# On ground states of spin-1 Bose-Einstein condensates with Ioffe-Pritchard magnetic field in 2D and 3D

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**Abstract**

New developments of optical trapping techniques make it possible to confine atoms independently of their spin orientation in experiments and thus result in spinor condensates. As a continuation of our previous work [36], we investigate physical states as well as qualitative properties describing the aggregation and extinction of atoms, of spin-1 Bose-Einstein condensate in Ioffe-Pritchard magnetic field, through two conserved quantities, the number of atoms and the total magnetization. Unlike the related free case which is well studied in [30, 36], the presence of the Ioffe-Pritchard magnetic field, which competes dramatically with the harmonic trapping, requires new ideas to capture the physical states and analyze their qualitative properties. Based on the ferromagnetic or antiferromagnetic characterization of spin-1 Bose-Einstein condensate, our results support some experimental observations in [26, 32] and some numerical analysis on ground states reported in [3–5].

**Keywords:** Spin-1 Bose-Einstein condensation, Ioffe-Pritchard magnetic field, Physical states, Qualitative analysis.

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## 1 Introduction

Einstein predicted in 1925 that massive noninteracting bosons at low temperature could occupy the same lowest energy single particle state and form the Bose-Einstein condensates (BEC). Until 1995, this prediction was realized experimentally by laser cooling technique for several alkali atomic dilute gases, such as $^{87}$Rb [1], $^{23}$Na [13], and $^{7}$Li [8]. In these earlier BEC experiments, atoms were confined in a magnetic trap, where their spin degree of freedom was frozen. Later the developments of optical trapping techniques enabled to confine atoms independently of their spin orientation and thus result in so-called spinor condensates. Spinor BEC has been achieved experimentally and attracted considerable

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interest for providing a unique possibility of exploring fundamental concepts of quantum mechanics in a remarkably controllable and tunable environment, see [25, 41, 42] for details.

Ioffe-Pritchard (IP) magnetic trap was introduced in [39] and then often used in spinor BEC experiments. For example, in [33], Coreless vortices were created in $F = 1$ spinor condensates held in a IP magnetic trap by adiabatically reducing the magnetic bias field along the trap axis to zero. Cross disgyration texture of spin-1 BEC of alkali-metal atoms appears in an IP trap, see [27] for details. For a more detailed account of backgrounds on IP magnetic trap, we refer the reader to [15, 39].

For bosonic atoms, the total spin number $F$ corresponding to the lowest energy has to be an integer with $2F + 1$ hyperfine states ($m_F = -F, -F + 1, ..., F - 1, F$). In the mean-field approximation, a spin-$F$ ($F \in \mathbb{N}$) condensate can be described by coupled Gross-Pitaevskii equations consisting of $2F+1$ equations, and each of them governs one of the $2F + 1$ states. For the alkali atoms $^{87}$Rb, $^{23}$Na and $^{7}$Li, $F = 1$. In this paper, we will focus our study on physical states of spin-1 BEC with IP magnetic field, which is described by the following Gross-Pitaevskii system

$$\begin{cases} -\Delta u_1 + V(x)u_1 = (\mu+\lambda)u_1 + (c_0+c_1)|u_1|^2u_1 + (c_0-c_1)u_1|u_2|^2 \\ \qquad\qquad\qquad\quad + (c_0+c_1)|u_0|^2u_1 + c_1\overline{u}_2u_0^2 + Bu_0, \\ -\Delta u_2 + V(x)u_2 = (\mu-\lambda)u_2 + (c_0+c_1)|u_2|^2u_2 + (c_0-c_1)|u_1|^2u_2 \\ \qquad\qquad\qquad\quad + (c_0+c_1)|u_0|^2u_2 + c_1\overline{u}_1u_0^2 + Bu_0, \\ -\Delta u_0 + V(x)u_0 = \mu u_0 + c_0|u_0|^2u_0 + (c_0+c_1)(|u_1|^2+|u_2|^2)u_0 \\ \qquad\qquad\qquad\quad + 2c_1u_1u_2\overline{u}_0 + B(u_1+u_2) \end{cases} \tag{1.1}$$

under the constraints

$$\int_{\mathbb{R}^d} \big(|u_1|^2 + |u_2|^2 + |u_0|^2\big)dx = N, \quad \int_{\mathbb{R}^d} \big(|u_1|^2 - |u_2|^2\big)dx = M, \tag{1.2}$$

where $N$ is the number of atoms and $M$ denotes the total magnetization. $\mu$ and $\lambda$ are the Lagrange multipliers arising from the constraint (1.2). $\overline{u}_i$ denotes the complex conjugate of $u_i (i = 0, 1, 2)$. The parameters $c_0$ and $c_1$ describe the mean-field interaction and spin-exchange interaction, respectively and they are both tunable in experiments. The mean-field interaction is attractive if $c_0 > 0$ and repulsive if $c_0 < 0$. The spin-1 BEC system is called ferromagnetic if $c_1 > 0$ and antiferromagnetic if $c_1 < 0$. $V(x) = |x|^2$ is a harmonic trapping in $\mathbb{R}^d$, real function $B(x)$ denotes the external IP magnetic field.

For spin-1 BEC without IP magnetic field (i.e. $B(x) \equiv 0$), according to the relations among $c_0$, $c_1$, $N$ and $M$, the existence, dynamics and numerical analysis of physical states of (1.1)-(1.2) have been studied by many authors, see [10–12, 30, 35, 36] and the references therein. For the one-dimensional (1D) case, Cao, Chern and Wei in [10] proved the existence of ground states of (1.1)-(1.2) with $V(x) \equiv 0$, $c_0 > 0$, $c_1 > 0$, by minimizing the corresponding energy functional under (1.2). For the two-dimensional (2D) case, motivated by the recent works [19, 20] on two-component attractive BEC, Kong, Wang and Zhao [30] gave the existence and detailed asymptotic behavior of ground states for (1.1)-(1.2) with harmonic trapping potentials. Turning to the three-dimensional (3D) case, ground states for (1.1)-(1.2) were investigated by Lin and Chern in [35], where $V(x)$ is a harmonic potential with Zeeman effect and $c_0 < 0$, $c_1 < 0$. Recently, in [36], we developed an exhaustive analysis on physical states for (1.1)-(1.2) in $\mathbb{R}^3$ in both ferromagnetic and antiferromagnetic cases. All these results together show that the characteristics of spin-1 BEC are different in 1D, 2D and 3D.

When a negative constant IP magnetic field is involved, following [10], via approximation, Luo, Lü and Liu in [37] proved the existence of ground states in the 1D case and showed that under some conditions, searching for ground states of ferromagnetic spin-1 BEC with an external IP magnetic field can be reduced to a one-component minimisation problem. Bao, Chern and Zhang in [3] proposed efficient numerical methods for computing ground states of spin-1 BEC with/without the IP magnetic field in both 1D and 2D cases. This is the first numerical study on ground states of spin-1 BEC with $B(x) \neq 0$. In addition, the authors in [3] also showed that ground states are always symmetric with respect to $x = 0$, which is the center of $V$, see Fig.1 there. Later, Hajaiej and Carles in [23] proved the existence and stability of ground states for repulsive and antiferromagnetic (1.1)-(1.2) in $\mathbb{R}^d$ $(d = 1, 2, 3)$, where $V(x)$ is a harmonic potential and $c_0 < 0,\ c_1 < 0$.

In the 2D or 3D case, there is no study on attractive spin-1 BEC with IP magnetic field in the literatures. In this case, the problem becomes more challenging. One reason is that, the energy functional is sign-indefinite when $c_0 > 0$, while it is positive definite and coercive on related physical mainifold, when $c_0 < 0,\ c_1 < 0$. The existence of physical states is related to the range of $c_0,\ c_1$ and the initial data, which determines the number of atoms $N$ and the total magnetization $M$. Secondly, the presence of the IP magnetic field, which competes dramatically with the harmonic trapping, requires new ideas to study the physical states.

Another motivation of this paper comes from recent experimental and computational results. The so-called single-mode approximation (SMA) phenomenon on spin-1 BEC with/without the IP magnetic field has been known for many years from numerical simulations in [3–5] and was also observed experimentally in [26, 32]. The first rigorous mathematical justification of SMA was given by Lin and Chern in [35] for repulsive spin-1 BEC, that is the case of $c_0 < 0$. The corresponding attractive case remains open and challenging.

In this present paper, we investigate ground states and their qualitative properties for attractive spin-1 BEC with IP magnetic field. Firstly, we introduce the working space and some notations. Let $H := H^1(\mathbb{R}^d, \mathbb{C}^3)$ and we define

$$\Lambda := \Big\{ \boldsymbol{u} = (u_1, u_2, u_0) \in H \Big| \int_{\mathbb{R}^d} |x|^2 |\boldsymbol{u}|^2 dx = \int_{\mathbb{R}^d} |x|^2 (|u_1|^2 + |u_2|^2 + |u_0|^2) dx < \infty \Big\}, \tag{1.3}$$

then $\Lambda$ is a Hilbert space equipped with the norm

$$\|\boldsymbol{u}\|_\Lambda := \Big( \int_{\mathbb{R}^d} (|\nabla \boldsymbol{u}|^2 + |\boldsymbol{u}|^2 + |x|^2 |\boldsymbol{u}|^2) dx \Big)^{\frac{1}{2}} \text{ for } \forall \boldsymbol{u} \in \Lambda.$$

Denote

$$\mathcal{M}(N) := \Big\{ \boldsymbol{u} = (u_1, u_2, u_0) \in \Lambda \Big| \int_{\mathbb{R}^d} \big( |u_1|^2 + |u_2|^2 + |u_0|^2 \big) dx = N,\ \int_{\mathbb{R}^d} \big( |u_1|^2 - |u_2|^2 \big) dx = M \Big\}, \tag{1.4}$$

then solutions to (1.1)-(1.2) can be found as critical points of the functional $I(\boldsymbol{u})$ on the manifold $\mathcal{M}(N)$, where

$$I(\boldsymbol{u}) = \frac{1}{2} A(\boldsymbol{u}) + \frac{1}{2} \int_{\mathbb{R}^d} |x|^2 |\boldsymbol{u}|^2 dx - \frac{1}{4} E(\boldsymbol{u}) - F(\boldsymbol{u}),$$

with

$$\begin{cases} A(\boldsymbol{u}) := \int_{\mathbb{R}^d} \left(|\nabla u_1|^2 + |\nabla u_2|^2 + |\nabla u_0|^2\right) dx, \\ B(\boldsymbol{u}) := \int_{\mathbb{R}^d} \Big( \big(c_0 + c_1\big)\big(|u_1|^4 + |u_2|^4\big) + c_0|u_0|^4 \Big) dx, \\ C(\boldsymbol{u}) := \int_{\mathbb{R}^d} \Big( \big(c_0 - c_1\big)|u_1|^2|u_2|^2 + \big(c_0 + c_1\big)\big(|u_1|^2 + |u_2|^2\big)|u_0|^2 \Big) dx, \\ D(\boldsymbol{u}) := c_1 Re \int_{\mathbb{R}^d} \overline{u}_1 \overline{u}_2 u_0^2 dx, \quad F(u) := Re \int_{\mathbb{R}^d} B(x)(\overline{u}_1 u_0 + \overline{u}_0 u_2) dx, \\ E(\boldsymbol{u}) := B(\boldsymbol{u}) + 2C(\boldsymbol{u}) + 4D(\boldsymbol{u}). \end{cases}$$

Before introducing the main results, we recall some definitions ( [7, 36]).

**Definition.** *We say that* $(v_1, v_2, v_0)$ *is a ground state of* (1.1)-(1.2) *if* $I'|_{\mathcal{M}(N)}(v_1, v_2, v_0) = 0$ *with*

$$I(v_1, v_2, v_0) = \inf \Big\{ I(u_1, u_2, u_0) \Big| \ I'|_{\mathcal{M}(N)}(u_1, u_2, u_0) = 0 \ \textit{and} \ (u_1, u_2, u_0) \in \mathcal{M}(N) \Big\};$$

*and* $(w_1, w_2, w_0)$ *is an excited state of* (1.1)-(1.2) *if* $I'|_{\mathcal{M}(N)}(w_1, w_2, w_0) = 0$ *with*

$$I(w_1, w_2, w_0) > \inf \Big\{ I(u_1, u_2, u_0) \Big| \ I'|_{\mathcal{M}(N)}(u_1, u_2, u_0) = 0 \ \textit{and} \ (u_1, u_2, u_0) \in \mathcal{M}(N) \Big\}.$$

Recall the following nonlinear equation in $\mathbb{R}^d$ $(d = 2, 3)$:

$$-\Delta u + u = u^3, \quad u \in H^1(\mathbb{R}^d), \tag{1.5}$$

from [31], there exists a unique positive solution $Q(x)$ for (1.5). By the related Pohozaev identity, we get

$$a^* := \int_{\mathbb{R}^d} |Q|^2 dx = \frac{4-d}{d} \int_{\mathbb{R}^d} |\nabla Q|^2 dx = \frac{4-d}{4} \int_{\mathbb{R}^d} |Q|^4 dx. \tag{1.6}$$

Moreover, we obtain from [21] that $Q(x)$ satisfies

$$Q(x), \ |\nabla Q(x)| = O\big(|x|^{-\frac{1}{2}} e^{-|x|}\big), \quad \text{as } |x| \to \infty.$$

Based on the fact that the characteristics of spin-1 BEC are different in 2D and 3D, we deal with them respectively. Firstly, we consider ground states of spin-1 BEC in 2D by the following minimization problem

$$m(N) := \inf_{\boldsymbol{u} \in \mathcal{M}} I(\boldsymbol{u}),$$

where $\mathcal{M} = \mathcal{M}(N)$ is defined in (1.4) with $d = 2$. Suppose $B(x) \in C^{\alpha}_{loc}(\mathbb{R}^2)$ is a function satisfying the conditions

$$B(x) \in L^\infty, \qquad \text{or} \qquad \frac{B(x)}{x} \in L^\infty, \tag{1.7}$$

or

$$\Big\| \frac{B(x)}{x^2} \Big\|_{L^\infty} < \frac{1}{2}, \tag{1.8}$$

and

$$B(\tau x) = \tau^p B(x), \quad \text{for } \tau > 0, \quad p \in \mathbb{R}^+. \tag{1.9}$$

Set

$$N^* := \min\Big\{\frac{a^*}{c_0+c_1}, \frac{a^*}{c_0}\Big\},$$

we have the following

**Theorem 1.** *Suppose* $c_0>0,\ c_0+c_1>0$, $B(x)$ *satisfies* (1.7) *or* (1.8), *then*
*(i)* $m(N)$ *has at least one minimizer if* $0<N<N^*$*;*
*(ii)* $m(N)$ *has no minimizer under one of the following three conditions*

$$\#_1 \begin{cases} c_1 \geq 0, \\ B(x) \text{ satisfies } (1.8), \\ N \geq N^*, \end{cases}$$

$$\#_2 \begin{cases} c_1 < 0, \\ B(x) \text{ satisfies } (1.8), \\ M = 0,\ N \geq N^*, \end{cases}$$

$$\#_3 \begin{cases} c_1 < 0, \\ M \neq 0,\ c_0N^2 - a^*N + c_1M^2 > 0; \end{cases}$$

*(iii) for any minimizer* $\boldsymbol{u}=(u_1,u_2,u_0)\in\mathcal{M}(N)$ *of* $m(N)$,

$$\Big\|\boldsymbol{u} - \big(l_1e^{-\frac{x^2}{2}}, l_2e^{-\frac{x^2}{2}}, l_0e^{-\frac{x^2}{2}}\big)\Big\|_\Lambda^2 = O(N), \quad \text{as } N\to 0^+, \tag{1.10}$$

*where*

$$l_i = \frac{1}{\pi}\int_{\mathbb{R}^2} u_i e^{-\frac{x^2}{2}}dx, \quad \text{for } i=1,2,0.$$

**Remark 1.1.** *Theorem 1 gives the existence and nonexistence of ground states along with qualitative properties describing extinction of atoms, of planar spin-1 BEC in IP magnetic field. Particularly, for the antiferromagnetic case* $c_1<0$, *if the total magnetization* $M\neq 0$ *with*

$$c_0N^2 - a^*N + c_1M^2 > 0,\ \text{i.e. } N > \frac{a^*+\sqrt{(a^*)^2-4c_0c_1M^2}}{2c_0},$$

*then* $m(N)$ *has no minimizer. Note that*

$$\frac{a^*+\sqrt{(a^*)^2-4c_0c_1M^2}}{2c_0} > \frac{a^*}{c_0} = N^*,$$

*it remains open that whether there exists a minimizer for* $m(N)$ *when*

$$N^* \leq N \leq \frac{a^*+\sqrt{(a^*)^2-4c_0c_1M^2}}{2c_0}.$$

*Precisely, on one hand, we don't know that in this case whether* $m(N)$ *is well defined, and on the other hand, it seems difficult to find a suitable test function to prove that* $m(N)=-\infty$ *due to the competition of the mean-field interaction and antiferromagnetic terms. We believe it is interesting to fulfill this gap.*

In the following, $C$ and $C'$ are indiscriminately used to denote various absolutely positive constants. $a \sim b$ means that $Cb \le a \le C'b$. Next, qualitative properties of ground states in 2D are analysed.

**Theorem 2.** *Let $c_0 > 0$, $c_1 > 0$, $N_n \nearrow N^*$ as $n \to \infty$ and $\boldsymbol{u_n} = (u_{1n}, u_{2n}, u_{0n}) \in \mathcal{M}(N_n)$ be a minimizer of $m(N_n)$. We have*
*(i) if $B(x) \ge 0$ satisfies (1.8), then as $n \to \infty$,*

$$m(N_n) \sim (N^* - N_n)^{\frac{1}{2}}. \tag{1.11}$$

*(ii) if $B(x) < 0$ satisfies (1.8) and (1.9), then as $n \to \infty$,*

$$m(N_n) \sim (N^* - N_n)^{\frac{p}{p+2}}, \quad \text{for } 0 < p < 2, \tag{1.12}$$

*and*

$$m(N_n) \sim (N^* - N_n)^{\frac{1}{2}}, \quad \text{for } p \ge 2. \tag{1.13}$$

*In addition, $\boldsymbol{u_n}$ satisfies*

$$\begin{cases} \lim\limits_{n\to\infty} \varepsilon_n u_{1n}(\varepsilon_n x + \tilde{z}_{1n}) = \dfrac{N^* + M}{2N^*\sqrt{c_0 + c_1}} Q(x), \\ \lim\limits_{n\to\infty} \varepsilon_n u_{2n}(\varepsilon_n x + \tilde{z}_{2n}) = \dfrac{N^* - M}{2N^*\sqrt{c_0 + c_1}} Q(x), \quad \text{strongly in } H^1(\mathbb{R}^2), \\ \lim\limits_{n\to\infty} \varepsilon_n u_{0n}(\varepsilon_n x + \tilde{z}_{0n}) = \dfrac{1}{N^*}\sqrt{\dfrac{(N^*)^2 - M^2}{2(c_0 + c_1)}} Q(x), \end{cases} \tag{1.14}$$

*where $\tilde{z}_{in}$ $(i = 0, 1, 2)$ is the unique maximum point of $u_{in}$ with*

$$\lim_{n\to\infty} \left|\frac{\tilde{z}_{in} - \tilde{z}_{jn}}{\varepsilon_n}\right| = 0 \ (i, j = 0, 1, 2, i \ne j), \quad \lim_{n\to\infty} |\tilde{z}_{in}| = 0$$

*and*

$$\varepsilon_n = \begin{cases} C\big(N^* - N_n\big)^{\frac{1}{4}}, & \text{if } B(x) \ge 0, \\ C\big(N^* - N_n\big)^{\frac{1}{p+2}}, & \text{if } B(x) < 0 \text{ and } 0 < p < 2, \\ C\big(N^* - N_n\big)^{\frac{1}{4}}, & \text{if } B(x) < 0 \text{ and } p \ge 2. \end{cases} \tag{1.15}$$

**Remark 1.2.** *The ground state energy is estimated accurately in Theorem 2 as atoms gather to a explicit threshold value, and then a related qualitative property of ground states for planar ferromagnetic spin-1 BEC in IP magnetic field follows. Indeed, we could get estimate (1.11) by assuming*

$$\int_{\mathbb{R}^2} B(x)Q^2(x)dx \ge 0 \text{ instead of } B(x) \ge 0,$$

*while (1.12)-(1.13) yield when replace*

$$B(x) < 0 \text{ by } \int_{\mathbb{R}^2} B(x)Q^2(x)dx < 0$$

*with an additional condition*

$$\int_{\mathbb{R}^2} B(x+x_2)Q^2(x)dx < 0, \text{ for some point } x_2 \in \mathbb{R}^2,$$

*see* (3.29) *for the definition of* $x_2$. *This extra condition guarantees a fine control of the linearly coupled term involving the IP magnetic field in the corresponding energy functional, then consistent upper and lower bound estimates* (1.12)-(1.13) *follows. Unfortunately, it seems difficult to determine the location of* $x_2$, *due to the presence of the abstract IP magnetic field. In particular, if* $x_2$ *is the origin, then qualitative analysis* (1.14) *of ground states for ferromagnetic spin-1 BEC would hold for sign-changing IP magnetic fields.*

Next, we give the asymptotic behavior of ground states for the antiferromagnetic ($c_1 < 0$) case.

**Theorem 3.** *Let* $c_0 + c_1 > 0,\ c_1 < 0$ *and* $M = 0$. *If* $N_n \nearrow N^*$ *as* $n \to \infty$ *and* $\boldsymbol{u_n} = (u_{1n}, u_{2n}, u_{0n}) \in \mathcal{M}(N_n)$ *is a minimizer of* $m(N_n)$, *then items (i)-(ii) of Theorem 2 hold. Moreover,* $\boldsymbol{u_n}$ *satisfies one of the following cases:*
*(i)* $u_{0n} \equiv 0$ *in* $\mathbb{R}^2$ *for* $n$ *large enough,* $u_{1n},\ u_{2n} > 0$, *and*

$$\begin{cases} \lim\limits_{n\to\infty} \varepsilon_n u_{1n}(\varepsilon_n x + \tilde{z}_{1n}) = \sqrt{\dfrac{1}{2c_0}}Q(x), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{strongly in } H^1(\mathbb{R}^2), \\ \lim\limits_{n\to\infty} \varepsilon_n u_{2n}(\varepsilon_n x + \tilde{z}_{2n}) = \sqrt{\dfrac{1}{2c_0}}Q(x), \end{cases}$$

*where* $\tilde{z}_{in}$ $(i = 1, 2)$ *is the unique maximum point of* $u_{in}$ *with*

$$\lim_{n\to\infty} \left|\frac{\tilde{z}_{1n} - \tilde{z}_{2n}}{\varepsilon_n}\right| = 0 \ (i, j = 1, 2, i \neq j), \quad \lim_{n\to\infty} |\tilde{z}_{in}| = 0$$

*and* $\varepsilon_n$ *satisfies* (1.15).

*(ii)* $u_{1n} \equiv 0,\ u_{2n} \equiv 0$ *in* $\mathbb{R}^2$ *for* $n$ *large enough,* $u_{0n} > 0$, *and*

$$\lim_{n\to\infty} \varepsilon_n u_{0n}(\varepsilon_n x + \tilde{z}_{0n}) = \sqrt{\frac{1}{c_0}}Q(x), \quad \text{strongly in } H^1(\mathbb{R}^2),$$

*where* $\tilde{z}_{0n}$ *is the unique maximum point of* $u_{0n}$ *with* $\lim\limits_{n\to\infty} |\tilde{z}_{0n}| = 0$ *and* $\varepsilon_n$ *satisfies* (1.15).

**Remark 1.3.** *Theorem 2 shows that for the ferromagnetic case* $c_1 > 0$, *if* $|M| \in [0, N)$, *any minimizer* $\boldsymbol{u}$ *of* $m(N)$ *in the case of* $N \nearrow N^*$ *is nontrivial. While for the antiferromagnetic case* $c_1 < 0$, *Theorem 3 shows that when* $M = 0$, *the minimizers* $\boldsymbol{u}$ *of* $m(N)$ *must be semi-trivial as* $N \nearrow N^*$, *and it is independent of* $c_1$. *These results not only show that spin-1 BEC has independent characteristics in both ferromagnetic and antiferromagnetic cases, but also support the so-called single-mode approximation (SMA) in experimental observations [26, 32] and numerical simulations [3–5], that is, each component of the ground state is a multiple of one single density function. For a related mathematical study on SMA, we refer the reader to [35]. Together with Proposition 4.2, Theorems 2-3 also show that the* $m_F = \pm 1$ *components in the ground states tend to have the same density functions when* $M = 0$ *in both ferromagnetic and antiferromagnetic cases. In the ferromagnetic cases, the ground state can be always described exactly by the SMA. While in the antiferromagnetic systems, the situations can be classified into two types, single-mode or two-component systems. Rigorous mathematical justifications of these conclusions are exactly what is expected in ( [3], Section 5).*

**Remark 1.4.** *The challenging point in analysing qualitative properties of ground states in the antiferromagnetic case, is to confirm that any minimizer of $m(N)$ is semi-trivial as $N \nearrow N^*$. When $M$ is non-zero, we fails to get a concentrated result like the case of $M = 0$ in Theorem 3 only depending upon $N$, mainly because it seems difficult to get a consistent upper and lower bound on the energy $m(N)$, which is the key in getting semi-trivial property of ground states by using corresponding linearization operators as in [30]. If we furthermore consider $c_1$ as a parameter and make it infinitesimal of the same order as $N^* - N$, after calculations similar to those in [30], we obtain some precise estimates of the energy and the vanishing property of ground states, see Appendix for details. When $B(x) \equiv 0$, the related results established in Proposition 4.2 also show that different from that in ferromagnetic cases, the ground state energy in an antiferromagnetic system depends on the constants $c_0$, $c_1$ and the magnetization $M$. For fixed $c_0$ and $c_1$, the ground state energy increases when the magnitude of $M$, i.e. $|M|$, increases, and the energy reaches its minimizer at $M = 0$. These have been observed numerically in ( [3], Fig.10). Therefore, All these elements reflect that the introduction of IP magnetic field term brings difficulties to confirm that any ground states of antiferromagnetic spin-1 BEC is semi-trivial as atoms gather to a threshold value.*

Global minimizers obtained in Theorem 1 are obvious ground states for (1.1)-(1.2) in $\mathbb{R}^2$. However, the functional $I(\boldsymbol{u})$ is no longer bounded from below on $\mathcal{M}(N)$ in 3D case. Hence, the global minimization method does not work. Instead, we consider a local minimization problem. Let

$$\|\boldsymbol{u}\|_\Lambda^2 := A(\boldsymbol{u}) + \int_{\mathbb{R}^3} |x|^2 |\boldsymbol{u}|^2 dx.$$

For any $r > 0$ and $N \leq \frac{r}{3}$, we define

$$\mathcal{B}(r) := \Big\{ \boldsymbol{u} = (u_1, u_2, u_0) \in \Lambda \Big| \|\boldsymbol{u}\|_\Lambda^2 \leq r \Big\} \text{ and } m(r, N) := \inf_{\boldsymbol{u} \in \mathcal{M}(N) \cap \mathcal{B}(r)} I(\boldsymbol{u}).$$

Our main results in this aspect are the following:

**Theorem 4.** *Suppose $c_0 > 0,\ c_0 + c_1 > 0$, $B(x)$ satisfies the conditions $\langle \nabla B(x), x \rangle \in L^\infty$ and one of* (1.7) *and*

$$\Big\| \frac{B(x)}{x^2} \Big\|_{L^\infty} < \frac{1}{8}, \tag{1.16}$$

*then*
*(i) for any $r > 0$ and $0 < N \leq \frac{r}{3}$, $m(r, N)$ has at least one minimizer;*
*(ii) for any $r > 0$, there exists a positive constant $N^{**} = N^{**}(r)$, such that when $0 < N < N^{**}$, each minimizer of $m(r, N)$ is a solution to* (1.1)-(1.2). *In particular, the solution is a ground state if $N$ is sufficiently small;*
*(iii) suppose $\boldsymbol{u} = (u_1, u_2, u_0) \in \mathcal{M}(N) \cap \mathcal{B}(r)$ is a minimizer of $m(r, N)$, then*

$$\Big\| \boldsymbol{u} - \big(k_1 e^{-\frac{x^2}{2}}, k_2 e^{-\frac{x^2}{2}}, k_0 e^{-\frac{x^2}{2}}\big) \Big\|_\Lambda^2 = O(N), \quad \text{as } N \to 0^+,$$

*where*

$$k_i = \frac{1}{\pi^{\frac{3}{2}}} \int_{\mathbb{R}^3} u_i e^{-\frac{x^2}{2}} dx, \quad \text{for } i = 1, 2, 0.$$

**Remark 1.5.** *It is worth mentioning that our results in Theorem 4 are not perturbative, indeed for any $r > 0$, the constant $N^{**}$ is determined by*

$$N^{**} = \begin{cases} \min\left\{\left(\frac{\sqrt{C_0^2 r^3 + 4\|B(x)\|_{L^\infty} r} - C_0 r^{\frac{3}{2}}}{8\|B(x)\|_{L^\infty}}\right)^2, \frac{r}{16\|B(x)\|_{L^\infty}}, \frac{r}{12}\right\}, & \text{if } B(x) \in L^\infty, \\ \min\left\{\left(\frac{\sqrt{C_0^2 r^3 + 16\|\frac{B(x)}{x}\|_{L^\infty}^2 r} - C_0 r^{\frac{3}{2}}}{64\|\frac{B(x)}{x}\|_{L^\infty}^2}\right)^2, \frac{3r}{512\|\frac{B(x)}{x}\|_{L^\infty}^2}, \frac{r}{12}\right\}, & \text{if } \frac{B(x)}{x} \in L^\infty, \\ \min\left\{\left(\frac{1 - 8\|\frac{B(x)}{x^2}\|_{L^\infty}}{4C_0 r^{\frac{1}{2}}}\right)^2, \frac{r}{12}\right\}, & \text{if } B(x) \text{ satisfies } (1.16), \end{cases}$$

*where $C_0 := \max\{c_0, 3c_0 + 4c_1\}C_*$ and $C_* = \frac{4\sqrt{3}}{9a^*}$ is the optimal constant of the Gagliardo-Nirenberg type inequality* (2.1) *for $d = 3$.*

Next, we give the symmetric property for the minimizers obtained in Theorems 1 and 4.

