chapter 10

1 (a). $f(x) \sim \frac{a_0}{2} + \sum \left(a_n \cos \frac{n\pi}{L}x + b_n \sin \frac{n\pi}{L}x\right)$

Here $2L = 2\pi \Rightarrow L = \pi$.

So

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi}{L}x\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} x\cos nx \cdot$$

$$= \frac{1}{\pi}\left(\frac{1}{n}x\sin nx + \frac{1}{n^2}\cos nx\Big|_{-\pi}^{0}\right)$$

$$= \frac{1}{n^2\pi}(1-\cos n\pi) = \frac{1-(-1)^n}{n^2\pi}$$

$$\int x\sin\beta x\,dx = -\frac{1}{\beta}x\cos\beta x + \frac{1}{\beta^2}\sin\beta x$$

$$\int x\cos\beta x\,dx = \frac{1}{\beta}x\sin\beta x + \frac{1}{\beta^2}\cos\beta x$$

$$\cancel{b_n = \frac{1}{L}\int_{-L}^{L}}$$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} x\,dx = \frac{\pi}{2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{0} x\sin nx = \frac{1}{\pi}\left(-\frac{1}{n}x\cos nx + \frac{1}{n^2}\sin nx\Big|_{-\pi}^{0}\right)$$

$$= -\frac{1}{n\pi}\left(0-(-\pi)\cos n\pi\right) = -\frac{(-1)^n}{n}$$

So $f(x) \sim \frac{\pi}{4} + \sum_{n=1}^{+\infty}\left(\frac{1-(-1)^n}{n^2\pi}\cos nx - \frac{(-1)^n}{n}\sin nx\right)$

(b) f is odd, $2L = 4 \Rightarrow L = 2$

So $f(x) \sim \sum_{n=1}^{+\infty} b_n \sin\left(\frac{n\pi}{L}x\right)$

$$\sim \sum_{n=1}^{+\infty} b_n \sin\left(\frac{n\pi}{2}x\right)$$

$$b_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi}{2}x\right)$$

$$= \int_0^2 f(x) \sin\left(\frac{n\pi}{2}x\right) dx$$

$$= \int_0^1 x \sin\left(\frac{n\pi}{2}x\right) dx.$$

$$= -\frac{2}{n\pi} x \cos\frac{n\pi}{2}x + \frac{4}{n^2\pi^2} \sin\left(\frac{n\pi}{2}x\right)\Big|_0^1$$

$$= -\frac{2}{n\pi}\cos\frac{n\pi}{2} + \frac{4}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)$$

$$f(x) \sim \sum_{n=1}^{+\infty}\left(-\frac{2}{n\pi}\cos\frac{n\pi}{2} + \frac{4}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\right)\sin\left(\frac{n\pi}{2}x\right).$$

(c) $2L = 4 \Rightarrow L = 2$

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{+\infty}\left(a_n \cos\frac{n\pi}{2}x + b_n \sin\left(\frac{n\pi}{2}x\right)\right)$$

$$a_0 = \frac{1}{2}\int_{-2}^{2} f(x)\, dx = \frac{1}{2}\left(\int_{-2}^{0}(x+2) + \int_0^2 (2-2x)\right)$$

$$= 1$$

$$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi}{2}x\,dx$$

$$= \frac{1}{2}\left[\int_{-2}^{0}(x+2)\cos\frac{n\pi}{2}x\,dx + \int_{0}^{2}(2-2x)\cos\frac{n\pi}{2}x\,dx\right]$$

$$= \frac{1}{2}\left[\int_{-2}^{0}x\cos\frac{n\pi}{2}x\,dx + \int_{0}^{2}(-2x)\cos\frac{n\pi}{2}x\,dx\right]$$

$$= -\frac{3}{2}\int_{0}^{2}x\cos\frac{n\pi}{2}x\,dx$$

$$= -\frac{3}{2}\left[\frac{2}{n\pi}x\sin\frac{n\pi}{2}x + \frac{4}{n^2\pi^2}\cos\frac{n\pi}{2}x\right]\Bigg|_{0}^{2}$$

