Final Review of MATH256-103, 2018-2019

Coverage of this course in the book "Elementary Differential Equations and Boundary value Problems" by Boyce and DiPrima:
Chapter 2: 2.1, 2.2, 2.3, 2.4, 2.5
Chapter 3: 3.1, 3.2, 3.3, 3.4, 3.5, 3.6, 3.7, 3.8
Chapter 4: 4.2, 4.3
Chapter 6: 61., 6.2, 6.3, 6.4, 6.5, 6.6
Chapter 7: 7.4, 7,5, 7.6, 7.7, 7.8, 7.9
Chapter 10: 10.1, 10.2, 10.3, 10.4, 10.5, 10.6, 10.7, 10.8

Formulas to be provided on your final exam paper:
I. Reduction of order formula: y_1 is given, then $y_2 = y_1 v$ and v satisfies $v' = \frac{W}{y_1^2}$
II. Laplace transform formulas (as in Midterm II)
III. Fourier series coefficients

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos(\frac{n\pi}{L}x) + b_n \sin(\frac{n\pi}{L}x))dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos(\frac{n\pi}{L}x)dx, n = 0, 1, ...; b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin(\frac{n\pi}{L}x)dx$$

1. First order equations
1.1. Homogeneous linear first order

$$y' + p(t)y = 0, \qquad \text{is} \qquad y = Ce^{-\int p(t)dt}$$

1.2. Inhomogeneous linear first order

$$y' + p(t)y = g(t)$$

$$\text{compute } \mu(t) = e^{\int p(t)dt}, \quad \int \mu(t)g(t)dt =$$

$$y(t) = \frac{1}{\mu(t)}(C + \int \mu(t)g(t)dt)$$

1.3. Separable equation

$$\frac{dy}{dt} = f(t)g(y), \qquad \int \frac{dy}{g(y)} = \int f(t)dt$$

1.4. Bernoulli equation

$$y' + p(t)y = q(t)y^n, \quad \text{let } v = y^{1-n}$$
$$v' + (1-n)p(t)v = (1-n)q(t)$$

1.5. Homogeneous equation

$$\frac{dy}{dx} = f(\frac{y}{x})$$

$$\text{let } v = \frac{y}{x},\ x\frac{dv}{dx} = f(v) - v$$

1.6. Interval of Existence: three factors a) The solution, b) The Equation, c) the Initial Condition
1.7. Difference between linear and nonlinear: for linear equation, existence is global and uniqueness is guaranteed; for nonlinear equation, existence and uniqueness depends on whether or not $|\frac{\partial f}{\partial y}(t_0, y_0)$ is bounded.
1.8. Applications in banking, falling objects, escaping velocity problem
1.9. Autonomous ODEs:

$$\frac{dy}{dt} = f(y)$$

Classification of critical points: $f(y_0) = 0, f'(y_0) < 0$ then y_0 is stable; $f(y_0) = 0, f'(y_0) > 0$ then y_0 is unstable. Population models.

2. Linear Second Order Equations

$$y'' + p(t)y' + q(t)y = g(t)$$

2.1. Homogeneous case

$$y'' + p(t)y' + q(t)y = 0$$

2.1.1. Wronskian $W[y_1, y_2](t) = y_1y_2' - y_1'y_2$.

$$\text{Abel's equation } W' + pW = 0$$

$$W(t) = Ce^{-\int p(s)ds}$$

2.1.2. Set of Fundamental Solutions y_1, y_2. All solutions are given by $y = c_1 y_1 + c_2 y_2$
2.1.3. Constant Coefficients:

$$ay'' + by' + cy = 0$$

$$\text{Characteristic equation} \quad ar^2 + br + c = 0$$

- $b^2 - 4ac > 0$, two unequal real roots $r_1 \neq r_2$.

$$y_1 = e^{r_1 t}, y_2 = e^{r_2 t}$$

- $b^2 - 4ac < 0$, two complex roots: $r_1 = \lambda + i\mu, r_2 = \lambda - i\mu$

$$y_1 = e^{\lambda t}\cos(\mu t), \quad y_2 = e^{\lambda t}\sin(\mu t)$$

- $b^2 - 4ac = 0$, two equal roots: $r_1 = r_2 = r$

$$y_1 = e^{rt}, \ y_2 = te^{rt}$$

2.1.4. Euler's type equation

$$at^2 y'' + bty' + cy = 0$$

$$\text{Characteristic equation} \quad ar(r-1) + br + c = 0, \quad ar^2 + (b-a)r + c = 0$$

