 (A minimum principle)**.** *Assume that* $\mathcal{W}$ *is modular invariant, i.e.,*

$$\mathcal{W}(\frac{az+b}{cz+d}) = \mathcal{W}(z), \ \ for \ all \ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}), \tag{3.1}$$

and

$$\mathcal{W}(-\overline{z}) = \mathcal{W}(z).$$

If

$$\begin{aligned} &\frac{\partial}{\partial y}\mathcal{W}(z) > 0, \;\; z \in \mathcal{D}_{\mathcal{G}} \cap \{y \geq a\} \;\; \textit{for some} \;\; a > \frac{\sqrt{3}}{2} \\ &\frac{\partial}{\partial x}\mathcal{W}(z) < 0, \;\; z \in \mathcal{D}_{\mathcal{G}} \cap \{y \geq b\} \;\; \textit{for some} \;\; b < \frac{\sqrt{3}}{2} \end{aligned} \tag{3.2}$$

and

$$\frac{a}{\frac{1}{4} + a^2} \geq b. \tag{3.3}$$

Here $\mathcal{D}_{\mathcal{G}}$ *is the fundamental domain corresponding to modular group* $SL_2(\mathbb{Z})$*, explicitly,* $\mathcal{D}_{\mathcal{G}} = \{z \in \mathbb{H} : |z| > 1, \; 0 < x < \frac{1}{2}\}$*. Then*

$$\min_{z\in\mathbb{H}} \mathcal{W}(z) = \min_{z\in\overline{\mathcal{D}_{\mathcal{G}}}} \mathcal{W}(z) \;\; \textit{is attained at} \;\; e^{i\frac{\pi}{3}} (\textit{hexagonal point}).$$

Proof. By the first part of (3.2), we have

$$\min_{z\in\overline{\mathcal{D}_{\mathcal{G}}}} \mathcal{W}(z) = \min_{z\in\overline{\mathcal{D}_{\mathcal{G}}}\cap\{y\leq a\}} \mathcal{W}(z)$$

We then assume $\min_{z\in\overline{\mathcal{D}_{\mathcal{G}}}\cap\{y\leq a\}} \mathcal{W}(z)$ is attained at $z_1 := (x_1, y_1)$. Then $y_1 \leq a$. Since $b < a$, by the second part of (3.2), we have

$$x_1 = \frac{1}{2}. \tag{3.4}$$

Taking $\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, we define $z_2 := \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} z_1$, it follows that

$$z_2 = \frac{1}{1 - z_1} = \frac{\frac{1}{2}}{\frac{1}{4} + y_1^2} + i\frac{y_1}{\frac{1}{4} + y_1^2}$$

and

$$\mathcal{W}(z_2) = \mathcal{W}(z_1).$$

This implies that z_2 still attains the minimum of $\min_{z\in\overline{\mathcal{D}_{\mathcal{G}}}} \mathcal{W}(z)$. Now we need an elementary inequality, namely,

$$\frac{u_2}{\frac{1}{4} + u_2^2} \geq \frac{u_1}{\frac{1}{4} + u_1^2} \;\; \text{if} \;\; \frac{1}{2} \leq u_2 \leq u_1.$$

From this equality, one has $\mathrm{Im}(z_2) \geq b$. In fact,

$$\mathrm{Im}(z_2) = \frac{y_1}{\frac{1}{4} + y_1^2} \geq \frac{a}{\frac{1}{4} + a^2} \;\; \text{if} \;\; \frac{1}{2} \leq y_1 \leq a.$$

By (3.3), we have $z_2 \in \overline{\mathcal{D}_{\mathcal{G}}} \cap \{y \geq b\}$. Still by the second part of (3.2) and z_2 is the minimum point, there must has $\mathrm{Re}(z_2) = \frac{1}{2}$, i.e., $\frac{\frac{1}{2}}{\frac{1}{4}+y_1^2} = \frac{1}{2}$. It yields that $y_1 = \frac{\sqrt{3}}{2}$. This and (3.4) yield the result. These complete the proof.

□

In many cases, the monotonicity estimates in (3.2) may not hold for such a large domain (cylinder, $y \geq$ or $y \geq b$). In fact, we can replace such a large domain (an infinite cylinder) to a finite rectangle domain. While we should add a comparison inequality as

$$\mathcal{W}(z) > \mathcal{W}(z_0) \text{ for some } z_0 \in \mathcal{D}_{\mathcal{G}} \cap \{y < c\}, \text{ and any } z \in \mathcal{D}_{\mathcal{G}} \cap \{y \geq c\}.$$

In practice, such a point z_0 can be choose to very special and easily calculated points like i or $e^{i\frac{\pi}{3}}$.

We state it precisely for application as follows.

Proposition 3.2 (A refined minimum principle). *Assume that $\mathcal{W}$ is modular invariant, i.e.,*

$$\mathcal{W}(\frac{az+b}{cz+d})=\mathcal{W}(z),\ \ \textit{for all}\ \left(\begin{array}{cc} a & b\\ c & d\end{array}\right)\in SL_2(\mathbb{Z}),$$

and

$$\mathcal{W}(-\overline{z})=\mathcal{W}(z).$$

If

(1) $\frac{\partial}{\partial y}\mathcal{W}(z)>0,\ \ z\in\mathcal{D}_{\mathcal{G}}\cap\{c\geq y\geq a\}$ *for some* $a>\frac{\sqrt{3}}{2}$;
(2) $\frac{\partial}{\partial x}\mathcal{W}(z)<0,\ \ z\in\mathcal{D}_{\mathcal{G}}\cap\{c\geq y\geq b\}$ *for some* $b<\frac{\sqrt{3}}{2}$;
(3) $\mathcal{W}(z)>\mathcal{W}(z_0)$ *for some* $z_0\in\mathcal{D}_{\mathcal{G}}\cap\{y<c\}$, *and any* $z\in\mathcal{D}_{\mathcal{G}}\cap\{y\geq c\}$ *where* $c>a$.

Here

$$\frac{a}{\frac{1}{4}+a^2}\geq b$$

and $\mathcal{D}_{\mathcal{G}}$ is the fundamental domain corresponding to modular group $SL_2(\mathbb{Z})$, explicitly, $\mathcal{D}_{\mathcal{G}}=\{z\in\mathbb{H}:|z|>1,\ 0<x<\frac{1}{2}\}$. Then

$$\min_{z\in\mathbb{H}}\mathcal{W}(z)=\min_{z\in\overline{\mathcal{D}_{\mathcal{G}}}}\mathcal{W}(z)\ \ \textit{is attained at}\ \ e^{i\frac{\pi}{3}}(\textit{hexagonal point}).$$

Proof. Item (3) implies that

$$\min_{z\in\overline{\mathcal{D}_{\mathcal{G}}}}\mathcal{W}(z)=\min_{z\in\overline{\mathcal{D}_{\mathcal{G}}}\cap\{y\leq c\}}\mathcal{W}(z).$$

The rest of the proof is similar to the proof of Proposition 3.1, hence we omit the details here. □

We shall use Proposition 3.2 to prove Theorem 1.2. To Proposition 3.2, we shall select suitably of the pair (a,b) satisfying $\frac{a}{\frac{1}{4}+a^2}\geq b$ and $a>\frac{\sqrt{3}}{2}$. It is crucial to select the pair (a,b). In the following, we choose $(a,b)=(\frac{4}{3},\frac{48}{73})$ and $c=2$. The corresponding estimates of (1), (2) and (3) are established in Lemmas 3.1-3.3 respectively.

We have the following computation at some particular point.

Lemma 3.1. *Assume that $s>1,\alpha\geq 2s$. Then*

$$\frac{\zeta(s,z)}{\theta^k(\alpha,z)}\ |_{\mathrm{Im}(z)\geq 2}\ /\frac{\zeta(s,z)}{\theta^k(\alpha,z)}\ |_{z=e^{i\frac{\pi}{3}}}>1.$$

This implies that for $\alpha\geq s+10, s\geq 2$, it holds that

$$\min_{z\in\mathbb{H}}\frac{\zeta(s,z)}{\theta^k(\alpha,z)}=\min_{z\in\overline{\mathcal{D}_{\mathcal{G}}}}\frac{\zeta(s,z)}{\theta^k(\alpha,z)}=\min_{z\in\overline{\mathcal{D}_{\mathcal{G}}}\cap\{y\leq 2\}}\frac{\zeta(s,z)}{\theta^k(\alpha,z)}.$$

Lemma 3.2. *Assume that $s>1,\alpha\geq 2s$. Then*

$$\frac{\partial}{\partial x}\frac{\zeta(s,z)}{\theta^k(\alpha,z)}<0\ \ \textit{for}\ \ z\in\mathcal{D}_{\mathcal{G}}\cap\{2\geq y\geq\frac{48}{73}\}.$$

Lemma 3.3. *Assume that $s>1,\alpha\geq 2s$. Then*

$$\frac{\partial}{\partial y}\frac{\zeta(s,z)}{\theta^k(\alpha,z)}>0\ \ \textit{for}\ \ z\in\mathcal{D}_{\mathcal{G}}\cap\{2\geq y\geq\frac{4}{3}\}.$$

3.2. **Summation formulas.** To prove Lemmas 3.1-3.3, we need some preliminary and auxiliary tools.

