sol'ns to Final Exam in April18, MATH256

1. $y' + 2\frac{\cos t}{\sin t}\, y = \sin^2 t\; y^2$, Bernoulli type

$v = y^{-1}$

$v' - \frac{2\cos t}{\sin t}\, y = -\sin^2 t$

$p = -\frac{2\cos t}{\sin t}$, $\mu = e^{\int p} = e^{-\int \frac{2\cos t}{\sin t}dt} = e^{-2\ln \sin t} = \frac{1}{\sin^2 t}$

$\int \mu g = -\int \sin^2 t \,\frac{1}{\sin^2 t}\, dt = -t$

$v = \frac{1}{\mu}\left(C + \int \mu g\right) = \sin^2 t\,(C - t)$

$v(\frac{\pi}{2}) = 1 \Rightarrow 1 = C - \frac{\pi}{2} \Rightarrow C = \frac{\pi}{2} + 1$

$v = \sin^2 t\,(\frac{\pi}{2} + 1 - t)$

$y = v^{-1} = \frac{1}{\sin^2 t\,(\frac{\pi}{2}+1-t)}$, $t \neq 0$, $t \neq \frac{\pi}{2}+1$

Interval of existence, $0 < t < \frac{\pi}{2}+1$

2. $y' = 2y^2(1+x)$, $\frac{y'}{y^2} = 2(1+x)$

$-\frac{1}{y}dy = 2x + x^2 + C$

$y = -\frac{1}{x^2+2x+C}$ $\quad y(0) = 1 \Rightarrow 1 = -\frac{1}{C} \Rightarrow C = -1$

$y = -\frac{1}{x^2+2x-1}$

$x^2+2x-1 = 0 \Rightarrow x = \frac{-2 \pm \sqrt{8}}{2} = -1 \pm \sqrt{2}$

So $x \neq -1 \pm \sqrt{2}$, $x_0 = 0$

Interval of Existence, $-1-\sqrt{2} < x < -1+\sqrt{2}$

3. (a) $y_1 = \sin x^2$

$y_2 = y_1 v$

$v' = \frac{W}{y_1^2}$

$W = e^{-\int(-\frac{1}{x})dx} = x$

$v' = \frac{x}{\sin^2 x^2}$, $v = \int \frac{x}{\sin^2 x^2} dx = -\frac{1}{2}\cot(x^2)$

$y_2 = -\frac{1}{2}\cos x^2$

(b) choose $y_1 = \sin x^2$, $y_2 = \cos x^2$

$y'' - \frac{1}{x}y' + 4x^2 y = 2x^2$

$y_p = y_1 u_1 + y_2 u_2$

$\sin x^2 u_1' + \cos x^2 u_2' = 0$

$2x\cos x^2 u_1' - 2x\sin x^2 u_2' = 2x^2 \Rightarrow \cos x^2 u_1' - \sin x^2 u_2' = x$

$\Rightarrow u_1' = +x\cos x^2$, $u_2' = -x\sin x^2$

$u_1 = \int x\cos x^2 = \frac{1}{2}\sin x^2$, $u_2 = +\frac{1}{2}\cos x^2$

Thus

$y_p = \frac{1}{2}\sin^2 x^2 + \frac{1}{2}\cos^2 x^2 = \frac{1}{2}$

4. (a) $u = A\cos 2t + B\sin 2t$

$u(0) = 1 \Rightarrow A = 1$, $u'(0) = 2 \Rightarrow 2B = 2 \Rightarrow B = 1$

$u = \cos 2t + \sin 2t = \sqrt{2}\cos(2t - \delta)$

$\sqrt{2}\cos\delta = 1$
$\sqrt{2}\sin\delta = 1$ $\Rightarrow \delta = \frac{\pi}{4}$

(b). (i) $u_p = A$ (ii) $u_p = Ae^t$ (iii) $u_p = (A\cos t + B\sin t) + Ce^{2t}$

(iv) $u_p = t(A\cos 2t + B\sin 2t)$

For case (iv), resonance phenomena occurs.