- $(b-a)^2 - 4ac > 0$, two unequal real roots $r_1 \neq r_2$.

$$y_1 = t^{r_1}, y_2 = t^{r_2}$$

- $(b-a)^2 - 4ac = 0$, two equal roots: $r_1 = r_2 = r$

$$y_1 = t^r, \ y_2 = t^r \log t$$

- $(b-a)^2 - 4ac < 0$, two complex roots: $r_1 = \lambda + i\mu, r_2 = \lambda - i\mu$

$$y_1 = t^{\lambda}\cos(\mu \log t), \quad y_2 = t^{\lambda}\sin(\mu \log t)$$

2.1.5. Reduction of Order

$$y'' + p(t)y' + q(t)y = 0$$

If y_1 is known, we can get y_2 by letting $y_2 = v(t)y_1$. Then v satisfies

$$v' = \frac{W}{y_1^2}$$

where $W = e^{-\int p(t)dt}$ is the Wronskian.

2.2 Inhomogeneous equations

$$y'' + py' + qy = h(t)$$

$$y = y_p(t) + c_1 y_1 + c_2 y_2$$

where y_p is a particular solution and y_1, y_2–set of fundamental solutions of homogeneous problem.

2.2.1 Method I: Method of Undetermined Coefficients. Works only for

$$ay'' + by' + cy = h(t)$$

- $h(t) = a_0 + a_1 t + ... + a_n t^n$

$$y_p = t^s(A_0 + A_1 t + ... + A_n t^n)$$

- $h(t) = e^{\alpha t}(a_0 + a_1 t + ... + a_n t^n)$

$$y_p = t^s e^{\alpha t}(A_0 + A_1 t + ... + A_n t^n)$$

- $h(t) = e^{\alpha t}(a_0 + a_1 t + ... + a_n t^n)\cos(\beta t)$ or $g(t) = e^{\alpha t}(a_0 + a_1 t + ... + a_n t^n)\sin(\beta t)$

$$y_p = t^s e^{\alpha t}[(A_0 + A_1 t + ... + A_n t^n)\cos(\beta t) + (B_0 + B_1 t + ... + B_n t^n)\sin(\beta t)]$$

- s equals either 0, or 1, or 2, is the least integer such that there are no solutions of the homogeneous problem in y_p

- $h(t) = h_1 + ... + h_m$

$$y_p = y_{p,1} + ... y_{p,m}$$

2.2.2. Method of Variation of Parameters

$$y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$$

where

$$\begin{cases} u_1^{'}y_1 + u_2^{'}y_2 = 0, \\ u_1^{'}y_1^{'} + u_2^{'}y_2 = g(t) \end{cases}$$

Formula:

$$u_1 = -\int \frac{y_2 g(t)}{W} dt, \quad u_2 = \int \frac{y_1 g(t)}{W} dt$$

$$y_p = -y_1(t) \int \frac{y_2 g(t)}{W} dt + y_2(t) \int \frac{y_1 g(t)}{W} dt$$

2.3. Applications: Spring-Mass System

$$mu^{''} + \gamma u^{'} + ku = F(t)$$

2.3.1. $\gamma = 0,\ F_0 = 0$

$$u = A\cos(\omega_0 t) + B\sin(\omega_0 t) = R\cos(\omega_0 t - \delta), R\cos\delta = A, R\sin\delta = B$$

2.3.2. $\gamma = 0, F_0 = F_0\cos(\omega t)$. If $\omega = \omega_0$, resonance, solution becomes unbounded

2.3.3. $\gamma \neq 0$. All solutions approach zero as $t \to +\infty$. $\gamma^2 > 4mk$–over-damping; $\gamma^2 = 4mk$ critical damping; $\gamma^2 < 4mk$ under damping.

2.4. Higher order ODEs with constant coefficients:

$$a_n y^((n)) + a_{n-1}y^{(n-1)} + ... + a_1 y^{'} + a_0 y = g(t)$$

2.4.1. Homogeneous case: solutions are sums of $t^s e^{rt}$, where $s = 0, 1, ..., m-1$ and m is the algebraic multiplicity of the root r.

2.4.2: Inhomogeneous case: t^s(same type), s=smallest integer so that no part of y_p is a solution of homogeneous ODE.

