Solutions to Practice Problem 7

1a. $\begin{pmatrix} -3 & \frac{3}{4} \\ 5 & 1 \end{pmatrix}$. $\lambda_1 = \frac{-2-\sqrt{31}}{2}$ $v_1 = \begin{pmatrix} -4-\sqrt{31} \\ 10 \end{pmatrix}$

$\lambda^2 + 2\lambda - \frac{27}{4} = 0$ $\lambda_2 = \frac{-2+\sqrt{31}}{2}$. $v_2 = \begin{pmatrix} -4+\sqrt{31} \\ 10 \end{pmatrix}$.

all multiplicities $= 1$.

□

b. $\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{pmatrix}$

$-\lambda^3 + 3\lambda^2 - 7\lambda + 5 = 0$

or $-(\lambda-1)[(\lambda-1)^2+4] = 0$

$\lambda_1 = 1$ $v_1 = \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix}$

$\lambda_2 = 1+2i$ $v_2 = \begin{pmatrix} 0 \\ i \\ 1 \end{pmatrix}$

$\lambda_3 = 1-2i$ $v_3 = \begin{pmatrix} 0 \\ -i \\ 1 \end{pmatrix}$.

all multiplicities $= 1$.

□

c. $\begin{pmatrix} 3 & 2 & 2 \\ 1 & 4 & 1 \\ -2 & -4 & -1 \end{pmatrix}$

$-\lambda^3 + 6\lambda^2 - 11\lambda + 6 = 0$

$-(\lambda-1)(\lambda-2)(\lambda-3) = 0$

$\lambda_1 = 1$ $v_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

$\lambda_2 = 2$ $v_2 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$

$\lambda_3 = 3$ $v_3 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$.

all multiplicities $= 1$.

□

2. $$W(t) = ce^{\int \mathrm{tr} A\, dt} = ce^{2t}$$ □

3a $\begin{pmatrix} 3 & -2 \\ -2 & 1 \end{pmatrix}$ $\lambda_1 = 2+\sqrt{5}$ $v_1 = \begin{pmatrix} -1-\sqrt{5} \\ 2 \end{pmatrix}$

$\lambda_2 = 2-\sqrt{5}$ $v_2 = \begin{pmatrix} -1+\sqrt{5} \\ 2 \end{pmatrix}$

$$x = c_1\begin{pmatrix} -1-\sqrt{5} \\ 2 \end{pmatrix} e^{(2+\sqrt{5})t} + c_2 \begin{pmatrix} -1+\sqrt{5} \\ 2 \end{pmatrix} e^{(2-\sqrt{5})t}$$ □

b. $\begin{pmatrix} 3 & 2 \\ -2 & 1 \end{pmatrix}$ $\lambda_1 = 2+\sqrt{3}\,i$ $v_1 = \begin{pmatrix} -1-\sqrt{3}\,i \\ 2 \end{pmatrix}$

$\lambda_2 = 2-\sqrt{3}\,i$ $v_2 = \begin{pmatrix} -1+\sqrt{3}\,i \\ 2 \end{pmatrix}$

$$x = c_1' \begin{pmatrix} -1-\sqrt{3}\,i \\ 2 \end{pmatrix} e^{(2+\sqrt{3}i)t} + c_2' \begin{pmatrix} -1+\sqrt{3}\,i \\ 2 \end{pmatrix} e^{(2-\sqrt{3}i)t}$$

$$= c_1 e^{2t}\left[\begin{pmatrix} -1 \\ 2 \end{pmatrix} \cos\sqrt{3}t - \begin{pmatrix} -\sqrt{3} \\ 2 \end{pmatrix} \sin\sqrt{3}t\right]$$

$$+ c_2 e^{2t}\left[\begin{pmatrix} -\sqrt{3} \\ 2 \end{pmatrix} \cos\sqrt{3}t + \begin{pmatrix} -1 \\ 2 \end{pmatrix} \sin\sqrt{3}t\right]$$ □

c. $\begin{pmatrix} 3 & 2 \\ -2 & -1 \end{pmatrix}$ $\lambda_1 = 1$ (mult. 2), $v_1 = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$

$$\begin{pmatrix} 2 & 2 \\ -2 & -2 \end{pmatrix} v_2 = \begin{pmatrix} 2 \\ -2 \end{pmatrix} \Rightarrow v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$x = c_1 \begin{pmatrix} 2 \\ -2 \end{pmatrix} e^t + c_2 \left[\begin{pmatrix} 2 \\ -2 \end{pmatrix} te^t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t\right]$$ □

3. d. $\begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix}$

$$\lambda_1 = -2,\ v_1 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix},\ \lambda_2 = 1,\ v_2 = \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix},\ \lambda_3 = 3,\ v_3 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

$$x = c_1 \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix} e^{t} + c_3 \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} e^{3t}$$

□

e. $\begin{pmatrix} -3 & 0 & 2 \\ 1 & -1 & 0 \\ -2 & -1 & 0 \end{pmatrix}$

$$\lambda_1 = -2,\ v_1 = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix},\ \lambda_2 = -1-\sqrt{2}i,\ v_2 = \begin{pmatrix} 2+\sqrt{2}i \\ -1+\sqrt{2}i \\ 3 \end{pmatrix}$$

$$\lambda_3 = -1+\sqrt{2}i,\ v_3 = \begin{pmatrix} 2-\sqrt{2}i \\ -1-\sqrt{2}i \\ 3 \end{pmatrix}$$

$$x = c_1' \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} e^{-2t} + c_2' \begin{pmatrix} 2+\sqrt{2}i \\ -1+\sqrt{2}i \\ 3 \end{pmatrix} e^{(-1-\sqrt{2}i)t} + c_3' \begin{pmatrix} 2-\sqrt{2}i \\ -1-\sqrt{2}i \\ 3 \end{pmatrix} e^{(-1+\sqrt{2}i)t}$$

$$= c_1 \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} e^{-2t} + c_2 e^{-t} \left[\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \cos\sqrt{2}t + \begin{pmatrix} \sqrt{2} \\ \sqrt{2} \\ 0 \end{pmatrix} \sin\sqrt{2}t \right]$$

$$+ c_3 e^{-t} \left[\begin{pmatrix} \sqrt{2} \\ \sqrt{2} \\ 0 \end{pmatrix} \cos\sqrt{2}t - \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \sin\sqrt{2}t \right]$$