We first recall some basic estimates. By skilfully using of the Euler-Maclaurin summation formula, Rankin deduced that in his paper implicitly

Lemma 3.4 (A summation formula, Rankin 1953 [42])**.** *Assume that $z \in \mathbb{H}$ and $s > 1$. Then*
$$\sum_{n\in\mathbb{Z}} \frac{1}{|mz+n|^{2s}} = \frac{\Gamma(\frac{1}{2})\Gamma(s-\frac{1}{2})}{\Gamma(s)} y^{1-2s} \frac{1}{m^{2s-1}} + \sigma \cdot \frac{s}{2} \frac{(2s+1)^{s+\frac{1}{2}}}{(2s+2)^{s+1}} y^{-(2s+1)} \frac{1}{m^{2s+1}}, \quad \sigma \in [-1,1].$$
Here $z = x+iy$ is an complex number in the upper half plane.

It looks that there is no exact and explicit summation formula for $\sum_{n\in\mathbb{Z}} \frac{1}{|mz+n|^{2s}}$, hence Lemma 3.4 is the best available one to use. In Lemma 3.4, one can see that
$$\sum_{n\in\mathbb{Z}} \frac{1}{|mz+n|^{2s}} = \text{approximate part} + \text{error part}.$$
At least for large y, we get
$$\begin{aligned} \text{approximate part} &:= \frac{\Gamma(\frac{1}{2})\Gamma(s-\frac{1}{2})}{\Gamma(s)} y^{1-2s} \frac{1}{m^{2s-1}} \\ \text{error part} &:= \sigma \cdot \frac{s}{2} \frac{(2s+1)^{s+\frac{1}{2}}}{(2s+2)^{s+1}} y^{-(2s+1)} \frac{1}{m^{2s+1}}. \end{aligned}$$
By Lemma 3.4, one has

Lemma 3.5. *Assume that $z \in \mathbb{H}$ and $s > 1$. Then*
$$\sum_{m=1}^{\infty}\sum_{n\in\mathbb{Z}} \frac{1}{|mz+n|^{2s}} = \xi(2s-1)\frac{\Gamma(\frac{1}{2})\Gamma(s-\frac{1}{2})}{\Gamma(s)} y^{1-2s} + \sigma \cdot \xi(2s+1)\frac{s}{2} \frac{(2s+1)^{s+\frac{1}{2}}}{(2s+2)^{s+1}} y^{-(2s+1)}, \quad \sigma \in [-1,1].$$

Finally, one obtains the approximate and error part of Zeta functions $\zeta(s,z)$.

Lemma 3.6. *Assume that $z \in \mathbb{H}$ and $s > 1$. Then*
$$\begin{aligned} \zeta(s,z) &= \sum_{(m,n)\in\mathbb{Z}^2\backslash\{0\}} \frac{y^s}{|mz+n|^{2s}} \\ &= 2\xi(2s)y^s + 2\xi(2s-1)\frac{\Gamma(\frac{1}{2})\Gamma(s-\frac{1}{2})}{\Gamma(s)} y^{1-s} + \sigma \cdot \xi(2s+1) s \frac{(2s+1)^{s+\frac{1}{2}}}{(2s+2)^{s+1}} y^{-(s+1)}, \quad \sigma \in [-1,1]. \end{aligned}$$

Proof. Recall that
$$\zeta(s,z) = \sum_{(m,n)\in\mathbb{Z}^2\backslash\{0\}} \frac{y^s}{|mz+n|^{2s}}. \tag{3.5}$$

We split the summation in terms of m into $m=0$, $m>0$, $m<0$. Note that when $m=0$, the double summation in (3.5) becomes
$$\sum_{n\in\mathbb{Z}\backslash\{0\}} \frac{y^s}{|n|^{2s}} = 2y^s \sum_{n=1}^{\infty} \frac{1}{n^{2s}} = 2\xi(2s)y^s.$$

Then it holds that
$$\begin{aligned} \zeta(s,z) = \sum_{(m,n)\in\mathbb{Z}^2\backslash\{0\}} \frac{y^s}{|mz+n|^{2s}} &= 2\xi(2s)y^s + \sum_{m=1}^{\infty}\sum_{n\in\mathbb{Z}} \frac{y^s}{|mz+n|^{2s}} + \sum_{m=-\infty}^{-1}\sum_{n\in\mathbb{Z}} \frac{y^s}{|mz+n|^{2s}} \\ &= 2\xi(2s)y^s + 2\sum_{m=1}^{\infty}\sum_{n\in\mathbb{Z}} \frac{y^s}{|mz+n|^{2s}}. \end{aligned} \tag{3.6}$$

Therefore, by Lemma 3.5, we have

$$\zeta(s,z)=2\xi(2s)y^s+2\xi(2s-1)\frac{\Gamma(\frac{1}{2})\Gamma(s-\frac{1}{2})}{\Gamma(s)}y^{1-s}+\sigma\cdot\xi(2s+1)s\frac{(2s+1)^{s+\frac{1}{2}}}{(2s+2)^{s+1}}y^{-(s+1)},\ \ \sigma\in[-1,1].$$

□

There is another useful tool. Rankin deduced that in his paper implicitly

Lemma 3.7 (A summation formula, Rankin 1953 [42])**.** *Assume that $z\in\mathbb{H}$ and $s>1$. Then*

$$\begin{aligned}\zeta_y(s,z)=&2s\Big[\xi(2s)y^{s-1}-\frac{s-1}{s}\xi(2s-1)\frac{\Gamma(\frac{1}{2})\Gamma(s-\frac{1}{2})}{\Gamma(s)}y^{-s}\\&+\xi(2s+1)\big(\sigma_1\cdot\frac{1}{2}s\frac{(2s+1)^{s+\frac{1}{2}}}{(2s+2)^{s+1}}+\sigma_2\cdot(s+1)\frac{(2s+3)^{s+\frac{3}{2}}}{(2s+4)^{s+2}}\big)y^{-(s+2)}\Big],\ \ \sigma_1,\sigma_2\in[-1,1].\end{aligned}$$

Proof. Recall that

$$\zeta(s,z)=\sum_{(m,n)\in\mathbb{Z}^2\setminus\{0\}}\frac{y^s}{|mz+n|^{2s}}.$$

A direct calculation yields that

$$\zeta_y(s,z)=\sum_{(m,n)\in\mathbb{Z}^2\setminus\{0\}}\frac{sy^{s-1}}{|mz+n|^{2s}}-\frac{2sy^{s+1}m^2}{|mz+n|^{2(s+1)}}.$$

Splitting the summation in terms of m into $m=0$, $m>0$, $m<0$, and by symmetry, we get that

$$\zeta_y(s,z)=2s\xi(2s)y^{s-1}+2sy^{s-1}\sum_{m=1}^{\infty}\sum_{n\in\mathbb{Z}}\frac{1}{|mz+n|^{2s}}-4sy^{s+1}\sum_{m=1}^{\infty}m^2\sum_{n\in\mathbb{Z}}\frac{1}{|mz+n|^{2(s+1)}}.$$

Note that the summation $\sum_{n\in\mathbb{Z}}\frac{1}{|mz+n|^{2s}}$ is studied in Lemma 3.4. The rest of the proof followed by Lemmas 3.4 and 3.5.

□

At the end of this section, we recall some estimates on one-dimensional Theta functions. Recall that in (2.7)

$$\vartheta(X;Y):=\sum_{n\in\mathbb{Z}}e^{-\pi n^2X}e^{2n\pi iY},$$

where $X>0$ and $Y\in\mathbb{R}$.

The following Lemmas 3.8 and 3.9 are proved in [25].

Lemma 3.8. [25]. *Assume $X>\frac{1}{5}$. If $\sin(2\pi Y)>0$, then*

$$-\overline{\vartheta}(X)\sin(2\pi Y)\leq\frac{\partial}{\partial Y}\vartheta(X;Y)\leq-\underline{\vartheta}(X)\sin(2\pi Y).$$

If $\sin(2\pi Y)<0$, then

$$-\underline{\vartheta}(X)\sin(2\pi Y)\leq\frac{\partial}{\partial Y}\vartheta(X;Y)\leq-\overline{\vartheta}(X)\sin(2\pi Y).$$

Here

$$\underline{\vartheta}(X):=4\pi e^{-\pi X}(1-\mu(X)),\ \ \overline{\vartheta}(X):=4\pi e^{-\pi X}(1+\mu(X)),$$

and

$$\mu(X):=\sum_{n=2}^{\infty}n^2e^{-\pi(n^2-1)X}. \tag{3.7}$$

Lemma 3.9. [25]. *Assume* $X < \min\{\frac{\pi}{\pi+2}, \frac{\pi}{4\log\pi}\} = \frac{\pi}{\pi+2}$. *If* $\sin(2\pi Y) > 0$, *then*

$$-\overline{\vartheta}(X)\sin(2\pi Y) \leq \frac{\partial}{\partial Y}\vartheta(X;Y) \leq -\underline{\vartheta}(X)\sin(2\pi Y).$$

If $\sin(2\pi Y) < 0$, *then*

$$-\underline{\vartheta}(X)\sin(2\pi Y) \leq \frac{\partial}{\partial Y}\vartheta(X;Y) \leq -\overline{\vartheta}(X)\sin(2\pi Y).$$

Here

$$\underline{\vartheta}(X) := \pi e^{-\frac{\pi}{4X}} X^{-\frac{3}{2}};\ \ \overline{\vartheta}(X) := X^{-\frac{3}{2}}.$$

4. Proof of Theorem 1.2 and its corollary

In this section, we give the proof of Theorem 1.2. By the minimum principle given by Proposition 3.2, it suffices to prove Lemmas 3.1-3.3. We prove Lemmas 3.2, 3.3, and 3.1 in Subsections 4.1, 4.2 and 4.3 respectively. In Subsection 4.4, we give the proof of Corollary 1.3.