**Theorem 5.** *Suppose $c_1 \geq 0$ and $B(x) \geq 0$, then every minimizer for $m(N)$ ($m(r, N)$), denoted as $\boldsymbol{w} = (w_1, w_2, w_0)$(which is in principle complex valued), is of the form*

$$w_j(x) = e^{i\theta_j}\rho_j(x), \quad j = 1, 2, 0,$$

*where $\theta_j \in \mathbb{R}$, $\theta_1 + \theta_2 - 2\theta_0 = 2k\pi$ ($k \in \mathbb{Z}$) and $(\rho_1, \rho_2, \rho_0)$ is a positive real valued minimizer. Furthermore, if $B(x)$ is symmetric-decreasing, then $\rho_j(x)$ is radial symmetric.*

**Remark 1.6.** *Theorem 5 supports some numerical results established in [3], that is the density functions of ground states for ferromagnetic spin-1 BEC with nonnegative IP magnetic field are always symmetric with respect to $x = 0$, which is the center of $V(x) = |x|^2$, see Fig.1 there. In particular, if $B(x) \equiv B$ is a non-negative constant function, then the minimizers obtained in Theorems 1 and 4 are radial. This indicates that the ground states of spin-1 BEC without IP magnetic field obtained in (Theorem 1.1, [30]) and (Theorem 1, [36]) are radial. As a special case, taking $c_1 = 0$ and $B = 0$, Theorem 5 also shows that the ground states for three coupled Schrödinger system considering in [29, 44] are radially symmetric, some slight modifications may be needed. However, for the antiferromagnetic case $c_1 < 0$, the method of Schwartz symmetrization fails, symmetric property of ground states for spin-1 BEC is not clear in theory. In terms of numerical calculation, Bao, Chern and Zhang in [3] show that radial ground states of spin-1 BEC with IP magnetic field exist in both ferromagnetic and antiferromagnetic cases.*

After finding ground states in the 3D case, we further search for an excited state, that is

**Theorem 6.** *Under the conditions of Theorem 4, for any $r > 0$ and $0 < N < N^{**}(r)$, there exists an excited state $\hat{\boldsymbol{u}} = (\hat{u}_1, \hat{u}_2, \hat{u}_0) \in \mathcal{M}(N)$ to* (1.1)-(1.2).

**Remark 1.7.** *Compared with ground states, the results on excited states of spin-1 BEC are much less in the literature. In the previous work [36], we prove the existence of a strongly unstable excited state of attractive spin-1 BEC without IP magnetic field in both ferromagnetic and antiferromagnetic cases. Numerical results on excited states and the corresponding energy of repulsive/attractive spin-1 BEC without IP magnetic field in both ferromagnetic and antiferromagnetic cases are presented in [11]. Furthermore, the authors in [11] reveal that the component separation and population transfer between the different hyperfine states can only occur in excited states due to the spin exchange interactions. We believe that theoretically proving similar qualitative and quantitative properties of excited states are interesting and challenging problems. In the following work, we will focus on these issues.*

**Remark 1.8.** *As an example, we can take*

$$B(x)=\begin{cases}\pm\frac{x^2}{4}, & d=2,\\ \frac{1}{1+|x|^{\frac{3}{2}}}, & d=3,\end{cases}$$

*which satisfies the conditions in the above theorems. In addition, our results are also valid for a remarkable logarithmic function model in physics $B(x)=\frac{1}{4}\ln(1+x^2)$, which is commonly used in charged particles transport and particularly in semiconductor physics, and with interesting mathematical properties. For more information about constraint minimization problem with logarithmic type potential functions, we refer the readers to some recent works [14, 18].*

The paper is organized as follows. In section 2, some preliminary results are introduced. In section 3, we consider the 2D case and prove Theorems 1-3. Finally, Theorems 4-6 will be proved in section 4.

## 2 Preliminaries

In this section, we introduce some preliminary results.

**Lemma 2.1.** *( [45] Compact embedding) Let $\Lambda$ be the space defined in (1.3), then $\Lambda\hookrightarrow\hookrightarrow L^t(\mathbb{R}^d)\times L^t(\mathbb{R}^d)\times L^t(\mathbb{R}^d)$ for any $t\in[2,2^*)$.*

Next, we give a Gagliardo-Nirenberg type inequality and the estimates of $E(\boldsymbol{u})$ without details, the proofs can be found in [30] and [36].

**Lemma 2.2.** *For $d=2,3$ and $\boldsymbol{u}=(u_1,u_2,u_0)\in H$, there holds*

$$\int_{\mathbb{R}^d}\big(|u_1|^2+|u_2|^2+|u_0|^2\big)^2dx\le C_*(d)\Big(A(\boldsymbol{u})\Big)^{\frac{d}{2}}\cdot\Big(\int_{\mathbb{R}^d}\big(|u_1|^2+|u_2|^2+|u_0|^2\big)dx\Big)^{\frac{4-d}{2}},\tag{2.1}$$

*where $C_*(d)$ is the best constant.*

In this paper, we just denote $C_*(d)$ as $C_*$ for simplicity. Particularly, for $d=2$, we see from [30] that $C_*=\frac{2}{a^*}$ is the best constant. Further, up to translations and suitable scalings, the equality holds only at

$$\big(Q\sin\varphi_1\cos\varphi_2,Q\sin\varphi_1\sin\varphi_2,Q\cos\varphi_1\big),\quad\text{for }\varphi_1,\ \varphi_2\in\big[0,\frac{\pi}{2}\big),$$

where $Q(x)$ is the unique positive solution for (1.5).

**Lemma 2.3.** *( [36]) For $d=2,3$, suppose $c_0>0$, $c_0+c_1>0$, then for any $\boldsymbol{u}=(u_1,u_2,u_0)\in\mathcal{M}$, there holds*

$$0\le E(\boldsymbol{u})\le\max\Big\{c_0,3c_0+4c_1\Big\}C_*\Big(A(\boldsymbol{u})\Big)^{\frac{d}{2}}N^{\frac{4-d}{2}}.\tag{2.2}$$

Finally, we give the pure point spectrum and the associated eigenvectors for harmonic oscillator $-\Delta+|x|^2$, which is useful for us to study the qualitative properties of solutions.

**Lemma 2.4.** *( [2]) The pure point spectrum of the harmonic oscillator $-\Delta+|x|^2$ is*

$$\sigma(-\Delta+|x|^2)=\big\{\xi_k=d+2k,\ k\in\mathbb{N}\big\},$$

*and the corresponding eigenvectors are given by Hermite functions (denoted by $\Psi_k$, associated to $\xi_k$), which form an orthogonal basis of $L^2(\mathbb{R}^d)$. Particularly, the first eigenvector is $\Psi_0 = \frac{1}{\pi^{\frac{d}{4}}} e^{-\frac{x^2}{2}}$ and further $\Psi_0$ satisfies the Pohozaev identity:*

$$(d-2)\int_{\mathbb{R}^d} |\nabla \Psi_0|^2 dx + (d+2)\int_{\mathbb{R}^d} |x|^2 \Psi_0^2 dx = d^2 \int_{\mathbb{R}^d} \Psi_0^2 dx,$$

*which follows that*

$$\int_{\mathbb{R}^d} |\nabla \Psi_0|^2 dx = \int_{\mathbb{R}^d} |x|^2 \Psi_0^2 dx = \frac{d}{2}. \tag{2.3}$$

# 3 The 2D case

In this section, we are going to prove Theorems 1 to 3, where the 2D case is studied. Suppose $c_0 > 0,\ c_0 + c_1 > 0$, $|M| < N$, we consider the following minimization problem

$$m(N) := \inf_{\boldsymbol{u} \in \mathcal{M}} I(\boldsymbol{u}),$$

where

$$\mathcal{M} = \mathcal{M}(N) := \Big\{ \boldsymbol{u} = (u_1, u_2, u_0) \in \Lambda \Big| \int_{\mathbb{R}^2} \big(|u_1|^2 + |u_2|^2 + |u_0|^2\big) dx = N,\ \int_{\mathbb{R}^2} \big(|u_1|^2 - |u_2|^2\big) dx = M \Big\}.$$

Let $a^*$ be the constant defined in (1.6) and set

$$N^* := \min \Big\{ \frac{a^*}{c_0 + c_1}, \frac{a^*}{c_0} \Big\}.$$

## 3.1 Proof of Theorem 1

**Lemma 3.1.** *Suppose $c_0 > 0,\ c_0 + c_1 > 0$ and $B(x)$ satisfies* (1.7) *or* (1.8)*, then $m(N)$ has at least one minimizer if $0 < N < N^*$.*

*Proof.* We first suppose $c_1 \geq 0$. For any $\boldsymbol{u} = (u_1, u_2, u_0) \in \mathcal{M}$, we get from (2.1) that if $B(x) \in L^\infty$, then

$$\begin{aligned}
I(\boldsymbol{u}) &\geq \frac{1}{2} A(\boldsymbol{u}) + \frac{1}{2}\int_{\mathbb{R}^2} |x|^2 |\boldsymbol{u}|^2 dx - \frac{c_0 + c_1}{4} \int_{\mathbb{R}^2} (|u_1|^2 + |u_2|^2 + |u_0|^2)^2 dx \\
&\quad + \frac{c_1}{4} \int_{\mathbb{R}^2} (|u_0|^2 - 2|u_1||u_2|)^2 dx - |F(\boldsymbol{u})| \\
&\geq \frac{1}{2} A(\boldsymbol{u}) + \frac{1}{2}\int_{\mathbb{R}^2} |x|^2 |\boldsymbol{u}|^2 dx - \frac{a^*}{4N^*} \int_{\mathbb{R}^2} (|u_1|^2 + |u_2|^2 + |u_0|^2)^2 dx \\
&\quad + \frac{c_1}{4} \int_{\mathbb{R}^2} (|u_0|^2 - 2|u_1||u_2|)^2 dx - \|B(x)\|_{L^\infty} N \qquad (3.1) \\
&\geq \frac{1}{2} A(\boldsymbol{u}) - \frac{a^*}{4N^*} \int_{\mathbb{R}^2} (|u_1|^2 + |u_2|^2 + |u_0|^2)^2 dx + \frac{1}{2}\int_{\mathbb{R}^2} |x|^2 |\boldsymbol{u}|^2 dx - \|B(x)\|_{L^\infty} N \\
&\geq \frac{1}{2} A(\boldsymbol{u}) - \frac{a^*}{4N^*} \cdot \frac{2N}{a^*} A(\boldsymbol{u}) + \frac{1}{2}\int_{\mathbb{R}^2} |x|^2 |\boldsymbol{u}|^2 dx - \|B(x)\|_{L^\infty} N \\
&= \frac{1}{2N^*}(N^* - N) A(\boldsymbol{u}) + \frac{1}{2}\int_{\mathbb{R}^2} |x|^2 |\boldsymbol{u}|^2 dx - \|B(x)\|_{L^\infty} N.
\end{aligned}$$

If $\frac{B(x)}{x} \in L^\infty$, by Young's inequality, we have for any $\varepsilon > 0$,

$$
\begin{aligned}
|F(\boldsymbol{u})| &\leq \int_{\mathbb{R}^2} |B(x)||(\overline{u}_1 u_0 + \overline{u}_0 u_2)|dx \leq \Big\| \frac{B(x)}{x} \Big\|_{L^\infty} \int_{\mathbb{R}^2} |x|\big(|u_1| + |u_2|\big)|u_0|dx \\
&\leq \Big\| \frac{B(x)}{x} \Big\|_{L^\infty} \varepsilon \int_{\mathbb{R}^2} |x|^2 \big(|u_1| + |u_2|\big)^2 dx + \frac{1}{4\varepsilon} \Big\| \frac{B(x)}{x} \Big\|_{L^\infty} \int_{\mathbb{R}^2} |u_0|^2 dx \\
&\leq 2 \Big\| \frac{B(x)}{x} \Big\|_{L^\infty} \varepsilon \int_{\mathbb{R}^2} |x|^2 |\boldsymbol{u}|^2 dx + \frac{1}{4\varepsilon} \Big\| \frac{B(x)}{x} \Big\|_{L^\infty} N.
\end{aligned}
\tag{3.2}
$$

Taking $\varepsilon$ small, such that for some positive constant $C$, there holds

$$
|F(\boldsymbol{u})| \leq \frac{1}{4} \int_{\mathbb{R}^2} |x|^2 |\boldsymbol{u}|^2 dx + CN.
$$

Then similar to (3.1), we conclude

$$
I(\boldsymbol{u}) \geq \frac{1}{2N^*}(N^* - N)A(\boldsymbol{u}) + \frac{1}{4} \int_{\mathbb{R}^2} |x|^2 |\boldsymbol{u}|^2 dx - CN. \tag{3.3}
$$

Since

$$
|F(\boldsymbol{u})| \leq \Big\| \frac{B(x)}{x^2} \Big\|_{L^\infty} \cdot \int_{\mathbb{R}^2} |x|^2 |\boldsymbol{u}|^2 dx,
$$

if $\big\| \frac{B(x)}{x^2} \big\|_{L^\infty} < \frac{1}{2}$, we get

$$
I(\boldsymbol{u}) \geq \frac{1}{2N^*}(N^* - N)A(\boldsymbol{u}) + \Big( \frac{1}{2} - \Big\| \frac{B(x)}{x^2} \Big\|_{L^\infty} \Big) \cdot \int_{\mathbb{R}^2} |x|^2 |\boldsymbol{u}|^2 dx. \tag{3.4}
$$

Let $\{\boldsymbol{u_n}\} \subset \mathcal{M}$ be the minimizing sequence of $m(N)$, then by (3.1), (3.3) and (3.4), $\{\boldsymbol{u_n}\}$ is bounded in $\Lambda$ if $B(x)$ satisfies (1.7) or (1.8). Applying Lemma 2.1, there exists $\boldsymbol{w} = (w_1, w_2, w_0) \in H$, such that up to a subsequence, as $n \to +\infty$,

$$
\begin{cases}
\boldsymbol{u_n} \rightharpoonup \boldsymbol{w}, & \text{in } H. \\
\boldsymbol{u_n} \to \boldsymbol{w}, & \text{in } L^t(\mathbb{R}^2) \times L^t(\mathbb{R}^2) \times L^t(\mathbb{R}^2), \ \forall t \in [2, +\infty). \\
\boldsymbol{u_n} \to \boldsymbol{w}, & a.e. \text{ in } \mathbb{R}^2.
\end{cases}
$$

Then $\boldsymbol{w} \in \mathcal{M}$. Further, by the lower semi-continuity of the norm in $H$, there holds

$$
m(N) \leq I(\boldsymbol{w}) \leq \lim_{n \to \infty} I(\boldsymbol{u_n}) = m(N).
$$

It yields $I(\boldsymbol{w}) = m(N)$, that is, $\boldsymbol{w} \in \mathcal{M}$ is a minimizer of $m(N)$ for any $N \in (0, N^*)$.

Next, suppose $c_1 < 0$. For any $\boldsymbol{u} = (u_1, u_2, u_0) \in \mathcal{M}$, we conclude if $B(x) \in L^\infty$, then

$$\begin{aligned} I(\boldsymbol{u}) \geq & \frac{1}{2}A(\boldsymbol{u}) + \frac{1}{2}\int_{\mathbb{R}^2} |x|^2|\boldsymbol{u}|^2 dx - \frac{a^*}{4N^*}\int_{\mathbb{R}^2} (|u_1|^2 + |u_2|^2 + |u_0|^2)^2 dx - \frac{c_1}{4}\int_{\mathbb{R}^2} (|u_1|^2 - |u_2|^2)^2 dx \\ & - \frac{c_1}{2}\int_{\mathbb{R}^2} (|u_1|^2 + |u_2|^2)|u_0|^2 dx - c_1 Re \int_{\mathbb{R}^d} \overline{u}_1 \overline{u}_2 u_0^2 dx - \|B(x)\|_{L^\infty} N \\ \geq & \frac{1}{2}A(\boldsymbol{u}) - \frac{a^*}{4N^*}\int_{\mathbb{R}^2} (|u_1|^2 + |u_2|^2 + |u_0|^2)^2 dx + \frac{1}{2}\int_{\mathbb{R}^2} |x|^2|\boldsymbol{u}|^2 dx \\ & - \frac{c_1}{2}\int_{\mathbb{R}^2} (|u_1|^2 + |u_2|^2)|u_0|^2 dx + c_1 \int_{\mathbb{R}^2} |u_1||u_2||u_0|^2 dx - \|B(x)\|_{L^\infty} N \\ \geq & \frac{1}{2}A(\boldsymbol{u}) - \frac{a^*}{4N^*}\cdot\frac{2N}{a^*}A(\boldsymbol{u}) + \frac{1}{2}\int_{\mathbb{R}^2} |x|^2|\boldsymbol{u}|^2 dx - \|B(x)\|_{L^\infty} N \\ = & \frac{1}{2N^*}(N^* - N)A(\boldsymbol{u}) + \frac{1}{2}\int_{\mathbb{R}^2} |x|^2|\boldsymbol{u}|^2 dx - \|B(x)\|_{L^\infty} N. \end{aligned} \tag{3.5}$$

Let $\{\boldsymbol{u_n}\} \subset \mathcal{M}$ be the minimizing sequence of $m(N)$, then $\{\boldsymbol{u_n}\}$ is bounded in $\Lambda$. Similarly, if $\frac{B(x)}{x} \in L^\infty$ or $\big\|\frac{B(x)}{x^2}\big\|_{L^\infty} < \frac{1}{2}$, the boundedness of $\{\boldsymbol{u_n}\}$ can be obtained as well. Then some procedures as the case of $c_1 > 0$ allow that there exists at least one minimizer for $m(N)$ if $N \in (0, N^*)$, we omit the details here. □

In the following, we introduce a property of $m(N)$ as $N \nearrow N^*$ and further give the nonexistence results of minimizers for $m(N)$.

**Lemma 3.2.** *Suppose $c_1 \geq 0$ and $B(x)$ satisfies* (1.8)*, then*

$$\lim_{N \nearrow N^*} m(N) = 0. \tag{3.6}$$

*Moreover, there has no minimizer for $m(N)$ if $N \geq N^*$.*

*Proof.* We first prove (3.6) by choosing some proper test functions. If $\int_{\mathbb{R}^2} B(x)Q^2(x)dx \geq 0$, for $\tau > 0$ and $\theta \in [M, N]$, we define $\boldsymbol{\Phi} = (\Phi_1, \Phi_2, \Phi_0) \in \mathcal{M}$ as

$$\Phi_1(x) := \sqrt{\frac{\theta + M}{2a^*}}\tau Q(\tau x), \quad \Phi_2(x) := \sqrt{\frac{\theta - M}{2a^*}}\tau Q(\tau x), \quad \Phi_0(x) := \sqrt{\frac{N - \theta}{a^*}}\tau Q(\tau x), \tag{3.7}$$

where $Q(x)$ is the unique positive solution of equation (1.5). By direct calculations, we get

$$\frac{1}{2}A(\boldsymbol{\Phi}) - \frac{a^*}{4N}\int_{\mathbb{R}^2} (\Phi_1^2 + \Phi_2^2 + \Phi_0^2)^2 dx = \frac{1}{2}\cdot N\tau^2 - \frac{a^*}{4N}\cdot\frac{2N^2\tau^2}{a^*} = 0,$$

$$\int_{\mathbb{R}^2} |x|^2\boldsymbol{\Phi}^2 dx = \frac{N}{a^*}\int_{\mathbb{R}^2} |x|^2\tau^2 Q^2(\tau x)dx = \frac{N\tau^{-2}}{a^*}\int_{\mathbb{R}^2} |x|^2 Q^2(x)dx$$

and

$$\begin{aligned} & \Big(\frac{a^*}{4N} - \frac{c_0 + c_1}{4}\Big)\int_{\mathbb{R}^2} (\Phi_1^2 + \Phi_2^2 + \Phi_0^2)^2 dx + \frac{c_1}{4}\int_{\mathbb{R}^2} (\Phi_0^2 - 2\Phi_1\Phi_2)^2 dx \\ = & \Big(\frac{a^*}{4N} - \frac{c_0 + c_1}{4}\Big)\cdot\frac{2N^2\tau^2}{a^*} + \frac{c_1\tau^2}{2a^*}\Big((N - \theta) - \sqrt{\theta^2 - M^2}\Big)^2 \\ = & \tau^2\cdot\bigg(\frac{N}{2} - \frac{(c_0 + c_1)N^2}{2a^*} + \frac{c_1}{2a^*}\Big((N - \theta) - \sqrt{\theta^2 - M^2}\Big)^2\bigg). \end{aligned}$$

Moreover, $F(\mathbf{\Phi}) \geq 0$ as $\int_{\mathbb{R}^2} B(x)Q^2(x)dx \geq 0$. Denote

$$K := \frac{N}{2} - \frac{(c_0+c_1)N^2}{2a^*} + \frac{c_1}{2a^*}\Big((N-\theta) - \sqrt{\theta^2 - M^2}\Big)^2,$$

then it follows that

$$\begin{aligned} I(\mathbf{\Phi}) &= \frac{1}{2}A(\mathbf{\Phi}) - \frac{a^*}{4N}\int_{\mathbb{R}^2}(\Phi_1^2+\Phi_2^2+\Phi_0^2)^2dx + \frac{c_1}{4}\int_{\mathbb{R}^2}(\Phi_0^2 - 2\Phi_1\Phi_2)^2dx - F(\mathbf{\Phi}) \\ &\quad + \frac{1}{2}\int_{\mathbb{R}^2}|x|^2\mathbf{\Phi}^2dx + \Big(\frac{a^*}{4N} - \frac{c_0+c_1}{4}\Big)\int_{\mathbb{R}^2}(\Phi_1^2+\Phi_2^2+\Phi_0^2)^2dx \\ &\leq \frac{N\tau^{-2}}{2a^*}\int_{\mathbb{R}^2}|x|^2Q^2(x)dx + K\tau^2. \end{aligned} \tag{3.8}$$

Taking $\theta = \frac{M^2+N^2}{2N}$, then $N - \theta - \sqrt{\theta^2 - M^2} = 0$, we conclude for any $\tau > 0$,

$$\begin{aligned} m(N) &\leq \frac{N\tau^{-2}}{2a^*}\int_{\mathbb{R}^2}|x|^2Q^2(x)dx + \Big(\frac{N}{2} - \frac{(c_0+c_1)N^2}{2a^*}\Big)\tau^2 \\ &= \frac{N\tau^{-2}}{2a^*}\int_{\mathbb{R}^2}|x|^2Q^2(x)dx + \frac{N}{2N^*}(N^*-N)\tau^2. \end{aligned} \tag{3.9}$$

Taking $\tau = \Big(\frac{N^*\int_{\mathbb{R}^2}|x|^2Q^2(x)dx}{a^*(N^*-N)}\Big)^{\frac{1}{4}}$, we get

$$m(N) \leq N\cdot\Big(\frac{\int_{\mathbb{R}^2}|x|^2Q^2(x)dx\cdot(N^*-N)}{a^*N^*}\Big)^{\frac{1}{2}} \to 0, \quad \text{as } N \nearrow N^*, \tag{3.10}$$

that is, $\lim_{N\nearrow N^*} m(N) \leq 0$. On the other hand, when $N \in (0, N^*)$, we obtain from (3.4) that $I(\boldsymbol{u}) \geq 0$ for any $\boldsymbol{u} = (u_1, u_2, u_0) \in \mathcal{M}$, which implies $\lim_{N\nearrow N^*} m(N) \geq 0$. Hence, (3.6) holds if $\int_{\mathbb{R}^2} B(x)Q^2(x)dx \geq 0$.

If $\int_{\mathbb{R}^2} B(x)Q^2(x)dx < 0$, the proof in this instant is similar to that of $(B1)$, we just sketch the differences. Let $\mathbf{\Phi} = (\Phi_1, \Phi_2, -\Phi_0) \in \mathcal{M}$ be the test function, where $\Phi_i$ $(i = 1, 2, 0)$ has been defined in (3.7), then $F(\mathbf{\Phi}) \geq 0$. It follows that there holds

$$m(N) \leq \frac{N\tau^{-2}}{2a^*}\int_{\mathbb{R}^2}|x|^2Q^2(x)dx + \frac{N}{2N^*}(N^*-N)\tau^2. \tag{3.11}$$

Then

$$m(N) \leq N\cdot\Big(\frac{(N^*-N)\int_{\mathbb{R}^2}|x|^2Q^2(x)dx}{a^*N^*}\Big)^{\frac{1}{2}}. \tag{3.12}$$

Hence, together with (3.4), we obtain (3.6) if $\int_{\mathbb{R}^2} B(x)Q^2(x)dx < 0$.

Next, we show that there has no minimizer for $m(N)$ if $N \geq N^*$. If $N > N^*$, let $\tau \to \infty$ in (3.9) and (3.11) respectively, then $m(N) \to -\infty$. Thus, there has no minimizer for $m(N)$.

If $N = N^*$, we argue by contradiction to show that there has no minimizer for $m(N^*)$. Suppose $\boldsymbol{u^*} = (u_1^*, u_2^*, u_0^*)$ is a minimizer of $m(N^*)$. From the proof of (3.4), we have

$$\begin{aligned} I(\boldsymbol{u^*}) &\geq \frac{1}{2}A(\boldsymbol{u^*}) - \frac{a^*}{4N^*}\int_{\mathbb{R}^2}\big(|u_1^*|^2+|u_2^*|^2+|u_0^*|^2\big)^2dx \\ &\quad + \Big(\frac{1}{2} - \Big\|\frac{B(x)}{x^2}\Big\|_{L^\infty}\Big)\int_{\mathbb{R}^2}|x|^2|\boldsymbol{u^*}|^2dx \geq 0. \end{aligned}$$

Together with (3.10) or (3.12), we get $m(N^*) = 0$. As a consequence,

$$\frac{1}{2}A(\boldsymbol{u^*}) = \frac{a^*}{4N^*}\int_{\mathbb{R}^2}\big(|u_1^*|^2+|u_2^*|^2+|u_0^*|^2\big)^2dx \tag{3.13}$$

and

$$\int_{\mathbb{R}^2}|x|^2|\boldsymbol{u^*}|^2dx = 0. \tag{3.14}$$

From (3.13), $\boldsymbol{u^*}$ is an optimal function of the Gagliardo-Nirenberg inequality (2.1) for $d = 2$. By Lemma 2.2, $\boldsymbol{u^*}$ can be formed as a scaling of $Q(x)$. However, this contradicts to (3.14). Therefore, there has no minimizer for $m(N^*)$ and we complete the proof. □

**Lemma 3.3.** *Suppose $c_1 < 0$ and $B(x)$ satisfies (1.8), then*
*(i) if $M = 0$, there holds $\lim\limits_{N\nearrow N^*} m(N) = 0$. Moreover, there has no minimizer for $m(N)$ if $N \geq N^*$;*
*(ii) if $M \neq 0$, then there has no minimizer for $m(N)$, when $c_0N^2 - a^*N + c_1M^2 > 0$.*

*Proof.* (i) if $M = 0$ and $\int_{\mathbb{R}^2} B(x)Q^2(x)dx \geq 0$, we choose $\theta = 0$ in (3.7), then

$$\Phi_1(x) = \Phi_2(x) = 0 \quad \text{and} \quad \Phi_0(x) := \sqrt{\frac{N}{a^*}}\tau Q(\tau x).$$

If $M = 0$ and $\int_{\mathbb{R}^2} B(x)Q^2(x)dx < 0$, let

$$\Phi_1(x) = \Phi_2(x) = 0 \quad \text{and} \quad \Phi_0(x) := -\sqrt{\frac{N}{a^*}}\tau Q(\tau x).$$

Noting that $N^* = \frac{a^*}{c_0}$ for $c_1 < 0$, similar to Lemma 3.2, we can show $\lim\limits_{N\nearrow N^*} m(N) = 0$ and further if $N \geq N^*$, there has no minimizer for $m(N)$.

