Practice Problems in Chapter 7

1. Find all eigenvalues and eigenvectors of the given matrix. Identify the algebraic multiplicity and geometric multiplicity

a. $\begin{pmatrix} -3 & 3/4 \\ 5 & 1 \end{pmatrix}$ b. $\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{pmatrix}$ c. $\begin{pmatrix} 3 & 2 & 2 \\ 1 & 4 & 1 \\ -2 & -4 & -1 \end{pmatrix}$

2. Consider

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} 1 & \sin t & \cos t \\ e^t & 0 & t^{-2} \\ e^{-t} & \cos t \, \sin t & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Compute the Wronskian $W(t)$

3. Solve the following system of ODEs : $X' = AX$

a. $X' = \begin{pmatrix} 3 & -2 \\ -2 & 1 \end{pmatrix} X$ b. $X' = \begin{pmatrix} 3 & 2 \\ -2 & 1 \end{pmatrix} X$

c. $X' = \begin{pmatrix} 3 & 2 \\ -2 & -1 \end{pmatrix} X$ d. $X' = \begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix} X$

e. $X' = \begin{pmatrix} -3 & 0 & 2 \\ 1 & -1 & 0 \\ -2 & -1 & 0 \end{pmatrix} X$ f. $X' = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{pmatrix} X$

g. $X' = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 3 & 2 \\ 3 & -2 & -1 \end{pmatrix} X$

4. For the following ODE, find a fundamental matrix $\Phi(t)$ such that $\Phi(0)=I$

a. $X' = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} X$

b. $X' = \begin{pmatrix} -1 & 4 \\ 1 & -1 \end{pmatrix} X$

c. $X' = \begin{pmatrix} 1 & -1 \\ 5 & -3 \end{pmatrix} X$

d. $X' = \begin{pmatrix} 2 & -3 \\ 1 & -2 \end{pmatrix} X$

5. Solve the initial value problem

a. $X' = \begin{pmatrix} 2 & -3 \\ 1 & -2 \end{pmatrix} X, \quad X(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

b. $X' = \begin{pmatrix} 1 & -1 \\ 5 & -3 \end{pmatrix} X, \quad X(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

6. Solve the following inhomogeneous problem with the method given

a. $X' = \begin{pmatrix} 2 & -1 \\ 2 & -1 \end{pmatrix} X + \begin{pmatrix} 1 \\ e^t \end{pmatrix}$ (method of undetermined coefficient)

b. $X' = \begin{pmatrix} 2 & -1 \\ 2 & -1 \end{pmatrix} X + \begin{pmatrix} 1 \\ e^t \end{pmatrix}$ (method of diagonalization)

c. $X' = \begin{pmatrix} 2 & -1 \\ 2 & -1 \end{pmatrix} X + \begin{pmatrix} e^t \\ e^{2t} \end{pmatrix}$ (method of variation of parameters)

d. $X' = \begin{pmatrix} 2 & -1 \\ \frac{9}{4} & -1 \end{pmatrix} X + \begin{pmatrix} 1+e^t \\ -e^t \end{pmatrix}$ (Method of undetermined coefficients)

e. $X' = \begin{pmatrix} 2 & -1 \\ \frac{9}{4} & -1 \end{pmatrix} X + \begin{pmatrix} 1+e^t \\ -e^t \end{pmatrix}$ (Method of variation of parameters)

f. $X' = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} X + \begin{pmatrix} \csc t \\ \sec t \end{pmatrix}$ (method of variation of parameters)

7. For the following ODE, find fundamental set of solutions by taking $X = t^r \vec{\xi}$

a. $tX' = \begin{pmatrix} 2 & -1 \\ 2 & -1 \end{pmatrix} X$

b. $tX' = \begin{pmatrix} 2 & -1 \\ \frac{9}{4} & -1 \end{pmatrix} X$

c. $tX' = \begin{pmatrix} 3 & 2 \\ -2 & 1 \end{pmatrix} X$

Practice Problems in Chapter 6.

