Sol'ns to MATH 256, 2014, Final Exam

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1. This is a separable equation

$$\frac{dy}{dx} = \left(\frac{y}{x}\right)^2 + 2\left(\frac{y}{x}\right)$$

Let $y = xv$. Then

$$y' = xv' + v = v^2 + 2v$$

$$xv' = v^2 + v$$

$$\frac{v'}{v^2+v} = \frac{1}{x}$$

$$\int \frac{dv}{v^2+v} = \int \frac{1}{x}\,dx$$

$$\int \left(\frac{1}{v} - \frac{1}{v+1}\right)dv = \ln x + c$$

$$\ln \frac{v}{v+1} = \ln x + c$$

$$\frac{v}{v+1} = cx$$

$$v = cx(v+1) \Rightarrow \qquad v = \frac{cx}{1-cx}$$

$$v(1) = 1 \quad \Rightarrow \quad 1 = \frac{c}{1-c} \Rightarrow c = \frac{1}{2}$$

So $v = \dfrac{\frac{1}{2}x}{1-\frac{x}{2}} = \dfrac{x}{2-x}$

$$y = \frac{x^2}{2-x}$$

domain of existence: $0 < x < 2$

2. $f(y) = (y^2-1)(e^y-1)$

$f(y) = 0 \Rightarrow y = \pm 1,\ y = 0 \Rightarrow$ critical points are $y = 0,\ y = \pm 1$

For $y=0$, $f'(y) = e^y(y^2-1) + 2y(e^y-1) = -1 < 0 \Rightarrow y = 0$ is stable

for $y=1$, $f'(y) = 2(e-1) > 0 \Rightarrow y = 1$ is unstable

for $y=-1$, $f'(y) = -2(e^{-1}-1) > 0 \Rightarrow y = -1$ is unstable

3. We use method of Undetermined Coefficient.

$y''-3y'+2y=0 \Rightarrow r^2-3r+2 \Rightarrow r_1=+1,\ r_2=2$

$y_h=c_1e^t+c_2e^{2t}$

$y_p=At^se^t=Ate^t \Rightarrow y_p''-3y_p'+2y_p=2Ae^t-3Ae^t=-Ae^t=e^t$

So $A=-1$, $y=-te^t+c_1e^t+c_2e^{2t}$

$$\left.\begin{array}{l} y(0)=1 \Rightarrow c_1+c_2=1 \\ y'(0)=0 \Rightarrow -1+c_1+2c_2=0 \end{array}\right\} c_2=0,\ c_1=1$$

$y=-te^t+e^t$

4. $y''-\frac{1}{t}y'+4t^2y=0$

$W=e^{-\int(-\frac{1}{t})dt}=t$

By the reduction of order formula, $y_1=\cos t^2$

$y_2=y_1v \Rightarrow v'=\frac{W}{y_1^2}=\frac{t}{\cos^2t^2}$

$v=\int\frac{t}{\cos^2t^2}dt=\frac{1}{2}\int\sec^2u\,du=\frac{1}{2}\tan t^2$

$y_2=\cos t^2\left(\frac{1}{2}\tan t^2\right)=\frac{1}{2}\sin t^2$

5. $\begin{vmatrix}1-r & -3\\ -2 & 2-r\end{vmatrix}=0 \Rightarrow r^2-3r+2-6=r^2-3r-4=(r-4)(r+1)=0$

$r_1=4,\ r_2=-1$

For $r_1=4 \Rightarrow \begin{pmatrix}1-4 & -3\\ -2 & 2-4\end{pmatrix}\begin{pmatrix}a_1\\a_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix} \Rightarrow \begin{pmatrix}a_1\\a_2\end{pmatrix}=\begin{pmatrix}1\\-1\end{pmatrix}$

For $r_2=-1 \Rightarrow \begin{pmatrix}2 & -3\\ -2 & 3\end{pmatrix}\begin{pmatrix}a_1\\a_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix} \Rightarrow \begin{pmatrix}a_1\\a_2\end{pmatrix}=\begin{pmatrix}3\\2\end{pmatrix}$

$x^{(1)}=e^{4t}\begin{pmatrix}1\\-1\end{pmatrix},\ x^{(2)}=e^{-t}\begin{pmatrix}3\\2\end{pmatrix}$

By Method of Undetermined Coefficients,

$x_p=At+B \Rightarrow A=a\begin{pmatrix}1 & -3\\ -2 & 2\end{pmatrix}t+\begin{pmatrix}1 & -3\\ -2 & 2\end{pmatrix}b+t\begin{pmatrix}1\\2\end{pmatrix}+\begin{pmatrix}-2\\0\end{pmatrix}$

$$a\begin{pmatrix}1 & -3\\ -2 & 2\end{pmatrix} + \begin{pmatrix}1\\2\end{pmatrix} = 0 \Rightarrow \left.\begin{array}{l} a_1 - 3a_2 + 1 = 0 \\ -2a_1 + 2a_2 + 2 = 0 \end{array}\right\} \Rightarrow a_2 = 1,\ a_1 = 2$$

$$a = \begin{pmatrix}1 & -3\\ -2 & 2\end{pmatrix} b + \begin{pmatrix}-2\\0\end{pmatrix} \Rightarrow \begin{pmatrix}2\\1\end{pmatrix} - \begin{pmatrix}-2\\0\end{pmatrix} = \begin{pmatrix}1 & -3\\ -2 & 2\end{pmatrix}\begin{pmatrix}b_1\\b_2\end{pmatrix}$$