(ii) We first suppose $\int_{\mathbb{R}^2} B(x)Q^2(x)dx \geq 0$, let $\boldsymbol{\Phi} = (\Phi_1, \Phi_2, \Phi_0) \in \mathcal{M}$ be the test function defined in (3.7), then choose $\theta = N$ in (3.8), we get

$$I(\boldsymbol{\Phi}) \leq \frac{N\tau^{-2}}{2a^*}\int_{\mathbb{R}^2}|x|^2Q^2(x)dx + \Big(\frac{N}{2} - \frac{c_0N^2}{2a^*} - \frac{c_1M^2}{2a^*}\Big)\tau^2. \tag{3.15}$$

Now, we suppose $\int_{\mathbb{R}^2} B(x)Q^2(x)dx < 0$ and let $\boldsymbol{\Phi} = (\Phi_1, \Phi_2, -\Phi_0) \in \mathcal{M}$ be the test function, where $\Phi_i$ $(i = 1, 2, 0)$ has been defined in (3.7). Then (3.15) holds as well.

Hence, if $c_0N^2 - a^*N + c_1M^2 > 0$, then $m(N) \to -\infty$ when we take $\tau \to \infty$, which implies that there has no minimizer for $m(N)$. Therefore, we complete the proof. □

***Proof of Theorem 1 (i)-(ii).*** The conclusions follow immediately from Lemmas 3.1-3.3. □

Let $\boldsymbol{u} = (u_1, u_2, u_0) \in \mathcal{M}(N)$ be a minimizer for $m(N)$ obtained above. Suppose $0 < N < N^*$, we then prove (iii) in Theorem 1. That is, we are going to show that

$$\|\boldsymbol{u} - (l_1\Psi_0, l_2\Psi_0, l_0\Psi_0)\|_\Lambda^2 = O(N), \quad \text{as } N \to 0^+,$$

where $\Psi_0 = \frac{1}{\sqrt{\pi}}e^{-\frac{x^2}{2}}$ is the first eigenvector of the harmonic oscillator $-\Delta + |x|^2$ (see Lemma 2.4) and $l_i = l_{i0} = \int_{\mathbb{R}^2} u_i\Psi_0 dx,$ for $i = 1, 2, 0$. Before that, we give an estimate for the least energy $m(N)$.

**Lemma 3.4.** *Suppose $c_0 > 0$ and $c_0 + c_1 > 0$, then there holds $m(N) < N$, for $N \in (0, N^*)$.*

*Proof.* Since $\Big(\sqrt{\frac{N+M}{2}}\Psi_0, \sqrt{\frac{N-M}{2}}\Psi_0, 0\Big) \in \mathcal{M}$, we get

$$\begin{aligned} m(N) &= \inf_{\boldsymbol{u}\in\mathcal{M}} I(\boldsymbol{u}) \leq I\Big(\sqrt{\frac{N+M}{2}}\Psi_0, \sqrt{\frac{N-M}{2}}\Psi_0, 0\Big) \\ &< \frac{N}{2}\int_{\mathbb{R}^2}\Big(|\nabla\Psi_0|^2 + |x|^2{\Psi_0}^2\Big)dx = \frac{N}{2}\|(\Psi_0, 0, 0)\|_{\Lambda}^2 = N. \end{aligned}$$

□

***Proof of Theorem 1 (iii).*** Set $l_{ik} = \int_{\mathbb{R}^2} u_i \Psi_k dx$, for $i = 1, 2, 0$, then

$$\boldsymbol{u} = \Big(\sum_{k=0}^{\infty} l_{1k}\Psi_k, \sum_{k=0}^{\infty} l_{2k}\Psi_k, \sum_{k=0}^{\infty} l_{0k}\Psi_k\Big).$$

Moreover, we conclude

$$N = \|(u_1, u_2, u_0)\|_{L^2}^2 = \sum_{k=0}^{\infty}(l_{1k}^2 + l_{2k}^2 + l_{0k}^2)\|\Psi_k\|_{L^2}^2 = \sum_{k=0}^{\infty}(l_{1k}^2 + l_{2k}^2 + l_{0k}^2) \tag{3.16}$$

and

$$\|\boldsymbol{u}\|_{\Lambda}^2 = \sum_{k=0}^{\infty}(l_{1k}^2 + l_{2k}^2 + l_{0k}^2)\|\Psi_k\|_{\Lambda}^2 = \sum_{k=0}^{\infty}\xi_k(l_{1k}^2 + l_{2k}^2 + l_{0k}^2).$$

If $B(x) \in L^\infty$, denote $M_0 := \frac{1}{2N^*}(N^* - N) \in \big(0, \frac{1}{2}\big)$, then by (3.1), we get

$$\begin{aligned} m(N) = I(\boldsymbol{u}) &\geq M_0 A(\boldsymbol{u}) + \frac{1}{2}\int_{\mathbb{R}^2}|x|^2\boldsymbol{u}^2 dx - \|B(x)\|_{L^\infty}N \\ &\geq M_0\|\boldsymbol{u}\|_{\Lambda}^2 - \|B(x)\|_{L^\infty}N = M_0 \cdot \sum_{k=0}^{\infty}\xi_k(l_{1k}^2 + l_{2k}^2 + l_{0k}^2) - \|B(x)\|_{L^\infty}N \\ &= M_0 \cdot \sum_{k=0}^{\infty}(\xi_k - \xi_0)(l_{1k}^2 + l_{2k}^2 + l_{0k}^2) + M_0 \cdot \sum_{k=0}^{\infty}\xi_0(l_{1k}^2 + l_{2k}^2 + l_{0k}^2) - \|B(x)\|_{L^\infty}N. \end{aligned}$$

By Lemma 3.4 and (3.16), we have

$$\begin{aligned} &(\xi_1 - \xi_0)\sum_{k=1}^{\infty}(l_{1k}^2 + l_{2k}^2 + l_{0k}^2) \leq \sum_{k=1}^{\infty}(\xi_k - \xi_0)(l_{1k}^2 + l_{2k}^2 + l_{0k}^2) \\ \leq\ & \frac{m(N) + \|B(x)\|_{L^\infty}N}{M_0} - \sum_{k=0}^{\infty}\xi_0(l_{1k}^2 + l_{2k}^2 + l_{0k}^2) \leq \Big(\frac{1 + \|B(x)\|_{L^\infty}}{M_0} - 2\Big)N, \end{aligned}$$

then

$$\sum_{k=1}^{\infty}(l_{1k}^2 + l_{2k}^2 + l_{0k}^2) \leq \Big(\frac{1 + \|B(x)\|_{L^\infty}}{M_0} - 2\Big) \cdot \frac{N}{\xi_1 - \xi_0}.$$

Thus

$$\begin{aligned}\sum_{k=1}^{\infty}\xi_k(l_{1k}^2+l_{2k}^2+l_{0k}^2) &= \sum_{k=1}^{\infty}(\xi_k-\xi_0)(l_{1k}^2+l_{2k}^2+l_{0k}^2)+\xi_0\sum_{k=1}^{\infty}(l_{1k}^2+l_{2k}^2+l_{0k}^2)\\ &\le \Big(\frac{1+\|B(x)\|_{L^\infty}}{M_0}-2\Big)N+\xi_0\Big(\frac{1+\|B(x)\|_{L^\infty}}{M_0}-2\Big)\cdot\frac{N}{\xi_1-\xi_0}\\ &= \frac{\xi_1}{\xi_1-\xi_0}\cdot\Big(\frac{1+\|B(x)\|_{L^\infty}}{M_0}-2\Big)N.\end{aligned}$$

For $N\to 0^+$, we can see that

$$\begin{aligned}&\|\boldsymbol{u}-(l_1\Psi_0,l_2\Psi_0,l_0\Psi_0)\|_{\dot{\Lambda}}^2=\Big\|\Big(\sum_{k=1}^{\infty}l_{1k}\Psi_k,\sum_{k=1}^{\infty}l_{2k}\Psi_k,\sum_{k=1}^{\infty}l_{0k}\Psi_k\Big)\Big\|_{\dot{\Lambda}}^2\\ &=\sum_{k=1}^{\infty}\xi_k(l_{1k}^2+l_{2k}^2+l_{0k}^2)=O(N)\end{aligned}$$

and

$$\begin{aligned}&\|\boldsymbol{u}-(l_1\Psi_0,l_2\Psi_0,l_0\Psi_0)\|_{L^2}^2=\Big\|\Big(\sum_{k=1}^{\infty}l_{1k}\Psi_k,\sum_{k=1}^{\infty}l_{2k}\Psi_k,\sum_{k=1}^{\infty}l_{0k}\Psi_k\Big)\Big\|_{L^2}^2\\ &=\sum_{k=1}^{\infty}(l_{1k}^2+l_{2k}^2+l_{0k}^2)=O(N).\end{aligned}$$

Therefore, it is obvious that (1.10) is valid.

When $\frac{B(x)}{x}\in L^\infty$ or $\big\|\frac{B(x)}{x^2}\big\|_{L^\infty}<\frac{1}{2}$, the conclusion holds as well if we set

$$M_0=\min\Big\{\frac{1}{2N^*}(N^*-N),\frac{1}{4}\Big\}\in\Big(0,\frac{1}{4}\Big)$$

or

$$M_0=\min\Big\{\frac{1}{2N^*}(N^*-N),\frac{1}{2}-\Big\|\frac{B(x)}{x^2}\Big\|_{L^\infty}\Big\}\in\Big(0,\frac{1}{2}\Big).$$

Hence, we complete the proof of (iii) in Theorem 1. □

### 3.2 Proof of Theorem 2

Next, we give the proof of Theorem 2. Assume $c_0>0$, $c_1>0$ and $N_n\nearrow N^*$ as $n\to\infty$, let $\boldsymbol{u_n}=(u_{1n},u_{2n},u_{0n})\in\mathcal{M}(N_n)$ be a minimizer for $m(N_n)$. Then $\boldsymbol{u_n}$ satisfies the following Euler-Lagrange system

$$\begin{cases}-\Delta u_{1n}+|x|^2u_{1n}=(\mu_n+\lambda_n)u_{1n}+(c_0+c_1)|u_{1n}|^2u_{1n}+(c_0-c_1)u_{1n}|u_{2n}|^2\\ \qquad\qquad+(c_0+c_1)|u_{0n}|^2u_{1n}+c_1\overline{u}_{2n}u_{0n}^2+B(x)u_{0n},\\ -\Delta u_{2n}+|x|^2u_{2n}=(\mu_n-\lambda_n)u_{2n}+(c_0+c_1)|u_{2n}|^2u_{2n}+(c_0-c_1)|u_{1n}|^2u_{2n}\\ \qquad\qquad+(c_0+c_1)|u_{0n}|^2u_{2n}+c_1\overline{u}_{1n}u_{0n}^2+B(x)u_{0n},\\ -\Delta u_{0n}+|x|^2u_{0n}=\mu_nu_{0n}+c_0|u_{0n}|^2u_{0n}+(c_0+c_1)(|u_{1n}|^2+|u_{2n}|^2)u_{0n}\\ \qquad\qquad+2c_1u_{1n}u_{2n}\overline{u}_{0n}+B(x)(u_{1n}+u_{2n}),\end{cases}\tag{3.17}$$

where $\mu_n$ and $\lambda_n$ are the corresponding Lagrange multipliers. Similar to (3.4), we have

$$\begin{aligned} I(\boldsymbol{u_n}) \geq \frac{1}{2}A(\boldsymbol{u_n}) - \frac{a^*}{4N^*}\int_{\mathbb{R}^2}(|u_{1n}|^2 + |u_{2n}|^2 + |u_{0n}|^2)^2dx + \frac{1}{2}\int_{\mathbb{R}^2}|x|^2|\boldsymbol{u_n}|^2dx \\ + \frac{c_1}{4}\int_{\mathbb{R}^2}(|u_{0n}|^2 - 2|u_{1n}||u_{2n}|)^2dx - F(\boldsymbol{u_n}) \geq 0. \end{aligned} \tag{3.18}$$

Combining with the fact that $\lim\limits_{N\nearrow N^*} m(N) = 0$, we can see that

$$\lim_{n\to\infty}\Big(\frac{1}{2}\int_{\mathbb{R}^2}|x|^2|\boldsymbol{u_n}|^2dx - F(\boldsymbol{u_n})\Big) = 0 \tag{3.19}$$

and

$$\lim_{n\to\infty}\frac{A(\boldsymbol{u_n})}{\int_{\mathbb{R}^2}(|u_{1n}|^2 + |u_{2n}|^2 + |u_{0n}|^2)^2dx} = \frac{a^*}{2N^*}. \tag{3.20}$$

We claim

$$\lim_{n\to\infty}A(\boldsymbol{u_n}) = \lim_{n\to\infty}\int_{\mathbb{R}^2}\big(|\nabla u_{1n}|^2 + |\nabla u_{2n}|^2 + |\nabla u_{0n}|^2\big)dx = +\infty.$$

Otherwise, suppose that there exists a positive constant $C$, such that $A(\boldsymbol{u_n}) \leq C$ for large $n$. Then $\{\boldsymbol{u_n}\}$ is a bounded sequence in $\Lambda$, which implies that there is a subsequence, still denoted by $\{\boldsymbol{u_n}\}$, such that

$$\boldsymbol{u_n} \to \boldsymbol{u^*} = (u_1^*, u_2^*, u_0^*) \text{ in } L^t(\mathbb{R}^2)\times L^t(\mathbb{R}^2)\times L^t(\mathbb{R}^2) \text{ with } t \in [2, +\infty).$$

Hence, we get

$$0 = \lim_{n\to\infty} I(\boldsymbol{u_n}) \geq I(\boldsymbol{u^*}) \geq m(N^*) = 0.$$

It shows that $\boldsymbol{u^*}$ is a minimizer of $m(N^*)$, which contradicts to Lemma 3.2. Thus, we obtain the claim. Further, we conclude from (3.20) that

$$\lim_{n\to\infty}\int_{\mathbb{R}^2}(|u_{1n}|^4 + |u_{2n}|^4 + |u_{0n}|^4)dx \geq \frac{1}{3}\cdot\lim_{n\to\infty}\int_{\mathbb{R}^2}(|u_{1n}|^2 + |u_{2n}|^2 + |u_{0n}|^2)^2dx = +\infty.$$

Now, define

$$\varepsilon_n := \sqrt{N^*}\Big(A(\boldsymbol{u_n})\Big)^{-\frac{1}{2}} = \sqrt{N^*}\Big(\int_{\mathbb{R}^2}\big(|\nabla u_{1n}|^2 + |\nabla u_{2n}|^2 + |\nabla u_{0n}|^2\big)dx\Big)^{-\frac{1}{2}}, \tag{3.21}$$

then it is easy to see that $\varepsilon_n \to 0$ as $n \to \infty$.

***Proof of Theorem 2.*** First, suppose $B(x) \geq 0$ and $N_n \nearrow N^*$. On the one hand, we obtain from (3.9) that

$$m(N_n) \leq \frac{N_n\tau^{-2}}{2a^*}\int_{\mathbb{R}^2}|x|^2Q^2(x)dx + \frac{N_n}{2N^*}(N^* - N_n)\tau^2.$$

By (3.10), it follows that

$$\lim_{n\to\infty}\frac{m(N_n)}{(N^* - N_n)^{\frac{1}{2}}} \leq \Big(\frac{N^*\int_{\mathbb{R}^2}|x|^2Q^2(x)dx}{a^*}\Big)^{\frac{1}{2}}. \tag{3.22}$$

On the other hand, let $\tilde{\boldsymbol{w}}_{\boldsymbol{n}} := (\tilde{w}_{1n}, \tilde{w}_{2n}, \tilde{w}_{0n})$ with $\tilde{w}_{in}(x) := \varepsilon_n u_{in}(\varepsilon_n x)$ $(i = 1, 2, 0)$, then $A(\tilde{\boldsymbol{w}}_{\boldsymbol{n}}) = \varepsilon_n^2 A(\boldsymbol{u_n}) = N^*$ and from (3.20), we have

$$\lim_{n\to\infty} \frac{A(\tilde{\boldsymbol{w}}_{\boldsymbol{n}})}{\int_{\mathbb{R}^2}(|\tilde{w}_{1n}|^2 + |\tilde{w}_{2n}|^2 + |\tilde{w}_{0n}|^2)^2 dx} = \lim_{n\to\infty} \frac{A(\boldsymbol{u_n})}{\int_{\mathbb{R}^2}(|u_{1n}|^2 + |u_{2n}|^2 + |u_{0n}|^2)^2 dx} = \frac{a^*}{2N^*}, \tag{3.23}$$

which yields that

$$\lim_{n\to\infty} \int_{\mathbb{R}^2} (|\tilde{w}_{1n}|^2 + |\tilde{w}_{2n}|^2 + |\tilde{w}_{0n}|^2)^2 dx = \frac{2(N^*)^2}{a^*}. \tag{3.24}$$

We claim that there exist $\{y_n\} \subset \mathbb{R}^2$ and $R_0$, $\eta > 0$, such that at least one $i \in \{0, 1, 2\}$ satisfies $\liminf_{n\to\infty} \int_{B_{R_0}(y_n)} |\tilde{w}_{in}|^2 dx \geq \eta > 0$. Otherwise, suppose for any $R > 0$, there has a subsequence $\{\tilde{w}_{in_k}\}$ $(i = 0, 1, 2)$, such that $\lim_{k\to\infty} \sup_{x\in\mathbb{R}^2} \int_{B_R(y)} |\tilde{w}_{in_k}|^2 dx = 0$. Then by Lion's vanishing Lemma, we conclude that $\tilde{w}_{in_k} \to 0$ $(i = 0, 1, 2)$ in $L^t(\mathbb{R}^2)$ for $t \in (2, \infty)$, which contradicts to (3.24). Hence, we obtain the claim. Now we define $\boldsymbol{w_n} := (w_{1n}, w_{2n}, w_{0n})$ with

$$w_{in}(x) := \tilde{w}_{in}(x + y_n) = \varepsilon_n u_{in}(\varepsilon_n x + \varepsilon_n y_n), \quad i = 1, 2, 0, \tag{3.25}$$

then

$$\lim_{n\to\infty} A(\boldsymbol{w_n}) = \lim_{n\to\infty} \int_{\mathbb{R}^2} (|w_{1n}|^2 + |w_{2n}|^2 + |w_{0n}|^2) dx = N^*$$

and

$$\lim_{n\to\infty} \int_{\mathbb{R}^2} (|w_{1n}|^2 + |w_{2n}|^2 + |w_{0n}|^2)^2 dx = \frac{2(N^*)^2}{a^*}.$$

Moreover, there exists some $i \in \{0, 1, 2\}$, such that

$$\liminf_{n\to\infty} \int_{B_{R_0}(0)} |w_{in}|^2 dx \geq \eta > 0. \tag{3.26}$$

It follows that

$$\lim_{n\to\infty} \frac{A(\boldsymbol{w_n}) \int_{\mathbb{R}^2}(|w_{1n}|^2 + |w_{2n}|^2 + |w_{0n}|^2) dx}{\int_{\mathbb{R}^2}(|w_{1n}|^2 + |w_{2n}|^2 + |w_{0n}|^2)^2 dx} = \frac{a^*}{2}. \tag{3.27}$$

By Lemma 2.2, $\{\boldsymbol{w_n}\}$ is a minimizing sequence for the following minimization problem:

$$j := \inf_{(0,0,0)\neq \boldsymbol{u}\in H} J(u_1, u_2, u_0),$$

where

$$J(u_1, u_2, u_0) := \frac{A(\boldsymbol{u}) \int_{\mathbb{R}^2}(|u_1|^2 + |u_2|^2 + |u_0|^2) dx}{\int_{\mathbb{R}^2}(|u_1|^2 + |u_2|^2 + |u_0|^2)^2 dx}.$$

Applying the arguments in [16] and [17], the minimizer $(w_1, w_2, w_0)$ must be

$$w_1(x) = \sqrt{\frac{N^*}{a^*}} Q(x) \sin\varphi_1 \cos\varphi_2, \quad w_2(x) = \sqrt{\frac{N^*}{a^*}} Q(x) \sin\varphi_1 \sin\varphi_2$$

and $w_0(x) = \sqrt{\frac{N^*}{a^*}} Q(x) \cos\varphi_1$, for $\varphi_1, \varphi_2 \in [0, \frac{\pi}{2}]$. Since $\int_{\mathbb{R}^2}(|w_1|^2 + |w_2|^2 + |w_0|^2) dx = N^*$, we get $w_{in} \to w_i$ in $L^2(\mathbb{R}^2)$ for $i = 1, 2, 0$. Further, using the interpolation inequality, there holds $w_{in} \to w_i$ in

$L^4(\mathbb{R}^2)$ for $i = 1, 2, 0$. From (3.27), we obtain

$$
\begin{aligned}
&\frac{a^*}{2}\int_{\mathbb{R}^2}(|w_1|^2+|w_2|^2+|w_0|^2)^2dx = N^*\int_{\mathbb{R}^2}\big(|\nabla w_1|^2+|\nabla w_2|^2+|\nabla w_0|^2\big)dx\\
\le& \lim_{n\to\infty} N_n\int_{\mathbb{R}^2}\big(|\nabla w_{1n}|^2+|\nabla w_{2n}|^2+|\nabla w_{0n}|^2\big)dx = \frac{a^*}{2}\lim_{n\to\infty}\int_{\mathbb{R}^2}(|w_{1n}|^2+|w_{2n}|^2+|w_{0n}|^2)^2dx\\
=& \frac{a^*}{2}\int_{\mathbb{R}^2}(|w_1|^2+|w_2|^2+|w_0|^2)^2dx,
\end{aligned}
$$

which gives that

$$
\lim_{n\to\infty}\int_{\mathbb{R}^2}\big(|\nabla w_{1n}|^2+|\nabla w_{2n}|^2+|\nabla w_{0n}|^2\big)dx = \int_{\mathbb{R}^2}\big(|\nabla w_1|^2+|\nabla w_2|^2+|\nabla w_0|^2\big)dx,
$$

that is, $\boldsymbol{w_n} \to (w_1, w_2, w_0)$ in $H$ as $n \to \infty$. Therefore, there exists some $x_1 \in \mathbb{R}^2$, such that

$$
\lim_{n\to\infty} w_{1n}(x) = \sqrt{\frac{N^*}{a^*}}Q(x-x_1)\sin\varphi_1\cos\varphi_2, \quad \lim_{n\to\infty} w_{2n}(x) = \sqrt{\frac{N^*}{a^*}}Q(x-x_1)\sin\varphi_1\sin\varphi_2
$$

and

$$
\lim_{n\to\infty} w_{0n}(x) = \sqrt{\frac{N^*}{a^*}}Q(x-x_1)\cos\varphi_1, \quad \text{for } \varphi_1,\varphi_2\in[0,\frac{\pi}{2}].
$$

By direct calculations, we obtain from (3.25) that

$$
\begin{aligned}
&\int_{\mathbb{R}^2}|x|^2|\boldsymbol{u_n}|^2dx = \sum_{i=0}^{2}\int_{\mathbb{R}^2}|x|^2\cdot\frac{1}{\varepsilon_n^2}\Big|w_{in}\Big(\frac{x-\varepsilon_n y_n}{\varepsilon_n}\Big)\Big|^2dx\\
=& \sum_{i=0}^{2}\int_{\mathbb{R}^2}|\varepsilon_n x+\varepsilon_n y_n|^2|w_{in}(x)|^2dx = \sum_{i=0}^{2}\varepsilon_n^2\int_{\mathbb{R}^2}\Big|x+y_n+x_1\Big|^2|w_{in}(x+x_1)|^2dx.
\end{aligned}
\tag{3.28}
$$

We now claim $\lim\limits_{n\to\infty}|y_n| \le C$ for arbitrary positive $C$. Otherwise, suppose that $\lim\limits_{n\to\infty}\big|y_n+x_1\big| = +\infty$, then it follows from (3.28) that for arbitrary $C_1 > 0$, there holds $\int_{\mathbb{R}^2}|x|^2|\boldsymbol{u_n}|^2dx \ge C_1\varepsilon_n^2$, as $n \to \infty$. By (3.23), we get

$$
\begin{aligned}
I(\boldsymbol{u_n}) \ge& \frac{1}{2}A(\boldsymbol{u_n}) - \frac{a^*}{4N_n}\int_{\mathbb{R}^2}(|u_{1n}|^2+|u_{2n}|^2+|u_{0n}|^2)^2dx + \frac{1}{2}\int_{\mathbb{R}^2}|x|^2|\boldsymbol{u_n}|^2dx - F(\boldsymbol{u_n})\\
&+\Big(\frac{a^*}{4N_n}-\frac{c_0+c_1}{4}\Big)\int_{\mathbb{R}^2}(|u_{1n}|^2+|u_{2n}|^2+|u_{0n}|^2)^2dx + \frac{c_1}{4}\int_{\mathbb{R}^2}(|u_{0n}|^2-2|u_{1n}||u_{2n}|)^2dx\\
\ge& \Big(\frac{1}{2}-\Big\|\frac{B(x)}{x^2}\Big\|_{L^\infty}\Big)\int_{\mathbb{R}^2}|x|^2|\boldsymbol{u_n}|^2dx + \Big(\frac{a^*}{4N_n}-\frac{c_0+c_1}{4}\Big)\int_{\mathbb{R}^2}(|u_{1n}|^2+|u_{2n}|^2+|u_{0n}|^2)^2dx\\
\ge& \Big(\frac{1}{2}-\Big\|\frac{B(x)}{x^2}\Big\|_{L^\infty}\Big)\cdot C_1\varepsilon_n^2 + \Big(\frac{a^*}{2N_n}-\frac{c_0+c_1}{2}\Big)\cdot\frac{(N^*)^2\varepsilon_n^{-2}}{a^*}+o(1)\\
=& \Big(\frac{1}{2}-\Big\|\frac{B(x)}{x^2}\Big\|_{L^\infty}\Big)\cdot C_1\varepsilon_n^2 + \frac{N^*}{2N_n}(N^*-N_n)\varepsilon_n^{-2}+o(1),
\end{aligned}
$$

where $o(1) \to 0$ as $n \to \infty$. Taking the infimum with respect to $\varepsilon_n > 0$, then we conclude

$$
\lim_{n\to\infty}\frac{m(N_n)}{(N^*-N_n)^{\frac{1}{2}}} \ge \Big(C_1(1-2\Big\|\frac{B(x)}{x^2}\Big\|_{L^\infty})\Big)^{\frac{1}{2}}.
$$

However, it contradicts to (3.22). Thus, there exists $x_2 \in \mathbb{R}^2$, such that

$$\lim_{n\to\infty} \Big(y_n + x_1\Big) = x_2, \tag{3.29}$$

which yields $\lim\limits_{n\to\infty} |y_n| \le C$. Therefore, by (3.23), (3.28) and Fatou's Lemma, we have

$$\begin{aligned} I(\boldsymbol{u_n}) &\ge \Big(\frac{1}{2} - \Big\|\frac{B(x)}{x^2}\Big\|_{L^\infty}\Big)\int_{\mathbb{R}^2} |x|^2|\boldsymbol{u_n}|^2 dx + \Big(\frac{a^*}{4N_n} - \frac{c_0+c_1}{4}\Big)\int_{\mathbb{R}^2}(|u_{1n}|^2 + |u_{2n}|^2 + |u_{0n}|^2)^2 dx \\ &\ge \Big(\frac{1}{2} - \Big\|\frac{B(x)}{x^2}\Big\|_{L^\infty}\Big)\frac{N^*\varepsilon_n^2}{a^*}\cdot\int_{\mathbb{R}^2}|x|^2Q^2(x)dx + \frac{N^*}{2N_n}(N^*-N_n)\varepsilon_n^{-2} + o(1). \end{aligned} \tag{3.30}$$

Then taking

$$\varepsilon_n = \Bigg(\frac{a^*(N^*-N_n)}{N_n\big(1-2\|\frac{B(x)}{x^2}\|_{L^\infty}\big)\int_{\mathbb{R}^2}|x|^2Q^2(x)dx}\Bigg)^{\frac{1}{4}},$$

we can see that

$$\lim_{n\to\infty}\frac{m(N_n)}{(N^*-N_n)^{\frac{1}{2}}} \ge \Big(1-2\Big\|\frac{B(x)}{x^2}\Big\|_{L^\infty}\Big)^{\frac{1}{2}}\cdot\Big(\frac{N^*\int_{\mathbb{R}^2}|x|^2Q^2(x)dx}{a^*}\Big)^{\frac{1}{2}}.$$

Combining (3.22), we conclude $m(N_n) \sim (N^*-N_n)^{\frac{1}{2}}$, as $n\to\infty$.