□

8.f. $\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{pmatrix}$

$\lambda_1 = 1,\ v_1 = \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix},\ \lambda_2 = 1+2i,\ v_2 = \begin{pmatrix} 0 \\ i \\ 1 \end{pmatrix},\ \lambda_3 = 1-2i,\ v_3 = \begin{pmatrix} 0 \\ -i \\ 1 \end{pmatrix}.$

$$x = c_1' \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix} e^{t} + c_2' \begin{pmatrix} 0 \\ i \\ 1 \end{pmatrix} e^{(1+2i)t} + c_3' \begin{pmatrix} 0 \\ -i \\ 1 \end{pmatrix} e^{(1-2i)t}$$

$$= c_1 \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix} e^{t} + c_2 e^{t} \left[\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \cos 2t - \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \sin 2t \right]$$

$$+ c_3 e^{t} \left[\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \cos 2t + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \sin 2t \right]$$

□

g. $\begin{pmatrix} 1 & 0 & 0 \\ 2 & 3 & 2 \\ 3 & -2 & -1 \end{pmatrix}$ $\lambda_1 = 1$ (multiplicity 3).

$$\begin{pmatrix} 0 & 0 & 0 \\ 2 & 2 & 2 \\ 3 & -2 & -2 \end{pmatrix} v_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow v_1 = \begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 2 & 2 & 2 \\ 3 & -2 & -2 \end{pmatrix} v_2 = \begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix} \Rightarrow v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 2 & 2 & 2 \\ 3 & -2 & -2 \end{pmatrix} v_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \Rightarrow v_3 = \begin{pmatrix} \frac{1}{5} \\ \frac{3}{10} \\ 0 \end{pmatrix}.$$

$$x = c_1 \begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix} e^{t} + c_2 e^{t} \left(\begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix} t + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right) + c_3 e^{t} \left[\begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix} \frac{t^2}{2} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} t + \begin{pmatrix} \frac{1}{5} \\ \frac{3}{10} \\ 0 \end{pmatrix} \right]$$

□

4.a. $\begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix}$

$\lambda_1 = i,\ v_1 = \begin{pmatrix} 2+i \\ 1 \end{pmatrix},\quad \lambda_2 = -i,\ v_2 = \begin{pmatrix} 2-i \\ 1 \end{pmatrix}.$

~~A fundamental matrix: $\Psi(t)$~~

$$x = c_1'\begin{pmatrix} 2+i \\ 1 \end{pmatrix}e^{it} + c_2'\begin{pmatrix} 2-i \\ 1 \end{pmatrix}e^{-it}$$

$$= c_1\left[\begin{pmatrix} 2 \\ 1 \end{pmatrix}\cos t - \begin{pmatrix} 1 \\ 0 \end{pmatrix}\sin t\right] + c_2\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}\cos t + \begin{pmatrix} 2 \\ 1 \end{pmatrix}\sin t\right]$$

$$= \begin{pmatrix} 2\cos t - \sin t & \cos t + 2\sin t \\ \cos t & \sin t \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$=: \Psi(t)\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

$$\Psi(t)^{-1} = \frac{1}{-1}\begin{pmatrix} \sin t & -\cos t - 2\sin t \\ -\cos t & 2\cos t - \sin t \end{pmatrix}$$

$$\Psi(0)^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix}$$

$$\Phi(t) = \begin{pmatrix} 2\cos t - \sin t & \cos t + 2\sin t \\ \cos t & \sin t \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix}.$$

$$= \begin{pmatrix} \cos t + 2\sin t & -5\sin t \\ \sin t & \cos t - 2\sin t \end{pmatrix}.$$

□.

b. $\begin{pmatrix} -1 & 4 \\ 1 & -1 \end{pmatrix}$.

$\lambda_1 = -3,\ v_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix},\quad \lambda_2 = 1,\ v_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$

$$\Psi(t) = \begin{pmatrix} -2e^{-3t} & 2e^{t} \\ e^{-3t} & e^{t} \end{pmatrix},\quad \Psi(0)^{-1} = \begin{pmatrix} -2 & 2 \\ 1 & 1 \end{pmatrix}^{-1} = \frac{1}{-4}\begin{pmatrix} 1 & -2 \\ -1 & -2 \end{pmatrix}.$$

$$\Phi(t) = \frac{1}{4}\begin{pmatrix} -2e^{-3t} & 2e^{t} \\ e^{-3t} & e^{t} \end{pmatrix}\begin{pmatrix} -1 & 2 \\ 1 & 2 \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} 2e^{-3t}+2e^{t} & -4e^{-3t}+4e^{t} \\ -e^{-3t}+e^{t} & 2e^{-3t}+2e^{t} \end{pmatrix}. \qquad \square.$$

4c. $\begin{pmatrix} 1 & -1 \\ 5 & -3 \end{pmatrix}$.

$\lambda_1 = -1+i,\ v_1 = \begin{pmatrix} 2+i \\ 5 \end{pmatrix},\ \lambda_2 = -1-i,\ v_2 = \begin{pmatrix} 2-i \\ 5 \end{pmatrix}.$

$$\Psi(t) = e^{-t}\begin{pmatrix} 2\cos t - \sin t & \cos t + 2\sin t \\ 5\cos t & 5\sin t \end{pmatrix}.$$

$$\Psi(0)^{-1} = \begin{pmatrix} 2 & 1 \\ 5 & 0 \end{pmatrix}^{-1} = \frac{1}{-5}\begin{pmatrix} 0 & -1 \\ -5 & 2 \end{pmatrix}$$

$$\Phi(t) = \frac{1}{5}e^{-t}\begin{pmatrix} 2\cos t - \sin t & \cos t + 2\sin t \\ 5\cos t & 5\sin t \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 5 & -2 \end{pmatrix}.$$

$$= \frac{1}{5}e^{-t}\begin{pmatrix} 5\cos t + 10\sin t & -5\sin t \\ 25\sin t & 5\cos t - 10\sin t \end{pmatrix}.$$

$$= e^{-t}\begin{pmatrix} \cos t + 2\sin t & -\sin t \\ 5\sin t & \cos t - 2\sin t \end{pmatrix} \qquad \square.$$

d. $\begin{pmatrix} 2 & -3 \\ 1 & -2 \end{pmatrix}$.