1. Find out the Laplace transform of

a. $2\cos t - 3\sin 2t$

b. $2t\cos t - 3t^2\sin 2t$

c. $e^{-t}\cos t$

d. $te^{-t}\sin t$

d. $f(t) = \begin{cases} t^2, & 0 \le t \le 1 \\ 1, & 1 < t \le 2 \\ 3-t, & 2 < t \le 3 \end{cases}$,

e. $f(t) = \begin{cases} t, & 0 \le t \le 1 \\ 3-t, & 1 < t \le 2 \\ 1, & 2 < t \le 3 \end{cases}$

g. $f(t) = t\,u_1(t) - 2u_2(t) + \delta_3(t)$

h. $f(t) = e^{t}u_1(t) - e^{-t}u_3(t) + \delta_{10}(t)\,e^{-t}$

2. Find out the inverse Laplace transform of

a. $\dfrac{1}{s^2+s+1}$

b. $\dfrac{s}{s^2+s+1}$

c. $\dfrac{e^{-s}}{s^2+s+1}$

d. $\dfrac{e^{-s}}{s(s^2+s+1)}$

e. $\dfrac{1}{s^2(s^2+2s+2)}$

f. $\dfrac{e^{-2s}}{s(s^2+2s+2)}$

3. Solve the differential equations using the Laplace transform

a. $y'' + 2y' + 2y = e^{t} + \sin t, \quad y(0) = 0, \; y'(0) = 1$

b. $y'' + 2y' + y = e^{-t} + \delta_1(t), \quad y(0) = 0, \; y'(0) = 0$

c. $\begin{cases} y'' + 4y = u_1(t) + \delta_1(t)\cos(\pi t) - \sin 2t \\ y(0) = y'(0) = 0 \end{cases}$

d. $2y'' + y' + 4y = \delta(t - \frac{\pi}{4}) \sin t, \quad y(0) = 0, \; y'(0) = 0$

e. $\begin{cases} y'' + y = g(t) = \begin{cases} t/2, & 0 \le t < 6 \\ 3, & t \geq 6 \end{cases} \\ y(0) = 0, \; y'(0) = 1 \end{cases}$

f. $\begin{cases} y'' + y' + \frac{5}{4} y = g(t), \quad g(t) = \begin{cases} \sin t, & 0 \le t < \pi \\ \cos t, & \pi \le t < 2\pi \\ -1, & 2\pi \le t < +\infty \end{cases} \\ y(0) = 0, \; y'(0) = 0 \end{cases}$

h. $y'' + 2y' + 3y = (t-5) u_5(t) + \delta_{10}(t) \sin t + e^{-t} \cos\sqrt{2} t$

i. $\begin{cases} y'' + y' + \frac{5}{4} y = t - u_{\pi}(t)(t - \frac{\pi}{2}) + e^{-\frac{t}{2}} \cos t + \delta_5(t) t^2 \\ y(0) = y'(0) = 0 \end{cases}$

4. Find a general formula for the following ODE using Laplace transform

(a) $y'' + 9y = g(t)$, $y(0) = y_0$, $y'(0) = y_1$

(b) $y'' + 2y' + 10y = g(t)$, $y(0) = y_0$, $y'(0) = y_1$

5. Find the Laplace and inverse Laplace transform for

a. $f(t) = \int_0^t (t-\tau)^2 \cos 2\tau \, d\tau$

b. $f(t) = \int_0^t (t-\tau) e^{\tau} d\tau$

c. $f(t) = \int_0^t e^{-(t-\tau)} \sin \tau \, d\tau$

d. $f(t) = \int_0^t (t-\tau) e^{-(t-\tau)} \sin \tau \, d\tau$

e. $F(s) = \dfrac{1}{s^4(s^2+1)}$

f. $F(s) = \dfrac{s}{(s+1)(s^2+4)}$

g. $F(s) = \dfrac{1}{(s^2+4)(s+1)^2}$

h. $F(s) = \dfrac{G(s)}{s^2+1}$