4.1. ∂_x **estimates.** By a direct computation and deformation, we have

$$\begin{aligned}\partial_x \frac{\zeta(s,z)}{\theta^k(\alpha,z)} &= \frac{\zeta(s,z)}{\theta^k(\alpha,z)} \cdot \Big(\frac{\zeta_x(s,z)}{\zeta(s,z)} - k\frac{\theta_x(\alpha,z)}{\theta(\alpha,z)}\Big)\\ &= -\frac{\zeta(s,z)}{\theta^k(\alpha,z)}\sin(2\pi x)\cdot\Big(\frac{-\zeta_x(s,z)}{\sin(2\pi x)\zeta(s,z)} - k\frac{-\theta_x(\alpha,z)}{\sin(2\pi x)\theta(\alpha,z)}\Big).\end{aligned}$$

Lemma 3.2 is equivalent to

Lemma 4.1. *Assume that* $\alpha \geq 2s, s > 1$. *Then*

$$\frac{-\zeta_x(s,z)}{\sin(2\pi x)\zeta(s,z)} - 2s\frac{-\theta_x(\alpha,z)}{\sin(2\pi x)\theta(\alpha,z)} > 0 \ \ for\ \ z \in \mathcal{D}_{\mathcal{G}} \cap \{2 \geq y \geq \frac{48}{73}\}.$$

In the rest of this subsection, we prove Lemma 4.1. To prove it, we estimate $\frac{-\zeta_x(s,z)}{\sin(2\pi x)\zeta(s,z)}$ and $\frac{-\theta_x(s,z)}{\sin(2\pi x)\theta(s,z)}$ separately. For $\frac{-\theta_x(s,z)}{\sin(2\pi x)\theta(s,z)}$, we use lower and upper bounds given by Lemmas 3.8 and 3.9. While for $\frac{-\zeta_x(s,z)}{\sin(2\pi x)\zeta(s,z)}$, getting a lower bound of it directly becomes complicated. To overcome it, we use a relation between $\zeta(s,z)$ and $\theta(\alpha,z)$. Namely, we use the identity

$$\zeta(s,z) = \frac{\pi^s}{\Gamma(s)}\int_0^\infty \big(\theta(\alpha,z)-1\big)\alpha^{s-1}d\alpha.$$

Then by taking derivative with respect to x, we get an identity of $\zeta_x(s,z)$ in terms of $\theta_x(\alpha,z)$

$$\zeta_x(s,z) = \frac{\pi^s}{\Gamma(s)}\int_0^\infty \theta_x(\alpha,z)\alpha^{s-1}d\alpha. \tag{4.1}$$

Using (4.1), we can get the bounds of $\zeta_x(s,z)$ by bounds of $\theta_x(\alpha,z)$. Together with summation formula (3.6), we can bound $\frac{-\zeta_x(s,z)}{\sin(2\pi x)\zeta(s,z)}$.

We now start the detailed proof. With the expression of theta function in Lemma 2.1, namely,

$$\theta(\alpha,z) = \sqrt{\frac{y}{\alpha}}\cdot\sum_{n\in\mathbb{Z}} e^{-\pi\alpha y n^2}\vartheta(\frac{y}{\alpha};nx). \tag{4.2}$$

Using bounds of 1-d Theta functions($\vartheta(X;Y)$) given by Lemmas 3.8 and 3.9, one has

Lemma 4.2 (An upper bound of $\frac{-\theta_x(\alpha,z)}{\sin(2\pi x)}$)**.** *Depending on the value of* $\frac{y}{\alpha}$, *it holds that*

- *for* $\frac{y}{\alpha} \geq \frac{1}{5}$,

$$\frac{-\theta_x(\alpha,z)}{\sin(2\pi x)} \leq 8\pi(1+\mu(\frac{y}{\alpha}))\sqrt{\frac{y}{\alpha}}\sum_{n=1}^\infty n e^{-\pi y(n^2\alpha+\frac{1}{\alpha})};$$

- *for* $\frac{y}{\alpha} \leq \frac{\pi}{\pi+2}$,

$$\frac{-\theta_x(\alpha, z)}{\sin(2\pi x)} \leq 2(\frac{y}{\alpha})^{-1} \sum_{n=1}^{\infty} n e^{-\pi\alpha y n^2}.$$

Here we denote

$$\mu(X) := \sum_{n=2}^{\infty} n^2 e^{-\pi(n^2-1)X}.$$

With Lemma 4.2, to bound $\frac{-\theta_x(s,z)}{\sin(2\pi x)\theta(s,z)}$, we need a lower bound of $\theta(\alpha, z)$. By (4.2), we have

Lemma 4.3 (Lower bounds of $\theta(\alpha, z)$)**.** *Assume that* $\alpha, y > 0$. *It holds that*

- *for* $\frac{y}{\alpha} \geq 1$, *then* $\theta(\alpha, z) \geq \sqrt{\frac{y}{\alpha}}$.
- *for* $\frac{y}{\alpha} \leq 1$, *then* $\theta(\alpha, z) \geq 1$.

Proof. The first part is trivial. The second part based on a duality formula of Jacobi theta function of third type. Recall that $\vartheta_3(x) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 x}$. Then $\theta(\alpha, z) \geq \sqrt{\frac{y}{\alpha}}\vartheta(\frac{y}{\alpha}; 0) = \sqrt{\frac{y}{\alpha}}\vartheta_3(\frac{y}{\alpha}) = \vartheta_3(\frac{\alpha}{y}) \geq 1$.

□

Combining Lemma 4.2 with Lemma 4.3, we get an upper bound of $\frac{-\theta_x(\alpha,z)}{\sin(2\pi x)\theta(\alpha,z)}$.

Lemma 4.4 (An upper bound of $\frac{-\theta_x(\alpha,z)}{\sin(2\pi x)\theta(\alpha,z)}$)**.** *Depending on the value of* $\frac{y}{\alpha}$, *it holds that*

- *for* $\frac{y}{\alpha} \geq \frac{1}{5}$,

$$\frac{-\theta_x(\alpha, z)}{\sin(2\pi x)\theta(\alpha, z)} \leq 8\pi(1 + \mu(\frac{y}{\alpha})) \sum_{n=1}^{\infty} n e^{-\pi y(n^2\alpha + \frac{1}{\alpha})};$$

- *for* $\frac{y}{\alpha} \leq \frac{\pi}{\pi+2}$,

$$\frac{-\theta_x(\alpha, z)}{\sin(2\pi x)\theta(\alpha, z)} \leq 2(\frac{y}{\alpha})^{-\frac{3}{2}} \sum_{n=1}^{\infty} n e^{-\pi\alpha y n^2}.$$

Here

$$\mu(X) := \sum_{n=2}^{\infty} n^2 e^{-\pi(n^2-1)X}.$$

We proceed to get the lower bound of $\frac{-\zeta_x(\alpha,z)}{\sin(2\pi x)\zeta(\alpha,z)}$. Using (4.1), we first estimate the lower bound of $\theta_x(\alpha, z)$. By (4.2) and bounds of 1-d Theta functions($\vartheta(X; Y)$) given by Lemmas 3.8 and 3.9, we have

Lemma 4.5 (A lower bound of $\frac{-\theta_x(\alpha,z)}{\sin(2\pi x)}$)**.** *Depending on the value of* $\frac{y}{\alpha}$, *it holds that*

- *for* $\frac{y}{\alpha} \geq \frac{1}{5}$,

$$\frac{-\theta_x(\alpha, z)}{\sin(2\pi x)} \geq 8\pi(1 - \mu(\frac{y}{\alpha}))\sqrt{\frac{y}{\alpha}} \sum_{n=1}^{\infty} n e^{-\pi y(n^2\alpha + \frac{1}{\alpha})};$$

- *for* $\frac{y}{\alpha} \leq \frac{\pi}{\pi+2}$,

$$\frac{-\theta_x(\alpha, z)}{\sin(2\pi x)} \geq 2\pi(\frac{y}{\alpha})^{-1} \sum_{n=1}^{\infty} n e^{-\pi(n^2 y + \frac{1}{4y})\alpha}.$$

Here

$$\mu(X) := \sum_{n=2}^{\infty} n^2 e^{-\pi(n^2-1)X}.$$

Proceeding by (4.1),

$$\begin{aligned}\frac{-\zeta_x(s,z)}{\sin(2\pi x)} &= \frac{\pi^s}{\Gamma(s)}\int_0^\infty \frac{-\theta_x(\alpha,z)}{\sin(2\pi x)}\alpha^{s-1}d\alpha\\ &= \frac{\pi^s}{\Gamma(s)}\Big(\int_0^y \frac{-\theta_x(\alpha,z)}{\sin(2\pi x)}\alpha^{s-1}d\alpha + \int_y^\infty \frac{-\theta_x(\alpha,z)}{\sin(2\pi x)}\alpha^{s-1}d\alpha\Big)\\ &\geq \frac{\pi^s}{\Gamma(s)}\int_y^\infty \frac{-\theta_x(\alpha,z)}{\sin(2\pi x)}\alpha^{s-1}d\alpha.\end{aligned} \tag{4.3}$$