$\lambda_1 = -1,\ v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix},\ \lambda_2 = 1,\ v_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

$$\Psi(t) = \begin{pmatrix} e^{-t} & 3e^{t} \\ e^{-t} & e^{t} \end{pmatrix},\ \Psi(0)^{-1} = \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix}^{-1} = \frac{1}{-2}\begin{pmatrix} 1 & -3 \\ -1 & 1 \end{pmatrix}.$$

$$\Phi(t) = \frac{1}{2}\begin{pmatrix} e^{-t} & 3e^{t} \\ e^{-t} & e^{t} \end{pmatrix}\begin{pmatrix} -1 & 3 \\ 1 & -1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} -e^{-t}+3e^{t} & 3e^{-t}-3e^{t} \\ -e^{-t}+e^{t} & 3e^{-t}-e^{t} \end{pmatrix}. \qquad \square.$$

5.a. by 4d.

$$x=\frac{1}{2}\begin{pmatrix}-e^{-t}+3e^{t} & 3e^{-t}-3e^{t}\\ -e^{-t}+e^{t} & 3e^{-t}-e^{t}\end{pmatrix}\begin{pmatrix}1\\2\end{pmatrix}$$

$$=\frac{1}{2}\begin{pmatrix}5e^{-t}-3e^{t}\\ 5e^{-t}-e^{t}\end{pmatrix}.$$

□.

b. by 4c.

$$x=e^{-t}\begin{pmatrix}\cos t+2\sin t & -\sin t\\ 5\sin t & \cos t-2\sin t\end{pmatrix}\begin{pmatrix}1\\-1\end{pmatrix}$$

$$=e^{-t}\begin{pmatrix}\cos t+3\sin t\\ 7\sin t-\cos t\end{pmatrix}.$$

□.

6.b. $\begin{pmatrix}2 & -1\\ 2 & -1\end{pmatrix}=\begin{pmatrix}1 & 1\\ 2 & 1\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}-1 & 1\\ 2 & -1\end{pmatrix}$. $\lambda_1=0,\ v_1=\begin{pmatrix}1\\2\end{pmatrix}$, $\lambda_2=1,\ v_2=\begin{pmatrix}1\\1\end{pmatrix}$.

$$u=\begin{pmatrix}-1 & 1\\ 2 & -1\end{pmatrix}x.$$

$$x'=\begin{pmatrix}2 & -1\\ 2 & -1\end{pmatrix}x+\begin{pmatrix}1\\ e^{t}\end{pmatrix}$$

$$=\begin{pmatrix}1 & 1\\ 2 & 1\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}-1 & 1\\ 2 & -1\end{pmatrix}x+\begin{pmatrix}1\\ e^{t}\end{pmatrix}.$$

$$u'=\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}u+\begin{pmatrix}-1+e^{t}\\ 2-e^{t}\end{pmatrix}.$$

i.e. $\begin{cases}u_1'=-1+e^{t}\\ u_2'=u_2+2-e^{t}.\end{cases}$

$\Rightarrow u_1=-t+e^{t}+c_1$

$$u_2'-u_2=2-e^{t}$$

$$(e^{-t}u_2)'=2e^{-t}-1$$

$$u_2=e^{t}(-2e^{-t}-t+c_2).$$

$$x=\begin{pmatrix}1&1\\2&1\end{pmatrix}\begin{pmatrix}c_1-t+e^t\\c_2e^t-2-te^t\end{pmatrix}$$
$$=c_1\begin{pmatrix}1\\2\end{pmatrix}+c_2\begin{pmatrix}1\\1\end{pmatrix}e^t+\begin{pmatrix}-t+e^t\ -2-te^t\\-2t+2e^t\ -2-te^t\end{pmatrix}$$
$$=c_1\begin{pmatrix}1\\2\end{pmatrix}+c_2\begin{pmatrix}1\\1\end{pmatrix}e^t+\begin{pmatrix}-te^t+e^t-t-2\\-te^t+2e^t-2t-2\end{pmatrix}$$
$$=c_1\begin{pmatrix}1\\2\end{pmatrix}+c_2\begin{pmatrix}1\\1\end{pmatrix}e^t-\begin{pmatrix}1\\1\end{pmatrix}te^t+\begin{pmatrix}1\\2\end{pmatrix}e^t-\begin{pmatrix}1\\2\end{pmatrix}t-2\begin{pmatrix}1\\1\end{pmatrix}\ \square$$

6. a. homogeneous equation has sol.

$$x_h=c_1\begin{pmatrix}1\\2\end{pmatrix}+c_2\begin{pmatrix}1\\1\end{pmatrix}e^t.$$

Let $x_p=(At+B)e^t+(Ct+D)$

$$x_p'=(At+A+B)e^t+C.$$

$$\Rightarrow\begin{cases}A=\begin{pmatrix}2&-1\\2&-1\end{pmatrix}A\\ A+B=\begin{pmatrix}2&-1\\2&-1\end{pmatrix}B+\begin{pmatrix}0\\1\end{pmatrix}.\\ 0=\begin{pmatrix}2&-1\\2&-1\end{pmatrix}C\\ C=\begin{pmatrix}2&-1\\2&-1\end{pmatrix}D+\begin{pmatrix}1\\0\end{pmatrix}\end{cases}$$

$$\Rightarrow A=A_0\begin{pmatrix}1\\1\end{pmatrix},$$

$$\begin{pmatrix}1&-1\\2&-2\end{pmatrix}B=\begin{pmatrix}A_0\\A_0-1\end{pmatrix}\Rightarrow(-1\ \ 1)B=1\Rightarrow A_0=-1,\ B=\begin{pmatrix}0\\1\end{pmatrix}$$

$$C=C_0\begin{pmatrix}1\\2\end{pmatrix}$$

$$\begin{pmatrix}2&-1\\2&-1\end{pmatrix}D=\begin{pmatrix}C_0-1\\2C_0\end{pmatrix}\Rightarrow(-2\ \ 1)D=2\Rightarrow C_0=-2,\ D=\begin{pmatrix}-2\\-2\end{pmatrix}.$$

$\Rightarrow$ same as 6b.

6.d. homo. eq: ~~has sol.~~

$$x_h' = \begin{pmatrix} 2 & -1 \\ \frac{9}{4} & -1 \end{pmatrix} x_h.$$

$$\begin{pmatrix} 2 & -1 \\ \frac{9}{4} & -1 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} & \frac{4}{9} \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{2} & 1 \\ 0 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ \frac{9}{4} & -\frac{3}{2} \end{pmatrix}$$

$$x_h = c_1\begin{pmatrix} 2 \\ 3 \end{pmatrix} e^{\frac{1}{2}t} + c_2\left[\begin{pmatrix} 2 \\ 3 \end{pmatrix} t e^{\frac{1}{2}t} + \begin{pmatrix} \frac{4}{3} \\ 1 \end{pmatrix} e^{\frac{1}{2}t}\right].$$

Let $x_p = Ae^t + B$.

$$\begin{cases} A = \begin{pmatrix} 2 & -1 \\ \frac{9}{4} & -1 \end{pmatrix} A + \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ 0 = \begin{pmatrix} 2 & -1 \\ \frac{9}{4} & -1 \end{pmatrix} B + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{cases}$$