3. Systems of Equations

3.1. General Theory

$$\mathbf{x}^{'}(t) = A(t)\mathbf{x} + \mathbf{g}(t)$$

$$x = x_h + x_p$$

3.2. Homogeneous Systems

$$\mathbf{x}^{'}(t) = A(t)\mathbf{x}$$

$$x_h = \sum_{j=1}^{n} c_j \mathbf{x}^{(\mathbf{j})}(t)$$

3.3. Homogeneous Systems with Constant Coefficients

$$\mathbf{x}^{'} = A\mathbf{x}$$

(2×2 case only)

3.3.1. Two linearly independent eigenvectors

$$x = c_1 \xi^1 e^{r_1 t} + c_2 \xi^2 e^{r_2 t}$$

3.3.2. Complex eigenvalue

$$x = c_1 Re(\xi e^{(\lambda+i\mu)t}) + c_2 Im(\xi e^{(\lambda+i\mu)t})$$

3.3.3. Repeated eigenvalues, only on eigenvector

$$x^1 = \xi e^{rt}, x^2 = \xi t e^{rt} + \eta, (A - rI)\eta = \xi$$

3.3.4. Types and stabilities and trajectories.

Types: unstable saddle, stable node, unstable node, improper node, stable spiral, unstable spiral, center

3.3.5. Euler type systems

$$t\mathbf{x}^{'} = A\mathbf{x}$$

$$\mathbf{x} = \xi t^r, A\xi = r\xi$$

3.3.6. Fundamental Matrix $\Psi(t)$ and $\Phi(t)$: $\Phi(t) = \Psi(t)(\Psi(0))^{-1}$.

3.4. Inhomogeneous Systems with Constant Coefficients

$$\mathbf{x}^{'} = A\mathbf{x} + \mathbf{g}(t)$$

Method I: Method of undetermined coefficients

If g is $ae^{\lambda t}\cos(\mu t)$ or polynomials or sum of these types, then

$$x_h = t^s(\text{the same type}) + \text{lower order terms}$$

Special case: r is an eigenvalue, $\mathbf{g} = e^{rt}\mathbf{a}_0$. $x_p = ate^{rt} + be^{rt}$

$$Aa = ra, Ab + \mathbf{a}_0 = rb + a$$

Method II: Method of variation of constants

$$x_p = \mathbf{\Psi c}(t), \mathbf{\Psi c}^{'} = \mathbf{g}$$

Method III: Diagonalization

$$x = Ty, y^{'} = T^{-1}AT + T^{-1}\mathbf{g}$$

where $T^{-1}AT$ is a diagonal matrix.

4. Laplace Transforms

4.1 The Laplace transform is defined by

$$\mathcal{L}[f](s) = \int_0^\infty e^{-st} f(t)dt$$

4.2 List of Formulas

$f(t)$	$\mathcal{L}[f](s)$	
1	$\frac{1}{s}$	
t	$\frac{1}{s^2}$	
e^{at}	$\frac{1}{s-a}$	
$\cos at$	$\frac{s}{s^2+a^2}$	
$\sin at$	$\frac{a}{s^2+a^2}$	
$e^{\lambda t}\cos(\mu t)$	$\frac{s-\lambda}{(s-\lambda)^2+\mu^2}$	
$e^{\lambda t}\sin(\mu t)$	$\frac{\mu}{(s-\lambda)^2+\mu^2}$	
$e^{at}f(t)$	$\mathcal{L}[f](s-a)$	
$tf(t)$	$-\frac{d}{ds}(\mathcal{L}[f](s))f(t-c)u_c(t)$	$e^{-cs}\mathcal{L}[f](s)$
$\delta(t-c)$	e^{-cs}	
$\int_0^t f(t-\tau)g(\tau)d\tau$	$\mathcal{L}[f](s)\mathcal{L}[g](s)$	
$f^{'}(t)$	$s\mathcal{L}[f](s) - f(0)$	
$f^{''}(t)$	$s^2\mathcal{L}[f](s) - sf(0) - f^{'}(0)$	

$$u_c(t) = H(t-c) = \begin{cases} 0, t < c \\ 1, t \geq c \end{cases}$$

4.3. Write piecewise continuous functions as sums of $H(t-c), f(t-c)H(t-c)$

4.4. $\delta(t-c)$ and covolutions

4.5. Use of Laplace transforms to solve:

$$a_n y^{(n)} + a_{n-1}y^{(n-1)} + ... + a_1 y' + a_0 y = g(t), y^{(j)}(0) = y_j, j = 0, ..., n-1$$

