Solutions to MATH 256-103-2016 Assignment 1 (written) $\boxed{\text{Total: 90 pts}}$

1 a) $v' + 2v = 9.8$, $\quad p(t) = 2$, $g = 9.8$, $\quad \mu = e^{2t}$, $\quad \int \mu g = 4.9 e^{2t}$

(10 points)

$$v = \frac{1}{\mu}\left(c + \int \mu g\right) = e^{-2t}\left(c + 4.9 e^{2t}\right) = c e^{-2t} + 4.9$$

$v(0) = 0 \Rightarrow c = -4.9$, $\quad v = 4.9 - 4.9 e^{-2t}$

b) $\frac{dx}{dt} = v = 4.9 - 4.9 e^{-2t}$, $\quad x(0) = 0$

$$\Rightarrow x = \int (4.9 - 4.9 e^{-2t})\, dt = 4.9t + \frac{1}{2} \times 4.9 e^{-2t} + C$$

$x(0) = 0 \Rightarrow C = -\frac{4.9}{2}$

$$x(t) = 4.9t + 2.45\left(e^{-2t} - 1\right)$$

c) The equation for T is

$$x(T) = 100 = 4.9T + 2.45\left(e^{-2T} - 1\right)$$

2. a) $y = e^{rt}$, $\quad y' = re^{rt} = ry$, $\quad y'' = r^2 y$

(10 points)

$y'' + 3y' - 4y = 0 \Rightarrow r^2 + 3r - 4 = 0 \Rightarrow r_1 = -4$, $r_2 = 1$

$y_1 = e^{-4t}$, $\quad y_2 = e^{t}$

b) $y = ae^{-4t} + be^{t}$

$$\left.\begin{array}{l} y(0) = 1 \Rightarrow a + b = 1 \\ y'(0) = -1 \Rightarrow -4a + b = -1 \end{array}\right\} \Rightarrow \begin{array}{l} a = \frac{2}{5} \\ b = \frac{3}{5} \end{array}$$

2'. a) $y = \frac{1}{e^{-2t}}\left(c + \frac{5}{2} e^{-2t}\right) = \frac{5}{2} + c e^{+2t}$, $\quad y \to \cancel{\frac{5}{2}}$ as $t \to +\infty$ as long as $y_0 \neq \frac{5}{2}$

(10 points)

$y(0) = y_0 \Rightarrow y = \frac{5}{2} + \left(y_0 - \frac{5}{2}\right)e^{+2t}$, $\quad$ if $y_0 > \frac{5}{2}$, $y \to +\infty$

If $y_0 = \frac{5}{2}$, $y \to \frac{5}{2}$

If $y_0 < \frac{5}{2}$, $y \to -\infty$

b) $y = \frac{1}{e^{8t}}\left(c + \frac{5}{4} e^{8t}\right) = \frac{5}{4} + c e^{-8t}$

$y(0) = y_0 \Rightarrow y = \frac{5}{4} + \left(y_0 - \frac{5}{4}\right)e^{-8t}$, $\quad y \to 0$ as $t \to +\infty$

3. a) $y = \frac{1}{e^t}(c+t) = te^{-t} + ce^{-t}$, $y(0)=1 \Rightarrow c=1$

(10 points)

$y = te^{-t} + e^{-t}$

b) $y' + \frac{1}{t}y = 3\cos 2t$

$\mu = e^{\int \frac{1}{t}dt} = t$, $\int \mu g = 3\int t\cos 2t = \frac{3}{2}t\sin 2t + \frac{3}{4}\cos 2t$

$y = \frac{1}{t}\left(c + \frac{3}{2}t\sin 2t + \frac{3}{4}\cos 2t\right)$

c) $y' + \frac{1}{2}y = \frac{3}{2}t^2 \Rightarrow \mu = e^{\frac{t}{2}}$, $\int e^{\frac{t}{2}}t^2 = t^2 2e^{t/2} - 8te^{t/2} + 16e^{t/2}$