~~$\begin{pmatrix} 1 & -1 \\ \frac{9}{4} & -2 \end{pmatrix} = 4\begin{pmatrix} -2 & 1 \\ -\frac{9}{4} & 1 \end{pmatrix}$~~

$$\Rightarrow \begin{cases} A = \begin{pmatrix} -8 & 4 \\ -9 & 4 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 12 \\ 13 \end{pmatrix}. \\ B = \begin{pmatrix} -4 & 4 \\ -9 & 8 \end{pmatrix}\begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 9 \end{pmatrix} \end{cases}$$

$$x = c_1\begin{pmatrix} 2 \\ 3 \end{pmatrix} e^{\frac{1}{2}t} + c_2\left[\begin{pmatrix} 2 \\ 3 \end{pmatrix} t e^{\frac{1}{2}t} + \begin{pmatrix} \frac{4}{3} \\ 1 \end{pmatrix} e^{\frac{1}{2}t}\right] + \begin{pmatrix} 12 \\ 13 \end{pmatrix} e^t + \begin{pmatrix} 4 \\ 9 \end{pmatrix} \square$$

c. Fund. matrix $\Phi(t) = \begin{pmatrix} 1 & e^t \\ 2 & e^t \end{pmatrix}$, ~~$\Phi(0)^{-1} = -\begin{pmatrix} 1 & -1 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix}$~~

~~$\Phi = \begin{pmatrix} 1 & e^t \\ 2 & e^t \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} -1+2e^t & 1-e^t \\ -2+2e^t & 2-e^t \end{pmatrix}$~~

~~$\Phi^{-1}(t) g(t) = \begin{pmatrix} -e^t & e^t \\ 2 & -1 \end{pmatrix}\begin{pmatrix} e^t \\ e^{2t} \end{pmatrix} = \begin{pmatrix} -e^{2t} + e^{3t} \\ 2e^t - e^{2t} \end{pmatrix}$.~~

$$x = \begin{pmatrix} 1 & e^t \\ 2 & e^t \end{pmatrix} \int^t \begin{pmatrix} -e^{2s} + e^{3s} \\ 2e^s - e^{2s} \end{pmatrix} ds$$

$$= \begin{pmatrix} 1 & e^t \\ 2 & e^t \end{pmatrix} \left[\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} -\frac{1}{2}e^{2t} + \frac{1}{3}e^{3t} \\ 2e^t - \frac{1}{2}e^{2t} \end{pmatrix} \right]$$

$$= c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t + \begin{pmatrix} -\frac{1}{2}e^{2t} + \frac{1}{3}e^{3t} + 2e^{2t} - \frac{1}{2}e^{3t} \\ -e^{2t} + \frac{2}{3}e^{3t} + 2e^{2t} - \frac{1}{2}e^{3t} \end{pmatrix}$$

$$= c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t + \begin{pmatrix} \frac{3}{2} \\ 1 \end{pmatrix} e^{2t} + \begin{pmatrix} -\frac{1}{6} \\ \frac{1}{6} \end{pmatrix} e^{3t}$$

$$\Psi^{-1}(t) = -e^{-t} \begin{pmatrix} e^t & -e^t \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 2e^{-t} & -e^{-t} \end{pmatrix}$$

$$x = \Psi(t) \int^t \begin{pmatrix} -1 & 1 \\ 2e^{-s} & -e^{-s} \end{pmatrix} \begin{pmatrix} e^s \\ e^{2s} \end{pmatrix} ds$$

$$= \begin{pmatrix} 1 & e^t \\ 2 & e^t \end{pmatrix} \int^t \begin{pmatrix} -e^s + e^{2s} \\ 2 - e^s \end{pmatrix} ds$$

$$= \begin{pmatrix} 1 & e^t \\ 2 & e^t \end{pmatrix} \begin{pmatrix} c_1 - e^t + \frac{1}{2}e^{2t} \\ c_2 + 2t - e^t \end{pmatrix}$$

$$= c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t + \begin{pmatrix} -e^t + \frac{1}{2}e^{2t} + 2te^t - e^{2t} \\ -2e^t + e^{2t} + 2te^t - e^t \end{pmatrix}$$

$$= c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t + \begin{pmatrix} 2 \\ 2 \end{pmatrix} te^t - \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^t + \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} e^{2t} \qquad \square$$

6e $$\Psi(t) = \begin{pmatrix} 2e^{\frac{1}{2}t} & (2t+\frac{4}{3})e^{\frac{1}{2}t} \\ 3e^{\frac{1}{2}t} & (3t+1)e^{\frac{1}{2}t} \end{pmatrix} = e^{\frac{1}{2}t} \begin{pmatrix} 2 & 2t+\frac{4}{3} \\ 3 & 3t+1 \end{pmatrix}$$

$$\Psi^{-1}(t) = \frac{1}{(6t+2-6t-4)e^t} \begin{pmatrix} (3t+1)e^{\frac{1}{2}t} & -(2t+\frac{4}{3})e^{\frac{1}{2}t} \\ -3e^{\frac{1}{2}t} & \end{pmatrix}$$

$$\Psi(t)^{-1} = e^{-\frac{1}{2}t} \cdot \frac{1}{-2} \begin{pmatrix} 3t+1 & -2t-\frac{4}{3} \\ -3 & 2 \end{pmatrix} = \frac{1}{2} e^{-\frac{1}{2}t} \begin{pmatrix} -3t-1 & 2t+\frac{4}{3} \\ 3 & -2 \end{pmatrix}$$

$$\Psi(t)^{-1} \begin{pmatrix} 1+e^t \\ -e^t \end{pmatrix} = \frac{1}{2} e^{-\frac{1}{2}t} \left[\begin{pmatrix} -3 & 2 \\ 0 & 0 \end{pmatrix} t + \begin{pmatrix} -1 & \frac{4}{3} \\ 3 & -2 \end{pmatrix} \right] \left[\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right]$$

$$= \frac{1}{2} \left[\begin{pmatrix} -5 \\ 0 \end{pmatrix} te^{\frac{1}{2}t} + \begin{pmatrix} -\frac{7}{3} \\ 5 \end{pmatrix} e^{\frac{1}{2}t} + \begin{pmatrix} -3 \\ 0 \end{pmatrix} te^{-\frac{1}{2}t} + \begin{pmatrix} -1 \\ 3 \end{pmatrix} e^{-\frac{t}{2}} \right]$$

$$\int^t \Phi(s)^{-1}\begin{pmatrix}1+e^s\\-e^s\end{pmatrix}ds$$ ~~$= \left((-5t+\frac{8}{3})e^t - \frac{3}{2}t^2 - t\right)$~~