$$= -\frac{3}{2}\times\frac{4}{n^2\pi^2}\left[\cos n\pi - 1\right]$$

$$= \frac{6}{n^2\pi^2}\left(1-(-1)^n\right)$$

$$b_n = \frac{1}{2}\int_{-2}^{2} f(x)\sin\left(\frac{n\pi}{2}x\right)dx$$

$$= \frac{1}{2}\left[\int_{-2}^{0}x\sin\frac{n\pi}{2}x\,dx + \int_{0}^{2}(-2x)\sin\left(\frac{n\pi}{2}x\right)dx\right]$$

$$= -\frac{1}{2}\int_{0}^{2}x\sin\left(\frac{n\pi}{2}x\right)dx$$

$$= -\frac{1}{2}\left[-\frac{2}{n\pi}x\cos\frac{n\pi}{2}x + \frac{4}{n^2\pi^2}\sin\frac{n\pi}{2}x\right]\Bigg|_{0}^{2}$$

$$= \frac{1}{n\pi}\left(2\cos n\pi\right) = \frac{(-1)^n}{n\pi}$$

2. Recall:

$$\frac{a_0}{2} + \sum_{n=1}^{+\infty}\left(a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right)\right) = \frac{1}{2}\left(f(x+) + f(x-)\right)$$

$x = -\frac{\pi}{2}$, $\quad f(-\frac{\pi}{2}+) = -\frac{\pi}{2}$, $\quad f(-\frac{\pi}{2}-) = -\frac{\pi}{2}$

So $\quad \frac{1}{2}\left(f(-\frac{\pi}{2}+) + f(-\frac{\pi}{2}-)\right) = -\frac{\pi}{2}$

$x = 0$, $\quad f(0+) = 0$, $\quad f(0+) = 0$.

So $\quad \frac{1}{2}\left(f(0+) + f(0-)\right) = 0$.

$x = \frac{\pi}{2}$, $\quad f(\frac{\pi}{2}+) = 0$, $\quad f(\frac{\pi}{2}-) = 0$

$\quad \frac{1}{2}\left(f(\frac{\pi}{2}+) + f(\frac{\pi}{2}-)\right) = 0$

3. We have Dirichlet BC, $\quad L = 2\pi$.

So $\quad u(x,t) = \sum_{n=1}^{+\infty} b_n e^{-(\frac{n\pi}{L})^2 t} \sin\left(\frac{n\pi}{L}x\right) = \sum_{n=1}^{+\infty} b_n \sin\left(\frac{n}{2}x\right) e^{-(\frac{n}{2})^2 t}$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx = \frac{2}{2\pi}\int_0^{2\pi} f(x)\sin\left(\frac{n}{2}x\right)dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} (\sin x - \sin 4x)\sin\left(\frac{n}{2}x\right)dx$$

$$= \begin{cases} 1, & n = 2 \\ -1, & n = 8 \\ 0 & n \neq 2, 8 \end{cases}$$

So $\quad u(x,t) = e^{-t}\sin x - e^{-16t}\sin 4x$

4. $$\begin{cases} u_{tt} = u_{xx}, & 0 < x < \pi \\ u(x,0) = \cos 2x, & u_t(x,0) = \cos^2 x \\ u_x(0,t) = 0, & u_x(\pi, t) = 0 \end{cases}$$

BC is Neumann BC

EVP: $X'' + \lambda X = 0, \quad X'(0) = 0, \quad X'(\pi) = 0$

$$\lambda = \lambda_n = \left(\frac{n\pi}{L}\right)^2 = \left(\frac{n\pi}{\pi}\right)^2 = n^2, \quad n = 0, 1, 2, \cdots$$

$$X_n = \cos\left(\frac{n\pi}{L}x\right) = \cos nx$$

$\cancel{u(x,t)}$

$$T'' + \lambda_n T = 0 \Rightarrow T'' + n^2 T = 0$$

$$\Rightarrow T_n = a\cos nt + b \sin nt$$

$n = 0$

$T = a_1 + b_2 t$

$$u(x,t) = \frac{a_0 + b_0 t}{2} + \sum_{n=1}^{+\infty} \left(a_n \cos nt + b_n \sin nt\right) \cos(nx)$$

