(15 points) 1. Consider the following ordinary differential equation

$$ty' + 2y = t^{-1}e^t, \qquad y(1) = 1$$

(2 points) (a) Write the equation in the following form

$$y' + p(t)y = g(t).$$

(6 points) (b) Compute $\mu(t) = e^{+\int p(t)dt}$ and $\int \mu(t)g(t)dt$.

(7 points) (c) Find the solution and state the Interval of Existence.

(a) $y' + \frac{2}{t}y = t^{-2}e^t$ — 2

$p(t) = +\frac{2}{t}$, $g(t) = \frac{1}{t^2}e^t$

(b) $\mu(t) = e^{\int \frac{2}{t}dt} = t^2$ — 3

$\int \mu(t)g(t)dt = \int t^2 \frac{1}{t^2}e^t dt = e^t + C$ +3

(c) $y = \frac{1}{\mu(t)}(C + \int \mu(t)g(t)dt)$ | 3

$= \frac{1}{t^2}(C + e^t)$

$y(1) = 1 \Rightarrow C + e = 1 \Rightarrow C = 1-e$ | 3

$y(t) = \frac{1}{t^2}e^t + \frac{1-e}{t^2}$

Interval of Existence: $(0, +\infty)$ | 1

(20 points) 2. Solve the following ordinary differential equation

$$y' = \frac{t}{2(y - y^3)}, \quad y(0) = -2$$

and state the Interval of Existence.

$$2(y \cancel{+} y^2)\,dy = t\,dt$$

$$\int 2(y - y^2)\,dy = \int t\,dt$$

$$y^2 - \frac{y^4}{2} = \frac{t^2}{2} + c$$

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$$y^4 - 2y^2 = -t^2 + c$$

$$t = 0,\ y = -2, \quad c = 8$$

$$y^4 - 2y^2 = 8 - t^2$$

$$(y^2 - 1)^2 = 9 - t^2$$

$$y^2 = 1 \pm \sqrt{9 - t^2}$$

4

$$y^2 = 1 + \sqrt{9 - t^2}$$

$$y = -\sqrt{1 + \sqrt{9 - t^2}}$$

Interval of existence: $-3 < t < 3$ | 4

(15 points) 3. Consider the following ordinary differential equation

$$y' = (y-4)\log y, \ \ y > 0.$$

(10 points) (a) Find all critical points and classify the stability/instability of these critical points.

(3 points) (b) Let $y(0) = \frac{1}{2}$. What is the asymptotic behavior of $y(t)$ as $t \to +\infty$?

(2 points) (c) Let $y(0) = 2$. What is the asymptotic behavior of $y(t)$ as $t \to +\infty$?

(a) $f(y) = (y-4)\log y$

$f(y) = 0 \Rightarrow y-4 = 0$ or $\log y = 0$

$y_1 = 4$, $y_2 = 1$ are critical points

$f'(y) = +\log y + (y-4)\frac{1}{y}$

$f'(y_1) = +\log 4 > 0 \Rightarrow y_1$ is unstable

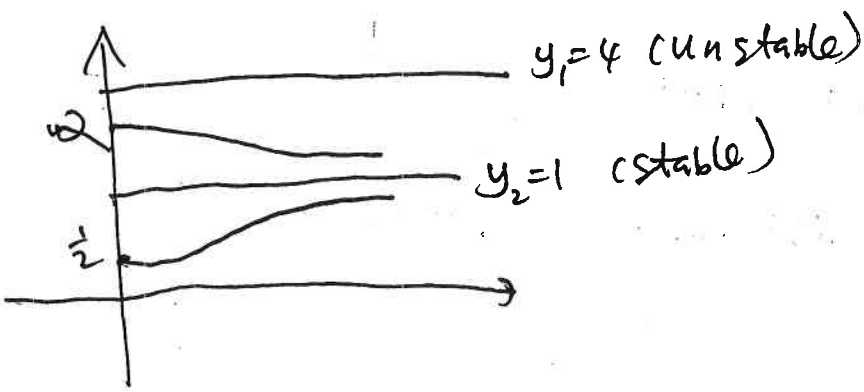
$f'(y_2) = -3 < 0 \Rightarrow y_2$ is stable

3

(b) $y(0) = \frac{1}{2} < 1 = y_2 \Rightarrow y(t) \to 1$ as $t \to +\infty$

2

(c) $y(0) = 2$, $f(y) < 0 \Rightarrow y(t) \to 1$ as $t \to +\infty$



.5 points) 4. Consider the following ordinary differential equation

$$t^2y'' - ty' + y = 0$$

2 points) (a) Write the equation in the following form:

$$y'' + p(t)y' + q(t)y = 0$$

5 points) (b) Find the Wronskian W.