Next, suppose $B(x) < 0$ satisfies (1.9) and $N_n \nearrow N^*$. The proof is similar to the case of $B(x)\ge 0$, we just sketch the differences here.

Let $\Phi = (\Phi_1, \Phi_2, \Phi_0)$ be the test function defined in (3.7), by (1.9) and direct calculations, we get

$$\begin{aligned} Re\int_{\mathbb{R}^2} B(x)(\overline{\Phi}_1\Phi_0 + \overline{\Phi}_0\Phi_2)dx &= \int_{\mathbb{R}^2} B(x)\Big(\sqrt{\frac{\theta+M}{2a^*}} + \sqrt{\frac{\theta-M}{2a^*}}\Big)\cdot\sqrt{\frac{N-\theta}{a^*}}\tau^2Q^2(\tau x)dx \\ &= \Big(\sqrt{\frac{\theta+M}{2a^*}} + \sqrt{\frac{\theta-M}{2a^*}}\Big)\cdot\sqrt{\frac{N-\theta}{a^*}}\int_{\mathbb{R}^2} B(\tau^{-1}x)Q^2(x)dx \\ &= \tau^{-p}\Big(\sqrt{\frac{\theta+M}{2a^*}} + \sqrt{\frac{\theta-M}{2a^*}}\Big)\cdot\sqrt{\frac{N-\theta}{a^*}}\int_{\mathbb{R}^2} B(x)Q^2(x)dx. \end{aligned}$$

Taking $\theta = \frac{M^2+N^2}{2N}$, then $A := \Big(\sqrt{\frac{\theta+M}{2a^*}} + \sqrt{\frac{\theta-M}{2a^*}}\Big)\cdot\sqrt{\frac{N-\theta}{a^*}} = \frac{\sqrt{2(N^2-M^2)}}{2a^*}$ and thus

$$m(N) \le \frac{N\tau^{-2}}{2a^*}\int_{\mathbb{R}^2}|x|^2Q^2(x)dx + \frac{N}{2N^*}(N^*-N)\tau^2 - A\tau^{-p}\int_{\mathbb{R}^2}B(x)Q^2(x)dx. \tag{3.31}$$

For $0 < p < 2$, we deduce

$$m(N) = \frac{N}{2N^*}(N^*-N)\tau^2 - A\tau^{-p}\int_{\mathbb{R}^2}B(x)Q^2(x)dx + o(1),$$

where $o(1)\to 0$ as $\tau\to\infty$. Taking

$$\tau = \Big(\frac{-pN^*A\int_{\mathbb{R}^2}B(x)Q^2(x)dx}{N(N^*-N)}\Big)^{\frac{1}{p+2}},$$

we get

$$m(N) \le \Big(\frac{p+2}{2}\Big)\Big(\frac{N(N^*-N)}{pN^*}\Big)^{\frac{p}{p+2}} \cdot \Big(-A\int_{\mathbb{R}^2} B(x)Q^2(x)dx\Big)^{\frac{2}{p+2}}, \tag{3.32}$$

that is, $\lim_{N\nearrow N^*} m(N) \le 0$. For $p=2$, denote

$$S := \frac{N}{2a^*}\int_{\mathbb{R}^2}|x|^2Q^2(x)dx - A\int_{\mathbb{R}^2} B(x)Q^2(x)dx,$$

we have $m(N) \le \frac{N}{2N^*}(N^*-N)\tau^2 + S\tau^{-2}$. Taking $\tau = \Big(\frac{2SN^*}{N(N^*-N)}\Big)^{\frac{1}{4}}$, then

$$m(N) \le \Big(\frac{2SN(N^*-N)}{N^*}\Big)^{\frac{1}{2}}. \tag{3.33}$$

For $p>2$, there holds

$$m(N) \le \frac{N\tau^{-2}}{2a^*}\int_{\mathbb{R}^2}|x|^2Q^2(x)dx + \frac{N}{2N^*}(N^*-N)\tau^2 + o(1),$$

where $o(1)\to 0$ as $\tau\to\infty$. Then

$$m(N) \le \Big(\frac{N^2(N^*-N)\int_{\mathbb{R}^2}|x|^2Q^2(x)dx}{a^*N^*}\Big)^{\frac{1}{2}}. \tag{3.34}$$

On the one hand, from (3.32)-(3.34), we obtain

$$\lim_{n\to\infty}\frac{m(N_n)}{(N^*-N_n)^{\frac{p}{p+2}}} \le \Big(\frac{p+2}{2}\Big)\Big(\frac{1}{p}\Big)^{\frac{p}{p+2}} \cdot \Big(-A\int_{\mathbb{R}^2} B(x)Q^2(x)dx\Big)^{\frac{2}{p+2}}, \quad \text{for } 0<p<2, \tag{3.35}$$

$$\lim_{n\to\infty}\frac{m(N_n)}{(N^*-N_n)^{\frac{1}{2}}} \le (2S)^{\frac{1}{2}}, \quad \text{for } p=2 \tag{3.36}$$

and

$$\lim_{n\to\infty}\frac{m(N_n)}{(N^*-N_n)^{\frac{1}{2}}} \le \Big(\frac{N^*\int_{\mathbb{R}^2}|x|^2Q^2(x)dx}{a^*}\Big)^{\frac{1}{2}}, \quad \text{for } p>2. \tag{3.37}$$

On the other hand, let $\boldsymbol{w_n} := (w_{1n}, w_{2n}, w_{0n})$ be the function defined in (3.25), then by (1.9) and some direct calculations, we get

$$\begin{aligned}
&Re\int_{\mathbb{R}^2}|B(x)|(\overline{u}_{1n}u_{0n} + \overline{u}_{0n}u_{2n})dx \\
&= \frac{1}{\varepsilon_n^2}Re\int_{\mathbb{R}^2}|B(x)|\Big(\overline{w}_{1n}\big(\frac{x-\varepsilon_n y_n}{\varepsilon_n}\big)w_{0n}\big(\frac{x-\varepsilon_n y_n}{\varepsilon_n}\big) + \overline{w}_{0n}\big(\frac{x-\varepsilon_n y_n}{\varepsilon_n}\big)w_{2n}\big(\frac{x-\varepsilon_n y_n}{\varepsilon_n}\big)\Big)dx \\
&= Re\int_{\mathbb{R}^2}|B(\varepsilon_n x + \varepsilon_n y_n)|(\overline{w}_{1n}w_{0n} + \overline{w}_{0n}w_{2n})dx \\
&= \varepsilon_n^p Re\int_{\mathbb{R}^2}\Big|B\Big(x + x_1 + y_n\Big)\Big|\big(\overline{w}_{1n}(x+x_1)w_{0n}(x+x_1) + \overline{w}_{0n}(x+x_1)w_{2n}(x+x_1)\big).
\end{aligned} \tag{3.38}$$

We now claim that $\lim_{n\to\infty}|y_n| \le C$ for arbitrary positive $C$. Otherwise, suppose $\lim_{n\to\infty}\big|y_n + x_1\big| = +\infty$. Since $B(x)$ satisfies (1.9), we get for arbitrary $C_1>0$, there holds

$$Re\int_{\mathbb{R}^2}|B(x)|(\overline{u}_{1n}u_{0n} + \overline{u}_{0n}u_{2n})dx \ge C_1\varepsilon_n^p, \quad \text{as } n\to\infty.$$

Moreover, from (3.28), for arbitrary $C_2 > 0$, there holds $\int_{\mathbb{R}^2} |x|^2 |\boldsymbol{u_n}|^2 dx \geq C_2 \varepsilon_n^2$, as $n \to \infty$. Then we get

$$\begin{aligned}
I(\boldsymbol{u_n}) \geq & \Big(\frac{a^*}{4N_n} - \frac{c_0+c_1}{4}\Big) \int_{\mathbb{R}^2} (|u_{1n}|^2 + |u_{2n}|^2 + |u_{0n}|^2)^2 dx + \frac{1}{2}\int_{\mathbb{R}^2} |x|^2 |\boldsymbol{u_n}|^2 dx \\
& + Re \int_{\mathbb{R}^2} |B(x)| (\overline{u}_{1n} u_{0n} + \overline{u}_{0n} u_{2n}) dx \\
\geq & \frac{N^*}{2N_n} (N^* - N_n) \varepsilon_n^{-2} + \frac{C_2}{2} \varepsilon_n^2 + C_1 \varepsilon_n^p + o(1),
\end{aligned}$$

where $o(1) \to 0$ as $n \to \infty$. Taking the infimum with respect to $\varepsilon_n > 0$, then we conclude for $0 < p < 2$,

$$\lim_{n\to\infty} \frac{m(N_n)}{(N^* - N_n)^{\frac{p}{p+2}}} \geq L_1,$$

and for $p \geq 2$,

$$\lim_{n\to\infty} \frac{m(N_n)}{(N^* - N_n)^{\frac{1}{2}}} \geq L_2,$$

for arbitrary large $L_1,\ L_2 > 0$. However, it contradicts to (3.35), (3.36) or (3.37). Thus, there exists $x_2 \in \mathbb{R}^2$, such that

$$\lim_{n\to\infty} \Big(y_n + x_1\Big) = x_2,$$

which yields $\lim\limits_{n\to\infty} |y_n| \leq C$. Therefore, we conclude from (3.38) that

$$\begin{aligned}
& \liminf_{n\to\infty} \varepsilon_n^{-p} Re \int_{\mathbb{R}^2} |B(x)| (\overline{u}_{1n} u_{0n} + \overline{u}_{0n} u_{2n}) dx \\
\geq & Re \int_{\mathbb{R}^2} \liminf_{n\to\infty} \bigg( \Big|B\Big(x + x_1 + y_n\Big)\Big| \Big(\overline{w}_{1n}(x+x_1) w_{0n}(x+x_1) + \overline{w}_{0n}(x+x_1) w_{2n}(x+x_1)\Big)\bigg) dx \\
= & \frac{N^*}{2a^*} \sin(2\varphi_1)(\cos\varphi_2 + \sin\varphi_2) \int_{\mathbb{R}^2} \big|B(x+x_2)\big| Q^2(x) dx.
\end{aligned}$$

Denote

$$T := \frac{N^*}{2a^*} \sin(2\varphi_1)(\cos\varphi_2 + \sin\varphi_2) \int_{\mathbb{R}^2} \big|B(x+x_2)\big| Q^2(x) dx, \tag{3.39}$$

we get for $n$ large enough,

$$\begin{aligned}
I(\boldsymbol{u_n}) \geq & \Big(\frac{a^*}{4N_n} - \frac{c_0+c_1}{4}\Big) \int_{\mathbb{R}^2} (|u_{1n}|^2 + |u_{2n}|^2 + |u_{0n}|^2)^2 dx + \frac{1}{2}\int_{\mathbb{R}^2} |x|^2 |\boldsymbol{u_n}|^2 dx \\
& + Re \int_{\mathbb{R}^2} |B(x)| (\overline{u}_{1n} u_{0n} + \overline{u}_{0n} u_{2n}) dx \\
\geq & \frac{N^*}{2N_n} (N^* - N_n) \varepsilon_n^{-2} + \frac{N^* \varepsilon_n^2}{2a^*} \int_{\mathbb{R}^2} |x|^2 Q^2(x) dx + T \varepsilon_n^p + o(1).
\end{aligned} \tag{3.40}$$

Taking

$$\varepsilon_n = \begin{cases}
\Big(\frac{N^*(N^*-N_n)}{N_n p T}\Big)^{\frac{1}{p+2}}, & \text{if } 0 < p < 2, \\
\Big(\frac{(c_0+c_1) N^*(N^*-N_n)}{N_n \big(\int_{\mathbb{R}^2} |x|^2 Q^2(x) dx + 2T(c_0+c_1)\big)}\Big)^{\frac{1}{4}}, & \text{if } p = 2, \\
\Big(\frac{(c_0+c_1) N^*(N^*-N_n)}{N_n \int_{\mathbb{R}^2} |x|^2 Q^2(x) dx}\Big)^{\frac{1}{4}}, & \text{if } p > 2,
\end{cases}$$

then we have for $0 < p < 2$,

$$\lim_{n\to\infty} \frac{m(N_n)}{(N^* - N_n)^{\frac{p}{p+2}}} \geq \Big(\frac{p+2}{2}\Big) \cdot \Big(\frac{1}{p}\Big)^{\frac{p}{p+2}} T^{\frac{2}{p+2}},$$

for $p = 2$,

$$\lim_{n\to\infty} \frac{m(N_n)}{(N^* - N_n)^{\frac{1}{2}}} \geq \Big(\frac{N^* \int_{\mathbb{R}^2} |x|^2 Q^2(x)dx}{a^*} + 2T\Big)^{\frac{1}{2}},$$

and for $p > 2$,

$$\lim_{n\to\infty} \frac{m(N_n)}{(N^* - N_n)^{\frac{1}{2}}} \geq \Big(\frac{N^* \int_{\mathbb{R}^2} |x|^2 Q^2(x)dx}{a^*}\Big)^{\frac{1}{2}}.$$

Together with (3.35)-(3.37), we obtain the conclusion.

Now, we are ready to prove the limit behavior of $\{\boldsymbol{u_n}\}$ as $n \to \infty$. Without loss of generality, we may assume $B(x) \geq 0$, the proof for the case $B(x) < 0$ can be obtained through slight modifications. By (3.18), (3.25) and the fact that $\lim\limits_{N\nearrow N^*} m(N) = 0$, we get

$$\lim_{n\to\infty} \frac{1}{\varepsilon_n^2} \int_{\mathbb{R}^2} (|w_{0n}|^2 - 2|w_{1n}||w_{2n}|)^2 dx = \lim_{n\to\infty} \int_{\mathbb{R}^2} (|u_{0n}|^2 - 2|u_{1n}||u_{2n}|)^2 dx = 0.$$

Since $w_{in} \to w_i$ $(i = 1, 2, 0)$ is strongly in $H^1(\mathbb{R}^2)$ and $\varepsilon_n \to 0$ as $n \to \infty$, we deduce

$$\int_{\mathbb{R}^2} (|w_0|^2 - 2|w_1||w_2|)^2 dx = 0.$$

By some direct calculations, we get

$$\begin{cases} \lim\limits_{n\to\infty} w_{1n}(x) = \dfrac{N^* + M}{2\sqrt{a^* N^*}} Q(x - x_1), \\ \lim\limits_{n\to\infty} w_{2n}(x) = \dfrac{N^* - M}{2\sqrt{a^* N^*}} Q(x - x_1), \\ \lim\limits_{n\to\infty} w_{0n}(x) = \sqrt{\dfrac{(N^*)^2 - M^2}{2a^* N^*}} Q(x - x_1). \end{cases} \tag{3.41}$$

Noting that $\boldsymbol{u_n}$ satisfies the Euler-Lagrange system (3.17), then

$$A(\boldsymbol{u_n}) + \int_{\mathbb{R}^2} |x|^2 |\boldsymbol{u_n}|^2 dx = \mu_n N_n + \lambda_n M + E(\boldsymbol{u_n}) + 2F(\boldsymbol{u_n}),$$

which implies that

$$\begin{aligned} &\mu_n N_n + \lambda_n M = A(\boldsymbol{u_n}) + \int_{\mathbb{R}^2} |x|^2 |\boldsymbol{u_n}|^2 dx - E(\boldsymbol{u_n}) - 2F(\boldsymbol{u_n}) \\ &= A(\boldsymbol{u_n}) + \int_{\mathbb{R}^2} |x|^2 |\boldsymbol{u_n}|^2 dx - 2\Big(A(\boldsymbol{u_n}) + \int_{\mathbb{R}^2} |x|^2 |\boldsymbol{u_n}|^2 dx - 2F(\boldsymbol{u_n}) - 2I(\boldsymbol{u_n})\Big) - 2F(\boldsymbol{u_n}) \\ &= 4I(\boldsymbol{u_n}) - A(\boldsymbol{u_n}) - \int_{\mathbb{R}^2} |x|^2 |\boldsymbol{u_n}|^2 dx + 2F(\boldsymbol{u_n}). \end{aligned}$$

Hence, by (3.6), (3.19) and (3.21), we get

$$\lim_{n\to\infty} \varepsilon_n^2\big(\mu_n N_n + \lambda_n M\big) = -N^*. \tag{3.42}$$

By (3.17) and (3.25), $\boldsymbol{w_n}$ satisfies the following system

$$\begin{cases} -\Delta w_{1n} + \varepsilon_n^2|\varepsilon_n x + \varepsilon_n y_n|^2 w_{1n} = (\mu_n + \lambda_n)\varepsilon_n^2 w_{1n} + (c_0+c_1)|w_{1n}|^2 w_{1n} + (c_0 - c_1) w_{1n}|w_{2n}|^2 \\ \qquad\qquad + (c_0+c_1)|w_{0n}|^2 w_{1n} + c_1 \overline{w}_{2n} w_{0n}^2 + \varepsilon_n^2 B(\varepsilon_n x + \varepsilon_n y_n) w_{0n}, \\ -\Delta w_{2n} + \varepsilon_n^2|\varepsilon_n x + \varepsilon_n y_n|^2 w_{2n} = (\mu_n - \lambda_n)\varepsilon_n^2 w_{2n} + (c_0+c_1)|w_{2n}|^2 w_{2n} + (c_0 - c_1)|w_{1n}|^2 w_{2n} \\ \qquad\qquad + (c_0+c_1)|w_{0n}|^2 w_{2n} + c_1 \overline{w}_{1n} w_{0n}^2 + \varepsilon_n^2 B(\varepsilon_n x + \varepsilon_n y_n) w_{0n}, \\ -\Delta w_{0n} + \varepsilon_n^2|\varepsilon_n x + \varepsilon_n y_n|^2 w_{0n} = \mu_n \varepsilon_n^2 w_{0n} + c_0|w_{0n}|^2 w_{0n} + (c_0+c_1)(|w_{1n}|^2 + |w_{2n}|^2) w_{0n} \\ \qquad\qquad + 2c_1 w_{1n} w_{2n} \overline{w}_{0n} + \varepsilon_n^2 B(\varepsilon_n x + \varepsilon_n y_n)(w_{1n} + w_{2n}). \end{cases} \tag{3.43}$$

Multiplying the first equation and the second equation in (3.43) by $w_{2n}$ and $w_{1n}$ respectively and then integrating by parts, we get

$$\begin{aligned} 0 = {}& 2\lambda_n \varepsilon_n^2 \int_{\mathbb{R}^2} w_{1n} w_{2n} dx + c_1 \int_{\mathbb{R}^2} \Big(2|w_{1n}|^2 w_{1n} w_{2n} - 2|w_{2n}|^2 w_{2n} w_{1n} + |w_{2n}|^2 w_{0n}^2 - |w_{1n}|^2 w_{0n}^2\Big) dx \\ & + \varepsilon_n^2 \int_{\mathbb{R}^2} B(\varepsilon_n x + \varepsilon_n y_n) w_{0n}(w_{2n} - w_{1n}) dx. \end{aligned} \tag{3.44}$$

From (3.41), we obtain

$$\lim_{n\to\infty} \int_{\mathbb{R}^2} w_{1n} w_{2n} dx > 0, \quad \lim_{n\to\infty} \int_{\mathbb{R}^2} \Big(2|w_{1n}|^2 w_{1n} w_{2n} - 2|w_{2n}|^2 w_{2n} w_{1n} + |w_{2n}|^2 w_{0n}^2 - |w_{1n}|^2 w_{0n}^2\Big) dx = 0. \tag{3.45}$$

In addition, we drive from (1.9) that

$$\begin{aligned} &\left|\int_{\mathbb{R}^2} B(\varepsilon_n x + \varepsilon_n y_n) w_{0n}(w_{2n} - w_{1n}) dx\right| \\ &\le \varepsilon_n^p \int_{\mathbb{R}^2} B\Big(x + x_1 + y_n\Big)\big(|w_{1n}(x+x_1)|^2 + |w_{2n}(x+x_1)|^2 + |w_{0n}(x+x_1)|^2\big) dx, \end{aligned}$$

then

$$\begin{aligned} &\limsup_{n\to\infty} \varepsilon_n^{-p} \left|\int_{\mathbb{R}^2} B(\varepsilon_n x + \varepsilon_n y_n) w_{0n}(w_{2n} - w_{1n}) dx\right| \\ \le {}& \limsup_{n\to\infty} \int_{\mathbb{R}^2} B\Big(x + x_1 + y_n\Big)\big(|w_{1n}(x+x_1)|^2 + |w_{2n}(x+x_1)|^2 + |w_{0n}(x+x_1)|^2\big) dx \\ \le {}& \int_{\mathbb{R}^2} \limsup_{n\to\infty} \bigg(B\Big(x + x_1 + y_n\Big)\big(|w_{1n}(x+x_1)|^2 + |w_{2n}(x+x_1)|^2 + |w_{0n}(x+x_1)|^2\big)\bigg) dx \\ = {}& \frac{N^*}{a^*} \int_{\mathbb{R}^2} B(x + x_2) Q^2(x) dx. \end{aligned}$$

It follows that

$$\limsup_{n\to\infty} \varepsilon_n^2 \left|\int_{\mathbb{R}^2} B(\varepsilon_n x + \varepsilon_n y_n) w_{0n}(w_{2n} - w_{1n}) dx\right| = 0. \tag{3.46}$$

Together with (3.42), (3.44), (3.45), we obtain $\lim\limits_{n\to\infty}\lambda_n\varepsilon_n^2=0$ and $\lim\limits_{n\to\infty}\mu_n\varepsilon_n^2=-1$. From (3.43), for sufficiently large $n$, there holds $-\Delta w_{in}\le b_{in}(x)w_{in}+f_{in}(x)$ for $i=1,2,0$, where

$$
\begin{aligned}
b_{1n}(x)&=(c_0+c_1)\big(|w_{1n}|^2+|w_{0n}|^2\big)+(c_0-c_1)|w_{2n}|^2;\\
b_{2n}(x)&=(c_0+c_1)\big(|w_{2n}|^2+|w_{0n}|^2\big)+(c_0-c_1)|w_{1n}|^2;\\
b_{0n}(x)&=c_0|w_{0n}|^2+(c_0+c_1)\big(|w_{1n}|^2+|w_{2n}|^2\big)+2c_1|w_{1n}||w_{2n}|;\\
f_{1n}(x)&=c_1|w_{2n}||w_{0n}|^2+\varepsilon_n^2B(\varepsilon_nx+\varepsilon_ny_n)|w_{0n}|;\\
f_{2n}(x)&=c_1|w_{1n}||w_{0n}|^2+\varepsilon_n^2B(\varepsilon_nx+\varepsilon_ny_n)|w_{0n}|;\\
f_{0n}(x)&=\varepsilon_n^2B(\varepsilon_nx+\varepsilon_ny_n)(|w_{1n}|+|w_{2n}|).
\end{aligned}
$$

Hence, by (3.46), we get $b_{in}(x),\ f_{in}(x)\in L^t(\mathbb{R}^2)$ for $t>2$. Using the De Giorgi-Nash-Moser theory (see for example Theorem 4.1 in [24] or Theorem 8.15 in [22]), we obtain for any $\xi\in\mathbb{R}^2$,

$$\sup_{B_1(\xi)} w_{in}\le C\big(\|w_{in}\|_{L^t(B_2(\xi))}+\|f_{in}\|_{L^t(B_2(\xi))}\big),$$

where $C>0$ is a constant. It follows that

$$w_{in}(x)\to 0,\quad \text{as } |x|\to\infty \text{ uniformly on } n, \tag{3.47}$$

and $\{w_{in}\}$ is bounded uniformly in $L^\infty(\mathbb{R}^2)$ for $i=1,2,0$. Then there exists at least one global maximum point for $w_{in}$ and $u_{in}$.

Let $\tilde z_{in}$ be a global maximum point of $u_{in}$ for $i=1,2,0$ and we define

$$\tilde w_{in}(x):=\varepsilon_nu_{in}(\varepsilon_nx+\tilde z_{in})=w_{in}\Big(x+\frac{\tilde z_{in}-\varepsilon_ny_n}{\varepsilon_n}\Big), \tag{3.48}$$

then $x=\frac{\tilde z_{in}-\varepsilon_ny_n}{\varepsilon_n}$ is the global maximum point of $w_{in}$. Moreover, $\big|\frac{\tilde z_{in}-\varepsilon_ny_n}{\varepsilon_n}\big|\le C$, as $n\to\infty$. Thus,

$$\lim_{n\to\infty}\Big|\frac{\tilde z_{in}-\tilde z_{jn}}{\varepsilon_n}\Big|\le C,\quad i,j=1,2,0. \tag{3.49}$$

From (3.41),

$$\tilde w_{in}(x)\to X_iQ(x+y_i-x_1),\quad \text{strongly in } H^1(\mathbb{R}^2), \tag{3.50}$$

where

$$X_1=\frac{N^*+M}{2\sqrt{a^*N^*}},\quad X_2=\frac{N^*-M}{2\sqrt{a^*N^*}},\quad X_0=\sqrt{\frac{(N^*)^2-M^2}{2a^*N^*}}$$

and $y_i=\lim\limits_{n\to\infty}\frac{\tilde z_{in}-\varepsilon_ny_n}{\varepsilon_n}$, $i=1,2,0$. Moreover, since $B(x)\in C^\alpha_{loc}(\mathbb{R}^2)$, we conclude by a standard elliptic regularity theory that

$$\tilde w_{in}(x)\to X_iQ(x+y_i-x_1),\quad \text{in } C^{2,\alpha}_{loc}(\mathbb{R}^2). \tag{3.51}$$

Suppose $x_{in}$ is an arbitrary local maximum point of $\tilde{w}_{in}$, then by (3.43) and (3.48), we obtain

$$\begin{cases} (c_0+c_1)|\tilde{w}_{1n}(x_{1n})|^2+(c_0-c_1)\Big|\tilde{w}_{2n}\Big(x_{1n}+\dfrac{\tilde{z}_{1n}-\tilde{z}_{2n}}{\varepsilon_n}\Big)\Big|^2+(c_0+c_1)\Big|\tilde{w}_{0n}\Big(x_{1n}+\dfrac{\tilde{z}_{1n}-\tilde{z}_{0n}}{\varepsilon_n}\Big)\Big|^2 \\ +\dfrac{c_1}{\tilde{w}_{1n}(x_{1n})}\overline{\tilde{w}}_{2n}\Big(x_{1n}+\dfrac{\tilde{z}_{1n}-\tilde{z}_{2n}}{\varepsilon_n}\Big)\tilde{w}_{0n}^2\Big(x_{1n}+\dfrac{\tilde{z}_{1n}-\tilde{z}_{0n}}{\varepsilon_n}\Big) \\ +\dfrac{\varepsilon_n^2}{\tilde{w}_{1n}(x_{1n})}B(\varepsilon_n x_{1n}+\tilde{z}_{1n})\tilde{w}_{0n}\Big(x_{1n}+\dfrac{\tilde{z}_{1n}-\tilde{z}_{0n}}{\varepsilon_n}\Big)\geq\dfrac{1}{2}, \\ (c_0+c_1)|\tilde{w}_{2n}(x_{2n})|^2+(c_0-c_1)\Big|\tilde{w}_{1n}\Big(x_{2n}+\dfrac{\tilde{z}_{2n}-\tilde{z}_{1n}}{\varepsilon_n}\Big)\Big)\Big|^2+(c_0+c_1)\Big|\tilde{w}_{0n}\Big(x_{2n}+\dfrac{\tilde{z}_{2n}-\tilde{z}_{0n}}{\varepsilon_n}\Big)\Big|^2 \\ +\dfrac{c_1}{\tilde{w}_{2n}(x_{2n})}\overline{\tilde{w}}_{1n}\Big(x_{2n}+\dfrac{\tilde{z}_{2n}-\tilde{z}_{1n}}{\varepsilon_n}\Big)\tilde{w}_{0n}^2\Big(x_{2n}+\dfrac{\tilde{z}_{2n}-\tilde{z}_{0n}}{\varepsilon_n}\Big) \\ +\dfrac{\varepsilon_n^2}{\tilde{w}_{2n}(x_{2n})}B(\varepsilon_n x_{2n}+\tilde{z}_{2n})\tilde{w}_{0n}\Big(x_{2n}+\dfrac{\tilde{z}_{2n}-\tilde{z}_{0n}}{\varepsilon_n}\Big)\geq\dfrac{1}{2}, \\ c_0|\tilde{w}_{0n}(x_{0n})|^2+(c_0+c_1)\bigg(\Big|\tilde{w}_{1n}\Big(x_{0n}+\dfrac{\tilde{z}_{0n}-\tilde{z}_{1n}}{\varepsilon_n}\Big)\Big|^2+\Big|\tilde{w}_{2n}\Big(x_{0n}+\dfrac{\tilde{z}_{0n}-\tilde{z}_{2n}}{\varepsilon_n}\Big)\Big|^2\bigg) \\ +2c_1\tilde{w}_{1n}\Big(x_{0n}+\dfrac{\tilde{z}_{0n}-\tilde{z}_{1n}}{\varepsilon_n}\Big)\tilde{w}_{2n}\Big(x_{0n}+\dfrac{\tilde{z}_{0n}-\tilde{z}_{2n}}{\varepsilon_n}\Big)\dfrac{\overline{\tilde{w}}_{0n}(x_{0n})}{\tilde{w}_{0n}(x_{0n})} \\ +\dfrac{\varepsilon_n^2}{\tilde{w}_{0n}(x_{0n})}B(\varepsilon_n x_{0n}+\tilde{z}_{0n})\bigg(\tilde{w}_{1n}\Big(x_{0n}+\dfrac{\tilde{z}_{0n}-\tilde{z}_{1n}}{\varepsilon_n}\Big)+\tilde{w}_{2n}\Big(x_{0n}+\dfrac{\tilde{z}_{0n}-\tilde{z}_{2n}}{\varepsilon_n}\Big)\bigg)\geq\dfrac{1}{2}. \end{cases}$$

From (3.26), (3.48) and (3.51), we deduce that $\lim\limits_{n\to\infty}\tilde{w}_{in}(x_{in})>0$. Further, by some similar arguments in [20], $\tilde{w}_{in}$ decays exponentially. It follows from (1.9) and (3.47)-(3.49) that $\{x_{in}\}$ is bounded uniformly as $n\to\infty$. Similar to Theorem 1.2 in [38], we obtain the uniqueness of maximum point for $\tilde{w}_{in}$ and the uniqueness of maximum point for $u_{in}$. Noting that the origin is the unique maximum point of $\tilde{w}_{in}$ and it is also the unique maximum point of $Q(x)$, then $y_i=x_1$ for $i=1,2,0$. Moreover, for $i,j=1,2,0, i\neq j$,

$$\lim_{n\to\infty}\Big|\frac{\tilde{z}_{in}-\tilde{z}_{jn}}{\varepsilon_n}\Big|=0,\quad \lim_{n\to\infty}\tilde{z}_{in}=\lim_{n\to\infty}\varepsilon_n y_n=0.$$

Therefore, by (3.48), (3.50) and the fact $a^*=(c_0+c_1)N^*$, we complete the proof. □

### 3.3 Proof of Theorem 3

In this subsection, we consider the limit behavior of the minimizer $\boldsymbol{u_n}=(u_{1n},u_{2n},u_{0n})$ for $m(N_n)$ as $n\to\infty$ when $c_1<0$, $N_n\nearrow N^*$ and $M=0$.