Using the incomplete gamma function $\Gamma(s,x)$, which is defined as

$$\Gamma(s,x)=\int_x^\infty t^{s-1}e^{-t}dt,$$

together with Lemma 4.5 and (4.3), we have

Lemma 4.6 (A lower bound of $\frac{-\zeta_x(s,z)}{\sin(2\pi x)}$)**.** *For $s>1, y>0$, it holds that*

$$\frac{-\zeta_x(s,z)}{\sin(2\pi x)} \geq \frac{\pi^s}{\Gamma(s)}\frac{2\pi}{y}\sum_{n=1}^\infty n\cdot\frac{\Gamma(s+1,\pi(n^2y^2+\frac14))}{(\pi(n^2y+\frac{1}{4y}))^{s+1}}.$$

Now we need a lower bound of the incomplete gamma function $\Gamma(s,x)$.
Integrating by parts, one has the recursion in s, i.e.,

$$\Gamma(s,x)=x^{s-1}e^{-x}+(s-1)\Gamma(s-1,x)\ \text{ for }\ s\geq 2. \tag{4.4}$$

We need a lower bound for incomplete gamma function $\Gamma(s,x)$. By using the recursion formula given by (4.4) and some monotonicity properties, Pinelis [27] deduced that

Lemma 4.7 (Lower-bound functions of incomplete gamma function $\Gamma(s,x)$)**.** *The incomplete gamma function $\Gamma(s,x)$ has the following*

$$\Gamma(s,x)\begin{cases} > \Big(\frac{(x+2)^s-x^s-2^s}{2s}+\Gamma(s)\Big)e^{-x} & \textit{for }\ s>3;\\ = (x^2+2x+2)e^{-x} & \textit{for }\ s=3;\\ > \Big(\frac{(x+2)^{s-1}+x^{s-1}-2^{s-1}}{2}+\Gamma(s)\Big)e^{-x} & \textit{for }\ s\in(2,3);\\ = (x+1)e^{-x} & \textit{for }\ s=2;\\ > \Big(\frac{(x+2)^s-x^s-2^s}{2s}+\Gamma(s)\Big)e^{-x} & \textit{for }\ s\in(1,2);\\ = e^{-x} & \textit{for }\ s=1.\end{cases}$$

Using Lemmas 4.7 and 4.6, we have

Lemma 4.8 (A lower bound of $\frac{-\zeta_x(s,z)}{\sin(2\pi x)}$)**.** *For $s>1, x\in[0,\frac12]$ and $y>0$, it holds that*

$$\frac{-\zeta_x(s,z)}{\sin(2\pi x)} \geq \frac{2s}{y}\big(y+\frac{1}{4y}\big)^{-(s+1)}e^{-\pi(y^2+\frac14)}.$$

Now we are ready to obtain an effective lower bound of $\frac{-\zeta_x(s,z)}{\sin(2\pi x)\zeta(s,z)}$. By Lemma 4.8 and the upper bound of $\zeta(s,z)$ given by the summation formula in Lemma 3.5, we obtain that

Lemma 4.9 (A lower bound of $\frac{-\zeta_x(s,z)}{\sin(2\pi x)\zeta(s,z)}$)**.** *For $s>1, x\in[0,\frac12]$ and $y>0$, it holds that*

$$\frac{-\zeta_x(s,z)}{\sin(2\pi x)\zeta(s,z)} \geq \frac{\frac{2s}{y}\big(y+\frac{1}{4y}\big)^{-(s+1)}e^{-\pi(y^2+\frac14)}}{2\xi(2s)y^s+2\xi(2s-1)\frac{\Gamma(\frac12)\Gamma(s-\frac12)}{\Gamma(s)}y^{1-s}+\xi(2s+1)s\frac{(2s+1)^{s+\frac12}}{(2s+2)^{s+1}}y^{-(s+1)}}.$$

Lemma 4.9 is quite useful when the parameter s is large, while when s is small, we have a more precise bound(Lemma 4.11). By Lemmas 4.2 and 4.9, to prove the main Lemma 4.1 for the cases $s\geq 4$, it suffices to prove that

Lemma 4.10 (An elementary inequality). *Assume that $s \geq 4, \alpha \geq 3s$. Then for $y \in [\frac{48}{73}, 2]$, it holds that*

$$\frac{\frac{2s}{y}\left(y+\frac{1}{4y}\right)^{-(s+1)}e^{-\pi(y^2+\frac{1}{4})}}{2\xi(2s)y^s+2\xi(2s-1)\frac{\Gamma(\frac{1}{2})\Gamma(s-\frac{1}{2})}{\Gamma(s)}y^{1-s}+\xi(2s+1)s\frac{(2s+1)^{s+\frac{1}{2}}}{(2s+2)^{s+1}}y^{-(s+1)}}$$

$$\geq \begin{cases} 3(\frac{y}{\alpha})^{-\frac{3}{2}}e^{-\pi\alpha y} & \text{for } \frac{y}{\alpha} \leq 1, \\ 9\pi e^{-\pi y(\alpha+\frac{1}{\alpha})} & \text{for } \frac{y}{\alpha} \geq 1. \end{cases}$$

Since $\alpha \geq 2s \geq 8$, the proof of Lemma 4.10 is trivial, hence we omit the details here.

It remains to prove the main Lemma 4.1 for the cases $s \in (1, 4]$. When s is small, we could deduce a precise lower bound for $\frac{-\zeta_x(s,z)}{\zeta(s,z)\sin(2\pi x)}$. We first have

Lemma 4.11 (A lower bound of $\frac{-\zeta_x(s,z)}{\sin(2\pi x)}$: $s \leq 4$). *Assume that $s \in (1, 4], x \in [0, \frac{1}{2}]$. Then for $y \geq \frac{48}{73}$, it holds that*

$$\frac{-\zeta_x(s,z)}{\sin(2\pi x)} \geq \frac{16}{3}\sqrt{y}\frac{\pi^{s+1}}{\Gamma(s)}K_{s-\frac{1}{2}}(2\pi y).$$

Here $K_s(z)$ is the modified Bessel function of the second kind and is defined as

$$K_s(y) = \frac{1}{2}\int_0^\infty t^{-(s+1)}e^{-\frac{1}{2}y(t+\frac{1}{t})}dt,$$

or

$$K_s(y) = \int_0^\infty e^{-y\cosh(t)}\cosh(st)dt. \tag{4.5}$$

See more details for $K_s(y)$ in Watson [43]. To keep the structure clear, we postpone the proof of Lemma 4.11 to the end of this subsection.

By Lemma 4.11 and the upper bound of $\zeta(s, z)$ given by the summation formula in Lemma 3.5, we obtain that

Lemma 4.12 (A lower bound of $\frac{-\zeta_x(s,z)}{\zeta(s,z)\sin(2\pi x)}$: $s \leq 4$). *Assume that $s \in (1, 4], x \in [0, \frac{1}{2}]$. Then for $y \geq \frac{48}{73}$, it holds that*

$$\frac{-\zeta_x(s,z)}{\zeta(s,z)\sin(2\pi x)} \geq \frac{\frac{16}{3}\sqrt{y}\frac{\pi^{s+1}}{\Gamma(s)}K_{s-\frac{1}{2}}(2\pi y)}{2\xi(2s)y^s+2\xi(2s-1)\frac{\Gamma(\frac{1}{2})\Gamma(s-\frac{1}{2})}{\Gamma(s)}y^{1-s}+\xi(2s+1)s\frac{(2s+1)^{s+\frac{1}{2}}}{(2s+2)^{s+1}}y^{-(s+1)}}.$$

By Lemmas 4.2 and 4.11, to prove the main Lemma 4.1 for the cases $s \in (1, 4]$, it suffices to prove that

Lemma 4.13 (An elementary inequality: (b)). *Assume that $s \in (1, 4], \alpha \geq 3s$. Then for $y \in [\frac{48}{73}, 2]$, it holds that*

$$\frac{\frac{16}{3}\sqrt{y}\frac{\pi^{s+1}}{\Gamma(s)}K_{s-\frac{1}{2}}(2\pi y)}{2\xi(2s)y^s+2\xi(2s-1)\frac{\Gamma(\frac{1}{2})\Gamma(s-\frac{1}{2})}{\Gamma(s)}y^{1-s}+\xi(2s+1)s\frac{(2s+1)^{s+\frac{1}{2}}}{(2s+2)^{s+1}}y^{-(s+1)}} \geq 3(\frac{y}{\alpha})^{-\frac{3}{2}}e^{-\pi\alpha y}.$$

Note that $K_{\frac{1}{2}}(2\pi y) = \frac{2}{\sqrt{y}}e^{-2\pi y}$ and $K_s(2\pi y)$ is increasing with respect to s by the expression given by (4.5). The proof of Lemma 4.13 is straightforward and elementary, we then omit the details here.

We now give the proof of Lemma 4.11. We use the Chowla-Selberg formula [14, 13].