$$= \begin{pmatrix}c_1\\c_2\end{pmatrix} + e^{-\frac{t}{2}}\begin{pmatrix}(-5t+\frac{23}{3})e^t+3t+7\\ \frac{5}{3}e^t-1\end{pmatrix}.$$

$$x = \Phi(t)\begin{pmatrix}c_1\\c_2\end{pmatrix} + \begin{pmatrix}2 & 2t+\frac{4}{3}\\ 3 & 3t+1\end{pmatrix}\begin{pmatrix}(-5t+\frac{23}{3})e^t+3t+7\\ \frac{5}{3}e^t-1\end{pmatrix}$$

$$= c_1\begin{pmatrix}2\\3\end{pmatrix}e^{\frac{1}{2}t} + c_2\left[\begin{pmatrix}2\\3\end{pmatrix}te^{\frac{1}{2}t} + \begin{pmatrix}\frac{4}{3}\\1\end{pmatrix}e^{\frac{1}{2}t}\right] + \begin{pmatrix}12\\13\end{pmatrix}e^t + \begin{pmatrix}4\\9\end{pmatrix} \quad \square$$

6.f. homo. eq. has sol.

$$x_h = c_1\begin{pmatrix}2\sin t+\cos t\\ \sin t\end{pmatrix} + c_2\begin{pmatrix}-5\sin t\\ \cos t-2\sin t\end{pmatrix}$$

so $$\Phi(t) = \begin{pmatrix}2\sin t+\cos t & -5\sin t\\ \sin t & \cos t-2\sin t\end{pmatrix}$$

$$\Phi(t)^{-1} = \frac{1}{\cos^2 t-4\sin^2 t+5\sin^2 t}\begin{pmatrix}\cos t-2\sin t & +5\sin t\\ -\sin t & 2\sin t+\cos t\end{pmatrix}$$

$$= \begin{pmatrix}\cos t-2\sin t & 5\sin t\\ -\sin t & 2\sin t+\cos t\end{pmatrix}$$

$$\Phi(t)^{-1}g(t) = \begin{pmatrix}\cos t-2\sin t & 5\sin t\\ -\sin t & 2\sin t+\cos t\end{pmatrix}\begin{pmatrix}\csc t\\ \sec t\end{pmatrix}$$

$$= \begin{pmatrix}\cot t-2+5\tan t\\ -1+2\tan t+1\end{pmatrix}$$

$$= \begin{pmatrix}\cot t-2+5\tan t\\ 2\tan t\end{pmatrix}.$$

$$\int^t \Phi(s)^{-1}g(s)ds = \begin{pmatrix}\log|\sin t|-2t-5\log|\cos t|\\ -2\log|\cos t|\end{pmatrix} + \begin{pmatrix}c_1\\c_2\end{pmatrix}.$$

~~$x = \begin{pmatrix}\log|\sin t|-5\log|\cos t|-2t+c_1\\ -2\log|\cos t|+c_2\end{pmatrix}$~~

$$x = \begin{pmatrix}2\sin t+\cos t & -5\sin t\\ \sin t & \cos t-2\sin t\end{pmatrix}\begin{pmatrix}\log|\sin t|-5\log|\cos t|+2t+c_1\\ -2\log|\cos t|+c_2\end{pmatrix} \quad \square$$

7a. $tx' = \begin{pmatrix} 2 & -1 \\ 2 & -1 \end{pmatrix} x, \quad x = t^r \xi$

$r\xi = \begin{pmatrix} 2 & -1 \\ 2 & -1 \end{pmatrix} \xi.$

$r_1 = 1, \; v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \; r_2 = 0, \; v_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$

$x = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} t + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$ □

b. $r_1 = \frac{1}{2}$ (mult. 2), $v_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \; \begin{pmatrix} \frac{3}{2} & -1 \\ \frac{9}{4} & -\frac{3}{2} \end{pmatrix} v_2 = v_1 \Rightarrow v_2 = \begin{pmatrix} \frac{4}{3} \\ 0 \end{pmatrix}.$

$x = c_1 \begin{pmatrix} 2 \\ 3 \end{pmatrix} t^{\frac{1}{2}} + c_2 \left[\begin{pmatrix} 2 \\ 3 \end{pmatrix} t^{\frac{1}{2}} \log t + \begin{pmatrix} \frac{4}{3} \\ 0 \end{pmatrix} t^{\frac{1}{2}} \right]$ □

c. $r_1 = 2 + \sqrt{3} i, \quad v_1 = \begin{pmatrix} -1 - \sqrt{3} i \\ 2 \end{pmatrix}$

$r_2 = 2 - \sqrt{3} i, \quad v_2 = \begin{pmatrix} -1 + \sqrt{3} i \\ 2 \end{pmatrix}$

$x = c_1' \begin{pmatrix} -1 - \sqrt{3} i \\ 2 \end{pmatrix} e^{(2+\sqrt{3} i)\log t} + c_2' \begin{pmatrix} -1 + \sqrt{3} i \\ 2 \end{pmatrix} e^{(2-\sqrt{3} i)\log t}$

~~$= c_1 \left[\begin{pmatrix} -1 \\ 2 \end{pmatrix} + \begin{pmatrix} -\sqrt{3} \\ 0 \end{pmatrix} i \right] t^2 (\cos\sqrt{3}t + i\sin\sqrt{3}t)$~~

$= c_1' \left[\begin{pmatrix} -1 \\ 2 \end{pmatrix} + \begin{pmatrix} -\sqrt{3} \\ 0 \end{pmatrix} i \right] t^2 \left(\cos(\sqrt{3}\log t) + i\sin(\sqrt{3}\log t) \right)$

$+ c_2' \left[\begin{pmatrix} -1 \\ 2 \end{pmatrix} - \begin{pmatrix} -\sqrt{3} \\ 0 \end{pmatrix} i \right] t^2 \left(\cos(\sqrt{3}\log t) - i\sin(\sqrt{3}\log t) \right)$

$= c_1 t^2 \left[\begin{pmatrix} -1 \\ 2 \end{pmatrix} \cos(\sqrt{3}\log t) + \begin{pmatrix} \sqrt{3} \\ 0 \end{pmatrix} \sin(\sqrt{3}\log t) \right]$

$+ c_2 t^2 \left[\begin{pmatrix} -\sqrt{3} \\ 0 \end{pmatrix} \cos(\sqrt{3}\log t) + \begin{pmatrix} -1 \\ 2 \end{pmatrix} \sin(\sqrt{3}\log t) \right]$ □