***Proof of Theorem 3.*** On the one hand, if $B(x)\geq 0$, taking $M=0$ and $\tau=\big(\frac{N_n}{2N^*}(N^*-N_n)\big)^{-\frac{1}{4}}$ in (3.15), we obtain

$$m(N_n)\leq I(\Phi)\leq\Big(\frac{N_n}{2N^*}(N^*-N_n)\Big)^{\frac{1}{2}}\cdot\Big(1+\frac{N_n}{2a^*}\int_{\mathbb{R}^2}|x|^2Q^2(x)dx\Big)\to 0,$$

as $N_n\nearrow N^*$. If $B(x)<0$, we can get the similar result. On the other hand, for any $\boldsymbol{u}\in\mathcal{M}(N_n)$, we have

$$I(\boldsymbol{u})\geq\frac{1}{2N^*}(N^*-N_n)A(\boldsymbol{u})+\Big(\frac{1}{2}-\Big\|\frac{B(x)}{x^2}\Big\|_{L^\infty}\Big)\int_{\mathbb{R}^2}|x|^2|\boldsymbol{u}|^2dx\geq 0.$$

It follows that for $c_1 < 0$ and $M = 0$, $\lim_{N_n \nearrow N^*} m(N_n) = 0$. Then similar to (3.19) and (3.20), we get

$$\lim_{n\to\infty} \Big( \frac{1}{2} \int_{\mathbb{R}^2} |x|^2 |\boldsymbol{u_n}|^2 dx - F(\boldsymbol{u_n}) \Big) = 0$$

and

$$\lim_{n\to\infty} \frac{A(\boldsymbol{u_n})}{\int_{\mathbb{R}^2} (|u_{1n}|^2 + |u_{2n}|^2 + |u_{0n}|^2)^2 dx} = \frac{a^*}{2N^*}.$$

Moreover,

$$\lim_{n\to\infty} A(\boldsymbol{u_n}) = \lim_{n\to\infty} \int_{\mathbb{R}^2} \big( |\nabla u_{1n}|^2 + |\nabla u_{2n}|^2 + |\nabla u_{0n}|^2 \big) dx = +\infty.$$

Let $\varepsilon_n$, $w_{in}$ and $\tilde{w}_{in}$ $(i = 1, 2, 0)$ be defined as in Theorem 2, then since $M = 0$, we can deduce for some $x_1 \in \mathbb{R}^2$, there holds

$$\lim_{n\to\infty} w_{1n}(x) = \lim_{n\to\infty} w_{2n}(x) = \sqrt{\frac{N^*}{2a^*}} Q(x - x_1) \sin\varphi_1$$

and

$$\lim_{n\to\infty} w_{0n}(x) = \sqrt{\frac{N^*}{a^*}} Q(x - x_1) \cos\varphi_1, \text{ for } \varphi_1 \in [0, \frac{\pi}{2}].$$

By the fact that $\lim_{N \nearrow N^*} m(N) = 0$ and the definition of $w_{in}$, we have

$$\begin{aligned} &\lim_{n\to\infty} \frac{1}{\varepsilon_n^2} \int_{\mathbb{R}^2} \big( (|w_{1n}|^2 - |w_{2n}|^2)^2 + 2(|w_{1n}| + |w_{2n}|)^2 |w_{0n}|^2 \big) dx \\ = &\lim_{n\to\infty} \int_{\mathbb{R}^2} \big( (|u_{1n}|^2 - |u_{2n}|^2)^2 + 2(|u_{1n}| + |u_{2n}|)^2 |u_{0n}|^2 \big) dx = 0, \end{aligned}$$

then it is easy to see that

$$\int_{\mathbb{R}^2} \big( (|w_1|^2 - |w_2|^2)^2 + 2(|w_1| + |w_2|)^2 |w_0|^2 \big) dx = 0,$$

where $w_i(x) = \lim_{n\to\infty} w_{in}(x)$ $(i = 1, 2, 0)$. Hence, there are two cases to be discussed:

$$\lim_{n\to\infty} w_{1n}(x) = \lim_{n\to\infty} w_{2n}(x) = \sqrt{\frac{N^*}{2a^*}} Q(x - x_1), \quad \lim_{n\to\infty} w_{0n} = 0, \quad \text{strongly in } H^1(\mathbb{R}^2), \tag{3.52}$$

or

$$\lim_{n\to\infty} w_{1n} = \lim_{n\to\infty} w_{2n} = 0, \quad \lim_{n\to\infty} w_{0n}(x) = \sqrt{\frac{N^*}{a^*}} Q(x - x_1), \quad \text{strongly in } H^1(\mathbb{R}^2). \tag{3.53}$$

If (3.52) holds, then similar to the proof of Theorem 2, we get $\lim_{n\to\infty} \lambda_n \varepsilon_n^2 = 0$ and $\lim_{n\to\infty} \mu_n \varepsilon_n^2 = -1$. Further, let $\tilde{z}_{in}$ $(i = 1, 2)$ be the maximum point of $u_{in}$, then $\lim_{n\to\infty} \frac{|\tilde{z}_{1n} - \tilde{z}_{2n}|}{\varepsilon_n} = 0$ and

$$\lim_{n\to\infty} \tilde{w}_{1n}(x) = \lim_{n\to\infty} \tilde{w}_{2n}(x) = \sqrt{\frac{N^*}{2a^*}} Q(x), \quad \lim_{n\to\infty} \tilde{w}_{0n} = 0, \quad \text{strongly in } H^1(\mathbb{R}^2).$$

In addition, we conclude $\tilde{w}_{1n}$ and $\tilde{w}_{2n}$ decay exponentially and thus for $i = 1, 2$,

$$\varepsilon_n u_{in}(\varepsilon_n x + \tilde{z}_{in}) = \tilde{w}_{in}(x) \to \sqrt{\frac{N^*}{2a^*}} Q(x), \text{ uniformly in } \mathbb{R}^2.$$

We next argue by contradiction to show that if (3.52) holds, then $u_{0n} \equiv 0$ in $\mathbb{R}^2$. Otherwise, suppose $u_{0n} \not\equiv 0$ in $\mathbb{R}^2$, we denote

$$\hat{u}_{0n}(x) := \frac{1}{Q_\infty \sigma_n} u_{0n}(\varepsilon_n x + \tilde{z}_{0n}),$$

where $\sigma_n = \|u_{0n}\|_{L^\infty} > 0$ and $Q_\infty = \frac{1}{\|Q(x)\|_{L^\infty}} > 0$. By (3.17) and (3.48), we obtain

$$\begin{aligned}
&-\Delta \hat{u}_{0n} + \varepsilon_n^2 |\varepsilon_n x + \tilde{z}_{0n}|^2 \hat{u}_{0n} = \mu_n \varepsilon_n^2 \hat{u}_{0n} + c_0 \varepsilon_n^2 (Q_\infty \sigma_n)^2 |\hat{u}_{0n}|^2 \hat{u}_{0n} \\
&+ (c_0 + c_1)\Big(\Big|\tilde{w}_{1n}\Big(x + \frac{\tilde{z}_{0n} - \tilde{z}_{1n}}{\varepsilon_n}\Big)\Big|^2 + \Big|\tilde{w}_{2n}\Big(x + \frac{\tilde{z}_{0n} - \tilde{z}_{2n}}{\varepsilon_n}\Big)\Big|^2\Big)\hat{u}_{0n} \\
&+ 2c_1 \tilde{w}_{1n}\Big(x + \frac{\tilde{z}_{0n} - \tilde{z}_{1n}}{\varepsilon_n}\Big)\tilde{w}_{2n}\Big(x + \frac{\tilde{z}_{0n} - \tilde{z}_{2n}}{\varepsilon_n}\Big)\overline{\hat{u}}_{0n} \\
&+ \frac{\varepsilon_n}{Q_\infty \sigma_n} B(\varepsilon_n x + \tilde{z}_{0n})\Big(\tilde{w}_{1n}\Big(x + \frac{\tilde{z}_{0n} - \tilde{z}_{1n}}{\varepsilon_n}\Big) + \tilde{w}_{2n}\Big(x + \frac{\tilde{z}_{0n} - \tilde{z}_{2n}}{\varepsilon_n}\Big)\Big).
\end{aligned}$$

Moreover, we deduce

$$\lim_{n\to\infty} \frac{|\tilde{z}_{0n} - \tilde{z}_{in}|}{\varepsilon_n} \le C, \quad i = 1, 2,$$

and further

$$\lim_{n\to\infty} \frac{\tilde{z}_{0n} - \tilde{z}_{1n}}{\varepsilon_n} = \lim_{n\to\infty} \frac{\tilde{z}_{0n} - \tilde{z}_{2n}}{\varepsilon_n} = x_1.$$

It then yields that the limit $\hat{u}_0(x) = \lim_{n\to\infty} \hat{u}_{0n}(x)$ satisfies

$$-\Delta \hat{u}_0 + \hat{u}_0 = \Big(1 + \frac{2N^* c_1}{a^*}\Big) Q^2(x) \hat{u}_0.$$

However, since $c_1 < 0$, the above equation contradicts to the fact (see [43]) that for any $u \in H^1(\mathbb{R}^2)$, there holds

$$\int_{\mathbb{R}^2} (|\nabla u|^2 + |u|^2) dx \ge \int_{\mathbb{R}^2} Q^2(x) |u|^2 dx. \tag{3.54}$$

Hence, we have proved that $u_{0n} \equiv 0$ in $\mathbb{R}^2$.

In the following, we consider the second case, that is if (3.53) holds, then by $M = 0$ and (3.42),

$$\lim_{n\to\infty} \mu_n \varepsilon_n^2 = -1, \qquad \lim_{n\to\infty} \lambda_n \varepsilon_n^2 \text{ is bounded.} \tag{3.55}$$

It yields that the limit $w_0(x) = \lim_{n\to\infty} w_{0n}(x)$ satisfies

$$-\Delta w_0 + w_0 = c_0 w_0^3 = \frac{N^*}{a^*} w_0^3.$$

Then we obtain

$$\lim_{n\to\infty} \tilde{w}_{1n} = \lim_{n\to\infty} \tilde{w}_{2n} = 0, \quad \lim_{n\to\infty} \tilde{w}_{0n}(x) = \sqrt{\frac{N^*}{a^*}} Q(x), \quad \text{strongly in } H^1(\mathbb{R}^2).$$

Moreover, $\tilde{w}_{0n}$ decays exponentially and thus $\tilde{w}_{0n}(x) \to \sqrt{\frac{N^*}{a^*}} Q(x)$ uniformly in $\mathbb{R}^2$.

Similarly, we argue by contradiction to show that if (3.53) holds, then $u_{1n} = u_{2n} \equiv 0$ in $\mathbb{R}^2$. Otherwise, suppose $u_{1n} \not\equiv 0$, $u_{2n} \not\equiv 0$ in $\mathbb{R}^2$ and define

$$\hat{u}_{in}(x) := \frac{1}{Q_\infty \sigma_{in}} u_{in}(\varepsilon_n x + \tilde{z}_{in}), \quad i = 1, 2,$$

where $\sigma_{in} = \|u_{in}\|_{L^\infty} > 0$. By (3.17) and (3.48), we get $(\hat{u}_{1n}, \hat{u}_{2n}, \tilde{w}_{0n})$ satisfies

$$\begin{cases} -\Delta \hat{u}_{1n} + \varepsilon_n^2 |\varepsilon_n x + \tilde{z}_{1n}|^2 \hat{u}_{1n} = (\mu_n + \lambda_n)\varepsilon_n^2 \hat{u}_{1n} + (c_0 + c_1)\varepsilon_n^2 (Q_\infty \sigma_{1n})^2 |\hat{u}_{1n}|^2 \hat{u}_{1n} \\ + (c_0 - c_1)\varepsilon_n^2 (Q_\infty \sigma_{2n})^2 \hat{u}_{1n} \Big| \hat{u}_{2n}(x + \frac{\tilde{z}_{1n} - \tilde{z}_{2n}}{\varepsilon_n}) \Big|^2 + (c_0 + c_1) \Big| \tilde{w}_{0n}(x + \frac{\tilde{z}_{1n} - \tilde{z}_{0n}}{\varepsilon_n}) \Big|^2 \hat{u}_{1n} \\ + c_1 \frac{\sigma_{2n}}{\sigma_{1n}} \overline{\hat{u}}_{2n}(x + \frac{\tilde{z}_{1n} - \tilde{z}_{2n}}{\varepsilon_n}) \tilde{w}_{0n}^2 (x + \frac{\tilde{z}_{1n} - \tilde{z}_{0n}}{\varepsilon_n}) + \frac{\varepsilon_n}{Q_\infty \sigma_{1n}} B(\varepsilon_n x + \tilde{z}_{1n}) \tilde{w}_{0n}(x + \frac{\tilde{z}_{1n} - \tilde{z}_{0n}}{\varepsilon_n}), \\ -\Delta \hat{u}_{2n} + \varepsilon_n^2 |\varepsilon_n x + \tilde{z}_{2n}|^2 \hat{u}_{2n} = (\mu_n - \lambda_n)\varepsilon_n^2 \hat{u}_{2n} + (c_0 + c_1)\varepsilon_n^2 (Q_\infty \sigma_{2n})^2 |\hat{u}_{2n}|^2 \hat{u}_{2n} \\ + (c_0 - c_1)\varepsilon_n^2 (Q_\infty \sigma_{1n})^2 \hat{u}_{2n} \Big| \hat{u}_{1n}(x + \frac{\tilde{z}_{2n} - \tilde{z}_{1n}}{\varepsilon_n}) \Big|^2 + (c_0 + c_1) \Big| \tilde{w}_{0n}(x + \frac{\tilde{z}_{2n} - \tilde{z}_{0n}}{\varepsilon_n}) \Big|^2 \hat{u}_{2n} \\ + c_1 \frac{\sigma_{1n}}{\sigma_{2n}} \overline{\hat{u}}_{1n}(x + \frac{\tilde{z}_{2n} - \tilde{z}_{1n}}{\varepsilon_n}) \tilde{w}_{0n}^2 (x + \frac{\tilde{z}_{2n} - \tilde{z}_{0n}}{\varepsilon_n}) + \frac{\varepsilon_n}{Q_\infty \sigma_{2n}} B(\varepsilon_n x + \tilde{z}_{2n}) \tilde{w}_{0n}(x + \frac{\tilde{z}_{2n} - \tilde{z}_{0n}}{\varepsilon_n}), \\ -\Delta \tilde{w}_{0n} + \varepsilon_n^2 |\varepsilon_n x + \tilde{z}_{0n}|^2 \tilde{w}_{0n} = \mu_n \varepsilon_n^2 \tilde{w}_{0n} + c_0 |\tilde{w}_{0n}|^2 \tilde{w}_{0n} \\ + (c_0 + c_1) Q_\infty^2 \varepsilon_n^2 \Big( \sigma_{1n}^2 \Big| \hat{u}_{1n}(x + \frac{\tilde{z}_{0n} - \tilde{z}_{1n}}{\varepsilon_n}) \Big|^2 + \sigma_{2n}^2 \Big| \hat{u}_{2n}(x + \frac{\tilde{z}_{0n} - \tilde{z}_{2n}}{\varepsilon_n}) \Big) \Big|^2 \tilde{w}_{0n} \\ + 2c_1 Q_\infty^2 \varepsilon_n^2 \sigma_{1n} \sigma_{2n} \hat{u}_{1n}(x + \frac{\tilde{z}_{0n} - \tilde{z}_{1n}}{\varepsilon_n}) \hat{u}_{2n}(x + \frac{\tilde{z}_{0n} - \tilde{z}_{2n}}{\varepsilon_n}) \overline{\tilde{w}}_{0n} \\ + \varepsilon_n^3 Q_\infty B(\varepsilon_n x + \tilde{z}_{0n}) \Big( \sigma_{1n} \hat{u}_{1n}(x + \frac{\tilde{z}_{0n} - \tilde{z}_{1n}}{\varepsilon_n}) + \sigma_{2n} \hat{u}_{2n}(x + \frac{\tilde{z}_{0n} - \tilde{z}_{2n}}{\varepsilon_n}) \Big), \end{cases} \tag{3.56}$$

where the Lagrange multipliers $\mu_n$ and $\lambda_n$ satisfy (3.55). Moreover, there exists nonnegative function $\hat{u}_i \in H^1(\mathbb{R}^2)$, such that $\hat{u}_{in} \to \hat{u}_i$ in $C^{2,\alpha}_{loc}(\mathbb{R}^2)$ and $\hat{u}_i(0) = \|Q\|_{L^\infty}$ for $i = 1, 2$.

We now claim that $\lambda_0 := \lim\limits_{n\to\infty} \lambda_n \varepsilon_n^2 = 0$. Indeed, suppose $\lim\limits_{n\to\infty} \frac{|\tilde{z}_{in} - \tilde{z}_{0n}|}{\varepsilon_n} = +\infty$ for $i = 1, 2$, since $c_1 < 0$, we obtain from (3.56) that

$$-\Delta \hat{u}_1 + (1 - \lambda_0)\hat{u}_1 \le 0 \quad \text{and} \quad -\Delta \hat{u}_2 + (1 + \lambda_0)\hat{u}_2 \le 0.$$

If $\lambda_0 \le 0$, then $\hat{u}_1 = 0$, which contradicts to the fact that $\hat{u}_1(0) = \|Q\|_{L^\infty} > 0$. Similarly, if $\lambda_0 \ge 0$, then $\hat{u}_2 = 0$, there is also a contradiction. Hence, without loss of generality, we may assume that $\lim\limits_{n\to\infty} \frac{|\tilde{z}_{1n} - \tilde{z}_{0n}|}{\varepsilon_n} = +\infty$ and $\lim\limits_{n\to\infty} \frac{|\tilde{z}_{2n} - \tilde{z}_{0n}|}{\varepsilon_n} \le C$ for some $C > 0$, then up to a subsequence, there exists a $y_2 \in \mathbb{R}^2$, such that $\lim\limits_{n\to\infty} \frac{\tilde{z}_{2n} - \tilde{z}_{0n}}{\varepsilon_n} = y_2$ and in this case, $\hat{u}_1,\ \hat{u}_2$ satisfies

$$-\Delta \hat{u}_1 + (1 - \lambda_0)\hat{u}_1 \le 0 \quad \text{and} \quad -\Delta \hat{u}_2 + (1 + \lambda_0)\hat{u}_2 \le Q^2(x + y_2)\hat{u}_2.$$

If $\lambda_0 \le 0$, then $\hat{u}_1 = 0$, a contradiction. If $\lambda_0 > 0$, then by (1.5), we get

$$\int_{\mathbb{R}^2} \hat{u}_2 Q(x + y_2) dx \le 0.$$

Since $\hat{u}_2 \in C^{2,\alpha}_{loc}(\mathbb{R}^2)$, then there exists a constant $R > 0$, such that $\hat{u}_2 > 0$ and $Q(x + y_2) > 0$ in $B_R$, which implies

$$\int_{\mathbb{R}^2} \hat{u}_2 Q(x + y_2) dx > \int_{B_R} \hat{u}_2 Q(x + y_2) dx > 0,$$

a contradiction as well. Suppose

$$\lim_{n\to\infty}\frac{|\tilde z_{1n}-\tilde z_{0n}|}{\varepsilon_n}\le C \text{ and } \lim_{n\to\infty}\frac{|\tilde z_{2n}-\tilde z_{0n}|}{\varepsilon_n}\le C \text{ for some } C>0,$$

then up to a subsequence, we may assume that there exist $y_1,\ y_2$, such that

$$\lim_{n\to\infty}\frac{\tilde z_{1n}-\tilde z_{0n}}{\varepsilon_n}=y_1 \text{ and } \lim_{n\to\infty}\frac{\tilde z_{2n}-\tilde z_{0n}}{\varepsilon_n}=y_2.$$

Under this circumstance, $\hat u_1$ and $\hat u_2$ satisfy

$$-\Delta\hat u_1+(1-\lambda_0)\hat u_1\le Q^2(x+y_1)\hat u_1 \quad\text{and}\quad -\Delta\hat u_2+(1+\lambda_0)\hat u_2\le Q^2(x+y_2)\hat u_2.$$

Similar to the above case, we can deduce a contradiction either $\lambda_0>0$ or $\lambda_0<0$. Hence,

$$\lambda_0=0 \text{ and } \lim_{n\to\infty}\frac{|\tilde z_{in}-\tilde z_{0n}|}{\varepsilon_n}\le C \text{ for } i=1,2.$$

Since $\left\{\frac{\tilde z_{in}-\tilde z_{0n}}{\varepsilon_n}\right\}$ is bounded uniformly in $\mathbb{R}^2$ for $i=1,2$, there exist $y_1,\ y_2\in\mathbb{R}^2$, such that up to a subsequence,

$$\lim_{n\to\infty}\frac{\tilde z_{1n}-\tilde z_{0n}}{\varepsilon_n}=y_1,\quad \lim_{n\to\infty}\frac{\tilde z_{2n}-\tilde z_{0n}}{\varepsilon_n}=y_2.$$

Moreover, $\hat u_i$ $(i=1,2)$ satisfies

$$-\Delta\hat u_i+\hat u_i\le(c_0+c_1)\cdot\frac{N^*}{a^*}Q^2(x+y_i)\hat u_i=\Big(1+\frac{c_1N^*}{a^*}\Big)Q^2(x+y_i)\hat u_i,$$

which means that

$$\int_{\mathbb{R}^2}(|\nabla\hat u_i|^2+|\hat u_i|^2)dx<\int_{\mathbb{R}^2}Q^2(x+y_2)|\hat u_i|^2dx.$$

It is a contradiction to (3.54). Therefore, $u_{1n}=u_{2n}\equiv0$ in $\mathbb{R}^2$ and we complete the proof. □

# 4 The 3D case

In this section, we are going to prove Theorems 4-6, where the 3D case of (1.1)-(1.2) is considered. Define the Pohozaev manifold of (1.1)-(1.2) as

$$\mathcal{P}:=\Big\{\boldsymbol{u}=(u_1,u_2,u_0)\in\mathcal{M}\Big|\ P(\boldsymbol{u})=0\Big\},$$

with

$$P(\boldsymbol{u})=A(\boldsymbol{u})-\int_{\mathbb{R}^3}|x|^2|\boldsymbol{u}|^2dx-\frac34E(\boldsymbol{u})+\int_{\mathbb{R}^3}\langle\nabla B(x),x\rangle(\overline{u}_1u_0+\overline{u}_0u_2)dx.$$

**Lemma 4.1.** *Suppose $\boldsymbol{u}=(u_1,u_2,u_0)\in\Lambda$ is a solution of* (1.1)-(1.2)*, then $P(\boldsymbol{u})=0$.*

*Proof.* Since $\boldsymbol{u}$ is a solution of (1.1)-(1.2), $\boldsymbol{u}$ satisfies the Pohozaev identity

$$\begin{aligned}&A(\boldsymbol{u})+5\int_{\mathbb{R}^3}|x|^2|\boldsymbol{u}|^2dx-\frac32E(\boldsymbol{u})-6F(\boldsymbol{u})-2\int_{\mathbb{R}^3}\langle\nabla B(x),x\rangle(\overline{u}_1u_0+\overline{u}_0u_2)dx\\&=3\Big((\mu+\lambda)\int_{\mathbb{R}^3}|u_1|^2dx+(\mu-\lambda)\int_{\mathbb{R}^3}|u_2|^2dx+\mu\int_{\mathbb{R}^3}|u_0|^2dx\Big).\end{aligned}$$

Multiplying the three equations in (1.1) by $\overline{u}_1,\ \overline{u}_2,\ \overline{u}_0$ and integrating by parts respectively, we then obtain

$$A(\boldsymbol{u})+\int_{\mathbb{R}^3}|x|^2|\boldsymbol{u}|^2dx-E(\boldsymbol{u})-2F(\boldsymbol{u})=(\mu+\lambda)\int_{\mathbb{R}^3}|u_1|^2dx+(\mu-\lambda)\int_{\mathbb{R}^3}|u_2|^2dx+\mu\int_{\mathbb{R}^3}|u_0|^2dx,$$

which follows that $P(\boldsymbol{u})=0$. Therefore, we have proved the lemma. □

In Theorems 4 and 6, we shall show the multiplicity of solutions to (1.1)-(1.2) with a local minimizer and a mountain pass solution. For simplicity, we just denote $\mathcal{M}(N)$ as $\mathcal{M}$. First of all, we prove a local minima structure for $I(\boldsymbol{u})$ on $\mathcal{M}$. Define

$$\|\boldsymbol{u}\|_{\Lambda}^2:=A(\boldsymbol{u})+\int_{\mathbb{R}^3}|x|^2|\boldsymbol{u}|^2dx,$$

then by Lemma 2.4, there holds

$$\|\boldsymbol{u}\|_{\Lambda}^2\geq 3\int_{\mathbb{R}^3}|\boldsymbol{u}|^2dx,\quad \text{for } \forall\boldsymbol{u}\in\Lambda.$$

For any $r>0$, let

$$\mathcal{B}(r):=\Big\{\boldsymbol{u}=(u_1,u_2,u_0)\in\Lambda\Big|\|\boldsymbol{u}\|_{\Lambda}^2\leq r\Big\}.$$

**Lemma 4.2.** *Suppose $c_0>0$ and $c_0+c_1>0$, then for any $r>0$, there holds*

$$\mathcal{M}\cap\mathcal{B}(r)\neq\varnothing,\quad \text{if } N\leq\frac{r}{3}. \tag{4.1}$$

*In addition, if we further assume $B(x)$ satisfies one of (1.7) and (1.16), then $I(\boldsymbol{u})$ is bounded from below on $\mathcal{M}\cap\mathcal{B}(r)$.*

*Proof.* For any $r>0$, by Lemma 2.4, it is easy to see that $\Big(\sqrt{\frac{N+M}{2}}\Psi_0,\sqrt{\frac{N-M}{2}}\Psi_0,0\Big)\in\mathcal{M}$. Moreover, if $N\leq\frac{r}{3}$,

$$\Big\|\Big(\sqrt{\frac{N+M}{2}}\Psi_0,\sqrt{\frac{N-M}{2}}\Psi_0,0\Big)\Big\|_{\Lambda}^2=N\int_{\mathbb{R}^3}\Big(|\nabla\Psi_0|^2+|x|^2|\Psi_0|^2\Big)dx=N\|(\Psi_0,0,0)\|_{\Lambda}^2=3N\leq r.$$

Hence, $\Big(\sqrt{\frac{N+M}{2}}\Psi_0,\sqrt{\frac{N-M}{2}}\Psi_0,0\Big)\in\mathcal{M}\cap\mathcal{B}(r)$. For any $\boldsymbol{u}\in\mathcal{M}\cap\mathcal{B}(r)$, by (2.2), we get if $B(x)\in L^\infty$,

$$\begin{aligned}I(\boldsymbol{u})&=\frac{1}{2}A(\boldsymbol{u})+\frac{1}{2}\int_{\mathbb{R}^3}|x|^2|\boldsymbol{u}|^2dx-\frac{1}{4}E(\boldsymbol{u})-F(\boldsymbol{u})\\&\geq-\frac{1}{4}\max\Big\{c_0,3c_0+4c_1\Big\}C_*\Big(A(\boldsymbol{u})\Big)^{\frac{3}{2}}N^{\frac{1}{2}}-\|B(x)\|_{L^\infty}N\\&\geq-\frac{1}{4}\max\Big\{c_0,3c_0+4c_1\Big\}C_*r^{\frac{3}{2}}N^{\frac{1}{2}}-\|B(x)\|_{L^\infty}N.\end{aligned}$$

If $\frac{B(x)}{x}\in L^\infty$, taking $\varepsilon=\frac{1}{32\|\frac{B(x)}{x}\|_{L^\infty}}$ in (3.2), then we obtain

$$|F(\boldsymbol{u})|\leq\frac{1}{16}\int_{\mathbb{R}^3}|x|^2|\boldsymbol{u}|^2dx+8\Big\|\frac{B(x)}{x}\Big\|_{L^\infty}^2N, \tag{4.2}$$

which follows that

$$I(\boldsymbol{u}) \geq -\frac{1}{4}\max\Big\{c_0, 3c_0+4c_1\Big\}C_* r^{\frac{3}{2}}N^{\frac{1}{2}} - 8\Big\|\frac{B(x)}{x}\Big\|^2_{L^\infty}N.$$

If $B(x)$ satisfies (1.16), then

$$|F(\boldsymbol{u})| \leq \Big\|\frac{B(x)}{x^2}\Big\|_{L^\infty}\int_{\mathbb{R}^3}|x|^2|\boldsymbol{u}|^2dx,$$

further

$$I(\boldsymbol{u}) \geq -\frac{1}{4}\max\Big\{c_0, 3c_0+4c_1\Big\}C_* r^{\frac{3}{2}}N^{\frac{1}{2}}.$$

Therefore, $I(\boldsymbol{u})$ is bounded from below on $\mathcal{M}\cap\mathcal{B}(r)$ if $B(x)$ satisfies one of (1.7) and (1.16). $\square$

For any $r>0$ and $N\leq \frac{r}{3}$, we consider the following local minimization problem:

$$m(r,N) := \inf_{\boldsymbol{u}\in\mathcal{M}\cap\mathcal{B}(r)} I(\boldsymbol{u}).$$

By Lemma 4.2, $m(r,N)$ is well defined.

