(20 points) 1. Solve the following system of ordinary differential equations

$$x' = \begin{pmatrix} 1 & -2 \\ 2 & 5 \end{pmatrix} x$$

$$x(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Solution: $A = \begin{pmatrix} 1 & -2 \\ 2 & 5 \end{pmatrix}$

$$\det(A - \lambda I) = \det\begin{pmatrix} 1-\lambda & -2 \\ 2 & 5-\lambda \end{pmatrix} = \lambda^2 - 6\lambda + 9 = 0$$ +2

$\lambda_1 = \lambda_2 = 3$

For $\lambda_1 = \lambda_2 = 3$, $(A - 3I)\xi = \begin{pmatrix} -2 & -2 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ +2

$\xi_1 + \xi_2 = 0$. choose $\xi_1 = 1$, $\xi_2 = -1$

eigenvector $\vec{\xi} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

+2

So $x^{(1)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{3t}$ +2

To look for $x^{(2)} = t\vec{\xi} e^{3t} + \eta e^{3t}$

we get $(A - 3I)\eta = \vec{\xi} \Rightarrow \begin{pmatrix} -2 & -2 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \rightarrow$ +3

$2\eta_1 + 2\eta_2 = -1$, $\eta_2 = 0 \Rightarrow \eta_1 = -\frac{1}{2}$ $\rightarrow$ +3

$\eta = \begin{pmatrix} -\frac{1}{2} \\ 0 \end{pmatrix}$ +2

So $x^{(2)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} t e^{3t} + \begin{pmatrix} -\frac{1}{2} \\ 0 \end{pmatrix} e^{3t}$ +2

$x = c_1 x^{(1)} + c_2 x^{(2)}$ +2 $\quad x(0) = c_1\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2\begin{pmatrix} -\frac{1}{2} \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow c_1 = -1,\ c_2 = -4$

Thus $x = -\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{3t} - 4\left(\begin{pmatrix} 1 \\ -1 \end{pmatrix} t e^{3t} + \begin{pmatrix} -\frac{1}{2} \\ 0 \end{pmatrix} e^{3t}\right) = \begin{pmatrix} -4 \\ 4 \end{pmatrix} t e^{3t} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t}$

(20 points) 2. Use the method of undetermined coefficients to obtain the general solutions of

$$x' = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} x + \begin{pmatrix} e^t \\ 0 \end{pmatrix}$$

Solution:

$A = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}$, $\det(A - \lambda I) = \lambda^2 - 3\lambda + 2 = 0$ +2

$\lambda_1 = 1$, $\lambda_2 = 2$

$(A - I)\xi^{(1)} = \begin{pmatrix} 0 & -1 \\ 0 & 2 \end{pmatrix}\xi^{(1)} = 0 \Rightarrow \xi^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ +2

$(A - 2I)\xi^{(2)} = \begin{pmatrix} -1 & -1 \\ 0 & 0 \end{pmatrix}\xi^{(2)} = 0 \Rightarrow \xi^{(2)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$x^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t$, $x^{(2)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t}$ —— +2

$x_p = \vec{a}\, t e^t + \vec{b} e^t$ since $g = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t$ — 4

$A x_p + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t = A\vec{a}\, t e^t + \left(A\vec{b} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) e^t$ +2

$x_p' = \vec{a}\, t e^t + (\vec{a} + \vec{b}) e^t$

$x_p' = A x_p + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t \Rightarrow$

$A\vec{a} = \vec{a}$ —— ① +2

$A\vec{b} = \vec{b} + \vec{a} - \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ — ② +2

① $\Rightarrow \vec{a} = k\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$A\vec{b} = \vec{b} + k\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & -1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} k-1 \\ 0 \end{pmatrix}$ +2

$-b_2 = k - 1$
$2b_2 = 0$ $\Rightarrow k = 1$, $b_2 = 0$
$\vec{b} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$x_p = \begin{pmatrix} 1 \\ 0 \end{pmatrix} t e^t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t$

$X = X_p + C_1 X^{(1)} + X^{(2)}$ +2

(20 points) 3. Use the method of variation of parameters to obtain the general solutions of

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\mathbf{x} + \begin{pmatrix} e^t \sec(t) \\ 0 \end{pmatrix}, \quad -\frac{\pi}{2} < t < \frac{\pi}{2}$$

where $\sec(t) = \frac{1}{\cos(t)}$.