$\cancel{\mu = e^{\int \frac{2}{t}dt} = t^2, \int \mu g = \int 3t^3 = \frac{3}{4}t^4}$, $y = 3(t^2-4t+8) + ce^{-t/2}$

$\cancel{y = \frac{1}{t^2}\left(c + \frac{3}{4}t^4\right)}$

d) $y' + \frac{(t+1)}{t}y = 1$

$\mu = e^{\int \frac{t+1}{t}dt} = te^t$, $\int \mu g = \int te^t = te^t - e^t$

$y = \frac{1}{te^t}\left(c + te^t - e^t\right)$

4. a) $y = \frac{1}{e^{-t}}(c+t^2) = ce^t + t^2e^t$, $y(0)=1 \Rightarrow c=1$

(10 pts)

$y = e^t + t^2e^t$, $-\infty < t < +\infty$

b) $\mu = t^2$, $\int \mu g = \int \cos t = \sin t$

$y = \frac{1}{t^2}(c + \sin t)$, $y(\pi)=0 \Rightarrow c=0$

$y = \frac{\sin t}{t^2}$. Interval of existence $0 < t < +\infty$

c) $\mu = e^{-t}$, $\int \mu g = \int e^{-t}(t - \sin t + e^{2t})$

$= -te^{-t} - e^{-t} + e^{-t}(\cos t + \sin t) + e^t$

$y = ce^t - t - 1 + \cos t + \sin t + e^{2t}$, $y(0)=0$

$\Rightarrow 0 = c - 1 + 1 + 1$ $\quad c = -2$, $\quad -\infty < t < +\infty$

d) $y' - \frac{2\cos t}{\sin t} y = \sin^2 t$

$$\mu = e^{-\int \frac{2\cos t}{\sin t} dt} = e^{-2\ln \sin t} = \frac{1}{\sin^2 t}$$

$$\int \mu g = t$$

$$y = \sin^2 t\,(c+t)$$

$y(\frac{\pi}{2}) = 1 \Rightarrow \quad c = 1-\frac{\pi}{2}$

$$y = \sin^2 t\left(1-\frac{\pi}{2}+t\right)$$

Interval of existence, $t_0 = \frac{\pi}{2}$, $\sin t \neq 0 \Rightarrow \quad 0 < t < \pi$

5. $\mu = e^{2t}$, $\int \mu g = \int e^{2t}(3+2\cos 2t) = \frac{3}{2}e^{2t} + \frac{1}{2}e^{2t}(\cos 2t + \sin 2t)$

(10pts)

$$y = ce^{-2t} + \frac{3}{2} + \frac{1}{2}(\cos 2t + \sin 2t)$$

$y(0)=0 \Rightarrow \quad 0 = c + \frac{3}{2} + \frac{1}{2} \Rightarrow c = -2$

$$y = -2e^{-2t} + \frac{3}{2} + \frac{1}{2}(\cos 2t + \sin 2t)$$

$$\boxed{\begin{aligned}\int e^{ax}\cos bx &= e^{ax}\frac{(a\cos bx + b\sin bx)}{a^2+b^2} \\ \int e^{ax}\sin bx &= \frac{e^{ax}(-b\cos bx + a\sin bx)}{a^2+b^2}\end{aligned}}$$

Note that $\cos 2t + \sin 2t = \sqrt{2}\cos(2t - \frac{\pi}{4})$

so y is oscillating between $\frac{3}{2}+\frac{\sqrt{2}}{2}$ and $\frac{3}{2}-\frac{\sqrt{2}}{2}$

6. a). $y = ce^{-t} - e^{-2t}$, $y \to 0$ as $t \to +\infty$

(10pts)

b) $y = ce^{t} + e^{-2t}(-\frac{1}{3})$

If $c \neq 0$, then $y \to +\infty$ as $t \to +\infty$. So $c = 0$

$y = -\frac{1}{3}e^{-2t}$ and $y(0) = -\frac{1}{3} = y_0$.