Solutions to practice problems in Chapter 6

1. a) $L\{2\cos t - 3\sin 2t\}$

$$= \frac{2s}{s^2+1} - \frac{3 \times 2}{s^2+4}$$

b) $L\{t\cos t\} = -\left(L\{\cos t\}\right)' = -\left(\frac{s}{s^2+1}\right)' = \frac{s^2-1}{(s^2+1)^2}$

$L\{t^2 \sin 2t\} = (-1)^2 \left(L\{\sin 2t\}\right)''$

$$= \left(\frac{2}{s^2+4}\right)'' = \frac{4(3s^2-4)}{(s^2+4)^3}$$

$$L\{2t\cos t - 3t^2 \sin 2t\} = \frac{2(s^2-1)}{(s^2+1)^2} - \frac{12(3s^2-4)}{(s^2+4)^3}$$

c) $L\{e^{-t}\cos t\} = \frac{s+1}{(s+1)^2+1}$

d) $L\{te^{-t}\sin t\} = -\left(L\{e^{-t}\sin t\}\right)' = -\left(\frac{1}{(s+1)^2+1}\right)'$

$$= \frac{2(s+1)}{\left((s+1)^2+1\right)^2}$$

d) $f(t) = t^2(1-u_1) + 1\cdot(u_1 - u_2) + (3-t)(u_2 - u_3)$

$= t^2 + (1-t^2)u_1 + (2-t)u_2 - (3-t)u_3$

$= t^2 - u_1(t)(t-1)(t-1+2) - u_2(t-2) + u_3(t-3)$

$$L\{f\} = \frac{2}{s^3} - e^{-s}L\{t(t+2)\}(s) - e^{-2s}L\{t\} + e^{-3s}L\{t\}$$

$$= \frac{2}{s^3} - e^{-s}\left(\frac{2}{s^3} + \frac{2}{s}\right) - e^{-2s}\frac{1}{s} + e^{-3s}\frac{1}{s}$$

e) $f(t) = t\,(1-u_1) + (3-t)\,(u_1 - u_2) + u_2 - u_3$

$$= t + (3-2t)\,u_1 + (t-2)\,u_2 - u_3$$
$$= t + (1-2(t-1))\,u_1 + (t-2)\,u_2 - u_3$$

$$L\{f\} = \frac{1}{s^2} + e^{-s}\left(\frac{1}{s} - \frac{2}{s^2}\right) + e^{-2s}\frac{1}{s^2} - \frac{e^{-2s}}{s}$$

g) $$L\{f\} = e^{-s}\left(\frac{1}{s} + \frac{1}{s^2}\right) - 2\frac{e^{-2s}}{s} + e^{-3s}$$

h) $$L\{f\} = e\,\frac{e^{-s}}{s} \cdot \frac{1}{s-1} - e^{-1}\,\frac{e^{-3s}}{s} \cdot \frac{1}{s+1}$$
$$+ e^{-10} \cdot e^{-10s}$$

2 a) $$\frac{1}{(s+\frac{1}{2})^2 + \frac{3}{4}} = \frac{1}{(s+\frac{1}{2})^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$L^{-1}\left\{\frac{1}{s^2+s+1}\right\} = e^{-\frac{1}{2}t}\left(\sin\frac{\sqrt{3}}{2}t\right) \cdot \frac{2}{\sqrt{3}}$$

b) $$\frac{s}{s^2+s+1} = \frac{s+\frac{1}{2}}{(s+\frac{1}{2})^2 + \left(\frac{\sqrt{3}}{2}\right)^2} - \frac{\frac{1}{2}}{(s+\frac{1}{2})^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$L^{-1}\left\{\frac{s}{s^2+s+1}\right\} = e^{-\frac{1}{2}t}\cos\frac{\sqrt{3}}{2}t - \frac{1}{\sqrt{3}}\,e^{-\frac{1}{2}t}\sin\frac{\sqrt{3}}{2}t$$

c). $L^{-1}\left\{\frac{e^{-s}}{s^2+s+1}\right\} = u_1(t)\, L^{-1}\left\{\frac{1}{s^2+s+1}\right\}(t-1)$

$= u_1(t)\, \frac{2}{\sqrt{3}}\, e^{-\frac{1}{2}(t-1)} \sin \frac{\sqrt{3}}{2}(t-1)$

d) $L^{-1}\left\{\frac{e^{-s}}{s(s^2+s+1)}\right\} = u_1(t)\, L^{-1}\left\{\frac{1}{s(s^2+s+1)}\right\}(t-1)$

$\frac{1}{s(s^2+s+1)} = \frac{A}{s} + \frac{B(s+\frac{1}{2})}{(s+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2} + \frac{C}{(s+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}$

$A=1,\ B=-1,\ C=-\frac{1}{2}$

$L^{-1}\left\{\frac{1}{s(s^2+s+1)}\right\} = 1 - e^{-\frac{1}{2}t} \cos \frac{\sqrt{3}}{2}t - \frac{1}{\sqrt{3}} e^{-\frac{1}{2}t} \sin \frac{\sqrt{3}}{2}t$

$L^{-1}\left\{\frac{e^{-s}}{s(s^2+s+1)}\right\} = u_1(t)\left(1 - e^{-\frac{1}{2}(t-1)} \cos \frac{\sqrt{3}}{2}(t-1) - \frac{1}{\sqrt{3}} e^{-\frac{1}{2}(t-1)} \sin \frac{\sqrt{3}}{2}(t-1)\right)$

e) $\frac{1}{s^2(s^2+2s+2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C(s+1)}{(s+1)^2+1} + \frac{D}{(s+1)^2+1}$

$As((s+1)^2+1) + B((s+1)^2+1) + Cs^2(s+1) + Ds^2 = 1$

$s=0 \Rightarrow B=1 \qquad A+2B=0 \Rightarrow A=-2$

$A+C=0 \Rightarrow C=2$

$s=-1 \qquad -A+B+D=1 \Rightarrow D=-2$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+2s+2)}\right\}$$

$$= -2 + t + 2e^{-t}\cos t - 2e^{-t}\sin t$$

f) $$\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s(s^2+2s+2)}\right\} = u_2(t)\,\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+2s+2)}\right\}(t-2)$$

$$\frac{1}{s(s^2+2s+2)} = \frac{A}{s} + \frac{B(s+1)}{(s+1)^2+1} + \frac{C}{(s+1)^2+1}$$

$$\Rightarrow A = \frac{1}{2} \quad C = \frac{1}{2} \quad B = -\frac{1}{2}$$

$$\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s(s^2+2s+2)}\right\} = u_2(t)\left(\frac{1}{2} + \frac{1}{2}e^{-(t-2)}\cos(t-2) - \frac{1}{2}e^{-(t-2)}\sin(t-2)\right)$$