Lemma 4.14 (Chowla-Selberg formula). *For $s > 1, y > 0$, it holds that*

$$\zeta(s, z) = a_0(s, y) + 2\sum_{n=1}^{\infty} a_n(s, y)\cos(2\pi nx),$$

where

$$
\begin{aligned}
a_0(s,y) &:= 2\xi(2s)y^s + 2\xi(2s-1)\frac{\Gamma(\frac{1}{2})\Gamma(s-\frac{1}{2})}{\Gamma(s)}y^{1-s},\\
a_n(s,y) &:= \frac{4\pi^s\sqrt{y}}{\Gamma(s)}n^{s-\frac{1}{2}}\sigma_{1-2s}(n)K_{s-\frac{1}{2}}(2\pi ny).
\end{aligned}
\tag{4.6}
$$

Here $\sigma_{1-2s}(n) = \sum_{d|n} d^{1-2s}$, *and* $K_s(y)$ *is the modified Bessel function.*

By Chowla-Selberg formula(Lemma 4.14), using $|\frac{\sin(2\pi nx)}{\sin(2\pi x)}| \leq n$ for $n \in \mathbb{Z}^+$ and $x \in \mathbb{R}$, one has

$$
\begin{aligned}
\frac{-\zeta_x(s,z)}{4\pi a_1(s,y)\sin(2\pi x)} &= 1 + \sum_{n=2}^{\infty} n\frac{a_n(s,y)}{a_1(s,y)}\frac{\sin(2\pi nx)}{\sin(2\pi x)}\\
&\geq 1 - \sum_{n=2}^{\infty} n^2\frac{a_n(s,y)}{a_1(s,y)}.
\end{aligned}
\tag{4.7}
$$

To prove Lemma 4.11, by (4.7), it suffices to prove that

$$
\sum_{n=2}^{\infty} n^2\frac{a_n(s,y)}{a_1(s,y)} \leq \frac{2}{3} \ \text{ for } \ s \in (1,4], \ \ y \geq \frac{48}{73}. \tag{4.8}
$$

By (4.6), (4.8) is equivalent to

$$
\sum_{n=2}^{\infty} n^{s+\frac{3}{2}}\sigma_{1-2s}(n)\frac{K_{s-\frac{1}{2}}(2\pi ny)}{K_{s-\frac{1}{2}}(2\pi y)} \leq \frac{2}{3} \ \text{ for } \ s \in (1,4], \ \ y \geq \frac{48}{73}. \tag{4.9}
$$

Now we need an estimate on quotient of modified Bessel function $K_s(y)$. This is done by the following

Lemma 4.15 (Baricz [3])**.** *If* $\nu > \frac{1}{2}$. *Then for* $y > x > 0$, *it holds that*

$$
\frac{K_\nu(y)}{K_\nu(x)} < e^{-(y-x)}(\frac{y}{x})^{-\frac{1}{2}}.
$$

By Lemma 4.15, to prove (4.9), it suffices to prove that

$$
\sum_{n=2}^{\infty} n^{s+1}\sigma_{1-2s}(n)e^{-2\pi(n-1)y} \leq \frac{2}{3} \ \text{ for } \ s \in (1,4], \ \ y \geq \frac{48}{73}. \tag{4.10}
$$

The proof of (4.10) is elementary, hence we omit the detail here. The proof is complete.

4.2. ∂_y **estimates.** By a direct calculation

$$
\partial_y\frac{\zeta(s,z)}{\theta^k(\alpha,z)} = \frac{\zeta(s,z)}{\theta^k(\alpha,z)}\cdot\Big(\frac{\zeta_y(s,z)}{\zeta(s,z)} - k\frac{\theta_y(\alpha,z)}{\theta(\alpha,z)}\Big).
$$

Lemma 3.3 is equivalent to

Lemma 4.16. *Assume that* $s > 1, \alpha \geq 3s$. *Then*

$$
\frac{\zeta_y(s,z)}{\zeta(s,z)} - 2s\frac{\theta_y(\alpha,z)}{\theta(\alpha,z)} > 0 \ \ \textit{for} \ \ z \in \mathcal{D}_{\mathcal{G}} \cap \{2 \geq y \geq \frac{4}{3}\}.
$$

To prove Lemma 4.16, we estimate $\frac{\zeta_y(s,z)}{\zeta(s,z)}$ and $\frac{\theta_y(\alpha,z)}{\theta(\alpha,z)}$ respectively. For $\frac{\zeta_y(s,z)}{\zeta(s,z)}$, we use the summation formulas in Lemmas 3.6 and 3.7. For $\frac{\theta_y(\alpha,z)}{\theta(\alpha,z)}$, we deduce by a careful study of properties of $\theta(\alpha,z)$.

By Lemmas 3.6 and 3.7, we have

Lemma 4.17 (A lower bound of $\frac{\zeta_y(s,z)}{\zeta(s,z)}$)**.** *Assume that $s>1$. Then*

$$\frac{\zeta_y(s,z)}{\zeta(s,z)} \geq \frac{s}{y}\Big(1-\mathcal{A}_s(y)\Big).$$

Here $\mathcal{A}_s(y)$ is small and explicitly,

$$\mathcal{A}_s(y) := \frac{\frac{2s-1}{s}\xi(2s-1)\frac{\Gamma(\frac{1}{2})\Gamma(s-\frac{1}{2})}{\Gamma(s)}y^{1-s} + \xi(2s+1)\big(\frac{3}{2}s\frac{(2s+1)^{s+\frac{1}{2}}}{(2s+2)^{s+1}} + (s+1)\frac{(2s+3)^{s+\frac{3}{2}}}{(2s+4)^{s+2}}\big)y^{-(s+1)}}{\xi(2s)y^s + \xi(2s-1)\frac{\Gamma(\frac{1}{2})\Gamma(s-\frac{1}{2})}{\Gamma(s)}y^{1-s} + \frac{1}{2}\xi(2s+1)s\frac{(2s+1)^{s+\frac{1}{2}}}{(2s+2)^{s+1}}y^{-(s+1)}}. \tag{4.11}$$

Table 1. **Evaluation of $\mathcal{A}_s(\frac{4}{3})$[Taking six digital numbers].**

$s=1$	$s=2$	$s=3$	$s=4$	$s=5$	$s=6$	$s=7$	$s=8$
0.886729	**0.772190**	**0.517878**	**0.324054**	**0.194742**	**0.114367**	**0.066316**	**0.038192**

For the function $\mathcal{A}_s(y)$ appeared in the lower bound of $\frac{\zeta_y(s,z)}{\zeta(s,z)}$, we have the following basic properties, whose proof is elementary hence we omit the details here.

Lemma 4.18.
- *For $s\geq 1$, $\mathcal{A}_s(y)$ is decreasing for $y\geq 1$.*
- *For $y\geq 1$, $\mathcal{A}_s(y)$ is decreasing for $s\geq 1$.*
- $\max_{y\geq a}\mathcal{A}_s(y)=\mathcal{A}_s(a)$ *for $a\geq 1$.*

For the upper bound of $\frac{\theta_y(\alpha,z)}{\theta(\alpha,z)}$, we have

Lemma 4.19 (An upper bound of $\frac{\theta_y(\alpha,z)}{\theta(\alpha,z)}$)**.** *Assume that $\alpha\geq 1$. Then*

(1) *For $\frac{y}{\alpha}\geq 1$,*

$$\frac{\theta_y(\alpha,z)}{\theta(\alpha,z)} \leq \frac{1}{2y}.$$

(2) *For $\frac{y}{\alpha}\leq 1$,*

$$\frac{\theta_y(\alpha,z)}{\theta(\alpha,z)} \leq \frac{2\pi\alpha}{y^2}e^{-\pi\frac{\alpha}{y}}.$$

We postpone the proof of Lemma 4.19 to the end of this subsection and give the proof of Lemma 4.16. By Lemmas 4.17 and 4.19, to prove Lemma 4.16, it suffices to the following

Lemma 4.20. *Assume that $s\geq 1$. Then*

$$1-\mathcal{A}_s(\frac{4}{3}) \geq 4\pi s e^{-\pi s}.$$

Here $\mathcal{A}_s(y)$ is defined in (4.11).

Now the proof of Lemma 4.20 is elementary hence we omit the details here.
It remains to prove Lemma 4.19.

Proof. Recall that $\theta(\alpha,z)=\sqrt{\frac{y}{\alpha}}\cdot\sum_{n\in\mathbb{Z}}e^{-\pi\alpha y n^2}\vartheta(\frac{y}{\alpha};nx)$. By a direct calculation

$$\begin{aligned}\theta_y(\alpha,z) =& \frac{1}{2\sqrt{\alpha y}}\sum_{n\in\mathbb{Z}}e^{-\pi\alpha y n^2}\vartheta(\frac{y}{\alpha};nx) - \pi\sqrt{\alpha y}\sum_{n\in\mathbb{Z}}n^2e^{-\pi\alpha y n^2}\vartheta(\frac{y}{\alpha};nx)\\ &+\frac{1}{\alpha}\sqrt{\frac{y}{\alpha}}\sum_{n\in\mathbb{Z}}e^{-\pi\alpha y n^2}\vartheta_X(\frac{y}{\alpha};nx).\end{aligned} \tag{4.12}$$

In the expression of $\theta_y(\alpha,z)$ given by (4.12), we have

$$\sum_{n\in\mathbb{Z}} n^2 e^{-\pi\alpha y n^2}\vartheta(\frac{y}{\alpha};nx)\geq 0 \tag{4.13}$$

and

$$\sum_{n\in\mathbb{Z}} e^{-\pi\alpha y n^2}\vartheta_X(\frac{y}{\alpha};nx)\leq 0. \tag{4.14}$$

Note that $\vartheta(X;Y)=1+2\sum_{n=1}^\infty e^{-\pi n^2 X}\cos(2n\pi Y)$. By the Poisson summation formula, one has

$$\vartheta(X;Y)=X^{-\frac{1}{2}}\sum_{n\in\mathbb{Z}} e^{-\frac{\pi(n-Y)^2}{X}}. \tag{4.15}$$