where

$$g(t) = e^{at}\cos(bt) \text{ or } e^{at}\sin(bt) \text{ or polynomials}$$

or

$$g(t) = \sum_{j=1}^{m} d_j(t) u_{c_j}(t), 0 \leq c_1 < c_2 < ... < c_m$$

or

$$g(t) = \delta(t-c)$$

5. Fourier Series and Method of Separation of Variables

5.1. Eigenvalue problems

$$y'' + \lambda y = 0, 0 < x < L, y(0) = y(L) = 0; \lambda = (\frac{n\pi}{L})^2, y = \sin(\frac{n\pi}{L}), n = 1, 2, ...$$

$$y'' + \lambda y = 0, 0 < x < L, y'(0) = y'(L) = 0; \lambda = (\frac{n\pi}{L})^2, y = \cos(\frac{n\pi}{L}), n = 0, 1, 2, ...$$

$$y'' + \lambda y = 0, -L < x < L, y(-L) = y(L), y'(-L) = y'(L); \lambda = (\frac{n\pi}{L})^2, y = c_1\cos(\frac{n\pi}{L}) + c_2\sin(\frac{n\pi}{L}), n = 0, 1,$$

5.2. Fourier Series

Let f be a function of $2L$ periodic, i.e., $f(x+2L) = f(x)$. Its Fourier Series is

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos(\frac{n\pi}{L}x) + b_n\sin(\frac{n\pi}{L}x))$$

where

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos(\frac{n\pi}{L}x)dx, n = 0, 1, 2, ...$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin(\frac{n\pi}{L}x)dx, n = 1, 2, ...$$

Convergence Formula:

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos(\frac{n\pi}{L}x)+b_n\sin(\frac{n\pi}{L}x))=\frac{1}{2}(f(x-)+f(x+))$$

5.3. Fourier Sine Series

Let f be an odd function of $2L$ periodic, i.e. $f(x)=-f(-x), f(x+2L)=f(x)$. Its Fourier Sine Series is

$$f(x)\sim\sum_{n=1}^{\infty}b_n\sin(\frac{n\pi}{L}x)$$

where

$$b_n=\frac{2}{L}\int_0^L f(x)\sin(\frac{n\pi}{L}x)dx, n=1,2,...$$

5.4. Fourier Cosine Series

Let f be an even function of $2L$ periodic, i.e. $f(x)=f(-x), f(x+2L)=f(x)$. Its Fourier Sine Series is

$$f(x)\sim\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos(\frac{n\pi}{L}x)$$

where

$$a_n=\frac{2}{L}\int_0^L f(x)\sin(\frac{n\pi}{L}x)dx, n=0,1,2,...$$

5.5. Even or Odd Extension

Let $f(x)$ be defined in $[0,L)$. We can extend it to an even $2L$-periodic function, or an odd $2L$-periodic function as follows

$$f_{even}(x)=\begin{cases} f(x), 0\le x<L,\\ f(-x), -L<x<0\end{cases}\quad f_{even}(x+2L)=f_{even}(x);$$

$$f_{odd}(x)=\begin{cases} f(x), 0\le x<L,\\ -f(-x), -L<x<0\end{cases}\quad f_{odd}(x+2L)=f_{odd}(x)$$

5.6. Method of Separation of Variables applied to Heat Equation

5.6.1. The solution to heat equation with Dirichlet boundary condition

$$\begin{cases} u_t=ku_{xx}, 0<x<L, t>0,\\ u(0,t)=0, u(L,t)=0\\ u(x,0)=f(x)\end{cases}$$

is

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-k(\frac{n\pi}{L})^2 t} \sin(\frac{n\pi}{L}x), \ \ a_n = \frac{2}{L}\int_0^L f(x)\sin(\frac{n\pi}{L}x)dx, n = 1, 2, ...$$

5.6.2. The solution to heat equation with Neumann boundary condition

$$\begin{cases} u_t = ku_{xx}, 0 < x < L, t > 0, \\ u_x(0,t) = 0, u_x(L,t) = 0 \\ u(x,0) = f(x) \end{cases}$$

is

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-k(\frac{n\pi}{L})^2 t} \cos(\frac{n\pi}{L}x), \ \ a_n = \frac{2}{L}\int_0^L f(x)\cos(\frac{n\pi}{L}x)dx, \ \ n = 0, 1, ...$$