**Lemma 4.3.** *Suppose $c_0>0$, $c_0+c_1>0$ and $B(x)\in L^\infty$, then for any $r>0$, there exists $N_0=N_0(r)$, such that*

$$m(r,N) = \inf_{\boldsymbol{u}\in\mathcal{M}\cap\mathcal{B}(\frac{r}{2})} I(\boldsymbol{u}), \quad \textit{for } N\in\Big(0, \min\big\{N_0, \frac{r}{12}, \frac{r}{16\|B(x)\|_{L^\infty}}\big\}\Big). \tag{4.3}$$

*Proof.* For any $r>0$, if $\mathcal{M}\cap\big(\mathcal{B}(r)\setminus\mathcal{B}(\frac{r}{2})\big)=\varnothing$, then (4.3) holds. If $\mathcal{M}\cap\big(\mathcal{B}(r)\setminus\mathcal{B}(\frac{r}{2})\big)\neq\varnothing$, then for any $\boldsymbol{u}\in\mathcal{M}\cap\big(\mathcal{B}(r)\setminus\mathcal{B}(\frac{r}{2})\big)$, we have

$$\begin{aligned} I(\boldsymbol{u}) &= \frac{1}{2}A(\boldsymbol{u}) + \frac{1}{2}\int_{\mathbb{R}^3}|x|^2|\boldsymbol{u}|^2dx - \frac{1}{4}E(\boldsymbol{u}) - F(\boldsymbol{u}) \\ &\geq \frac{1}{2}\|\boldsymbol{u}\|^2_{\Lambda} - \frac{1}{4}\max\Big\{c_0, 3c_0+4c_1\Big\}C_*\|(u_1,u_2,u_0)\|^3_{\Lambda}N^{\frac{1}{2}} - \|B(x)\|_{L^\infty}N \\ &\geq \frac{r}{4} - \frac{1}{4}\max\Big\{c_0, 3c_0+4c_1\Big\}C_* r^{\frac{3}{2}}N^{\frac{1}{2}} - \|B(x)\|_{L^\infty}N. \end{aligned}$$

For $N>0$, we define a function $g(N)$ as

$$g(N) := \frac{r}{4} - \frac{1}{4}\max\Big\{c_0, 3c_0+4c_1\Big\}C_* r^{\frac{3}{2}}N^{\frac{1}{2}} - \|B(x)\|_{L^\infty}N,$$

then it is easy to see that there exists a constant $N_0=N_0(r)$, such that $g(N)\geq \frac{3}{16}r$ for $0<N\leq N_0$. It follows that

$$\inf_{\boldsymbol{u}\in\mathcal{M}\cap\big(\mathcal{B}(r)\setminus\mathcal{B}(\frac{r}{2})\big)} I(\boldsymbol{u}) \geq \frac{3}{16}r, \quad \text{for } N\in(0,N_0].$$

For any $r>0$, by (4.1),

$$\mathcal{M}\cap\mathcal{B}\Big(\frac{r}{4}\Big)\neq\varnothing, \quad \text{if } N\leq\frac{r}{12}.$$

Since $N<\frac{r}{16\|B(x)\|_{L^\infty}}$, for any $\boldsymbol{u}\in\mathcal{M}\cap\mathcal{B}(\frac{r}{4})$, we have

$$I(\boldsymbol{u}) \leq \frac{1}{2}\|\boldsymbol{u}\|^2_{\Lambda} + \|B(x)\|_{L^\infty}N < \frac{3r}{16} \leq \inf_{\boldsymbol{u}\in\mathcal{M}\cap\big(\mathcal{B}(r)\setminus\mathcal{B}(\frac{r}{2})\big)} I(\boldsymbol{u}).$$

Hence, if

$$0 < N < \min\Big\{N_0, \frac{r}{12}, \frac{r}{16\|B(x)\|_{L^\infty}}\Big\},$$

we conclude

$$m(r,N) \leq \inf_{\boldsymbol{u}\in\mathcal{M}\cap\mathcal{B}(\frac{r}{4})} I(\boldsymbol{u}) < \inf_{\boldsymbol{u}\in\mathcal{M}\cap\big(\mathcal{B}(r)\setminus\mathcal{B}(\frac{r}{2})\big)} I(\boldsymbol{u}).$$

Therefore, we complete the proof. □

**Remark 4.1.** *Indeed, the constant $N_0$ in Lemma 4.3 is defined by*

$$N_0 = \bigg(\frac{\sqrt{C_0^2 r^3 + 4\|B(x)\|_{L^\infty} r} - C_0 r^{\frac{3}{2}}}{8\|B(x)\|_{L^\infty}}\bigg)^2,$$

*where $C_0 := \max\{c_0, 3c_0 + 4c_1\}C_*$ and $C_* = \frac{4\sqrt{3}}{9a^*}$ is the optimal constant of the Gagliardo-Nirenberg type inequality* (2.1) *for $d = 3$. Further, suppose $c_0 > 0$, $c_0 + c_1 > 0$, then for any $r > 0$, there also exists $N_1 = N_1(r)$, such that*

$$m(r,N) = \inf_{\boldsymbol{u}\in\mathcal{M}\cap\mathcal{B}(\frac{r}{2})} I(\boldsymbol{u}), \quad \textit{for } N \in (0, N_1).$$

*Here, if $\frac{B(x)}{x} \in L^\infty$, then*

$$N_1 = \min\bigg\{N_0, \frac{r}{12}, \frac{3r}{512\|\frac{B(x)}{x}\|^2_{L^\infty}}\bigg\} \quad \textit{and} \quad N_0 = N_0(r) = \bigg(\frac{\sqrt{C_0^2 r^3 + 16\|\frac{B(x)}{x}\|^2_{L^\infty} r} - C_0 r^{\frac{3}{2}}}{64\|\frac{B(x)}{x}\|^2_{L^\infty}}\bigg)^2.$$

*If $B(x)$ satisfies* (1.16), *then*

$$N_1 = \min\big\{N_0, \frac{r}{12}\big\} \quad \textit{and} \quad N_0 = N_0(r) = \Big(\frac{1 - 8\|\frac{B(x)}{x^2}\|_{L^\infty}}{4C_0 r^{\frac{1}{2}}}\Big)^2.$$

*The proofs are similar to Lemma 4.3, we omit the details here.*

**Lemma 4.4.** *Suppose $c_0 > 0$, $c_0 + c_1 > 0$ and $B(x)$ satisfies one of* (1.7) *and* (1.16), *then for any $r > 0$, there exists $N^{**} = N^{**}(r)$, such that*

$$\inf_{\boldsymbol{u}\in\mathcal{M}\cap\mathcal{B}(\frac{r}{4})} I(\boldsymbol{u}) < \inf_{\boldsymbol{u}\in\mathcal{M}\cap\big(\mathcal{B}(r)\setminus\mathcal{B}(\frac{r}{2})\big)} I(\boldsymbol{u}), \quad \textit{for } N \in (0, N^{**}). \tag{4.4}$$

*Proof.* From the proof of Lemma 4.3, it is sufficient to show that $\mathcal{M}\cap\big(\mathcal{B}(r)\setminus\mathcal{B}(\frac{r}{2})\big) \neq \varnothing$ for small $N$. First, we assume $B(x) \in L^\infty$. For any $\tau > 0$ and $u \in H^1(\mathbb{R}^3)$, we define

$$\tau \star u = e^{\frac{3}{2}\tau} u(e^\tau x), \tag{4.5}$$

then

$$\boldsymbol{U} = (U_1, U_2, U_0) := \tau \star \bigg(\sqrt{\frac{N+M}{2}}\Psi_0, \sqrt{\frac{N-M}{2}}\Psi_0, 0\bigg) \in \mathcal{M}\cap\mathcal{B}(r),$$

and by direct calculations, we get from (2.3) that

$$\|\boldsymbol{U}\|^2_\Lambda = e^{2\tau} N \int_{\mathbb{R}^3} |\nabla\Psi_0|^2 dx + e^{-2\tau} N \int_{\mathbb{R}^3} |x|^2|\Psi_0|^2 dx = \frac{3N}{2}\big(e^{2\tau} + e^{-2\tau}\big) \geq 3N.$$

Hence for any $r > 0$, if we choose $N \leq \frac{r}{4}$, then there exists $\tau > 0$, such that $\|\boldsymbol{U}\|_\Lambda^2 = \frac{3}{4}r$, that is $\boldsymbol{U} \in \mathcal{M} \cap \big(\mathcal{B}(r) \setminus \mathcal{B}(\frac{r}{2})\big)$. Let

$$N^{**} := \min\Big\{N_0, \frac{r}{12}, \frac{r}{4}, \frac{r}{16\|B(x)\|_{L^\infty}}\Big\} = \min\Big\{N_0, \frac{r}{12}, \frac{r}{16\|B(x)\|_{L^\infty}}\Big\},$$

we get (4.4). If $\frac{B(x)}{x} \in L^\infty$ or $B(x)$ satisfies (1.16), we set

$$N^{**} := \min\Big\{N_0, \frac{r}{12}, \frac{r}{4}, \frac{3r}{512\|\frac{B(x)}{x}\|_{L^\infty}^2}\Big\} = \min\Big\{N_0, \frac{r}{12}, \frac{3r}{512\|\frac{B(x)}{x}\|_{L^\infty}^2}\Big\}$$

or

$$N^{**} := \min\Big\{N_0, \frac{r}{12}, \frac{r}{4}\Big\} = \min\Big\{N_0, \frac{r}{12}\Big\},$$

then we obtain the conclusion. □

**Lemma 4.5.** *Suppose $c_0 > 0$, $c_0 + c_1 > 0$ and $B(x)$ satisfies one of* (1.7) *and* (1.16)*, then for any $r > 0$, there holds*

$$m(r, N) < \frac{3N}{2}, \qquad \textit{for } N \in (0, N^{**}).$$

*Proof.* From the proof of Lemma 4.2, we get $\Big(\sqrt{\frac{N+M}{2}}\Psi_0, \sqrt{\frac{N-M}{2}}\Psi_0, 0\Big) \in \mathcal{M} \cap \mathcal{B}(r)$. Thus

$$\begin{aligned} m(r, N) &= \inf_{\boldsymbol{u} \in \mathcal{M} \cap B(r)} I(\boldsymbol{u}) \leq I\Big(\sqrt{\frac{N+M}{2}}\Psi_0, \sqrt{\frac{N-M}{2}}\Psi_0, 0\Big) \\ &< \frac{N}{2}\int_{\mathbb{R}^3} \Big(|\nabla\Psi_0|^2 + |x|^2|\Psi_0|^2\Big)dx = \frac{N}{2}\|(\Psi_0, 0, 0)\|_\Lambda^2 = \frac{3N}{2}. \end{aligned}$$

□

## 4.1 Proof of Theorem 4

***Proof of Theorem 4.*** (i) For any $r > 0$ and $0 < N \leq \frac{r}{3}$, let $\{\boldsymbol{u_n}\} := \{(u_{1n}, u_{2n}, u_{0n})\}$ be a minimizing sequence of $m(r, N)$, i.e. $I(\boldsymbol{u_n}) \to m(r, N)$ as $n \to \infty$. Then $\|\boldsymbol{u_n}\|_\Lambda^2 = \|\boldsymbol{u_n}\|_{\dot\Lambda}^2 + \|\boldsymbol{u_n}\|_{L^2}^2 \leq r + N$, which implies that $\{\boldsymbol{u_n}\}$ is bounded in $\Lambda$. Therefore, there exists $\tilde{\boldsymbol{u}} := (\tilde{u}_1, \tilde{u}_2, \tilde{u}_0) \in \Lambda$, such that up to a subsequence, as $n \to \infty$,

$$\begin{cases} \boldsymbol{u_n} \rightharpoonup \tilde{\boldsymbol{u}}, & \text{in } \Lambda. \\ \boldsymbol{u_n} \to \tilde{\boldsymbol{u}}, & \text{in } L^t(\mathbb{R}^3) \times L^t(\mathbb{R}^3) \times L^t(\mathbb{R}^3),\ \forall t \in [2, 2^*). \\ \boldsymbol{u_n} \to \tilde{\boldsymbol{u}}, & a.e. \text{ in } \mathbb{R}^3. \end{cases}$$

Then we get $\tilde{\boldsymbol{u}} \in \mathcal{M} \cap \mathcal{B}(r)$. Further, by the lower semi-continuity of the norm in $\Lambda$, there holds

$$m(r, N) \leq I(\tilde{\boldsymbol{u}}) \leq \lim_{n\to\infty} I(\boldsymbol{u_n}) = m(r, N).$$

It yields that $I(\tilde{\boldsymbol{u}}) = m(r, N)$. Hence, $m(r, N)$ has at least one minimizer for any $r > 0$ and $N \leq \frac{r}{3}$.
(ii) For any $r > 0$ and $0 < N < N^{**}$, by (4.3), we can see that $\tilde{\boldsymbol{u}} \in \mathcal{B}(\frac{r}{2})$, which follows that $\tilde{\boldsymbol{u}}$ stays away from the boundary of $\mathcal{B}(r)$. Thus, $\tilde{\boldsymbol{u}}$ is indeed a critical point of $I(\boldsymbol{u})$ restricted to $\mathcal{M}$ and further $\tilde{\boldsymbol{u}}$ is a weak solution of (1.1)-(1.2).

Next, we argue by contradiction to show that $\tilde{\boldsymbol{u}}$ is a ground state solution of (1.1)-(1.2) when $N$ is sufficiently small. Let $N_n := \min\{\frac{1}{n}, N_0\}$, suppose there exist $r_0 > 0$ and $\{\boldsymbol{v_n}\} = \{(v_{1n}, v_{2n}, v_{0n})\} \subset \mathcal{M}(N_n)$, such that

$$I'|_{\mathcal{M}}(\boldsymbol{v_n}) = 0 \quad \text{and} \quad I(\boldsymbol{v_n}) < m(r_0, N_n). \tag{4.6}$$

Then by Lemma 4.1, we get $P(\boldsymbol{v_n}) = 0$ and further if $B(x) \in L^\infty$,

$$I(\boldsymbol{v_n}) \geq \frac{1}{6}A(\boldsymbol{v_n}) + \frac{5}{6}\int_{\mathbb{R}^3} |x|^2|\boldsymbol{v_n}|^2 dx - \frac{1}{3}\int_{\mathbb{R}^3} \langle \nabla B(x), x\rangle(\overline{v}_{1n}v_{0n} + \overline{v}_{0n}v_{2n})dx - \|B(x)\|_{L^\infty} N_n.$$

If $\frac{B(x)}{x} \in L^\infty$, we can show

$$I(\boldsymbol{v_n}) \geq \frac{1}{6}\|\boldsymbol{v_n}\|_\Lambda^2 - \frac{1}{3}\int_{\mathbb{R}^3} \langle \nabla B(x), x\rangle(\overline{v}_{1n}v_{0n} + \overline{v}_{0n}v_{2n})dx - 8\Big\|\frac{B(x)}{x}\Big\|_{L^\infty}^2 N_n,$$

and if $B(x)$ satisfies (1.16), then

$$I(\boldsymbol{v_n}) \geq \frac{1}{6}A(\boldsymbol{v_n}) + \Big(\frac{5}{6} - \Big\|\frac{B(x)}{x^2}\Big\|_{L^\infty}\Big)\int_{\mathbb{R}^3} |x|^2|\boldsymbol{v_n}|^2 dx - \frac{1}{3}\int_{\mathbb{R}^3} \langle \nabla B(x), x\rangle(\overline{v}_{1n}v_{0n} + \overline{v}_{0n}v_{2n})dx.$$

Thus, by (4.6), Lemma 4.5 and the fact that $N_n \to 0$ as $n \to \infty$, we conclude

$$\|\boldsymbol{v_n}\|_\Lambda^2 = A(\boldsymbol{v_n}) + \int_{\mathbb{R}^3} |x|^2|\boldsymbol{v_n}|^2 dx \to 0, \quad \text{as } n \to \infty,$$

then $\boldsymbol{v_n} \in \mathcal{M}(N_n) \cap \mathcal{B}(r_0)$. We can see that $I(\boldsymbol{v_n}) \geq m(r_0, N_n)$, which is a contradiction. Therefore, $\tilde{\boldsymbol{u}}$ is a ground state of (1.1)-(1.2).
(iii) Suppose $B(x) \in L^\infty$, we shall show that

$$\|\boldsymbol{u} - (k_1\Psi_0, k_2\Psi_0, k_0\Psi_0)\|_\Lambda^2 = O(N), \quad \text{as } N \to 0^+, \tag{4.7}$$

where $k_i = k_{i0} = \int_{\mathbb{R}^3} u_i\Psi_0 dx$ for $i = 1, 2, 0$ and $\Psi_0 = \frac{1}{\pi^{\frac{3}{4}}}e^{-\frac{x^2}{2}}$. Set

$$k_{ij} = \int_{\mathbb{R}^3} u_i\Psi_j dx, \text{ for } i = 1, 2, 0,$$

then

$$\boldsymbol{u} = \Big(\sum_{j=0}^\infty k_{1j}\Psi_j, \sum_{j=0}^\infty k_{2j}\Psi_j, \sum_{j=0}^\infty k_{0j}\Psi_j\Big).$$

Moreover, we conclude

$$N = \|(u_1, u_2, u_0)\|_{L^2}^2 = \sum_{j=0}^\infty (k_{1j}^2 + k_{2j}^2 + k_{0j}^2)\|\Psi_j\|_{L^2}^2 = \sum_{j=0}^\infty (k_{1j}^2 + k_{2j}^2 + k_{0j}^2) \tag{4.8}$$

and

$$\|\boldsymbol{u}\|_\Lambda^2 = \sum_{j=0}^\infty (k_{1j}^2 + k_{2j}^2 + k_{0j}^2)\|\Psi_j\|_\Lambda^2 = \sum_{j=0}^\infty \xi_j(k_{1j}^2 + k_{2j}^2 + k_{0j}^2).$$

By Lemma 4.1, we can see that

$$\begin{aligned}
&m(r,N)=I(\boldsymbol{u})=\frac{1}{2}A(\boldsymbol{u})+\frac{1}{2}\int_{\mathbb{R}^3}|x|^2|\boldsymbol{u}|^2dx-\frac{1}{4}E(\boldsymbol{u})-F(\boldsymbol{u})\\
&=\frac{1}{6}A(\boldsymbol{u})+\frac{5}{6}\int_{\mathbb{R}^3}|x|^2|\boldsymbol{u}|^2dx-\frac{1}{3}\int_{\mathbb{R}^3}\langle\nabla B(x),x\rangle(\overline{u}_1u_0+\overline{u}_0u_2)dx-F(\boldsymbol{u})\\
&\geq\frac{1}{6}\|\boldsymbol{u}\|_{\dot{\Lambda}}^2-\frac{1}{3}\int_{\mathbb{R}^3}\langle\nabla B(x),x\rangle(\overline{u}_1u_0+\overline{u}_0u_2)dx-F(\boldsymbol{u}).
\end{aligned}$$

Then

$$\begin{aligned}
\|\boldsymbol{u}\|_{\dot{\Lambda}}^2&\leq 6m(r,N)+2\int_{\mathbb{R}^3}\langle\nabla B(x),x\rangle(\overline{u}_1u_0+\overline{u}_0u_2)dx+6F(\boldsymbol{u})\\
&\leq\Big(9+2\big\|\langle\nabla B(x),x\rangle\big\|_{L^\infty}+6\|B(x)\|_{L^\infty}\Big)N.
\end{aligned}$$

Together with (2.2), we get

$$\begin{aligned}
&m(r,N)=I(\boldsymbol{u})=\frac{1}{2}\|\boldsymbol{u}\|_{\dot{\Lambda}}^2-\frac{1}{4}E(\boldsymbol{u})-F(\boldsymbol{u})\\
&\geq\frac{1}{2}\|\boldsymbol{u}\|_{\dot{\Lambda}}^2-\frac{1}{4}\max\big\{c_0,3c_0+4c_1\big\}C_*\Big(A(\boldsymbol{u})\Big)^{\frac{3}{2}}N^{\frac{1}{2}}-\|B(x)\|_{L^\infty}N\\
&\geq\frac{1}{2}\|\boldsymbol{u}\|_{\dot{\Lambda}}^2-\frac{1}{4}\max\big\{c_0,3c_0+4c_1\big\}C_*\|\boldsymbol{u}\|_{\dot{\Lambda}}^3N^{\frac{1}{2}}-\|B(x)\|_{L^\infty}N\\
&\geq\frac{1}{2}\sum_{j=0}^{\infty}\xi_j(k_{1j}^2+k_{2j}^2+k_{0j}^2)-\|B(x)\|_{L^\infty}N\\
&\quad-\frac{1}{4}\max\big\{c_0,3c_0+4c_1\big\}C_*\Big(9+2\big\|\langle\nabla B(x),x\rangle\big\|_{L^\infty}+6\|B(x)\|_{L^\infty}\Big)^{\frac{3}{2}}N^2\\
&=\frac{1}{2}\sum_{j=0}^{\infty}(\xi_j-\xi_0)(k_{1j}^2+k_{2j}^2+k_{0j}^2)+\frac{1}{2}\sum_{j=0}^{\infty}\xi_0(k_{1j}^2+k_{2j}^2+k_{0j}^2)-\|B(x)\|_{L^\infty}N\\
&\quad-\frac{1}{4}\max\big\{c_0,3c_0+4c_1\big\}C_*\Big(9+2\big\|\langle\nabla B(x),x\rangle\big\|_{L^\infty}+6\|B(x)\|_{L^\infty}\Big)^{\frac{3}{2}}N^2.
\end{aligned}$$

Then by Lemma 4.5 and (4.8), we have

$$\begin{aligned}
&(\xi_1-\xi_0)\sum_{j=1}^{\infty}(k_{1j}^2+k_{2j}^2+k_{0j}^2)\leq\sum_{j=1}^{\infty}(\xi_j-\xi_0)(k_{1j}^2+k_{2j}^2+k_{0j}^2)\\
&\leq\frac{1}{2}\max\big\{c_0,3c_0+4c_1\big\}C_*\Big(9+2\big\|\langle\nabla B(x),x\rangle\big\|_{L^\infty}+6\|B(x)\|_{L^\infty}\Big)^{\frac{3}{2}}N^2\\
&\quad-\sum_{j=0}^{\infty}\xi_0(k_{1j}^2+k_{2j}^2+k_{0j}^2)+2\|B(x)\|_{L^\infty}N+2m(r,N)\\
&\leq\frac{1}{2}\max\big\{c_0,3c_0+4c_1\big\}C_*\Big(9+2\big\|\langle\nabla B(x),x\rangle\big\|_{L^\infty}+6\|B(x)\|_{L^\infty}\Big)^{\frac{3}{2}}N^2+2\|B(x)\|_{L^\infty}N.
\end{aligned}$$

Denote

$$M_0:=\frac{1}{2}\max\big\{c_0,3c_0+4c_1\big\}C_*\Big(9+2\big\|\langle\nabla B(x),x\rangle\big\|_{L^\infty}+6\|B(x)\|_{L^\infty}\Big)^{\frac{3}{2}},$$

then

$$\sum_{j=1}^{\infty}(k_{1j}^2+k_{2j}^2+k_{0j}^2)\le \frac{M_0N^2+2\|B(x)\|_{L^\infty}N}{\xi_1-\xi_0}.$$

Thus

$$\begin{aligned}\sum_{j=1}^{\infty}\xi_j(k_{1j}^2+k_{2j}^2+k_{0j}^2)&=\sum_{j=1}^{\infty}(\xi_j-\xi_0)(k_{1j}^2+k_{2j}^2+k_{0j}^2)+\xi_0\sum_{j=1}^{\infty}(k_{1j}^2+k_{2j}^2+k_{0j}^2)\\&\le M_0N^2+2\|B(x)\|_{L^\infty}N+\xi_0\cdot\frac{M_0N^2+2\|B(x)\|_{L^\infty}N}{\xi_1-\xi_0}\\&=\frac{\xi_1}{\xi_1-\xi_0}\cdot\Big(M_0N^2+2\|B(x)\|_{L^\infty}N\Big).\end{aligned}$$

For $N\to 0^+$, we can see that

$$\begin{aligned}&\|\boldsymbol{u}-(k_1\Psi_0,k_2\Psi_0,k_0\Psi_0)\|_{\dot{\Lambda}}^2=\left\|\Big(\sum_{j=1}^{\infty}k_{1j}\Psi_j,\sum_{j=1}^{\infty}k_{2j}\Psi_j,\sum_{j=1}^{\infty}k_{0j}\Psi_j\Big)\right\|_{\dot{\Lambda}}^2\\&=\sum_{j=1}^{\infty}\xi_j(k_{1j}^2+k_{2j}^2+k_{0j}^2)=O(N)\end{aligned}$$

and

$$\begin{aligned}&\|\boldsymbol{u}-(k_1\Psi_0,k_2\Psi_0,k_0\Psi_0)\|_{L^2}^2=\left\|\Big(\sum_{j=1}^{\infty}k_{1j}\Psi_j,\sum_{j=1}^{\infty}k_{2j}\Psi_j,\sum_{j=1}^{\infty}k_{0j}\Psi_j\Big)\right\|_{L^2}^2\\&=\sum_{j=1}^{\infty}(k_{1j}^2+k_{2j}^2+k_{0j}^2)=O(N).\end{aligned}$$

Therefore, it is obvious that (4.7) holds. If $\frac{B(x)}{x}\in L^\infty$ or $B(x)$ satisfies (1.16), we can prove (4.7) with small modifications, here we omit the details. Hence, we complete the proof of Theorem 4 (iii). $\square$

Next, we prove Theorem 5, that is the minimizers obtained in Theorems 1 and 4 are radial symmetric if $c_1\ge 0$ and $B(x)\ge 0$.