(b). $\gamma^2 - 4\cdot 4 > 0 \Rightarrow \gamma > 4$

(d) $u'' + 5u' + 4u = 10\sin 2t$

$$u_p = A\cos 2t + B\sin 2t$$

$$u_p' = -2A\sin 2t + 2B\cos 2t$$

So $\quad -10A\sin 2t + 10B\cos 2t = 10\sin 2t$

$-10A = 10 \quad , \quad 10B = 0 \quad \Rightarrow A = -1, \; B = 0$

$$u_p = -\cos 2t$$

$$u = -\cos 2t + A e^{-t} + B e^{-4t}$$

$u(0) = 1 \Rightarrow \quad 1 = -1 + A + B$

$u'(0) = 0 \Rightarrow \quad 0 = -A - 4B \quad \Rightarrow A + 4B = 0$

$A = \frac{8}{3} \qquad B = -\frac{2}{3}$

$$u = -\cos 2t + \frac{8}{3}e^{-t} - \frac{2}{3}e^{-4t}$$

As $t \to +\infty$, u oscillates between -1 and $+1$.

5. $\begin{vmatrix} 1-r & 5 \\ -2 & -5-r \end{vmatrix} = 0 \Rightarrow r^2 + 4r + 5 = 0 \qquad (r+2)^2 + 1 = 0$

$r = -2 \pm i, \quad r_1 = -2 + i \qquad\qquad -2a_1 - (3+i)a_2 = 0$

$$\begin{pmatrix} 1-(-2+i) & 5 \\ -2 & -5-(-2+i) \end{pmatrix} = \begin{pmatrix} 3-i & 5 \\ -2 & -3-i \end{pmatrix}$$

$a_2 = 2, \quad a_1 = -(3+i)$

$$\begin{pmatrix} -(3+i) \\ 2 \end{pmatrix} e^{-2t}(\cos t + i\sin t) = \begin{pmatrix} -(3+i)e^{-2t}(\cos t + i\sin t) \\ 2e^{-2t}\cos t + i2e^{-2t}\sin t \end{pmatrix}$$

$$x^{(1)} = \begin{pmatrix} -3e^{-2t}\cos t + e^{-2t}\sin t \\ 2e^{-2t}\cos t \cancel{+2e^{-2t}\sin t} \end{pmatrix}, \quad x^{(2)} = \begin{pmatrix} -3e^{-2t}\sin t - e^{-2t}\cos t \\ 2e^{-2t}\sin t \end{pmatrix}$$

We use Method of Undetermined coefficients

$$x_p = A\cos t + B\sin t$$

$$-a\sin t + b\cos t = Pa\cos t + Pb\sin t + \cos t\begin{pmatrix}1\\0\end{pmatrix}$$

$$Pa + \begin{pmatrix}1\\0\end{pmatrix} = b$$

$$Pb = -a$$

$$-P^2 b + \begin{pmatrix}1\\0\end{pmatrix} = b \Rightarrow (P^2+I)\,b = \begin{pmatrix}1\\0\end{pmatrix}$$

$$P^2 = \begin{pmatrix}1 & 5\\-2 & -5\end{pmatrix}\begin{pmatrix}1 & 5\\-2 & -5\end{pmatrix} = \begin{pmatrix}-9 & -20\\8 & 15\end{pmatrix}$$

$$\begin{pmatrix}-9 & -20\\8 & 15\end{pmatrix}\begin{pmatrix}b_1\\b_2\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} \Rightarrow \begin{matrix}-9b_1-20b_2=1\\8b_1+15b_2=0\end{matrix} \Rightarrow \begin{matrix}(-160+135)b_2 = 8\\ b_2 = -\frac{8}{25}\end{matrix}$$

$$(-9\times15+8\times20)\,b_1 = 15 \Rightarrow b_1 = \frac{15}{25} = \frac{3}{5}$$

$$a = -Pb = -\begin{pmatrix}1 & 5\\-2 & -5\end{pmatrix}\begin{pmatrix}\frac{3}{5}\\-\frac{8}{25}\end{pmatrix} = -\begin{pmatrix}3-\frac{8}{25}\\-\frac{6}{5}+\frac{8}{5}\end{pmatrix}$$

6. $\begin{pmatrix}\alpha-r & 1\\-1 & 0-r\end{pmatrix} = 0 \qquad r^2-\alpha r+1=0$

$\alpha^2>4 \Rightarrow$ two real roots

$\alpha^2=4 \Rightarrow$ equal real roots

$\alpha^2<4 \Rightarrow$ complex

(i) $\alpha<-2 \Rightarrow$ two real roots, $\lambda_1+\lambda_2<0$, node, stable (sink)