6.5

3. a)

$$s^2 Y(s) - 1 + 2(sY(s) - 0) + 2Y(s) = \frac{1}{s-1} + \frac{1}{s^2+1}$$

$$(s^2+2s+2)Y(s) = \frac{s}{s-1} + \frac{1}{s^2+1}$$

$$Y(s) = \frac{s}{(s-1)(s^2+2s+2)} + \frac{1}{(s^2+1)(s^2+2s+2)}$$

$$\frac{s}{(s-1)(s^2+2s+2)} = \frac{A}{s-1} + \frac{B(s+1)}{(s+1)^2+1} + \frac{C}{(s+1)^2+1}$$

$$s = A(s^2+2s+2) + B(s^2-1) + C(s-1)$$

$s=1 \Rightarrow \quad 1 = 5A \quad \Rightarrow A = \frac{1}{5}$

• $A + B = 0 \Rightarrow B = -\frac{1}{5}$

$2A - B - C = 0 \Rightarrow C = 2A - B = \frac{3}{5}$

$$\frac{1}{(s^2+1)(s^2+2s+2)} = \frac{As}{s^2+1} + \frac{B}{s^2+1} + \frac{C(s+1)}{(s+1)^2+1} + \frac{D}{(s+1)^2+1}$$

$$As(s^2+2s+2) + B(s^2+2s+2) + C(s^2+1)(s+1) + D(s^2+1) = 1$$

$$\left.\begin{aligned} A + C &= 0 \\ 2A + B + C + D &= 0 \\ 2A + 2B + C &= 0 \\ 2B + C + D &= 1 \end{aligned}\right\} \Rightarrow \begin{cases} A = -\frac{2}{5} \\ B = \frac{1}{5} \\ C = \frac{2}{5} \\ D = \frac{1}{5} \end{cases}$$

$$y(t) = \frac{1}{5}e^{t} - \frac{1}{5}e^{-t}\cos t + \frac{3}{5}e^{-t}\sin t$$

$$- \frac{2}{5}\cos t + \frac{1}{5}\sin t + \frac{2}{5}e^{-t}\cos t + \frac{1}{5}e^{-t}\sin t$$

b. $(s^2+2s+1)Y(s) = \frac{1}{s+1} + e^{-s}$

$$Y(s) = \frac{1}{(s+1)^3} + \frac{e^{-s}}{(s+1)^2}$$

$$y(t) = \frac{1}{2} e^{-t} \frac{1}{t^2} - u_1(t)\, L^{-1}\left\{\frac{1}{(s+1)^2}\right\}(t-1)$$

$$= \frac{1}{2} e^{-t} \frac{1}{t^2} - u_1(t)\, e^{-(t-1)} \frac{1}{t-1}$$

c. $(s^2+4)Y(s) = \frac{e^{-s}}{s} + \cos\pi\, e^{-s} - \frac{2}{s^2+4}$

$$Y(s) = \frac{e^{-s}}{s(s^2+4)} - \frac{e^{-s}}{s^2+4} - \frac{2}{(s^2+4)^2}$$

To find inverse Laplace transform of $\frac{2}{(s^2+4)^2}$, we note that

$$\left(\frac{s}{s^2+4}\right)' = \frac{1}{s^2+4} - \frac{2s^2}{(s^2+4)^2} = \frac{4-s^2}{(s^2+4)^2}$$

$$= \frac{8}{(s^2+4)^2} - \frac{1}{(s^2+4)}$$

$$\frac{2}{(s^2+4)^2} = \frac{\frac{1}{4}}{(s^2+4)} + \frac{1}{4}\left(\frac{s}{s^2+4}\right)'$$

$$L^{-1}\left\{\frac{2}{(s^2+4)^2}\right\} = \frac{1}{8}\sin 2t - \frac{1}{4} t\cos 2t$$

d. $(2s^2+s+4)Y(s) = \sin\frac{\pi}{4} \quad e^{-\frac{\pi}{4}s}$

$$Y(s) = \frac{\frac{\sqrt{2}}{2} e^{-\frac{\pi}{4}s}}{2(s+\frac{1}{4})^2 + \frac{31}{8}} = \frac{\sqrt{3}}{4} \quad \frac{e^{-\frac{\pi}{4}s}}{(s+\frac{1}{4})^2 + (\frac{\sqrt{31}}{4})^2}$$

$$y(t) = \frac{\sqrt{3}}{4} \; u_{\frac{\pi}{4}}(t) \; L^{-1}\left\{\frac{1}{(s+\frac{1}{4})^2+(\frac{\sqrt{31}}{4})^2}\right\}(t-\frac{\pi}{4})$$

$$= \frac{\sqrt{3}}{2} \; u_{\frac{\pi}{4}}(t) \; \frac{4}{\sqrt{31}} \cdot e^{-\frac{1}{4}(t-\frac{\pi}{4})} \sin\left(\frac{\sqrt{31}}{4}(t-\frac{\pi}{4})\right)$$

e. $g(t) = \frac{t}{2}(1-u_6) + 3u_6$

$$= \frac{t}{2} + (3-\frac{t}{2})u_6$$

$$= \frac{t}{2} - \frac{t-6}{2} u_6(t)$$

$$L\{g\} = \frac{1}{2s^2} - \frac{e^{-6s}}{2s^2}$$

$$s^2 Y(s) - 1 + Y(s) = g(t)$$

$$Y(s) = \frac{1}{s^2+1} + \frac{1}{2s^2(s^2+1)} - \frac{e^{-6s}}{2s^2(s^2+1)}$$

$$\frac{1}{s^2(s^2+1)} = \frac{1}{s^2} - \frac{1}{s^2+1}, \quad L^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\} = t - \sin t$$

$$y(t) = \sin t + \frac{1}{2}(t - \sin t) - \frac{1}{2} u_6(t)\left((t-6) - \sin(t-6)\right)$$

f) $g(t) = \sin t(1 - u_\pi) + \cos t(u_\pi - u_{2\pi}) - u_{2\pi}$

$= \sin t + (\cos t - \sin t)\, u_\pi - (1 + \cos t)\, u_{2\pi}$

$= \sin t + (-\cos(t-\pi) + \sin(t-\pi))\, u_\pi$

$\quad - (1 + \cos(t-2\pi))\, u_{2\pi}$

$$L\{g\} = \frac{1}{s^2+1} + e^{-\pi s}\left(-\frac{s}{s^2+1} + \frac{1}{s^2+1}\right) - e^{-2\pi s}\left(\frac{1}{s} + \frac{s}{s^2+1}\right)$$

$$Y(s) = \frac{1}{(s^2+s+\frac{5}{4})(s^2+1)} + e^{-\pi s}\left(\frac{1-s}{(s^2+s+\frac{5}{4})(s^2+1)}\right) - \frac{e^{-2\pi s}(2s^2+1)}{s(s^2+1)(s^2+s+\frac{5}{4})}$$