It follows by (4.15) that

$$\vartheta(X;Y)\geq 0 \ \text{ for } \ X>0, Y\in\mathbb{R}. \tag{4.16}$$

This proves (4.13). To prove (4.14), we first notice that $|\vartheta_X(X;Y)|\leq 2\pi\sum_{n=1}^\infty n^2 e^{-\pi n^2 X}=-\vartheta_X(X;0)$. Splitting the summation to $n=0$ and $n\neq 0$, one gets that

$$\begin{aligned}
-\sum_{n\in\mathbb{Z}} e^{-\pi\alpha y n^2}\vartheta_X(\frac{y}{\alpha};nx) &= -\vartheta_X(\frac{y}{\alpha};0)-2\sum_{n=1}^\infty e^{-\pi\alpha y n^2}\vartheta_X(\frac{y}{\alpha};nx)\\
&\geq -\vartheta_X(\frac{y}{\alpha};0)-2\sum_{n=1}^\infty e^{-\pi\alpha y n^2}\big(-\vartheta_X(\frac{y}{\alpha};0)\big)\\
&= -\vartheta_X(\frac{y}{\alpha};0)\Big(1-2\sum_{n=1}^\infty e^{-\pi\alpha y n^2}\Big)\\
&> 0.
\end{aligned}$$

Combining (4.12) with (4.13) and with (4.14), one gets

$$\begin{aligned}
\theta_y(\alpha,z)\leq&\frac{1}{2\sqrt{\alpha y}}\sum_{n\in\mathbb{Z}} e^{-\pi\alpha y n^2}\vartheta(\frac{y}{\alpha};nx)\\
=&\frac{1}{2y}\sqrt{\frac{y}{\alpha}}\sum_{n\in\mathbb{Z}} e^{-\pi\alpha y n^2}\vartheta(\frac{y}{\alpha};nx)\\
=&\frac{1}{2y}\theta(\alpha,z).
\end{aligned}$$

This proves item (1) of Lemma 4.19.

It remains to prove item (2) of Lemma 4.19. Splitting the summation by $n=0$ and $n\neq 0$, one has

$$\theta(\alpha,y)=\vartheta_3(\frac{\alpha}{y})+2\sqrt{\frac{y}{\alpha}}\sum_{n=1}^\infty e^{-\pi\alpha y n^2}\vartheta(\frac{y}{\alpha};nx).$$

Using (4.16), one has

$$\theta(\alpha,y)\geq\vartheta_3(\frac{\alpha}{y})\geq 1+2e^{-\pi\frac{\alpha}{y}}. \tag{4.17}$$

While for $\theta_y(\alpha,y)$, we have

$$\theta_y(\alpha,y)=\partial_y(\vartheta_3(\frac{\alpha}{y}))+2\partial_y\Big(\sqrt{\frac{y}{\alpha}}\sum_{n=1}^\infty e^{-\pi\alpha y n^2}\vartheta(\frac{y}{\alpha};nx)\Big). \tag{4.18}$$

To deal with the second term in $\theta_y(\alpha, y)$, a direct calculation and regrouping the terms, one gets

$$\partial_y\Big(\sqrt{\frac{y}{\alpha}}\sum_{n=1}^{\infty}e^{-\pi\alpha yn^2}\vartheta(\frac{y}{\alpha};nx)\Big) = -\frac{1}{2\sqrt{\alpha y}}\sum_{n=1}^{\infty}(2\pi\alpha y-1)e^{-\pi\alpha yn^2}\vartheta(\frac{y}{\alpha};nx) + \frac{1}{\alpha}\sqrt{\frac{y}{\alpha}}\sum_{n=1}^{\infty}e^{-\pi\alpha yn^2}\vartheta_X(\frac{y}{\alpha};nx).$$

By (4.16), we have

$$\partial_y\Big(\sqrt{\frac{y}{\alpha}}\sum_{n=1}^{\infty}e^{-\pi\alpha yn^2}\vartheta(\frac{y}{\alpha};nx)\Big) \leq \frac{1}{\alpha}\sqrt{\frac{y}{\alpha}}\sum_{n=1}^{\infty}e^{-\pi\alpha yn^2}\vartheta_X(\frac{y}{\alpha};nx). \tag{4.19}$$

On the other hand, using the Jacobi Theta function, we get

$$|\frac{1}{\alpha}\sqrt{\frac{y}{\alpha}}\sum_{n=1}^{\infty}e^{-\pi\alpha yn^2}\vartheta_X(\frac{y}{\alpha};nx)| \leq |\frac{1}{\alpha}\sqrt{\frac{y}{\alpha}}\sum_{n=1}^{\infty}e^{-\pi\alpha yn^2}\vartheta_X(\frac{y}{\alpha};0)| = -\sqrt{\frac{y}{\alpha}}\sum_{n=1}^{\infty}e^{-\pi\alpha yn^2}\partial_y(\vartheta_3(\frac{y}{\alpha})). \tag{4.20}$$

By the transformation formula, $\sqrt{\frac{y}{\alpha}}\vartheta_3(\frac{y}{\alpha}) = \vartheta_3(\frac{\alpha}{y})$. By taking derivative with respect to y, one has

$$-\sqrt{\frac{y}{\alpha}}\partial_y(\vartheta_3(\frac{y}{\alpha})) = \frac{1}{2y}\vartheta_3(\frac{\alpha}{y}) - \partial_y(\vartheta_3(\frac{\alpha}{y})). \tag{4.21}$$

Since $\partial_y(\vartheta_3(\frac{\alpha}{y})) = \frac{\pi\alpha}{y^2}\sum_{n\in\mathbb{Z}}n^2e^{-\pi n^2\frac{\alpha}{y}} \geq 0$, then we have

$$-\sqrt{\frac{y}{\alpha}}\partial_y(\vartheta_3(\frac{y}{\alpha})) \leq \frac{1}{2y}\vartheta_3(\frac{\alpha}{y}). \tag{4.22}$$

Combining (4.18) with (4.19)-(4.22), we have

$$\theta_y(\alpha, y) \leq \partial_y(\vartheta_3(\frac{\alpha}{y})) + \frac{1}{\alpha y}\vartheta_3(\frac{\alpha}{y})\sum_{n=1}^{\infty}e^{-\pi\alpha n^2 y}. \tag{4.23}$$

By (4.17) and (4.23), one gets

$$\frac{\theta_y(\alpha, z)}{\theta(\alpha, z)} \leq \frac{\partial_y(\vartheta_3(\frac{\alpha}{y})) + \frac{1}{\alpha y}\vartheta_3(\frac{\alpha}{y})\sum_{n=1}^{\infty}e^{-\pi\alpha n^2 y}}{1+2e^{-\pi\frac{\alpha}{y}}} := \frac{A_1+A_2}{B_1+B_2}, \tag{4.24}$$

where

$$A_1 := \frac{2\pi\alpha}{y^2}e^{-\frac{\pi\alpha}{y}},\quad A_2 := \sum_{n=2}^{\infty}\frac{n^2\pi\alpha}{y^2}e^{-\pi n^2\frac{\alpha}{y}} + \sum_{n=1}^{\infty}\frac{1}{\alpha y}\vartheta_3(\frac{\alpha}{y})e^{-\pi n^2\alpha y}$$
$$B_1 := 1,\qquad B_2 := 2e^{-\pi\frac{\alpha}{y}}.$$

By direct computation, one has

$$\frac{A_2}{B_2} < \frac{A_1}{B_1}. \tag{4.25}$$

Then by an elementary inequality, it follows by (4.25) that

$$\frac{A_1+A_2}{B_1+B_2} < \frac{A_1}{B_1}. \tag{4.26}$$

Noting that $\frac{A_1}{B_1} = \frac{2\pi\alpha}{y^2}e^{-\frac{\pi\alpha}{y}}$, (4.23) and (4.26) yield the desired result. □

4.3. **Proof of Lemma 3.1.** Since

$$\frac{\zeta(s,z)}{\theta^k(\alpha,z)}/\frac{\zeta(s,z)}{\theta^k(\alpha,z)}\mid_{z=e^{i\frac{\pi}{3}}}=\frac{\zeta(s,z)}{\zeta(s,e^{i\frac{\pi}{3}})}\cdot\Big(\frac{\theta(\alpha,e^{i\frac{\pi}{3}})}{\theta(\alpha,z)}\Big)^k,$$

we shall estimate $\frac{\zeta(s,z)}{\zeta(s,e^{i\frac{\pi}{3}})}$ and $\frac{\theta(\alpha,e^{i\frac{\pi}{3}})}{\theta(\alpha,z)}$ respectively.