## 4.2 Proof of Theorem 5

***Proof of Theorem 5.*** By a standard regularity bootstrap argument, we conclude $\boldsymbol{w}$ is of class $C^1$. For $c_1\ge 0$ and $B(x)\ge 0$, it is easy to see that $F(|\boldsymbol{w}|)\ge F(\boldsymbol{w})$ and $I(|\boldsymbol{w}|)\le I(\boldsymbol{w})$. Hence, $|\boldsymbol{w}|$ is also a minimizer of $m(N)$. Applying the maximum principle, $|\boldsymbol{w}|>0$. Since $\boldsymbol{w}$ and $|\boldsymbol{w}|$ are minimizers, we conclude

$$\frac{1}{2}\Big(A(\boldsymbol{w})-A(|\boldsymbol{w}|)\Big)+\frac{1}{4}\Big(E(|\boldsymbol{w}|)-E(\boldsymbol{w})\Big)+\Big(F(|\boldsymbol{w}|)-F(\boldsymbol{w})\Big)=0,$$

which implies that $A(|\boldsymbol{w}|)=A(\boldsymbol{w})$. Using Theorem 5 of appendix B in [6], we get $w_j(x)=e^{i\theta_j}\rho_j(x),\ j=1,2,0$, where $\theta_j\in\mathbb{R}$ is a constant and $\boldsymbol{\rho}=(\rho_1,\rho_2,\rho_0)$ is a real valued minimizer with $\rho_j(x)>0$ for every $x \in \mathbb{R}^d$. Moreover, by direct calculations, we obtain

$$\begin{aligned}
Re \int_{\mathbb{R}^d} \overline{w}_1 \overline{w}_2 w_0^2 dx &= Re \int_{\mathbb{R}^d} \Big( \rho_1\rho_2 \cos(\theta_1+\theta_2) - i\rho_1\rho_2 \sin(\theta_1+\theta_2) + \rho_0^2 \cos(2\theta_0) + i\rho_0^2 \sin(2\theta_0) \Big) dx \\
&= \int_{\mathbb{R}^d} \rho_1\rho_2\rho_0^2 \cos(2\theta_0 - \theta_1 - \theta_2) dx
\end{aligned}$$

and

$$Re \int_{\mathbb{R}^d} B(x)(\overline{w}_1 w_0 + \overline{w}_0 w_2) dx = \int_{\mathbb{R}^d} B(x) \Big( \rho_1\rho_0 \cos(\theta_0 - \theta_1) + \rho_2\rho_0 \cos(\theta_0 - \theta_2) \Big) dx.$$

It yields that

$$\begin{aligned}
I(\boldsymbol{w}) = &\frac{1}{2} A(\boldsymbol{\rho}) + \frac{1}{2} \int_{\mathbb{R}^d} |x|^2 |\boldsymbol{\rho}|^2 dx - \frac{1}{4} \int_{\mathbb{R}^d} \Big( (c_0 + c_1)\big(|\rho_1|^4 + |\rho_2|^4\big) + c_0 |\rho_0|^4 \Big) dx \\
&- \frac{1}{2} \int_{\mathbb{R}^d} \Big( \big(c_0 - c_1\big) |\rho_1|^2 |\rho_2|^2 + \big(c_0 + c_1\big)(|\rho_1|^2 + |\rho_2|^2) |\rho_0|^2 \Big) dx \\
&- c_1 \int_{\mathbb{R}^d} \rho_1\rho_2\rho_0^2 \cos(2\theta_0 - \theta_1 - \theta_2) dx - \int_{\mathbb{R}^d} B(x) \Big( \rho_1\rho_0 \cos(\theta_0 - \theta_1) + \rho_2\rho_0 \cos(\theta_0 - \theta_2) \Big) dx,
\end{aligned}$$

by the fact that $\boldsymbol{w}$ is a minimizer, we then conclude

$$2\theta_0 - \theta_1 - \theta_2 = 2k_1\pi, \ \theta_0 - \theta_1 = 2k_2\pi \ \text{ and } \ \theta_0 - \theta_2 = 2k_3\pi \text{ for } k_j \in \mathbb{Z} \ (j = 1, 2, 3),$$

which implies $\theta_1 + \theta_2 - 2\theta_0 = 2k\pi$ $(k \in \mathbb{Z})$. Denote $\boldsymbol{\rho}^* = (\rho_1^*, \rho_2^*, \rho_0^*)$ as the Schwarz symmetrization of $\boldsymbol{\rho}$, by [34], we have

$$\begin{cases}
A(\boldsymbol{\rho}^*) \le A(\boldsymbol{\rho}), \quad \int_{\mathbb{R}^d} |x|^2 (\boldsymbol{\rho}^*)^2 dx \le \int_{\mathbb{R}^d} |x|^2 \boldsymbol{\rho}^2 dx, \\
\int_{\mathbb{R}^d} (\rho_j^*)^4 dx = \int_{\mathbb{R}^d} \rho_j^4 dx, \quad j = 1, 2, 0, \\
\int_{\mathbb{R}^d} (\rho_j^*)^2 (\rho_k^*)^2 dx \ge \int_{\mathbb{R}^d} \rho_j^2 \rho_k^2 dx, \quad j, k = 1, 2, 0, \\
\int_{\mathbb{R}^d} \rho_1^* \rho_2^* (\rho_0^*)^2 dx \ge \int_{\mathbb{R}^d} \rho_1 \rho_2 \rho_0^2 dx.
\end{cases}$$

Since $B(x) \ge 0$, again by [34], we get

$$\int_{\mathbb{R}^d} B(x)(\rho_1 + \rho_2)\rho_0 dx \le \int_{\mathbb{R}^d} B(x)(\rho_1^* + \rho_2^*)\rho_0^* dx.$$

Then, it is obvious that $I(\boldsymbol{\rho}^*) \le I(\boldsymbol{\rho})$ and as a consequence $\boldsymbol{\rho}^*$ is also a minimizer. Since $\boldsymbol{\rho}$ is a minimizer, it yields that

$$\int_{\mathbb{R}^d} |x|^2 (\boldsymbol{\rho}^*)^2 dx = \int_{\mathbb{R}^d} |x|^2 \boldsymbol{\rho}^2 dx.$$

Applying Theorem 4 in appendix of [6] with $V(x) = |x|^2$, then $\boldsymbol{\rho} = \boldsymbol{\rho}^*$ and hence the minimizer is symmetric. Hence, we complete the proof. □

### 4.3 Proof of Theorem 6

In the end of this section, we shall prove Theorem 6, that is there exists an excited state solution $\hat{\boldsymbol{u}} := (\hat{u}_1, \hat{u}_2, \hat{u}_0) \in \mathcal{M}(N)$ to system (1.1)-(1.2) with some $\hat{\mu}$, $\hat{\lambda}$ as Lagrange multipliers.

For any $r > 0$ and $0 < N \leq N^{**}$, suppose $\tilde{\boldsymbol{u}} = (\tilde{u}_1, \tilde{u}_2, \tilde{u}_0) \in \mathcal{M} \cap \mathcal{B}(r)$ is the solution of (1.1)-(1.2) obtained in Theorem 4 (ii), then $\tilde{\boldsymbol{u}} \in \mathcal{B}(\frac{r}{2})$. For any $\tau > 0$ and $u \in H^1(\mathbb{R}^3)$, let $\tau \star u$ be the operation defined in (4.5). Then for any $\boldsymbol{u} = (u_1, u_2, u_0) \in \mathcal{M}$, when $c_0 > 0$ and $c_0 + c_1 > 0$, we get if $B(x) \in L^\infty$,

$$\begin{aligned} I(\tau \star \boldsymbol{u}) &= \frac{1}{2} e^{2\tau} A(\boldsymbol{u}) + \frac{1}{2} e^{-2\tau} \int_{\mathbb{R}^3} |x|^2 |\boldsymbol{u}|^2 dx - \frac{1}{4} e^{3\tau} E(\boldsymbol{u}) - Re \int_{\mathbb{R}^3} B(e^{-\tau} x)(\overline{u}_1 u_0 + \overline{u}_0 u_2) dx \\ &\leq \frac{1}{2} e^{2\tau} A(\boldsymbol{u}) + \frac{1}{2} e^{-2\tau} \int_{\mathbb{R}^3} |x|^2 |\boldsymbol{u}|^2 dx - \frac{1}{4} e^{3\tau} E(\boldsymbol{u}) + \|B(x)\|_{L^\infty} N. \end{aligned}$$

If $\frac{B(x)}{x} \in L^\infty$, then

$$I(\tau \star \boldsymbol{u}) \leq \frac{1}{2} e^{2\tau} A(\boldsymbol{u}) + \frac{9}{16} e^{-2\tau} \int_{\mathbb{R}^3} |x|^2 |\boldsymbol{u}|^2 dx - \frac{1}{4} e^{3\tau} E(\boldsymbol{u}) + 8 \Big\| \frac{B(x)}{x} \Big\|_{L^\infty}^2 N.$$

If $B(x)$ satisfies (1.16), then

$$I(\tau \star \boldsymbol{u}) \leq \frac{1}{2} e^{2\tau} A(\boldsymbol{u}) + \Big( \frac{1}{2} + \Big\| \frac{B(x)}{x^2} \Big\|_{L^\infty} \Big) e^{-2\tau} \int_{\mathbb{R}^3} |x|^2 |\boldsymbol{u}|^2 dx - \frac{1}{4} e^{3\tau} E(\boldsymbol{u}).$$

It means that for any $\boldsymbol{u} \in \mathcal{M}$, there holds $\lim_{\tau \to +\infty} I\big(\tau \star \boldsymbol{u}\big) = -\infty$ if $B(x)$ satisfies (1.7) or (1.16). Thus there exists a large $\tau_1 > 0$, such that $\big\|(\tau_1 \star \tilde{\boldsymbol{u}})\big\|_\Lambda^2 > r$ and $I(\tau_1 \star \tilde{\boldsymbol{u}}) < 0$. We now define a path as

$$\Gamma := \Big\{ \boldsymbol{g} \in C\big([0,1], \mathcal{M}\big) \big| \boldsymbol{g}(0) = \tilde{\boldsymbol{u}},\ g(1) = \tau_1 \star \tilde{\boldsymbol{u}} \Big\},$$

then for $t \in [0,1]$, it is easy to see that $\boldsymbol{g}(t) := \big((1-t) + t\tau_1\big) \star \tilde{\boldsymbol{u}} \in \Gamma$, that is $\Gamma \neq \varnothing$. Hence, the minimax value

$$\sigma := \inf_{\boldsymbol{g} \in \Gamma} \max_{t \in [0,1]} I\big(\boldsymbol{g}(t)\big)$$

is well defined.

**Proposition 4.1.** *Suppose $c_0 > 0$ and $c_0 + c_1 > 0$, then for any $r > 0$ and $0 < N \leq N^{**}$, there exists a bounded Palais-Smale sequence $\{\boldsymbol{u_n}\} = \{(u_{1n}, u_{2n}, u_{0n})\}$ for $I$ restricted to $\mathcal{M}$ at level $\sigma$. In addition,*

$$P(\boldsymbol{u_n}) = A(\boldsymbol{u_n}) - \int_{\mathbb{R}^3} |x|^2 |\boldsymbol{u_n}|^2 dx - \frac{3}{4} E(\boldsymbol{u_n}) + \int_{\mathbb{R}^3} \langle \nabla B(x), x \rangle (\overline{u}_{1n} u_{0n} + \overline{u}_{0n} u_{2n}) dx = o(1), \quad \text{as } n \to \infty.$$

*Proof.* First, we are going to prove the existence of Palais-Smale sequence $\{\boldsymbol{u_n}\}$ with $P(\boldsymbol{u_n}) = o(1)$, as $n \to \infty$. Now, we define an auxiliary functional as $\tilde{I}\big(\tau, \boldsymbol{u}\big) = I\big(\tau \star \boldsymbol{u}\big)$. Let

$$\tilde{\Gamma} = \Big\{ \tilde{\boldsymbol{h}} \in C\big([0,1], \mathbb{R} \times \mathcal{M}\big) \big|\ \tilde{\boldsymbol{h}}(0) = \big(0, \tilde{\boldsymbol{u}}\big), \tilde{\boldsymbol{h}}(1) = \big(0, \tau_1 \star \tilde{\boldsymbol{u}}\big) \Big\},$$

then it is easy to see that

$$\tilde{\sigma} = \inf_{\tilde{\boldsymbol{h}} \in \tilde{\Gamma}} \max_{t \in [0,1]} \tilde{I}\big(\tilde{\boldsymbol{h}}(t)\big) = \sigma.$$

Taking a sequence $\{\boldsymbol{g_n}\} := \{(g_{1n}, g_{2n}, g_{0n})\} \subset \Gamma$, such that

$$\max_{t\in[0,1]} I\big(\boldsymbol{g_n}(t)\big) \le \sigma + \frac{1}{n}.$$

Let $\tilde{\boldsymbol{g}}_{\boldsymbol{n}} := (0, \boldsymbol{g_n}) \subset \tilde{\Gamma}$, then we get

$$\max_{t\in[0,1]} \tilde{I}\big(\tilde{\boldsymbol{g}}_{\boldsymbol{n}}(t)\big) = \max_{t\in[0,1]} I\big(\boldsymbol{g_n}(t)\big) \le \sigma + \frac{1}{n} = \tilde{\sigma} + \frac{1}{n}.$$

By Lemma 2.3 in [28], there exists a sequence $\big\{\big(\tau_n, \tilde{\boldsymbol{u}}_{\boldsymbol{n}}\big)\big\} := \big\{\big(\tau_n, (\tilde{u}_{1n}, \tilde{u}_{2n}, \tilde{u}_{0n})\big)\big\} \subset \mathbb{R} \times \mathcal{M}$, such that

(1) $\lim\limits_{n\to+\infty} \tilde{I}\big(\tau_n, \tilde{\boldsymbol{u}}_{\boldsymbol{n}}\big) = \tilde{\sigma} = \sigma$;

(2) $\lim\limits_{n\to+\infty} |\tau_n| + dist\big(\tilde{\boldsymbol{u}}_{\boldsymbol{n}}, \boldsymbol{g_n}(t)\big) = 0$;

(3) let $E := \mathbb{R} \times H$ and $E^{-1}$ denote the dual space of $E$, then there holds

$$\Big\| \tilde{I}'|_{\mathbb{R}\times H}\big(\tau_n, \tilde{\boldsymbol{u}}_{\boldsymbol{n}}\big) \Big\|_{E^{-1}} \le 2\sqrt{\frac{1}{n}}.$$

That is,

$$\big|\big\langle \tilde{I}'\big(\tau_n, \tilde{\boldsymbol{u}}_{\boldsymbol{n}}\big), \big(\tau, \boldsymbol{u}^*\big)\big\rangle\big| \le 2\sqrt{\frac{1}{n}} \|(\tau, \boldsymbol{u}^*)\|_E,$$

for all $\big(\tau, \boldsymbol{u}^*\big) := \big(\tau, (u_1^*, u_2^*, u_0^*)\big) \in \tilde{T}_{(\tau_n, \tilde{\boldsymbol{u}}_{\boldsymbol{n}})}$, where

$$\tilde{T}_{(\tau_n, \tilde{\boldsymbol{u}}_{\boldsymbol{n}})} := \Big\{ \big(\tau, \boldsymbol{u}^*\big) \in E \Big| \int_{\mathbb{R}^3} \Big( \overline{u}_1^* \tilde{u}_{1n} + \overline{u}_2^* \tilde{u}_{2n} + \overline{u}_0^* \tilde{u}_{0n} \Big) dx = 0, \\ \int_{\mathbb{R}^3} \Big( \overline{u}_1^* \tilde{u}_{1n} - \overline{u}_2^* \tilde{u}_{2n} \Big) dx = 0 \Big\}.$$

Let

$$\boldsymbol{u_n} := (u_{1n}, u_{2n}, u_{0n}) := \tau_n \star \tilde{\boldsymbol{u}}_{\boldsymbol{n}} = (\tau_n \star \tilde{u}_{1n}, \tau_n \star \tilde{u}_{2n}, \tau_n \star \tilde{u}_{0n}),$$

then by point (1), we obtain

$$I(\boldsymbol{u_n}) = I(\tau_n \star \tilde{\boldsymbol{u}}_{\boldsymbol{n}}) = \tilde{I}\big(\tau_n, \tilde{\boldsymbol{u}}_{\boldsymbol{n}}\big) \to \sigma, \quad \text{as } n \to \infty. \tag{4.9}$$

Further, by direct calculations, we conclude from (3) that

$$P(\boldsymbol{u_n}) = P\big(\tau_n \star \tilde{\boldsymbol{u}}_{\boldsymbol{n}}\big) = e^{2\tau_n} A(\tilde{\boldsymbol{u}}_{\boldsymbol{n}}) - \frac{3}{4} e^{3\tau_n} E(\tilde{\boldsymbol{u}}_{\boldsymbol{n}}) + \int_{\mathbb{R}^3} \big\langle \nabla B(e^{-\tau_n} x), e^{-\tau_n} x \big\rangle (\overline{\tilde{u}}_{1n} \tilde{u}_{0n} + \overline{\tilde{u}}_{0n} \tilde{u}_{2n}) dx \\ = \big\langle \tilde{I}'\big(\tau_n, \tilde{\boldsymbol{u}}_{\boldsymbol{n}}\big), \big(1, (0,0,0)\big)\big\rangle \to 0, \quad \text{as } n \to \infty. \tag{4.10}$$

Next, we are going to show that $I'\big|_{\mathcal{M}}(\boldsymbol{u_n}) \to 0$ as $n \to \infty$. For this purpose, it is sufficient to show that there exists a certain constant $C > 0$, such that

$$\big|\langle I'(\boldsymbol{u_n}), \boldsymbol{u}^* \rangle\big| \le \frac{C}{\sqrt{n}} \|\boldsymbol{u}^*\|, \tag{4.11}$$

for all

$$\boldsymbol{u}^* \in T_{\boldsymbol{u_n}} := \Big\{\boldsymbol{u}^* \in H \Big| \int_{\mathbb{R}^3} \Big(\overline{u}_1^* u_{1n} + \overline{u}_2^* u_{2n} + \overline{u}_0^* u_{0n}\Big) dx = 0, \int_{\mathbb{R}^3} \Big(\overline{u}_1^* u_{1n} - \overline{u}_2^* u_{2n}\Big) dx = 0\Big\}.$$

For any $\boldsymbol{u}^* \in T_{\boldsymbol{u_n}}$, we set $\tilde{\boldsymbol{u}}^* := (\tilde{u}_1^*, \tilde{u}_2^*, \tilde{u}_0^*) := (-\tau_n) \star \boldsymbol{u}^*$, then $(0, \tilde{\boldsymbol{u}}^*) \in \tilde{T}_{(\tau_n, \tilde{\boldsymbol{u}}_{\boldsymbol{n}})}$ and

$$\big\langle \tilde{I}'\big(\tau_n, \tilde{\boldsymbol{u}}_{\boldsymbol{n}}\big), (0, \tilde{\boldsymbol{u}}^*)\big\rangle = \langle I'(\boldsymbol{u_n}), \boldsymbol{u}^*\rangle.$$

By point (2), we may assume that $\tau_n \to 0$ as $n \to \infty$, then

$$\begin{aligned}\|(0, \tilde{\boldsymbol{u}}^*)\|_E^2 = \|\tilde{\boldsymbol{u}}^*\|^2 &= \int_{\mathbb{R}^3} \big(|\nabla u_1^*|^2 + |\nabla u_2^*|^2 + |\nabla u_0^*|^2\big) dx + \int_{\mathbb{R}^3} \big(|\tilde{u}_1^*|^2 + |\tilde{u}_2^*|^2 + |\tilde{u}_0^*|^2\big) dx \\ &= e^{-2\tau_n} \int_{\mathbb{R}^3} \big(|\nabla u_1^*|^2 + |\nabla u_2^*|^2 + |\nabla u_0^*|^2\big) dx + \int_{\mathbb{R}^3} \big(|u_1^*|^2 + |u_2^*|^2 + |u_0^*|^2\big) dx \le 2\|\boldsymbol{u}^*\|^2.\end{aligned}$$

For any $\boldsymbol{u}^* \in T_{\boldsymbol{u_n}}$, set $\tilde{\boldsymbol{u}}^* := (\tilde{u}_1^*, \tilde{u}_2^*, \tilde{u}_0^*) := (-\tau_n) \star \boldsymbol{u}^*$, then

$$(0, \tilde{\boldsymbol{u}}^*) \in \tilde{T}_{(\tau_n, \tilde{\boldsymbol{u}}_{\boldsymbol{n}})} \text{ and } \big\langle \tilde{I}'\big(\tau_n, \tilde{\boldsymbol{u}}_{\boldsymbol{n}}\big), (0, \tilde{\boldsymbol{u}}^*)\big\rangle = \langle I'(\boldsymbol{u_n}), \boldsymbol{u}^*\rangle.$$

By point (2), we may assume that $\tau_n \to 0$ as $n \to \infty$, then

$$\begin{aligned}\|(0, \tilde{\boldsymbol{u}}^*)\|_E^2 = \|\tilde{\boldsymbol{u}}^*\|^2 &= \int_{\mathbb{R}^3} \big(|\nabla u_1^*|^2 + |\nabla u_2^*|^2 + |\nabla u_0^*|^2\big) dx + \int_{\mathbb{R}^3} \big(|\tilde{u}_1^*|^2 + |\tilde{u}_2^*|^2 + |\tilde{u}_0^*|^2\big) dx \\ &= e^{-2\tau_n} \int_{\mathbb{R}^3} \big(|\nabla u_1^*|^2 + |\nabla u_2^*|^2 + |\nabla u_0^*|^2\big) dx + \int_{\mathbb{R}^3} \big(|u_1^*|^2 + |u_2^*|^2 + |u_0^*|^2\big) dx \\ &\le 2\|\boldsymbol{u}^*\|^2.\end{aligned}$$

Then by point (3), we get

$$\big|\langle I'(\boldsymbol{u_n}), \boldsymbol{u}^*\rangle\big| = \big|\langle \tilde{I}'\big(\tau_n, \tilde{\boldsymbol{u}}_{\boldsymbol{n}}\big), \big(0, \tilde{\boldsymbol{u}}^*\big)\rangle\big| \le 2\sqrt{\frac{1}{n}}\|(1, \tilde{\boldsymbol{u}}^*)\| \le 2\sqrt{\frac{2}{n}}\|\boldsymbol{u}^*\|,$$

thus (4.11) holds. Together with (4.9), (4.10), $\{\boldsymbol{u_n}\}$ is a Palais-Smale sequence for $I$ restricted to $\mathcal{M}$. Finally, we show $\{\boldsymbol{u_n}\} \subset \mathcal{M}$ is bounded in $\Lambda$. Indeed, direct calculation gives

$$\begin{aligned} I(\boldsymbol{u_n}) &= I(\boldsymbol{u_n}) - \frac{1}{3}P(\boldsymbol{u_n}) + o(1) \\ &= \frac{1}{6}A(\boldsymbol{u_n}) + \frac{5}{6}\int_{\mathbb{R}^3} |x|^2|\boldsymbol{u_n}|^2 dx - \frac{1}{3}\int_{\mathbb{R}^3} \langle \nabla B(x), x\rangle(\overline{u}_{1n}u_{0n} + \overline{u}_{0n}u_{2n}) dx - F(\boldsymbol{u_n}) + o(1).\end{aligned}$$

It follows that if $B(x) \in L^\infty$, then

$$\begin{aligned}\frac{1}{6}\|\boldsymbol{u_n}\|_\Lambda^2 &\le I(\boldsymbol{u_n}) + \frac{1}{3}\int_{\mathbb{R}^3} \langle \nabla B(x), x\rangle(\overline{u}_{1n}u_{0n} + \overline{u}_{0n}u_{2n}) dx + F(\boldsymbol{u_n}) + o(1) \\ &\le \sigma + \frac{1}{3}\|\langle \nabla B(x), x\rangle\|_{L^\infty} N + \|B(x)\|_{L^\infty} N + o(1).\end{aligned}$$

If $\frac{B(x)}{x} \in L^\infty$, then

$$\frac{1}{6}\|\boldsymbol{u_n}\|_\Lambda^2 \leq \sigma + \frac{1}{3}\|\langle \nabla B(x), x\rangle\|_{L^\infty} N + 8\left\|\frac{B(x)}{x}\right\|_{L^\infty}^2 N + o(1)$$

and if $B(x)$ satisfies (1.16), then

$$\frac{1}{6}\|\boldsymbol{u_n}\|_\Lambda^2 \leq \sigma + \frac{1}{3}\|\langle \nabla B(x), x\rangle\|_{L^\infty} N + o(1).$$

Since $\{\boldsymbol{u_n}\} \subset \mathcal{M}$, then we get the boundedness of $\{\boldsymbol{u_n}\}$ in $\Lambda$. Therefore, we complete the proof. □

**Lemma 4.6.** *Suppose $c_0 > 0$ and $c_0 + c_1 > 0$, for any $r > 0$ and $0 < N \leq N^{**}$, let $\{\boldsymbol{u_n}\} \subset \mathcal{M}$ be the Palais-Smale sequence obtained in Proposition 4.1, then there exists $\hat{\boldsymbol{u}} = (\hat{u}_1, \hat{u}_2, \hat{u}_0) \in \mathcal{M}$, such that $\boldsymbol{u_n} \to \hat{\boldsymbol{u}}$ is strongly in $\Lambda$.*

*Proof.* By Lemma 2.1 and Proposition 4.1, there exists $\hat{\boldsymbol{u}} = (\hat{u}_1, \hat{u}_2, \hat{u}_0) \in \Lambda$, such that up to a subsequence, as $n \to +\infty$,

$$\begin{cases} \boldsymbol{u_n} \rightharpoonup \hat{\boldsymbol{u}}, & \text{in } \Lambda. \\ \boldsymbol{u_n} \to \hat{\boldsymbol{u}}, & \text{in } L^t(\mathbb{R}^3) \times L^t(\mathbb{R}^3) \times L^t(\mathbb{R}^3),\ \forall t \in [2, 2^*). \\ \boldsymbol{u_n} \to \hat{\boldsymbol{u}}, & a.e. \text{ in } \mathbb{R}^3. \end{cases} \tag{4.12}$$

Since $I'\big|_{\mathcal{M}}(\boldsymbol{u_n}) \to 0$, then there exist two sequences $\{\mu_n\},\ \{\lambda_n\} \subset \mathbb{R}$, such that

$$\begin{aligned} &\left\langle \frac{1}{2}A'(\boldsymbol{u_n}) - \frac{1}{4}E'(\boldsymbol{u_n}) - F'(\boldsymbol{u_n}), \boldsymbol{\phi} \right\rangle + \int_{\mathbb{R}^3} |x|^2 \Big(u_{1n}\overline{\phi}_1 + u_{2n}\overline{\phi}_2 + u_{0n}\overline{\phi}_0\Big) dx \\ =\ & \mu_n \int_{\mathbb{R}^3} \Big(u_{1n}\overline{\phi}_1 + u_{2n}\overline{\phi}_2 + u_{0n}\overline{\phi}_0\Big) dx + \lambda_n \int_{\mathbb{R}^3} (u_{1n}\overline{\phi}_1 - u_{2n}\overline{\phi}_2) dx \\ =\ & (\mu_n + \lambda_n) \int_{\mathbb{R}^3} u_{1n}\overline{\phi}_1 dx + (\mu_n - \lambda_n) \int_{\mathbb{R}^3} u_{2n}\overline{\phi}_2 dx + \mu_n \int_{\mathbb{R}^3} u_{0n}\overline{\phi}_0 dx + o(1), \end{aligned} \tag{4.13}$$

for any $\boldsymbol{\phi} = (\phi_1, \phi_2, \phi_0) \in \Lambda$. Since $\{\boldsymbol{u_n}\} \subset \mathcal{M}$ is bounded in $\Lambda$ by Proposition 4.1, taking $\boldsymbol{\phi} = \boldsymbol{u_n}$ in (4.13), then $\{\mu_n\},\ \{\lambda_n\}$ are two bounded sequences in $\mathbb{R}$. Suppose that $\mu_n \to \hat{\mu}$, $\lambda_n \to \hat{\lambda}$ as $n \to \infty$. Taking $\boldsymbol{\phi} = \boldsymbol{u_n} - \hat{\boldsymbol{u}}$ in (4.13), we get

$$\begin{aligned} &\left\langle \frac{1}{2}A'(\boldsymbol{u_n}) - \frac{1}{4}E'(\boldsymbol{u_n}) - F'(\boldsymbol{u_n}), \boldsymbol{u_n} - \hat{\boldsymbol{u}} \right\rangle \\ &+ \int_{\mathbb{R}^3} |x|^2 \Big(u_{1n}(\overline{u}_{1n} - \overline{\hat{u}}_1) + u_{2n}(\overline{u}_{2n} - \overline{\hat{u}}_2) + u_{0n}(\overline{u}_{0n} - \overline{\hat{u}}_0)\Big) dx \\ =\ & (\mu_n + \lambda_n) \int_{\mathbb{R}^3} u_{1n}(\overline{u}_{1n} - \overline{\hat{u}}_1) dx + (\mu_n - \lambda_n) \int_{\mathbb{R}^3} u_{2n}(\overline{u}_{2n} - \overline{\hat{u}}_2) dx \\ &+ \mu_n \int_{\mathbb{R}^3} u_{0n}(\overline{u}_{0n} - \overline{\hat{u}}_0) dx + o(1). \end{aligned} \tag{4.14}$$

By (4.12), we get $\hat{\boldsymbol{u}}$ satisfies (1.1)-(1.2). Thus using $(\overline{u}_{1n} - \overline{\hat{u}}_1, \overline{u}_{2n} - \overline{\hat{u}}_1, \overline{u}_{0n} - \overline{\hat{u}}_0)$ as a test function in

(1.1), we then obtain

$$\begin{aligned}&\Big\langle \frac{1}{2}A'(\hat{\boldsymbol{u}})-\frac{1}{4}E'(\hat{\boldsymbol{u}})-F'(\hat{\boldsymbol{u}}),(\overline{u}_{1n}-\overline{\hat{u}}_1,\overline{u}_{2n}-\overline{\hat{u}}_1,\overline{u}_{0n}-\overline{\hat{u}}_0)\Big\rangle\\&+\int_{\mathbb{R}^3}|x|^2\Big(\hat{u}_1(\overline{u}_{1n}-\overline{\hat{u}}_1)+\hat{u}_2(\overline{u}_{2n}-\overline{\hat{u}}_2)+\hat{u}_0(\overline{u}_{0n}-\overline{\hat{u}}_0)\Big)dx\\=&(\hat{\mu}+\hat{\lambda}_n)\int_{\mathbb{R}^3}\hat{u}_1(\overline{u}_{1n}-\overline{\hat{u}}_1)dx+(\hat{\mu}-\hat{\lambda})\int_{\mathbb{R}^3}\hat{u}_2(\overline{u}_{2n}-\overline{\hat{u}}_2)dx\\&+\hat{\mu}\int_{\mathbb{R}^3}\hat{u}_0(\overline{u}_{0n}-\overline{\hat{u}}_0)dx+o(1).\end{aligned}$$

Together with (4.12), (4.14), we can see that

$$A(\boldsymbol{u_n}-\hat{\boldsymbol{u}})+\int_{\mathbb{R}^3}|x|^2|\boldsymbol{u_n}-\hat{\boldsymbol{u}}|^2dx=o(1),$$

which gives

$$A(\boldsymbol{u_n})\to A(\hat{\boldsymbol{u}})\quad\text{and}\quad\int_{\mathbb{R}^3}|x|^2|\boldsymbol{u_n}|^2dx\to\int_{\mathbb{R}^3}|x|^2|\hat{\boldsymbol{u}}|^2dx,\quad\text{as } n\to\infty.$$

Therefore, we get the strong convergence of $\boldsymbol{u_n}\to\hat{\boldsymbol{u}}$ in $\Lambda$ as $n\to\infty$. □

***Proof of Theorem 6***. The proof follows from Proposition 4.1 and Lemma 4.6. □

# Appendix

In this section, suppose $N_n\nearrow N^*$, $c_0+c_{1n}>0$ and $c_{1n}\nearrow 0$, then we investigate the limit behavior of $\boldsymbol{u_n}=(u_{1n},u_{2n},u_{0n})$ as $n\to\infty$.