7. a) $\frac{dy}{y^2} = -\sin x \, dx$

(10 pts)

$-\frac{1}{y} = \cos x + C \qquad y = -\frac{1}{\cos x + C}$

b) $(y + e^{y}) dy = (x - e^{-x}) dx$

$\frac{y^2}{2} + e^{y} = \frac{x^2}{2} + e^{-x} + C$

c) $(1+y^2)^{-1} dy = \frac{1}{(2+x^2)} dx$

$\tan^{-1} y = \sqrt{2} \tan^{-1} \frac{x}{\sqrt{2}} + C$

$y = \tan\left(\sqrt{2} \tan^{-1} \frac{x}{\sqrt{2}} + C\right)$

d) $yy' = x dx \rightarrow y^2 = x^2 + C \Rightarrow y = \pm\sqrt{x^2 + C}$

8. a). $-\frac{1}{y} = x - x^2 + C, \; y(0) = 1 \Rightarrow C = -1$

(10pts)

$y = -\frac{1}{x - x^2 - 1} = \frac{1}{x^2 - x + 1}$

$x^2 - x + 1 > 0, \forall x \Rightarrow$ Interval of existence : $-\infty < x < +\infty$

b) $\frac{dy}{y^3} = \frac{x}{\sqrt{1+x^2}} dx \Rightarrow -\frac{1}{2y^2} = \frac{1}{2}\left(\sqrt{1+x^2}\right) + C$

$\Rightarrow \frac{1}{y^2} = C - \sqrt{1+x^2}, \quad y^2 = \frac{1}{C - \sqrt{1+x^2}}$

$y(0) = 2, \quad 4 = \frac{1}{C-1} \Rightarrow C = \frac{5}{4}$

$y^2 = \frac{1}{\frac{5}{4} - \sqrt{1+x^2}}, \quad y = \pm\sqrt{\frac{1}{\frac{5}{4} - \sqrt{1+x^2}}}$

Since $y(0) = 2 \Rightarrow y = \sqrt{\frac{1}{\frac{5}{4} - \sqrt{1+x^2}}}$

Interval of existence, $\frac{5}{4} - \sqrt{1+x^2} > 0 \Rightarrow x^2 < \frac{9}{16} \Leftrightarrow -\frac{3}{4} < x < \frac{3}{4}$

Inteval of existence : $-\frac{3}{4} < x < \frac{3}{4}$

8 c).

$$(2y-5)\,dy = (3x^2 - e^x)\,dx$$

$$y^2 - 5y = x^3 - e^x + C$$

$$y(0)=1 \Rightarrow \quad -4 = -1 + C \Rightarrow C = -3$$

$$y^2 - 5y = x^3 - e^x - 3$$

$$\left(y-\tfrac{5}{2}\right)^2 = x^3 - e^x + \frac{13}{4}$$

$$y = \frac{5}{2} \pm \sqrt{x^3 - e^x + \frac{13}{4}}$$

$$y(0)=1 \Rightarrow y = \frac{5}{2} - \sqrt{x^3 - e^x + \frac{13}{4}}$$

Interval of existence: $x^3 - e^x + \frac{13}{4} > 0$.

d)

$$\sin(2x)\,dx + \cos(3y)\,dy = 0$$

$$\int \cos 3y\,dy = \int \sin 2x\,dx$$

$$+\frac{1}{3}\sin 3y = \frac{1}{2}\cos 2x + C$$

$$\sin 3y = \frac{3}{2}\cos 2x + C, \qquad y\left(\tfrac{\pi}{2}\right) = \frac{\pi}{3}$$

$$\sin 3y = \frac{3}{2}\cos 2x + \frac{3}{2}$$

[No need to compute Interval of Existence, but if you do, here is the answer]:

We need $\cos 3y \neq 0$ for the Interval of Existence:

$$\left(\cos^{-1}\left(\tfrac{1}{\sqrt{3}}\right),\ \pi - \cos^{-1}\left(\tfrac{1}{\sqrt{3}}\right)\right)$$