IC: $u(x,0) = \cos 2x \Rightarrow$

$$\frac{a_0}{2} + \sum_{n=1}^{+\infty} a_n \cos nx = \cos 2x$$

$$a_n = \frac{2}{L}\int_0^L (\cos 2x) \cos nx \, dx = \frac{2}{\pi}\int_0^\pi (\cos 2x) \cos nx \, dx$$

$$= \begin{cases} 0, & \text{if } n \neq 2 \\ 1, & \text{if } n = 2 \end{cases}$$

$u_t(x,0) = \cos^2 x$

$= \frac{1 - \cos 2x}{2} = \frac{1}{2} - \frac{1}{2}\cos 2x$

$u_t(x,0) = \frac{b_0}{2} + \sum_{n=1}^{+\infty} (n\, b_n) \cos nx$

$b_0 = \frac{2}{L}\int_0^L g(x)\,dx = \frac{2}{\pi}\int_0^{\pi} \left(\frac{1}{2} - \frac{1}{2}\cos 2x\right) dx$

$= 1$

$n b_n = \frac{2}{\pi}\int_0^{\pi} g(x) \cos nx \, dx$

$= \frac{2}{\pi}\int_0^{\pi} \left(\frac{1}{2} - \frac{1}{2}\cos 2x\right)\cos nx \, dx$

$= -\frac{2}{\pi}$ if $n = 2$.

0 if $n \neq 2$

$\Rightarrow 2b_2 = -\frac{2}{\pi} \Rightarrow b_2 = -\frac{1}{\pi}$

$b_n = 0$ if $n \neq 2$

$u(x,t) = a_2 \cos 2t \cos 2x + \frac{t}{2} + b_2 \sin 2t \sin 2x$

$= \cos 2t \cos 2x + \frac{t}{2} - \frac{1}{\pi} \sin 2t \cdot \sin 2x$

5. (EVP) $\begin{cases} X'' + \lambda X = 0 \\ X'(0) = 0, \quad X(\pi) = 0 \end{cases}$

$$\lambda_n = \left(\frac{2n-1}{2\pi}\pi\right)^2 = \left(n-\frac{1}{2}\right)^2, \quad n = 1, 2, \cdots$$

$$X_n = \cos\left(n-\frac{1}{2}\right)x$$

$$T' + \left(n-\frac{1}{2}\right)^2 T = 0 \qquad T_n = a\, e^{-\left(n-\frac{1}{2}\right)^2 t}$$

$$u(x,t) = \sum_{n=1}^{+\infty} a_n\, e^{-\left(n-\frac{1}{2}\right)^2 t} \cos\left(n-\frac{1}{2}\right)x$$

$$u(x,0) = \sin^2 x = \frac{1-\cos 2x}{2} = \sum_{n=1}^{+\infty} a_n \cos\left(n-\frac{1}{2}\right)x$$

$$a_n = \frac{2}{\pi}\int_0^{\pi}\left(\frac{1-\cos 2x}{2}\right)\cos\left(n-\frac{1}{2}\right)x$$

$$= \frac{2}{\pi}\left[\frac{\sin\left(n-\frac{1}{2}\right)\pi}{2\left(n-\frac{1}{2}\right)} - \frac{1}{4}\int_0^{\pi}\left(\cos\left(n+\frac{3}{2}\right)x + \cos\left(n-\frac{5}{2}\right)x\right)\right]$$

$$= \frac{2}{\pi}\left[\frac{(-1)^{n+1}}{2\left(n-\frac{1}{2}\right)} - \frac{\sin\left(n+\frac{3}{2}\right)\pi}{4\left(n+\frac{3}{2}\right)} - \frac{\sin\left(n-\frac{5}{2}\right)\pi}{4\left(n-\frac{5}{2}\right)}\right]$$

$$= \frac{2}{\pi}\left[\frac{(-1)^{n+1}}{2\left(n-\frac{1}{2}\right)} - \frac{(-1)^n}{4\left(n+\frac{3}{2}\right)} - \frac{(-1)^{n-1}}{4\left(n-\frac{5}{2}\right)}\right]$$