$$\Rightarrow \begin{array}{l} b_1 - 3b_2 = 4 \\ -2b_1 + 2b_2 = 1 \end{array} \Rightarrow b_2 = \frac{-9}{4},\ b_1 = -\frac{11}{4}$$

$$x = x_p + x_h = \begin{pmatrix}2\\1\end{pmatrix}t + \begin{pmatrix}-\frac{9}{4}\\ -\frac{11}{4}\end{pmatrix} + c_1 e^{4t}\begin{pmatrix}1\\-1\end{pmatrix} + c_2 e^{-t}\begin{pmatrix}3\\2\end{pmatrix}$$

6. $L[y] = Y(s)$

$$s^2Y - sy(0) - y'(0) + 2(sY - y(0)) + 2Y(s) = 2\frac{e^{-s}}{s} + e^2 \cdot e^{-2s}$$

$$Y = \frac{s+2}{s^2+2s+2} + \frac{2e^{-s}}{s(s^2+2s+2)} + e^2 \frac{1}{s^2+2s+2} e^{-2s}$$

$$\frac{s+2}{s^2+2s+2} = \frac{s+1}{(s+1)^2+1} + \frac{1}{(s+1)^2+1}$$

$$\frac{1}{s(s^2+2s+2)} = \frac{A}{s} + \frac{B(s+1)}{(s+1)^2+1} + \frac{C}{(s+1)^2+1}$$

$$= \frac{A(s^2+2s+1) + Bs(s+1) + Cs}{s((s+1)^2+1)}$$

$$A + B = 0,\ 2A + B + C = 0,\ A = 1 \Rightarrow B = -1,\ C = -1$$

$$\frac{1}{s(s^2+2s+2)} = \frac{1}{s} - \frac{s+1}{(s+1)^2+1} - \frac{1}{(s+1)^2+1}$$

$$y = \frac{s+1}{(s+1)^2+1} + \frac{1}{(s+1)^2+1} + \left(\frac{1}{s} - \frac{s+1}{(s+1)^2+1} - \frac{1}{(s+1)^2+1}\right)2e^{-s} + e^2\left(\frac{1}{(s+1)^2+1}\right)e^{-2s}$$

$$y = e^{-t}\cos t + e^{-t}\sin t + 2H(t-1)\left[1 - e^{-(t-1)}\cos(t-1) - e^{-(t-1)}\sin(t-1)\right]$$
$$+ e^2 H(t-2)\, e^{-(t-2)} \sin(t-2).$$

7. (a) $L=1$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx = \int_{-1}^{1} f(x)\,dx = \int_{-1}^{0} 1 + \int_0^1 x = 1+\frac{1}{2} = \frac{3}{2}$$

$$a_1 = \int_{-1}^{1} f(x)\cos n\pi x\,dx = \int_{-1}^{0}\cos n\pi x + \int_0^1 x\cos n\pi x\,dx$$

$$= \frac{1}{n\pi}(+\sin n\pi x)\Big|_{-1}^{0} + \frac{1}{n\pi}x\sin(n\pi) + \frac{1}{(n\pi)^2}\cos(n\pi x)\Big|_0^1$$

$$= \frac{1}{(n\pi)^2}\left((-1)^n - 1\right) = \frac{1}{\pi^2}(-1-1) = -\frac{2}{\pi^2}$$

$$b_1 = \int_{-1}^{1} f(x)\sin \pi x\,dx = \int_{-1}^{0}\sin \pi x\,dx + \int_0^1 x\sin \pi x\,dx$$

$$= -\frac{1}{\pi}(1+1) + \frac{1}{\pi}x\cos\pi x - \frac{1}{\pi^2}\sin\pi x\Big|_0^1$$

$$= -\frac{2}{\pi} \; \frac{1}{\pi} = -\frac{3}{\pi}$$

(b) By the Theorem,

$$\frac{a_0}{2} + \sum_{n=1}^{+\infty}(a_n\cos n\pi x + b_n\sin n\pi x) = \frac{1}{2}\left[f(x-) + f(x+)\right]$$

So for $x=-\frac{1}{2}$, f is continuous, so $= f(-\frac{1}{2}) = 1$

for $x=0$, $f(0-)=1$, $f(0+)=0 \Rightarrow = \frac{1}{2}$

for $x=\frac{1}{2}$, f is continuous, so $= f(\frac{1}{2}) = \frac{1}{2}$

8. $\phi(x) = 2\cos^2 x = \cos 2x - 1$, $L=\pi$, $k=1$.

Neumann problem

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{+\infty} a_n \cos\frac{n\pi}{L}x\; e^{-(\frac{n\pi}{L})^2 t}$$

$$= \frac{a_0}{2} + \sum_{n=1}^{+\infty} a_n\cos nx\; e^{-n^2 t}$$

$$u(x,0) = \cos 2x - 1 = \frac{a_0}{2} + \sum_{n=1}^{+\infty} a_n\cos nx$$

$a_0=-2$, $a_2=1$, $a_n=0$ for $n\neq 0,2$

Hence

$$u(x,t) = -1 + \cos 2x\; e^{-4t}$$