8 points) (c) Let $y_1 = t$ be a solution. Use reduction of order to find $y_2(t) = v(t)y_1(t)$. Hint: you may use the formula: $v' = \frac{W}{y_1^2}$.

(a) $y'' - \frac{1}{t}y' + \frac{1}{t^2}y = 0$

$p = -\frac{1}{t}, \quad q = \frac{1}{t^2}$

(b) $W' + pW = 0 \Rightarrow W = e^{-\int p(t)\,dt} = ct$

~~Take c~~.

(c) $v' = \frac{W}{y_1^2} = \frac{ct}{t^2}$

$v = c\ln t$

$y_2 = c(\ln t)\, t$

(20 points) 5. Consider the following second order ordinary differential equation:

$$y'' - y' - 2y = h(t)$$

(5 points) (a) Find the solutions to the homogeneous problem

$$y'' - y' - 2y = 0.$$

(5 points) (b) Suppose $h(t) = \cos(t) + 2e^t$. Use the method of undetermined coefficients to find the form of the special solution y_p. Do not attempt to find the coefficients.

(5 points) (c) Suppose $h(t) = te^{2t}$. Use the method of undetermined coefficients to find the form of the special solution y_p. Do not attempt to find the coefficients.

(5 points) (d) Solve the following second order differential equation

$$y'' - y' - 2y = t, y(0) = 0, y'(0) = 1$$

(a) $r^2 - r - 2 = 0 \Rightarrow r_1 = 2, \; r_2 = -1$

$y = c_1 e^{2t} + c_2 e^{-t}$

(b) $y_p = A\cos t + B\sin t + Ce^t$

(c) $y_p = t^s (At+B)e^{2t}$, $s = 1$

(d) $y_p = At + B$

$-A - 2At - 2B = t \Rightarrow A = -\frac{1}{2}, \; B = \frac{1}{4}$

$y = c_1 e^{2t} + c_2 e^{-t} - \frac{1}{2}t + \frac{1}{4}$

$y(0) = 0 \Rightarrow c_1 + c_2 = -\frac{1}{4}$
$y'(0) = 1 \Rightarrow 2c_1 - c_2 - \frac{1}{2} = 1$
$\Rightarrow c_1 = \frac{5}{12}, \; c_2 = -\frac{8}{12} = -\frac{2}{3}$

$y = \frac{5}{12}e^{2t} - \frac{2}{3}e^{-t} - \frac{1}{2}t + \frac{1}{4}$

(15 points) 6. Use the method of variation of parameters to solve the inhomogeneous problem

$$y'' + 9y = \frac{3}{\cos(3t)}, -\frac{\pi}{6} < t < \frac{\pi}{6}.$$

Hint: You may use the formula $\int \frac{\sin u}{\cos u} du = -\log \cos u + C$.

$y'' + 9y = 0 \Rightarrow r^2 + 9 = 0 \Rightarrow r = \pm 3i$

$\Rightarrow y_1 = \cos 3t, \quad y_2 = \sin 3t$

3

2

$y_p = u_1 y_1 + u_2 y_2$

3

$$\begin{cases} u_1' y_1 + u_2' y_2 = 0 \\ u_1' y_1' + u_2' y_2' = g(t) \end{cases}$$

$$\Rightarrow \begin{cases} u_1' \cos 3t + u_2' \sin 3t = 0 \\ u_1' (-3 \sin 3t) + u_2' (3 \cos 3t) = \frac{3}{\cos 3t} \end{cases}$$

6

$u_1' \cos 3t + u_2' \sin 3t = 0$

$u_1' (-\sin 3t) + u_2' (\cos 3t) = \frac{1}{\cos 3t}$

$u_1' = -\frac{\sin 3t}{\cos 3t} \Rightarrow u_1 = -\int \frac{\sin 3t}{\cos 3t} \overset{u=3t}{=} -\frac{1}{3} \int \frac{\sin u}{\cos u} = +\frac{1}{3} \ln \cos 3t$

$u_2' = 1 \Rightarrow u_2 = t$

$y_p = \frac{1}{3} \ln \cos 3t \cos 3t + t \sin 3t$

$y = C_1 \cos 3t + C_2 \sin 3t + \frac{1}{3} \ln \cos 3t \cos 3t + t \sin$