(ii) $-2<\alpha<0 \Rightarrow$ complex, $r=\frac{\alpha\pm\sqrt{4-\alpha^2}}{2}$, $\lambda>0 \Rightarrow$ spiral, unstable

(iii) $0<\alpha<2 \Rightarrow$ complex, $r=\frac{\alpha\pm\sqrt{4-\alpha^2}}{2}$, $\lambda<0 \Rightarrow$ spiral, stable

(iv) $\alpha=0$, $r^2+1=0 \Rightarrow r=\pm i$, center, stability undetermined

7 (a) $f(t) = 1 + (t-1)H(t-1) + \left(1+\sin \pi t - (t-1)\right)H(t-2)$

$$= 1 + (t-1)H(t-1) + \left(2 - t + \sin(\pi t)\right)H(t-2)$$

$$= 1 + (t-1)H(t-1) + \left(-(t-2) + \sin(\pi(t-2))\right)H(t-2)$$

$$L[f](s) = \frac{1}{s} + \frac{e^{-s}}{s^2} + \left(-\frac{1}{s} - \frac{1}{s^2} + \frac{\pi}{s^2+\pi^2}\right)e^{-2s}$$

(b) $f(t) = \int_0^t \sin(t-\tau)\, e^{2\tau}\, d\tau$

$$f(t) = \sin t, \quad g = e^{2t}$$

$$L[f] = L[\sin t]\, L[e^{2t}] = \frac{1}{s^2+1} \cdot \frac{1}{s-2}$$

8. (a) $Y(s) = \dfrac{s+1}{(s+1)^2+1} + \dfrac{1}{(s+1)^2+1} + \dfrac{2}{s((s+1)^2+1)}e^{-s} + \dfrac{1}{(s+1)^2+1}e^{-2s}$

$$= \frac{s+1}{(s+1)^2+1} + \frac{1}{(s+1)^2+1} + \left(\frac{1}{s} - \frac{s+1}{(s+1)^2+1} - \frac{1}{(s+1)^2+1}\right)e^{-s} + \frac{1}{(s+1)^2+1}e^{-2s}$$

$$y = e^{-t}\cos t + e^{-t}\sin t + H(t-1)\left(1 - e^{-(t-1)}\cos(t-1) - e^{-(t-1)}\sin(t-1)\right)$$

$$+ H(t-2)\, e^{-(t-2)}\sin(t-2)$$

9. First we solve the steady-state problem

$$4u^0_{xx} = 0, \quad u^0(0) = 0,\ u^0(\pi) = \pi \Rightarrow u^0 = A + Bx \Rightarrow u^0 = x$$

$$u = u^0 + v \Rightarrow \begin{cases} v_{tt} = 4v_{xx} \\ v(0,t) = 0,\ v(\pi,t) = 0 \\ v(x,0) = u(x,0) - u^0 = \begin{cases} -x, & 0 < x < \frac{\pi}{2} \\ 0, & \frac{\pi}{2} < x < \pi \end{cases} \\ v_t(x,0) = u_t(x,0) = \sin 3x \end{cases}$$

$$c^2 = 4 \Rightarrow c = 2, \quad L = \pi$$

$$V = \sum_{n=1}^{+\infty} \sin(nx) \left(a_n \cos(2nt) + b_n \sin(2nt) \right)$$

$$V(x,0) = f(x) = \sum_{n=1}^{+\infty} a_n \sin nx$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} (-x) \sin nx \, dx$$

$$= \frac{2}{\pi} \left(\frac{1}{n} x \cos nx - \frac{1}{n^2} \sin nx \right) \Big|_0^{\frac{\pi}{2}}$$

$$= \frac{2}{\pi} \left(\frac{1}{n} \frac{\pi}{2} \cos \frac{n\pi}{2} - \frac{1}{n^2} \sin \frac{n}{2}\pi \right)$$

$$V_t(x,0) = \sin 3x = \sum_{n=1}^{+\infty} (2n) \, b_n \sin nx$$

$$\Rightarrow \quad 2 \times 3 \, b_3 = 1 \quad \Rightarrow b_3 = \frac{1}{6}, \quad b_n = 0 \text{ for } n \neq 3$$

So

$$u = x + v(x,t)$$

$$= x + \sum_{n=1}^{+\infty} \left(\frac{1}{n} \cos \frac{n\pi}{2} - \frac{2}{\pi n^2} \sin \frac{n}{2}\pi \right) \sin nx$$

$$+ \frac{1}{6} \sin 3x \sin 6t$$