To estimate $\frac{\theta(\alpha,e^{i\frac{\pi}{3}})}{\theta(\alpha,z)}$, we already have lower bounds of $\theta(\alpha,z)$ in Lemma 4.3. Using the Jacobi Theta function of the third type, we have an upper bound of $\theta(\alpha,z)$, i.e.,

Lemma 4.21. *For any $\alpha>0, y>0$, it holds that $\theta(\alpha,z)\leq\theta(\alpha,iy)=\vartheta_3(\frac{\alpha}{y})\vartheta_3(\alpha y)$.*

Proof. By (4.2), $\theta(\alpha,z)=\sqrt{\frac{y}{\alpha}}\cdot\sum_{n\in\mathbb{Z}}e^{-\pi\alpha yn^2}\vartheta(\frac{y}{\alpha};nx)\leq\sqrt{\frac{y}{\alpha}}\cdot\sum_{n\in\mathbb{Z}}e^{-\pi\alpha yn^2}\vartheta(\frac{y}{\alpha};0)=\sqrt{\frac{y}{\alpha}}\vartheta_3(\frac{y}{\alpha})\cdot\sum_{n\in\mathbb{Z}}e^{-\pi\alpha yn^2}=\vartheta_3(\frac{\alpha}{y})\vartheta_3(\alpha y)$. □

By Lemmas 4.3 and 4.28, one has

Lemma 4.22. *Assume that $\alpha,y>0$. It holds that $\frac{\theta(\alpha,e^{i\frac{\pi}{3}})}{\theta(\alpha,z)}\geq\frac{1}{\theta(\alpha,iy)}=\frac{1}{\vartheta_3(\frac{\alpha}{y})\vartheta_3(\alpha y)}$.*

For $\frac{\zeta(s,z)}{\zeta(s,e^{i\frac{\pi}{3}})}$. Due to its difficulty, we divide it into two cases, i.e., (1): $s\in(1,4]$ and (2): $s>4$. For case (1), we use Chowla-Selberg formula given by Lemma 4.14, while for case (2), we use summation formula for $\zeta(s,z)$ given by Lemma 3.6.

We first have

Lemma 4.23. *For $s\in(1,4]$ and $y\geq2$, it holds that $\frac{\zeta(s,z)}{\zeta(s,e^{i\frac{\pi}{3}})}\geq\frac{a_0(s,y)-2a_1(s,y)}{a_0(s,\frac{\sqrt{3}}{2})}$. Or equivalently,*

$$\frac{\zeta(s,z)}{\zeta(s,e^{i\frac{\pi}{3}})}\geq\mathcal{B}_a(s,y),$$

where

$$\mathcal{B}_a(s,y):=\frac{2\xi(2s)y^s+2\xi(2s-1)\frac{\Gamma(\frac{1}{2})\Gamma(s-\frac{1}{2})}{\Gamma(s)}y^{1-s}-\frac{8\pi^s\sqrt{y}}{\Gamma(s)}K_{s-\frac{1}{2}}(2\pi y)}{2\xi(2s)(\frac{\sqrt{3}}{2})^s+2\xi(2s-1)\frac{\Gamma(\frac{1}{2})\Gamma(s-\frac{1}{2})}{\Gamma(s)}(\frac{\sqrt{3}}{2})^{1-s}}.$$

Proof. We first use Rankin's Lemma(or Montegomery's Lemma [29, 42]), i.e., $\frac{\partial}{\partial x}\zeta(s,z)<0$ for $\{\mathrm{Im}(z)\geq\frac{\sqrt{3}}{2}\}\cap z\in\mathcal{D}$ and $s>1$. Then $\zeta(s,z)\geq\zeta(s,\frac{1}{2}+iy)$. We then use Chowla-Selberg formula given by Lemma 4.14. Indeed,

$$\zeta(s,\frac{1}{2}+iy)=a_0(s,y)+2\sum_{n=1}^{\infty}(-1)^na_n(s,y). \tag{4.27}$$

See a_0,a_n in Lemma 4.14. Using Lemma 4.15, one has

$$\begin{aligned}\frac{a_{n+1}(y)}{a_n(y)}&=(1+\frac{1}{n})^{s-\frac{1}{2}}\frac{\sigma_{1-2s}(n+1)}{\sigma_{1-2s}(n)}\frac{K_{s-\frac{1}{2}}(2\pi(n+1)y)}{K_{s-\frac{1}{2}}(2\pi ny)}\\&\leq(1+\frac{1}{n})^{s-1}\frac{\sigma_{1-2s}(n+1)}{\sigma_{1-2s}(n)}e^{-2\pi y}.\end{aligned}$$

In the range $s\in(1,4]$ and $y\geq\frac{\sqrt{3}}{2}$, then $\frac{a_{n+1}(y)}{a_n(y)}<1$ for all $n\geq1$. Hence $\sum_{n=1}^{\infty}(-1)^na_n(s,\frac{\sqrt{3}}{2})$ is an alternating series. Then $\zeta(s,z)\geq a_0(s,y)-2a_1(s,y)$. For $\zeta(s,e^{i\frac{\pi}{3}})$, still using (4.27) and the series is alternating, then $\zeta(s,e^{i\frac{\pi}{3}})\leq a_0(s,y)$. These yield the result. □

The lower bound function $\mathcal{B}_a(s,y)$ is monotone on s and y directions. It follows that

Lemma 4.24. *It holds that $\min_{s\in[1,4],y\geq2}\mathcal{B}_a(s,y)=\mathcal{B}_a(1,2)$. Here $\mathcal{B}_a(1,2)=1.133290376\cdots$.*

Now we are ready to prove the case $s\in(1,4]$ of Lemma 3.1. Namely,

Lemma 4.25. *For $s \in (1, 4]$,$y \geq 2, \alpha \geq 3s$ and $k \leq 2s$, it holds that*

$$\frac{\zeta(s,z)}{\zeta(s,e^{i\frac{\pi}{3}})} \cdot \Big(\frac{\theta(\alpha,e^{i\frac{\pi}{3}})}{\theta(\alpha,z)}\Big)^k > 1.$$

Proof. By Lemmas 4.22, 4.23 and 4.24,

$$\begin{aligned}\frac{\zeta(s,z)}{\zeta(s,e^{i\frac{\pi}{3}})} \cdot \Big(\frac{\theta(\alpha,e^{i\frac{\pi}{3}})}{\theta(\alpha,z)}\Big)^k \geq & \mathcal{B}_a(s,y)\Big(\frac{\theta(\alpha,e^{i\frac{\pi}{3}})}{\theta(\alpha,z)}\Big)^{2s} \geq \mathcal{B}_a(s,y)\Big(\frac{1}{\theta(3s,iy)}\Big)^{2s}\\ \geq & \mathcal{B}_a(1,2)\Big(\frac{1}{\theta(3,2i)}\Big)^2 = 1.093639371\cdots > 1.\end{aligned}$$

□

For large s, the Chowla-Selberg formula given by Lemma 4.14 does not work well due to the property of $K_s(y)$. Instead, we use summation formula by Lemma 3.6. A direct consequence of Lemma 3.6 gives that

Lemma 4.26. *For $s \geq 4$ and $y \geq 2$, it holds that*

$$\frac{\zeta(s,z)}{\zeta(s,e^{i\frac{\pi}{3}})} \geq \mathcal{B}_b(s,y).$$

Here

$$\mathcal{B}_b(s,y) := \frac{2\xi(2s)y^s + 2\xi(2s-1)\frac{\Gamma(\frac{1}{2})\Gamma(s-\frac{1}{2})}{\Gamma(s)}y^{1-s} - \xi(2s+1)s\frac{(2s+1)^{s+\frac{1}{2}}}{(2s+2)^{s+1}}y^{-(s+1)}}{2\xi(2s)(\frac{\sqrt{3}}{2})^s + 2\xi(2s-1)\frac{\Gamma(\frac{1}{2})\Gamma(s-\frac{1}{2})}{\Gamma(s)}(\frac{\sqrt{3}}{2})^{1-s} + \xi(2s+1)s\frac{(2s+1)^{s+\frac{1}{2}}}{(2s+2)^{s+1}}(\frac{\sqrt{3}}{2})^{-(s+1)}}.$$

Now we are ready to prove the case $s > 4$ of Lemma 3.1. Namely,

Lemma 4.27. *For $s > 4$,$y \geq 2, \alpha \geq 3s$ and $k \leq 2s$, it holds that*

$$\frac{\zeta(s,z)}{\zeta(s,e^{i\frac{\pi}{3}})} \cdot \Big(\frac{\theta(\alpha,e^{i\frac{\pi}{3}})}{\theta(\alpha,z)}\Big)^k > 1.$$

Proof. For $s > 4$ and $y \geq 2$, one trivially has $\mathcal{B}_b(s,y) > \frac{y^s}{s(\frac{\sqrt{3}}{2})^{s+1}}$. By Lemmas 4.22 and 4.26,

$$\frac{\zeta(s,z)}{\zeta(s,e^{i\frac{\pi}{3}})} \cdot \Big(\frac{\theta(\alpha,e^{i\frac{\pi}{3}})}{\theta(\alpha,z)}\Big)^k \geq \mathcal{B}_b(s,y)\Big(\frac{1}{\theta(3s,iy)}\Big)^{2s} \geq \frac{y^s}{s(\frac{\sqrt{3}}{2})^{s+1}}\Big(\frac{1}{\theta(3,2i)}\Big)^8 > \sqrt{3}.$$

□

By Lemmas 4.25 and 4.27, we complete the proof of Lemma 3.1.

4.4. Proof of Corollary 1.3.

By a deformation

$$\zeta(s,z) - \theta^k(\alpha,z) = \theta^k(\alpha,z) \cdot \big(\frac{\zeta(s,z)}{\theta^k(\alpha,z)} - 1\big),$$

Theorem 1.2 and Montgomery's Theorem B([29]), to prove the first part of Corollary 1.3, it suffices to prove that

Lemma 4.28. *Assume that $s \in (1, 12]$, $\alpha \geq 3s$. Then for $k \in (0, 2s]$, it holds that $\frac{\zeta(s,e^{i\frac{\pi}{3}})}{\theta^k(\alpha,e^{i\frac{\pi}{3}})} \geq 1$.*

Since $\theta(\alpha, z) \geq 1$ by Lemma 4.3, to prove Lemma 4.28, it suffices to prove that

Lemma 4.29. *Assume that $s \in (1, 12]$. Then we have $\frac{\zeta(s,e^{i\frac{\pi}{3}})}{((\vartheta_3(2\sqrt{3}s)\vartheta(\frac{3\sqrt{3}}{2}s)))^{2s}} \geq 1$.*

Note that $(\vartheta_3(2\sqrt{3}s)\vartheta(\frac{3\sqrt{3}}{2}s))$ is very close to 1, and $(\vartheta_3(2\sqrt{3})\vartheta(\frac{3\sqrt{3}}{2})) = 1.000112671\cdots$ for $s > 1$. To prove Lemma 4.29, we use Chowla-Selberg formula given by Lemma 4.14. The proof is similar to that in Lemma 4.23, we omit the details here.