**Proposition 4.2.** *Suppose $c_0+c_{1n}>0,\ c_{1n}\nearrow 0$. Let $N_n\nearrow N^*$,*

$$\lim_{n\to\infty}\frac{c_{1n}}{N_nc_0(N^*-N_n)}=\eta<0\ \text{as } n\to\infty$$

*and $\boldsymbol{u_n}=(u_{1n},u_{2n},u_{0n})\in\mathcal{M}(N_n)$ be a corresponding minimizer of $m(N_n)$, then*
*(i) if $B(x)$ satisfies (1.8) with $B(x)\geq 0$, then*

$$M_1(N^*-N_n)^{\frac{1}{2}}\leq m(N_n)\leq M_2(N^*-N_n)^{\frac{1}{2}},\quad \text{as } n\to\infty.$$

*(ii) if $B(x)$ satisfies (1.8) and (1.9) with $B(x)<0$, then for $0<p<2$,*

$$M_3(N^*-N_n)^{\frac{p}{p+2}}\leq m(N_n)\leq M_4(N^*-N_n)^{\frac{p}{p+2}},\quad \text{as } n\to\infty,$$

*for $p=2$,*

$$M_5(N^*-N_n)^{\frac{1}{2}}\leq m(N_n)\leq M_6(N^*-N_n)^{\frac{1}{2}},\quad \text{as } n\to\infty$$

*and for $p>2$,*

$$\frac{m(N_n)}{(N^*-N_n)^{\frac{1}{2}}}\to\Big(\frac{1}{c_0}\int_{\mathbb{R}^2}|x|^2Q^2(x)dx\Big)^{\frac{1}{2}}(1-\eta M^2)^{\frac{1}{2}},\quad \text{as } n\to\infty,$$

*where*

$$M_1=\Big(\Big(1-2\Big\|\frac{B(x)}{x^2}\Big\|_{L^\infty}\Big)\frac{1}{c_0}\int_{\mathbb{R}^2}|x|^2Q^2(x)dx\Big)^{\frac{1}{2}}(1-\eta M^2)^{\frac{1}{2}},$$

$$M_2=\Big(\frac{1}{c_0}\int_{\mathbb{R}^2}|x|^2Q^2(x)dx\Big)^{\frac{1}{2}}(1-\eta M^2)^{\frac{1}{2}},$$

$$M_3=\frac{p+2}{2}\Big(\frac{1-\eta M^2}{p}\Big)^{\frac{p}{p+2}}T^{\frac{2}{p+2}},\quad M_4=\frac{p+2}{2}\Big(\frac{1-\eta M^2}{p}\Big)^{\frac{p}{p+2}}\Big(A\int_{\mathbb{R}^2}|B(x)|Q^2(x)dx\Big)^{\frac{2}{p+2}},$$

$$M_5=\Big(\frac{1}{c_0}\int_{\mathbb{R}^2}|x|^2Q^2(x)dx+2T\Big)^{\frac{1}{2}}(1-\eta M^2)^{\frac{1}{2}}$$

*and*

$$M_6=\Big(\frac{1}{c_0}\int_{\mathbb{R}^2}|x|^2Q^2(x)dx+2A\int_{\mathbb{R}^2}|B(x)|Q^2(x)dx\Big)^{\frac{1}{2}}(1-\eta M^2)^{\frac{1}{2}},$$

*here $T$ has been defined in* (3.39). *In addition, if $B(x)\equiv 0$, then $\boldsymbol{u_n}$ satisfies*

$$\begin{cases}\lim\limits_{n\to\infty}\varepsilon_n u_{1n}(\varepsilon_n x+\tilde{z}_{1n})=\sqrt{\dfrac{N^*+M}{2N^*c_0}}Q(x),\\ \lim\limits_{n\to\infty}\varepsilon_n u_{2n}(\varepsilon_n x+\tilde{z}_{2n})=\sqrt{\dfrac{N^*-M}{2N^*c_0}}Q(x),\quad \textit{strongly in } H^1(\mathbb{R}^2),\\ u_{0n}\equiv 0 \textit{ in } \mathbb{R}^2, \textit{when } n>0 \textit{ is large enough,}\end{cases}$$

*where $\tilde{z}_{in}$ $(i=1,2)$ is the unique maximum point of $u_{in}$ with*

$$\lim_{n\to\infty}\Big|\frac{\tilde{z}_{1n}-\tilde{z}_{2n}}{\varepsilon_n}\Big|=0\ (i,j=1,2,i\neq j),\ \lim_{n\to\infty}|\tilde{z}_{in}|=0$$

*and* $\varepsilon_n=C\Big(N^*-N_n\Big)^{\frac{1}{4}}$.

*Proof.* Without loss of generality, we may assume $B(x)\geq 0$. Similar to (3.30), we can see that

$$\begin{aligned}I(\boldsymbol{u_n})&\geq\frac{1}{2}\int_{\mathbb{R}^2}|x|^2|\boldsymbol{u_n}|^2dx+\Big(\frac{a^*}{4N_n}-\frac{c_0}{4}\Big)\int_{\mathbb{R}^2}(|u_{1n}|^2+|u_{2n}|^2+|u_{0n}|^2)^2dx-F(\boldsymbol{u_n})\\ &\geq\Big(\frac{1}{2}-\Big\|\frac{B(x)}{x^2}\Big\|_{L^\infty}\Big)\int_{\mathbb{R}^2}|x|^2|\boldsymbol{u_n}|^2dx+\Big(\frac{a^*}{4N_n}-\frac{c_0}{4}\Big)\int_{\mathbb{R}^2}(|u_{1n}|^2+|u_{2n}|^2+|u_{0n}|^2)^2dx\\ &\geq\Big(\frac{1}{2}-\Big\|\frac{B(x)}{x^2}\Big\|_{L^\infty}\Big)\frac{N^*\varepsilon_n^2}{a^*}\int_{\mathbb{R}^2}|x|^2Q^2(x)dx+\Big(\frac{a^*}{2N}-\frac{c_0}{2}\Big)\frac{(N^*)^2\varepsilon_n^{-2}}{a^*}+o(1).\end{aligned}\tag{4.15}$$

From (3.15), we have

$$m(N_n)\leq\frac{N_n\tau^{-2}}{2a^*}\int_{\mathbb{R}^2}|x|^2Q^2(x)dx+\Big(\frac{N_n}{2}-\frac{c_0N_n^2}{2a^*}-\frac{c_{1n}M^2}{2a^*}\Big)\tau^2.$$

Taking $\tau=\Big(\frac{N_n\int_{\mathbb{R}^2}|x|^2Q^2(x)dx}{N_nc_0(N^*-N_n)-c_{1n}M^2}\Big)^{\frac{1}{4}}$ and noting that

$$\lim_{n\to\infty}\frac{c_{1n}}{N_nc_0(N^*-N_n)}=\eta<0,\tag{4.16}$$

then we get

$$\lim_{n\to\infty}\frac{m(N_n)}{(N^*-N_n)^{\frac{1}{2}}}\le\Big(\frac{1}{c_0}\int_{\mathbb{R}^2}|x|^2Q^2(x)dx\Big)^{\frac{1}{2}}(1-\eta M^2)^{\frac{1}{2}}. \tag{4.17}$$

Therefore, $\lim_{n\to\infty} m(N_n)=0$, which implies that

$$\lim_{n\to\infty}\frac{A(\boldsymbol{u_n})}{\int_{\mathbb{R}^2}(|u_{1n}|^2+|u_{2n}|^2+|u_{0n}|^2)^2dx}=\frac{a^*}{2N^*}.$$

Further, we also have

$$\lim_{n\to\infty}A(\boldsymbol{u_n})=\lim_{n\to\infty}\int_{\mathbb{R}^2}\big(|\nabla u_{1n}|^2+|\nabla u_{2n}|^2+|\nabla u_{0n}|^2\big)dx=+\infty.$$

Let $\varepsilon_n$, $w_{in}$ and $\bar{w}_{in}$ $(i=1,2,0)$ be defined as Theorem 2, then by scaling, we conclude

$$\lim_{n\to\infty}\int_{\mathbb{R}^2}\big(|\nabla w_{1n}|^2+|\nabla w_{2n}|^2+|\nabla w_{0n}|^2\big)dx=N^*$$

and

$$\lim_{n\to\infty}\int_{\mathbb{R}^2}(|w_{1n}|^2+|w_{2n}|^2+|w_{0n}|^2)^2dx=\frac{2(N^*)^2}{a^*}.$$

Moreover, there exists some $x_1\in\mathbb{R}^2$, such that

$$\lim_{n\to\infty}w_{1n}(x)=\sqrt{\frac{N^*}{a^*}}Q(x-x_1)\sin\varphi_1\cos\varphi_2,\quad \lim_{n\to\infty}w_{2n}(x)=\sqrt{\frac{N^*}{2a^*}}Q(x-x_1)\sin\varphi_1\sin\varphi_2$$

and

$$\lim_{n\to\infty}w_{0n}(x)=\sqrt{\frac{N^*}{a^*}}Q(x-x_1)\cos\varphi_1,\quad \text{for } \varphi_1,\varphi_2\in[0,\frac{\pi}{2}].$$

Set $t=\sin^2\varphi_1\in[\frac{M}{N^*},1]$, then by the fact that $\boldsymbol{u_n}\in\mathcal{M}$, that is $\|u_{1n}\|_{L^2}^2-\|u_{2n}\|_{L^2}^2=M$ and some direct calculations, we obtain

$$\lim_{n\to\infty}w_{1n}(x)=\sqrt{\frac{N^*t+M}{2a^*}}Q(x-x_1),\quad \lim_{n\to\infty}w_{2n}(x)=\sqrt{\frac{N^*t-M}{2a^*}}Q(x-x_1)$$

and

$$\lim_{n\to\infty}w_{0n}(x)=\sqrt{\frac{N^*(1-t)}{a^*}}Q(x-x_1)$$

are strongly convergent in $H^1(\mathbb{R}^2)$. Denote

$$M_2:=\Big(\frac{1}{c_0}\int_{\mathbb{R}^2}|x|^2Q^2(x)dx\Big)^{\frac{1}{2}}(1-\eta M^2)^{\frac{1}{2}},$$

then by (4.15) and (4.17), for $n$ large enough, we have

$$\Big(\frac{1}{2}-\Big\|\frac{B(x)}{x^2}\Big\|_{L^\infty}\Big)\frac{N^*\varepsilon_n^2}{a^*}\int_{\mathbb{R}^2}|x|^2Q^2(x)dx\le M_2(N^*-N_n)^{\frac{1}{2}}$$

and

$$\Big(\frac{a^*}{2N}-\frac{c_0}{2}\Big)\frac{(N^*)^2\varepsilon_n^{-2}}{a^*}\le M_2(N^*-N_n)^{\frac{1}{2}}.$$

It follows that for $n$ large enough,

$$\Big(\frac{N^*(N^*-N_n)^{\frac12}}{2NM_2}\Big)^{\frac12}\le\varepsilon_n\le\Big(\frac{M_2(N^*-N_n)^{\frac12}}{(\frac12-\|\frac{B(x)}{x^2}\|_{L^\infty})\frac{1}{c_0}\int_{\mathbb{R}^2}|x|^2Q^2(x)dx}\Big)^{\frac12}.$$

Then up to a subsequence, we may assume that

$$\lim_{n\to\infty}\frac{\varepsilon_n}{(N^*-N_n)^{\frac14}}=\xi. \tag{4.18}$$

By direct calculations, we get

$$\begin{aligned}
&\lim_{n\to\infty}\frac{1}{(N^*-N_n)^{\frac12}}\Big\{\Big(\frac{a^*}{N_n}-c_0\Big)\int_{\mathbb{R}^2}(|u_{1n}|^2+|u_{2n}|^2+|u_{0n}|^2)^2dx-c_{1n}\int_{\mathbb{R}^2}(|u_{1n}|^2-|u_{2n}|^2)^2dx\Big\}\\
&=\lim_{n\to\infty}(N^*-N_n)^{\frac12}\Big\{\frac{a^*(\frac{1}{N_n}-\frac{1}{N^*})}{(N^*-N_n)}\int_{\mathbb{R}^2}(|u_{1n}|^2+|u_{2n}|^2+|u_{0n}|^2)^2dx\\
&\quad-\frac{c_{1n}}{N^*-N_n}\int_{\mathbb{R}^2}(|u_{1n}|^2-|u_{2n}|^2)^2dx\Big\}\\
&=\lim_{n\to\infty}\frac{(N^*-N_n)^{\frac12}}{\varepsilon_n^2}\Big\{\frac{a^*(\frac{1}{N_n}-\frac{1}{N^*})}{(N^*-N_n)}\int_{\mathbb{R}^2}(|w_{1n}|^2+|w_{2n}|^2+|w_{0n}|^2)^2dx\\
&\quad-\frac{c_{1n}}{N^*-N_n}\int_{\mathbb{R}^2}(|w_{1n}|^2-|w_{2n}|^2)^2dx\Big\}\\
&=\lim_{n\to\infty}\frac{(N^*-N_n)^{\frac12}}{\varepsilon_n^2}\Big\{\frac{a^*(\frac{1}{N_n}-\frac{1}{N^*})}{(N^*-N_n)}\cdot\frac{2(N^*)^2}{a^*}-\frac{c_{1n}}{N^*-N_n}\cdot\frac{2M^2}{a^*}\Big\}\\
&=\frac{2}{\xi^2}(1-\eta M^2).
\end{aligned}$$

Further,

$$\begin{aligned}
m(N_n)\ge&\Big(\frac12-\Big\|\frac{B(x)}{x^2}\Big\|_{L^\infty}\Big)\int_{\mathbb{R}^2}|x|^2|\boldsymbol{u_n}|^2dx+\Big(\frac{a^*}{4N_n}-\frac{c_0}{4}\Big)\int_{\mathbb{R}^2}(|u_{1n}|^2+|u_{2n}|^2+|u_{0n}|^2)^2dx\\
&-\frac{c_{1n}}{4}\int_{\mathbb{R}^2}(|u_{1n}|^2-|u_{2n}|^2)^2dx\\
\ge&\Big(\frac12-\Big\|\frac{B(x)}{x^2}\Big\|_{L^\infty}\Big)\frac{N^*\varepsilon_n^2}{a^*}\int_{\mathbb{R}^2}|x|^2Q^2(x)dx\\
&+\frac14\Big\{\Big(\frac{a^*}{N_n}-c_0\Big)\int_{\mathbb{R}^2}(|u_{1n}|^2+|u_{2n}|^2+|u_{0n}|^2)^2dx-c_{1n}\int_{\mathbb{R}^2}(|u_{1n}|^2-|u_{2n}|^2)^2dx\Big\}\\
=&\Big(\frac12-\Big\|\frac{B(x)}{x^2}\Big\|_{L^\infty}\Big)\frac{N^*\varepsilon_n^2}{a^*}\int_{\mathbb{R}^2}|x|^2Q^2(x)dx+\frac{N^*-N_n}{2\varepsilon_n^2}(1-\eta M^2).
\end{aligned}$$

Hence, it implies that

$$
\begin{aligned}
\liminf_{n\to\infty} \frac{m(N_n)}{(N^*-N_n)^{\frac12}} &\geq \liminf_{n\to\infty} \frac{\varepsilon_n^2}{(N^*-N_n)^{\frac12}}\Big(\frac12 - \Big\|\frac{B(x)}{x^2}\Big\|_{L^\infty}\Big)\frac{N^*}{a^*}\int_{\mathbb{R}^2}|x|^2Q^2(x)dx \\
&\quad + \liminf_{n\to\infty}\frac{(N^*-N_n)^{\frac12}}{2\varepsilon_n^2}(1-\eta M^2) \\
&= \xi^2\Big(\frac12 - \Big\|\frac{B(x)}{x^2}\Big\|_{L^\infty}\Big)\frac{N^*}{a^*}\int_{\mathbb{R}^2}|x|^2Q^2(x)dx + \frac{\xi^{-2}}{2}(1-\eta M^2) \\
&\geq \Big(\Big(1-2\Big\|\frac{B(x)}{x^2}\Big\|_{L^\infty}\Big)\frac{1}{c_0}\int_{\mathbb{R}^2}|x|^2Q^2(x)dx\Big)^{\frac12}(1-\eta M^2)^{\frac12}.
\end{aligned}
\tag{4.19}
$$

On the other hand, by (4.17) and denote

$$
M_1 := \Big(\Big(1-2\Big\|\frac{B(x)}{x^2}\Big\|_{L^\infty}\Big)\frac{1}{c_0}\int_{\mathbb{R}^2}|x|^2Q^2(x)dx\Big)^{\frac12}(1-\eta M^2)^{\frac12},
$$

we conclude that

$$
M_1(N^*-N_n)^{\frac12} \leq m(N_n) \leq M_2(N^*-N_n)^{\frac12} \text{ as } n\to\infty.
$$

If $B(x)\equiv 0$, then

$$
\lim_{n\to\infty}\frac{m(N_n)}{(N^*-N_n)^{\frac12}} = \Big(\frac{1}{c_0}\int_{\mathbb{R}^2}|x|^2Q^2(x)dx\Big)^{\frac12}(1-\eta M^2)^{\frac12}. \tag{4.20}
$$

Now, we assume that $B(x)<0$ and $p>2$ in (1.9). By direct calculations, we obtain

$$
\begin{aligned}
m(N_n) &\leq \frac{N_n\tau^{-2}}{2a^*}\int_{\mathbb{R}^2}|x|^2Q^2(x)dx + \Big(\frac{N_n}{2}-\frac{c_0N_n^2}{2a^*}-\frac{c_1M^2}{2a^*}\Big)\tau^2 + A\tau^{-p}\int_{\mathbb{R}^2}|B(x)|Q^2(x)dx \\
&= \frac{N_n\tau^{-2}}{2a^*}\int_{\mathbb{R}^2}|x|^2Q^2(x)dx + \Big(\frac{N_n}{2}-\frac{c_0N_n^2}{2a^*}-\frac{c_1M^2}{2a^*}\Big)\tau^2 + o(1).
\end{aligned}
$$

Then by (4.17),

$$
\lim_{n\to\infty}\frac{m(N_n)}{(N^*-N_n)^{\frac12}} \leq \Big(\frac{1}{c_0}\int_{\mathbb{R}^2}|x|^2Q^2(x)dx\Big)^{\frac12}(1-\eta M^2)^{\frac12}.
$$

On the other hand, similar to (3.40), we get

$$
\begin{aligned}
I(\boldsymbol{u_n}) &\geq \Big(\frac{a^*}{4N_n}-\frac{c_0}{4}\Big)\int_{\mathbb{R}^2}(|u_{1n}|^2+|u_{2n}|^2+|u_{0n}|^2)^2dx + \frac12\int_{\mathbb{R}^2}|x|^2|\boldsymbol{u_n}|^2dx - F(\boldsymbol{u_n}) \\
&\geq \frac{N^*}{2N_n}(N^*-N_n)\varepsilon_n^{-2} + \frac{N^*\varepsilon_n^2}{2a^*}\int_{\mathbb{R}^2}|x|^2Q^2(x)dx + T\varepsilon_n^p + o(1) \\
&= \frac{N^*}{2N_n}(N^*-N_n)\varepsilon_n^{-2} + \frac{N^*\varepsilon_n^2}{2a^*}\int_{\mathbb{R}^2}|x|^2Q^2(x)dx + o(1).
\end{aligned}
$$

Following the produces for $B(x)\geq 0$, we can deduce that

$$
\begin{aligned}
\liminf_{n\to\infty}\frac{m(N_n)}{(N^*-N_n)^{\frac12}} &\geq \frac{N^*\xi^2}{2a^*}\int_{\mathbb{R}^2}|x|^2Q^2(x)dx + \frac{\xi^{-2}}{2}(1-\eta M^2) \\
&\geq \Big(\frac{1}{c_0}\int_{\mathbb{R}^2}|x|^2Q^2(x)dx\Big)^{\frac12}(1-\eta M^2)^{\frac12}.
\end{aligned}
\tag{4.21}
$$

That is, (4.20) holds as well in this case. In addition, if $B(x) < 0$ and $0 < p \le 2$ in (1.9), we can similar obtain the energy estimations. Precisely, for $0 < p < 2$,

$$M_3(N^* - N_n)^{\frac{p}{p+2}} \le m(N_n) \le M_4(N^* - N_n)^{\frac{p}{p+2}}, \quad \text{as } n \to \infty$$

and for $p = 2$,

$$M_5(N^* - N_n)^{\frac{1}{2}} \le m(N_n) \le M_6(N^* - N_n)^{\frac{1}{2}}, \quad \text{as } n \to \infty.$$

By (4.20), the inequalities (4.19) and (4.21) hold if and only if $t = 1$ and $\xi = \Big(\frac{c_0(1-\eta M^2)}{\int_{\mathbb{R}^2} |x|^2 Q^2(x)dx}\Big)^{\frac{1}{4}}$. Then

$$\lim_{n\to\infty} \frac{\varepsilon_n}{(N^* - N_n)^{\frac{1}{4}}} = \Big(\frac{c_0(1-\eta M^2)}{\int_{\mathbb{R}^2} |x|^2 Q^2(x)dx}\Big)^{\frac{1}{4}}.$$

Since $t = 1$, we have

$$\lim_{n\to\infty} w_{1n}(x) = \sqrt{\frac{N^*+M}{2a^*}} Q(x - x_1), \lim_{n\to\infty} w_{2n}(x) = \sqrt{\frac{N^*-M}{2a^*}} Q(x - x_1) \text{ and } \lim_{n\to\infty} w_{0n}(x) = 0.$$

In addition, similar to the proofs of Theorems 2 and 3, we conclude as $n \to \infty$,

$$\tilde{w}_{1n}(x) \to \sqrt{\frac{N^*+M}{2a^*}} Q(x), \quad \tilde{w}_{2n}(x) \to \sqrt{\frac{N^*-M}{2a^*}} Q(x) \quad \text{and} \quad \tilde{w}_{0n}(x) \to 0$$

uniformly in $\mathbb{R}^2$.

To end the proof, we are going to show that $u_{0n} \equiv 0$ in $\mathbb{R}^2$ by contradiction in the case of $B(x) \equiv 0$ or $B(x) < 0$ with $p > 3$ in (1.9). Indeed, the conclusion can be proved as Theorem 1.3 in [30] with small modifications, we just sketch the differences here. Suppose $u_{0n} \not\equiv 0$ in $\mathbb{R}^2$ and define

$$\hat{u}_{in} := \frac{1}{A_i}\varepsilon_n u_{in}(\varepsilon_n x + \tilde{z}_{1n}),\ i = 1, 2, \quad \hat{u}_{0n}(x) := \frac{1}{Q_\infty \sigma_n} u_{0n}(\varepsilon_n x + \tilde{z}_{1n}),$$

where $\sigma_n = \|u_{0n}\|_{L^\infty} > 0$, $Q_\infty = \frac{1}{\|Q\|_{L^\infty}} > 0$ and

$$A_1 := \sqrt{\frac{N^*+M}{2a^*}}, \quad A_2 := \sqrt{\frac{N^*-M}{2a^*}}.$$

Using the linearized operators defined as in [30] and some calculations, we can obtain

$$\begin{aligned}&\lambda_n\varepsilon_n^2 a^* + 2c_{1n}M + c_{1n}\varepsilon_n^2(Q_\infty\sigma_n)^2 a^*\Big(\frac{A_2}{A_1} - \frac{A_1}{A_2}\Big)\\ &+ \frac{Q_\infty\sigma_n\varepsilon_n^{3+p}}{2}\int_{\mathbb{R}^N} B(x)Q^2(x)dx\Big(\frac{1}{A_1} - \frac{1}{A_2}\Big) = 0\end{aligned}$$

and

$$\begin{aligned}&\lambda_n\varepsilon_n^2 a^* + 2c_{1n}\varepsilon_n^2(Q_\infty\sigma_n)^2 a^* + 2c_{1n}(M - N^*) + 2c_{1n}\frac{A_2}{A_1}\varepsilon_n^2(Q_\infty\sigma_n)^2 a^*\\ &- 4c_{1n}A_1A_2a^* + \frac{Q_\infty\sigma_n\varepsilon_n^{3+p}}{A_1}\int_{\mathbb{R}^N} B(x)Q^2(x)dx - \frac{(A_1 + A_2)\varepsilon_n^{1+p}}{Q_\infty\sigma_n}\int_{\mathbb{R}^N} B(x)Q^2(x)dx = 0.\end{aligned}$$

It follows that

$$
\begin{aligned}
&-2c_{1n}\varepsilon_n^2(Q_\infty\sigma_n)^2a^* + 2c_{1n}N^* + c_{1n}\varepsilon_n^2(Q_\infty\sigma_n)^2a^*\Big(\frac{A_2}{A_1} - \frac{A_1}{A_2}\Big)\\
&-2c_{1n}\frac{A_2}{A_1}\varepsilon_n^2(Q_\infty\sigma_n)^2a^* - \frac{Q_\infty\sigma_n\varepsilon_n^{3+p}}{2}\int_{\mathbb{R}^N} B(x)Q^2(x)dx\Big(\frac{1}{A_1} + \frac{1}{A_2}\Big)\\
&+4c_{1n}A_1A_2a^* + \frac{(A_1+A_2)\varepsilon_n^{1+p}}{Q_\infty\sigma_n}\int_{\mathbb{R}^N} B(x)Q^2(x)dx = 0.
\end{aligned}
$$

If $B(x) \equiv 0$ or $B(x) < 0$ satisfies (1.9) for $p > 3$, then it yields from (4.16), (4.18) and $c_{1n} < 0$ that $N^* + 2A_1A_2a^* = 0$. However, it is impossible as $N^* + 2A_1A_2a^* > 0$. Therefore, $u_{0n} \equiv 0$ in $\mathbb{R}^2$ and we complete the proof. □

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