5.6.3. The solution to heat equation with periodic boundary condition

$$\begin{cases} u_t = \alpha^2 u_{xx}, -L < x < L, t > 0, \\ u(-L,t) = u(L,t), u_x(-L,t) = u_x(L,t) \\ u(x,0) = f(x) \end{cases}$$

is

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} e^{-k(\frac{n\pi}{L})^2 t}(a_n \cos(\frac{n\pi}{L}x) + b_n \sin(\frac{n\pi}{L}x),$$

$$a_n = \frac{1}{L}\int_{-L}^L f(x)\cos(\frac{n\pi}{L}x)dx, \ \ n = 0, 1, ..., b_n = \frac{1}{L}\int_{-L}^L \sin(\frac{n\pi}{L}x)dx, n = 1, ...$$

5.6.4. The solution to

$$\begin{cases} u_t = ku_{xx} + f(x), 0 < x < L, t > 0, \\ u(0,t) = T_1, u(L,t) = T_2 \\ u(x,0) = f(x) \end{cases}$$

is

$$u(x,t) = u^0(x) + \sum_{n=1}^{\infty} a_n e^{-k(\frac{n\pi}{L})^2 t} \sin(\frac{n\pi}{L}x)$$

where u^0 satisfies the steady-state: $ku^0_{xx} + f(x) = 0, u^0(0) = T_0, u^0(L) = T_1$. When $f = 0$, we have

$$u^0(x) = T_1 + \frac{T_2 - T_1}{L}x, \ \ a_n = \frac{2}{L}\int_0^L (f(x) - v(x))\sin(\frac{n\pi}{L}x)dx, n = 1, 2, ...$$

5.6. Method of Separation of Variables applied to Wave Equation

5.6.1. The solution to

$$\begin{cases} u_{tt} = ac^2 u_{xx}, 0 < x < L, t > 0, \\ u(0,t) = 0, u(L,t) = 0 \\ u(x,0) = f(x), u_t(x,0) = g(x) \end{cases}$$

is

$$u(x,t) = \sum_{n=1}^{\infty} (a_n \cos(\frac{n\pi}{L} ct) + b_n \sin(\frac{n\pi}{L} ct)) \sin(\frac{n\pi}{L} x)$$

$$a_n = \frac{2}{L} \int_0^L f(x) \sin(\frac{n\pi}{L} x) dx, b_n \frac{n\pi}{L} c = \frac{2}{L} \int_0^L f(x) \sin(\frac{n\pi}{L} x) dx, n = 1, 2, ...,$$

5.6.2. The solution to

$$\begin{cases} u_{tt} = c^2 u_{xx}, 0 < x < L, t > 0, \\ u_x(0,t) = 0, u_x(L,t) = 0 \\ u(x,0) = f(x), u_t(x,0) = g(x) \end{cases}$$

is

$$u(x,t) = \frac{a_0 + b_0 t}{2} + \sum_{n=1}^{\infty} (a_n \cos(\frac{n\pi}{L} ct) + b_n \sin(\frac{n\pi}{L} ct)) \cos(\frac{n\pi}{L} x)$$

$$a_n = \frac{2}{L} \int_0^L f(x) \cos(\frac{n\pi}{L} x) dx, n = 0, 1, ..., b_0 = \frac{2}{L} \int_0^L g(x) dx, b_n \frac{n\pi}{L} c = \frac{2}{L} \int_0^L g(x) \cos(\frac{n\pi}{L} x) dx, n = 1, 2, ...,$$

For inhomogeneous boundary conditions, we subtract a steady-state first $u = u^0(x) + v(x,t)$.

5.7. Method of Separation of Variables applied to Laplace Equation

5.7.1. The solution to

$$\begin{cases} u_{xx} + u_{yy} = 0, 0 < x < a, 0 < y < b \\ u(x,0) = 0, u(x,b) = 0 \\ u(0,y) = 0, u(a,y) = g(y) \end{cases}$$

is

$$u(x,y) = \sum_{n=1}^{\infty} a_n \sinh(\frac{n\pi}{b} x) \sin(\frac{n\pi}{b} y)$$

$$a_n \sinh(\frac{n\pi}{b} a) = \frac{2}{b} \int_0^b g(y) \sin(\frac{n\pi}{b} y) dy, n = 1, ...$$

5.7.2. For the problem

$$\begin{cases} u_{xx} + u_{yy} = 0, 0 < x < a, 0 < y < b \\ u(x,0) = f_1(x), u(x,b) = f_2(x) \\ u(0,y) = g_1(y), u(a,y) = g_2(y) \end{cases}$$

we decompose into four problems: each of them having three homogeneous boundary conditions and one inhomogeneous boundary condition.