Acknowledgements. The research of S. Luo is partially supported by NSFC(Nos. 12261045, 12001253) and double thousands plan of Jiangxi(jxsq2019101048). The research of J. Wei is partially supported by GRF of Hong Kong. S. Luo is grateful to Professor W.M. Zou(Tsinghua University) for his constant support and would like to say special thanks to Professors H.J. Zhao(Wuhan University).

Statements and Declarations: there is no conflict of interest.

Data availability: the manuscript has no associated data.

References

[1] Abrikosov, A. A., Nobel Lecture: Type-II superconductors and the vortex lattice. *Reviews of modern physics* 76(2004), no.3, p. 975.

[2] Alvarez-Gaumé, L.; Moore, G.; Vafa, C., Theta functions, modular invariance, and strings. *Comm. Math. Phys.* 106 (1986), no. 1, 1-40.

[3] Baricz, Á., Bounds for modified Bessel functions of the first and second kinds. *Proc. Edinb. Math. Soc.* (2) 53 (2010), no. 3, 575C599.

[4] Barreal, A.; Damir, M. T.; Freij-Hollanti, R.; Hollanti, C., An approximation of theta functions with applications to communications. *SIAM J. Appl. Algebra Geom.* 4 (2020), no. 4, 471-501.

[5] Benjamin, N.; Keller, C. A.; Ooguri, H.; Zadeh, I. G., Narain to Narnia. *Comm. Math. Phys.* 390 (2022), no. 1, 425-470.

[6] Bétermin, L., Two-dimensional theta functions and crystallization among Bravais lattices, *SIAM Journal on Mathematical Analysis*, 48(5) (2016), 3236-269.

[7] Bershadsky, M.; Klebanov, I. R., Partition functions and physical states in two-dimensional quantum gravity and supergravity. *Nuclear Phys. B* 360 (1991), no. 2-3, 559-585.

[8] Blanc, X. and Lewin, M., The Crystallization Conjecture: A Review. *EMS Surveys in Mathematical Sciences, EMS* 2(2)2015, 255-306.

[9] Chen, X. and Oshita, Y., An application of the modular function in nonlocal variational problems. *Arch. Rat. Mech. Anal.*, 186(1) (2007), 109-132.

[10] Cassels, J., On a problem of Rankin about the Epstein Zeta function, *Proc. Glasgow Math. Assoc.* 4(1959), 73-80. (Corrigendum, ibid. 6 (1963), 116.)

[11] Cohn, H.; de Courcy-Ireland, M., The Gaussian core model in high dimensions. *Duke Math. J.* 167 (2018), no. 13, 2417-2455.

[12] Conway, J. H.; Sloane, N. J. A., Sphere packings, lattices and groups. With contributions by E. Bannai, J. Leech, S. P. Norton, A. M. Odlyzko, R. A. Parker, L. Queen and B. B. Venkov. Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 290. Springer-Verlag, New York, 1988. xxviii+663 pp. ISBN: 0-387-96617-X.

[13] Chowla, S. and Selberg, A., On Epsteins Zeta-function, *J. Reine Angew. Math.* 227 (1967), 86-110.

[14] Selberg, A. and Chowla, S., On Epsteins Zeta function (I), *Proc. Nat. Acad. Sci.* 35 (1949) 371-74.

[15] Cohen, H., Number theory. Vol. II. Analytic and modern tools. Graduate Texts in Mathematics, 240. Springer, New York, 2007. xxiv+596 pp. ISBN: 978-0-387-49893-5.

[16] Diananda, P. H., Notes on two lemmas concerning the Epstein zeta-function, *Proc. Glasgow Math. Assoc.* 6 (1964), 202-204.

[17] Ennola, V., A lemma about the Epstein Zeta function, *Proc. Glasgow Math. Assoc.* 6 (1964), 198-201.

[18] Faulhuber, M., Minimal Frame Operator Norms via Minimal Theta Functions. *Journal of Fourier Analysis and Applications*, 24(2)(2018), 545-559.

[19] Folkins, I., Functions of two-dimensional Bravais lattices. *J. Math. Phys.* 32, 1965-1969 (1991).

[20] Goldman, D., Muratov, C. B. and Serfaty, S., The Gamma-limit of the two-dimensional Ohta-Kawasaki energy. I. droplet density. *Arch. Rat. Mech. Anal.* 210(2)(2013), 581-613.

[21] Ho, T.L., Bose-Einstein condensates with large number of vortices. *Physical Review Letters* 87(2001), 604031-604034

[22] Afkhami-Jeddi, N., Cohn, H., Hartman, T., Tajdini, A.: Free partition functions and an averaged holographic duality. *JHEP* 01, 130 (2021).

[23] Luo, S., Ren, X. and Wei, J., Non-hexagonal lattices from a two species interacting system, *SIAM J. Math. Anal.*, 52(2) (2020), 1903-1942.

[24] Luo, S., Wei, J., On lattice hexagonal crystallization for non-monotone potentials, *J. Math. Phys.* 65 (2024), no. 7, Paper No. 071901, 28 pp.

[25] Luo, S., Wei, J., On minima of sum of theta functions and application to Mueller-Ho conjecture. *Arch. Ration. Mech. Anal.* 243 (2022), no. 1, 139-199.

[26] Luo, S., Wei, J., On minima of difference of theta functions and application to hexagonal crystallization, *Math. Ann.* 387 (2023), no. 1-2, 499-539.

[27] Pinelis, I., Exact lower and upper bounds on the incomplete gamma function. *Math. Inequal. Appl.* 23 (2020), no. 4, 1261-1278.

[28] Maloney, A., Witten, E.: Averaging over Narain moduli space. *JHEP* 10, 187 (2020).

[29] Montgomery, H., Minimal theta functions. *Glasgow Math. J.* 30 (1988), 75-85.

[30] Mumford, D., Tata lectures on theta. I. With the assistance of C. Musili, M. Nori, E. Previato and M. Stillman. Progress in Mathematics, 28. Birkhäuser Boston, Inc., Boston, MA, 1983. xiii+235 pp. ISBN: 3-7643-3109-7.

[31] Nakayama Y., Liouville field theory: a decade after the revolution, *International Journal of Modern Physics A* Vol. 19, No. 17n18, pp. 2771-2930 (2004).

[32] Osgood B., Phillips R., and Sarnak P., Extremals of determinants of Laplacians, *Journal of functional analysis* 80, 148-211 (1988).

[33] Prestipino S., Saija F., Giaquinta P.V., Hexatic Phase in the Two-Dimensional Gaussian-Core Model, *Phys. Rev. Lett.* 106, 235701 C Published 10 June 2011.

[34] Regev, O.; Stephens-Davidowitz, N., A reverse Minkowski theorem. *Ann. of Math.* (2) 199 (2024), no. 1, 1-49.

[35] Regev, O., Some questions related to the reverse Minkowski theorem. ICM-International Congress of Mathematicians. Vol. 6. Sections 12-14, 4898-4912, EMS Press, Berlin, 2023.

[36] Sandier, E. and Serfaty, S., From the Ginzburg-Landau model to vortex lattice problems. *Comm. Math. Phys.* 313(2012), 635-743.

[37] Sarnak, P.; Strömbergsson, A., Minima of Epstein's zeta function and heights of flat tori. *Invent. Math.* 165 (2006), no. 1, 115-151.

[38] Serfaty, S., Systems of points with Coulomb interactions. *Proceedings of the International Congress of MathematiciansRio de Janeiro 2018.* Vol. I. Plenary lectures, 935-977, World Sci. Publ., Hackensack, NJ, 2018.

[39] Viazovska, M.S., The sphere packing problem in dimension 8, *Annals of Mathematics* 2017, 991-1015.

[40] Francesco, P., Mathieu P., Sénéchal D., Conformal Field Theory. *Graduate Texts in Contemporary Physics*, New York, NY: Springer New York. p. 104. ISBN 978-1-4612-2256-9, 1997.

[41] Mussardo G., Statistical Field Theory: An Introduction to Exactly Solved Models in Statistical Physics, Second Edition, *Oxford Graduate Texts*, ISBN: 9780198788102, 2020.

[42] Rankin, R. A., A minimum problem for the Epstein Zeta function, *Proc. Glasgow Math. Assoc.* 1 (1953), 149-158.

[43] Watson, G. N., A treatise on the theory of Bessel functions. Reprint of the second (1944) edition. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 1995. viii+804 pp. ISBN: 0-521-48391-3.

(S. Luo) School of Mathematics and statistics, Jiangxi Normal University, Nanchang, 330022, China

(J. Wei) Department of Mathematics, Chinese University of Hong Kong, Shatin, NT, Hong Kong
E-mail address, S. Luo: `luosp1989@163.com`
E-mail address, J. Wei: `wei@math.cuhk.edu.hk`