In this note, I will explain Riemann sum for double integral $\int\int_R f(x,y)dxdy$, where $R=[a,b]\times[c,d]$. Now slice

$$a = x_0 < x_1 < ... < x_n = b,\ \Delta x_i = x_i - x_{i-1}$$
$$c = y_0 < y_1 < ... < y_m = d,\ \Delta y_j = y_j - y_{j-1}$$

so that $R = \cup_i \cup_j [x_{i-1}, x_i] \times [y_{j-1}, y_j]$.

Choose $x_i^* \in [x_{i-1}, x_i], y_j^* \in [y_{j-1}, y_j]$. Then a Riemann sum is

$$\sum_{i=1}^{n}\sum_{j=1}^{m} f(x_i^*, y_j^*)\Delta x_i \Delta y_j$$

Theorem: $\int\int_R f(x,y)dxdy = \lim_{|\Delta x_i|+|\Delta y_j|\to 0}\sum_{i=1}^{n}\sum_{j=1}^{m} f(x_i^*, y_j^*)\Delta x_i \Delta y_j$.

The Riemann sum depends very much on the choices of (x_i^*, y_j^*) in each sub rectangle $R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$. There are special choices of (x_i^*, y_j^*):

Case 1: left hand corner, $(x_i^*, y_j^*) = (x_{i-1}, y_{j-1})$
Case 2: right hand corner, $(x_i^*, y_j^*) = (x_i, y_j)$
Case 3: maximum point on each $R_{i,j}$, $f(x_i^*, y_j^*) = \max_{R_{i,j}} f(x,y)$. This gives an upper bound on Riemmann sum as well as the double integral

$$\int\int_D f(x,.y)dxdy \le \sum_i\sum_j \max_{R_{i,j}} f(x,y)\Delta x_i \Delta y_j$$

Case 4: minimum point on each $R_{i,j}$, $f(x_i^*, y_j^*) = \min_{R_{i,j}} f(x,y)$. This gives a lower bound on Riemmann sum as well as the double integral

$$\int\int_D f(x,.y)dxdy \ge \sum_i\sum_j \min_{R_{i,j}} f(x,y)\Delta x_i \Delta y_j$$

case 5: Average points. In each $R_{i,j}$ we take $f(x_i^*, y_j^*) = \frac{1}{4}[f(x_{i-1}, y_{j-1}) + f(x_i, y_{j-1}) + f(x_{i-1}, y_j) + f(x_i, y_j)]$. This gives a good approximation.

$$\int\int_D f(x,.y)dxdy \sim \sum_i\sum_j \frac{1}{4}[f(x_{i-1}, y_{j-1}) + f(x_i, y_{j-1}) + f(x_{i-1}, y_j) + f(x_i, y_j)]\Delta x_i \Delta y_j$$

A special case of Case 3 and Case 4 is

$$\min_R f(x,y)(b-a)(d-c) \le \int\int_D f(x,y)dxdy \le \max_R f(x,y)(b-a)(d-c)$$

Example: Let $f(x,y) = \sqrt{xy}$ and $R = [0,4]\times[0,4]$.

(a) Without dividing into subrectangles, find an upper and lower bound for $\int\int_R f(x,y)dxdy$.
Solution: In R, $0 \le f \le \sqrt{4\times 4} = 4$. So $\min_R f = 0, \max_R f = 4$ and

$$0 \le \int\int_R f(x,y)dxdy \le 4 \times |R| = 4 \times 14 = 64$$

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(b) Divide the rectangle into four subrectangles: $0 = x_0 < x_1 = 2 < x_4, 0 = y_0 < y_1 = 2 < y_2 = 4$, $R_{1,1} = [0,2]\times[0,2], R_{2,1} = [2,4]\times[0,2], R_{1,2} = [0,2]\times[2,4], R_{2,2} = [2,4]\times[2,4]$. Use the left hand corner to compute the Riemann sum
Solution: The Riemann sum using the left hand corner points is

$$f(0,0)|R_{1,1}|+f(2,0)|R_{2,1}|+f(0,2)|R_{1,2}|+f(2,2)|R_{2,2}| = 0\times4+0\times4+0\times4+2\times4 = 8$$