Solution: $A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$, $\det(A - \lambda I) = \det\begin{pmatrix} 1-\lambda & 1 \\ -1 & 1-\lambda \end{pmatrix}$ 2

$= (\lambda - 1)^2 + 1$

$\lambda_1 = 1 + i, \quad \lambda_2 = 1 - i$

Eigenvector: $(A - \lambda_1 I)\xi^{(1)} = \begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ 2

$\xi_2 = i\xi_1, \quad \xi_1 = 1 \Rightarrow \xi_2 = i$

$\xi^{(1)} = \begin{pmatrix} 1 \\ i \end{pmatrix}$

$x^{(1)} = \xi^{(1)} e^{\lambda_1 t} = \begin{pmatrix} 1 \\ i \end{pmatrix} e^{(1+i)t} = \begin{pmatrix} e^t \cos t + i e^t \sin t \\ -e^t \sin t + i e^t \cos t \end{pmatrix}$ 2

$= \begin{pmatrix} e^t \cos t \\ -e^t \sin t \end{pmatrix} + i \begin{pmatrix} e^t \sin t \\ e^t \cos t \end{pmatrix}$

So $x = c_1 \begin{pmatrix} e^t \cos t \\ -e^t \sin t \end{pmatrix} + c_2 \begin{pmatrix} e^t \sin t \\ e^t \cos t \end{pmatrix}$ is the solution for homogeneous problem 2

$\Psi(t) = \begin{pmatrix} e^t \cos t & e^t \sin t \\ -e^t \sin t & e^t \cos t \end{pmatrix}$ 2

$x_p = \Psi(t)\, u(t) \Rightarrow \Psi(t)\, u'(t) = g(t)$ 2

$\begin{pmatrix} e^t \cos t & e^t \sin t \\ -e^t \sin t & e^t \cos t \end{pmatrix}\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} e^t \sec(t) \\ 0 \end{pmatrix}$

$\cos t\, u_1' + \sin t\, u_2' = \sec(t)$ 2

$-\sin t\, u_1' + \cos t\, u_2' = 0$

$\Rightarrow \begin{cases} u_1' = 1 \\ u_2' = \tan t \end{cases}$ 2 $\Rightarrow$ $u_1 = t$, $u_2 = -\ln(c$ 2

$x_p = \begin{pmatrix} e^t \cos t & e^t \sin t \\ -e^t \sin t & e^t \cos t \end{pmatrix}\begin{pmatrix} t \\ -\ln(\cos t) \end{pmatrix} = \begin{pmatrix} e^t t \cos t - e^t \sin t \ln(\cos t) \\ -e^t t \sin t - e^t \cos t \ln(\cos t) \end{pmatrix}$ 2

(40 points) 4. Use the method of Laplace transform to solve

$$y'' + y = \cos(t) + g(t) + t\delta(t-4), \quad y(0) = 0, y'(0) = 0$$

where

$$g(t) = \begin{cases} 1, & 0 \le t < 1 \\ -1, & 1 \le t < 2 \\ t-3, & 2 \le t < 3 \end{cases}$$

Solution: We compute

$$L\{\cos t\} = \frac{s}{s^2+1} \qquad 2$$

$$g(t) = u_0 - u_1 - (u_1 - u_2) + (t-3)(u_2 - u_3) \qquad 2$$

$$= 1 - 2u_1 + (t-2)u_2 - (t-3)u_3$$

$$L\{g\} = \frac{1}{s} - \frac{2e^{-s}}{s} + \frac{e^{-2s}}{s^2} - \frac{e^{-3s}}{s^2} \qquad -4$$

$$L\{t\delta(t-4)\} = 4e^{-4s} \qquad 2$$

Let $Y(s) = L\{y\}$. we have

$$(s^2+1)Y(s) = \frac{s}{(s^2+1)} + \frac{1}{s} - \frac{2e^{-s}}{s} + \frac{e^{-2s}}{s^2} - \frac{e^{-3s}}{s^2} + 4e^{-4s}$$

$$Y(s) = \frac{s}{(s^2+1)^2} + \frac{1}{s(s^2+1)} - \frac{2e^{-s}}{s(s^2+1)} + \frac{e^{-2s}}{s^2(s^2+1)} - \frac{e^{-3s}}{s^2(s^2+1)} + \frac{4e^{-4s}}{s^2+1} \qquad 2$$

2

Now: $\frac{s}{(s^2+1)^2} = -\frac{1}{2}\left(\frac{1}{s^2+1}\right)'$, $\frac{1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs}{s^2+1} + \frac{C}{s^2+1}$ $A=1$, $B=-1$, $C=0$

+4

4 $\frac{1}{s^2(s^2+1)} = \frac{1}{s^2} - \frac{1}{s^2+1}$ +4 4

$L^{-1}\left\{\frac{s}{s^2+1}\right\} = \frac{1}{2} t L^{-1}\left\{\frac{1}{s^2+1}\right\} = \frac{1}{2} t \sin t$, $L^{-1}\left\{\frac{1}{s(s^2+1)}\right\} = 1 - \cos t$

$L^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\} = t - \sin t$ +4

so $y(t) = \frac{1}{2} t\sin t + 1 - \cos t - 2u_1(1-\cos(t-1)) + u_2(t-2-\sin(t-2)) - u_3(t-3-\sin(t$
$+ 4u_4 \sin(t-4)$ +6