Method of partial fractions:

$$\frac{1}{(s^2+s+\frac{5}{4})(s^2+1)} = \frac{As}{s^2+1} + \frac{B}{s^2+1} + \frac{C(s+\frac{1}{2})}{s^2+s+\frac{5}{4}} + \frac{D}{s^2+s+\frac{5}{4}}$$

$$\frac{1-s}{(s^2+s+\frac{5}{4})(s^2+1)} = \frac{As}{s^2+1} + \frac{B}{s^2+1} + \frac{C(s+\frac{1}{2})}{s^2+s+\frac{5}{4}} + \frac{D}{s^2+s+\frac{5}{4}}$$

$$\frac{2s^2+1}{s(s^2+1)(s^2+s+\frac{5}{4})} = \frac{A}{s} + \frac{Bs}{s^2+1} + \frac{C}{s^2+1} + \frac{D(s+\frac{1}{2})}{s^2+s+\frac{1}{4}} + \frac{E}{s^2+s+\frac{1}{4}}$$

h. $(s^2+2s+3)Y(s) = e^{-5s}\frac{1}{s^2} + \sin 10\, e^{-10s} + \frac{(s+1)}{(s+1)^2+2}$

$$Y(s) = \frac{e^{-5s}}{s^2(s^2+2s+3)} + \frac{\sin 10\, e^{-10s}}{s^2+2s+3} + \frac{s+1}{((s+1)^2+2)^2}$$

$$\frac{1}{s^2(s^2+2s+3)} = \frac{-\frac{2}{9}}{s} + \frac{\frac{1}{3}}{s^2} + \frac{\frac{2}{9}(s+1)}{(s+1)^2+2} \quad \frac{\frac{1}{9}}{(s+1)^2+2}$$

$$y(t) = u_5(t)\left(-\frac{2}{9} + \frac{1}{3}(t-5) + \frac{2}{9}e^{-(t-5)}\cos\sqrt{2}(t-5) - \frac{1}{9\sqrt{2}}\sin\sqrt{2}(t-5)\, e^{-(t-5)}\right)$$
$$+ \sin 10\, u_{10}(t)\frac{1}{\sqrt{2}}\sin\sqrt{2}(t-5) + \frac{1}{2}t\,\sin\sqrt{2}t\; e^{-t})$$

i) $\left(s^2+s+\frac{5}{4}\right)Y(s) = \frac{1}{s} - e^{-\pi s}\left(\frac{1}{s^2} + \frac{\pi}{2}\frac{1}{s}\right) + \frac{s+\frac{1}{2}}{(s+\frac{1}{2})^2+1} + 25e^{-5s}$

$$Y(s) = \frac{1}{s(s^2+s+\frac{5}{4})} - e^{-\pi s}\frac{1+\frac{\pi}{2}s}{s^2(s^2+s+\frac{5}{4})} + \frac{(s+\frac{1}{2})}{(s^2+s+\frac{5}{4})^2} + \frac{25e^{-5s}}{(s^2+s+\frac{5}{4})}$$

$$y(t) = \frac{4}{5} - \frac{4}{5}e^{-\frac{t}{2}}\cos t \quad + \cdots$$

6.10

4 a). $s^2 Y(s) - sy_0 - y_1 + 9Y(s) = g(s)$

$$Y(s) = \frac{sy_0 + y_1}{s^2+9} + \frac{1}{s^2+9} g(s)$$

$$y(t) = L^{-1}\{Y\} = y_0 \cos 3t + \frac{1}{3} y_1 \sin 3t + \int_0^t \sin 3(t-\tau)\, g(\tau)\, d\tau$$

b). $s^2 Y(s) - sy_0 - y_1 + 2(sY(s) - y_0) + 10Y(s) = G(s)$

$$Y(s) = \frac{sy_0 + 2y_0 + y_1}{s^2+2s+10} + \frac{1}{s^2+2s+10} G(s)$$

$$y(t) = L^{-1}\{Y\} = L^{-1}\left\{\frac{(s+1)y_0 + y_0 + y_1}{(s+1)^2+9}\right\} + L^{-1}\left\{\frac{1}{(s+1)^2+9} g(s)\right\}$$

$$= y_0 e^{-t} \cos 3t + \frac{(y_0+y_1)}{3} e^{-t} \sin 3t$$

$$+ \int_0^t \frac{e^{-(t-\tau)}}{3} \sin 3(t-\tau)\, g(\tau)\, d\tau$$

5. a) $L\{f\} = L\{t^2\} L\{\cos 2t\} = \frac{2}{s^3} \cdot \frac{s}{s^2+4} = \frac{2}{s^2(s^2+4)}$

b) $L\{f\} = L\{t\} L\{e^t\} = \frac{1}{s} \cdot \frac{1}{s-1}$

c) $L\{f\} = L\{e^{-t}\} L\{\sin t\} = \frac{1}{s+1} \frac{1}{s^2+1}$

d) $L\{f\} = L\{te^{-t}\} L\{\sin t\}$

$= -\left(L\{e^{-t}\}\right)' L\{\sin t\}$

$= -\left(\frac{1}{s+1}\right)' \cdot \frac{1}{s^2+1}$

$= \frac{1}{(s+1)^2 (s^2+1)}$

e) $L^{-1}\{F\} = \int_0^t L^{-1}\{\frac{1}{s^4}\}(t-\tau)\, L^{-1}\{\frac{1}{s^2+1}\}$

$= \int_0^t \frac{1}{3!}(t-\tau)^3 \sin \tau \, d\tau$

f) $L^{-1}\{F\} = L^{-1}\{\frac{1}{s+1} \frac{s}{s^2+4}\} = \int_0^t e^{-(t-\tau)} \cos 2\tau \, d\tau$

g) $F(s) = \frac{1}{s^2+4} \cdot \frac{1}{(s+1)^2}$

$L^{-1}\{F\} = \int_0^t e^{-(t-\tau)} (t-\tau) \frac{1}{2} \sin 2\tau = \frac{1}{2}\int_0^t e^{-(t-\tau)} (t-\tau) \sin 2\tau \, d\tau$

h) $L^{-1}\{F\} = \int_0^t \sin(t-\tau)\, g(\tau)\, d\tau$