(c) Divide the rectangle into four subrectangles: $0 = x_0 < x_1 = 2 < x_4, 0 = y_0 < y_1 = 2 < y_2 = 4$, $R_{1,1} = [0,2]\times[0,2], R_{2,1} = [2,4]\times[0,2], R_{1,2} = [0,2]\times[2,4], R_{2,2} = [2,4]\times[2,4]$. Use the right hand corner to compute the Riemann sum
Solution: The Riemann sum using the right hand corner points is

$$f(2,2)|R_{1,1}|+f(4,2)|R_{2,1}|+f(2,4)|R_{1,2}|+f(4,4)|R_{2,2}| = 2\times4+\sqrt{8}\times4+\sqrt{8}\times4+4\times4 = 24+16\sqrt{2}$$

(d) Divide the rectangle into four subrectangles: $0 = x_0 < x_1 = 2 < x_4, 0 = y_0 < y_1 = 2 < y_2 = 4$, $R_{1,1} = [0,2]\times[0,2], R_{2,1} = [2,4]\times[0,2], R_{1,2} = [0,2]\times[2,4], R_{2,2} = [2,4]\times[2,4]$. Find an upper bound on $\int\int_R f(x,y)dxdy$.

Solution: In each subrectangle we need to find the maximum of f. The maximum of f is attained at the right hand corner point. So the upper bound is

$$\int\int_R f(x,y)dxdy \le f(2,2)|R_{1,1}|+f(4,2)|R_{2,1}|+f(2,4)|R_{1,2}|+f(4,4)|R_{2,2}| = 2\times4+\sqrt{8}\times4+\sqrt{8}\times4+4\times4 = 24+16$$

(e) Divide the rectangle into four subrectangles: $0 = x_0 < x_1 = 2 < x_4, 0 = y_0 < y_1 = 2 < y_2 = 4$, $R_{1,1} = [0,2]\times[0,2], R_{2,1} = [2,4]\times[0,2], R_{1,2} = [0,2]\times[2,4], R_{2,2} = [2,4]\times[2,4]$. Find a lower bound on $\int\int_R f(x,y)dxdy$.

Solution: In each subrectangle we need to find the minimum of f. The minimum of f is attained at the left hand corner point. So the lower bound is

$$\int\int_R f(x,y)dxdy \ge f(0,0)|R_{1,1}|+f(2,0)|R_{2,1}|+f(0,2)|R_{1,2}|+f(2,2)|R_{2,2}| = 0\times4+0\times4+0\times4+2\times4 = 8$$

(f) Divide the rectangle into four subrectangles: $0 = x_0 < x_1 = 2 < x_4, 0 = y_0 < y_1 = 2 < y_2 = 4$, $R_{1,1} = [0,2]\times[0,2], R_{2,1} = [2,4]\times[0,2], R_{1,2} = [0,2]\times[2,4], R_{2,2} = [2,4]\times[2,4]$. Find an approximation of $\int\int_R f(x,y)dxdy$.

Solution: In each subrectnagle we need to find the average. In $R_{1,1}$ the average is $\frac{1}{4}[0+0+0+2] = \frac{1}{2}$. In $R_{2,1}$ the average is $\frac{1}{4}[0+2+2\sqrt{2}+0] = \frac{1}{2}(1+\sqrt{2})$. In $R_{1,2}$ the average is $\frac{1}{2}(1+\sqrt{2})$. In $R_{2,2}$ the average is $\frac{1}{4}[2+2\sqrt{2}+4] = \frac{1}{2}[3+\sqrt{2}]$. Hence

$$\int\int_R f(x,y)dxdy \sim \frac{1}{2}\times4+\frac{1}{2}(1+\sqrt{2})\times4+\frac{1}{2}(1+\sqrt{2})\times4\frac{1}{2}(3+\sqrt{2}) = 2(6+3\sqrt{2})$$