List of Formulas in Chapter 7 and Chapter 6

Chapter 7: We study

$$X' = P(t)X + \vec{g}(t), \quad X(0) = X_0$$

(1) General Theory: General solution is

$$X_p(t) + c_1 X^{(1)} + \cdots + c_n X^{(n)}$$

where $X^{(1)}, \cdots, X^{(n)}$ are solutions of homogeneous problem which form a fundament set of solutions

(2) $\{X^{(1)}(t), \cdots, X^{(n)}\}$ fundamental set of solutions if and only if

$$W(t) = \det\left(X^{(1)} \cdots X^{(n)}\right) \neq 0$$

~~Consta~~

(3) (Abel's Formula): $\dfrac{dW}{dt} = (P_{11}(t) + \cdots + P_{nn}(t))\,W$

Constant Coefficient Case $P(t) = A$

$$X' = AX$$

Suppose A is a $n \times n$ matrix.

General Case

(4). Case 1: A has n linearly independent eigenvectors $\vec{\xi}^{(1)}, \dots, \vec{\xi}^{(n)}$

Then $X^{(1)} = \vec{\xi}^{(1)} e^{r_1 t}, \dots, X^{(n)} = \vec{\xi}^{(n)} e^{r_n t}$

forms a fundamental set of solutions

Case 2: A has repeated roots $r_1 = r_2 = r$.

Then $X^{(1)} = \vec{\xi} e^{rt}$

$X^{(2)}$ can be obtained by

$$X^{(2)} = t\vec{\xi} e^{rt} + \vec{\eta} e^{rt}$$

$$(A - rI)\vec{\eta} = \vec{\xi}$$

Case 3: A has complex eigenvalues, $r = \lambda \pm i\mu$

For $r = \lambda + i\mu$, $\xi = a + i b$ is the eigenvector

Then Real part of $\vec{\xi} e^{rt}$ and Imaginary part of $\vec{\xi} e^{rt}$ give two solutions

(5) If A is a 2×2 matrix, then there are 3 cases only.

<u>Case 1</u>: A has two linearly independent eigenvectors

$$\xi^{(1)} e^{r_1 t}, \quad \xi^{(2)} e^{r_2 t}$$

<u>Case 2</u>: A has two complex eigenvalues

$$r = \lambda \pm i\mu, \quad r_1 = \lambda + i\mu \Rightarrow \xi^{(1)}$$

$$\mathrm{Re}\left(\xi^{(1)} e^{r_1 t}\right) = x^{(1)}, \quad \mathrm{Im}\left(\xi^{(1)} e^{r_1 t}\right) = x^{(2)}$$

Case 3: A has repeated eigenvalues but one eigenvector

$$r_1 = r_2 = r, \quad \xi^{(1)}$$

$$x^{(1)} = \xi^{(1)} e^{rt}, \quad x^{(2)} = \xi^{(1)} t e^{rt} + \eta e^{rt}$$

$$(A - rI)\vec{\eta} = \vec{\xi}^{(1)}$$

(6) $\Psi(t) = \left(x^{(1)} \cdots x^{(n)}\right)$ is a fundamental matrix

Any solution of

$$X' = AX, \quad X(0) = X_0$$

is given by

$$X = \Psi(t)\left(\Psi(0)\right)^{-1} X_0$$

(7) For inhomogeneous case

$$X' = Ax + \vec{g}(t)$$

<u>Method of Diagonalization:</u> If A is diagonalizable

$$T^{-1}AT = \begin{pmatrix} r_1 & & 0 \\ & \ddots & \\ 0 & & r_n \end{pmatrix}$$

Then let $x = TY$, $h = T^{-1}\vec{g}(t)$

$$Y' = \begin{pmatrix} r_1 & & \\ & \ddots & \\ & & r_n \end{pmatrix} Y + \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix}$$

$$Y_i' = r_i Y_i + h_i, \quad i = 1, \cdots, n$$

<u>Method of Undetermined Coefficients:</u> If g is the form

$e^{\alpha t}\vec{a}$, $e^{\alpha t}\cos\beta t\,\vec{a}$, we try

$$x_p = t^s\left[(\text{polynomial}) \times e^{\alpha t}\cos\beta t + (\text{polynomial}) \times e^{\alpha t}\sin\beta t\right]$$

$+$ lower order polynomial $e^{\alpha t}\cos\beta t$

$+$ lower order polynomial $e^{\alpha t}\sin\beta t$

Method of Variation of Parameters: Let $x_p = \Psi(t)\vec{u}$

Then $\Psi(t)\vec{u}\,' = \vec{g} \Rightarrow \vec{u}\,' = (\Psi(t))^{-1}g \Rightarrow \vec{u}$

Chapter 6: We use Laplace transform to solve

$$ay'' + by' + cy' = g(t), \quad y(0) = y_0, \; y'(0) = y_1$$

where $g(t)$ is a combination of

- $t^n e^{\alpha t} \cos \beta t$
- $u_c(t) f(t-c)$
- $\delta_c(t) f(t)$.

1)

$$Y(s) = L\{y\} = \int_0^{+\infty} e^{-st} y(t)\, dt$$

$$y(t) \longleftrightarrow Y(s).$$

$$a(s^2 Y(s) - s y_0 - y_1) + b(s Y(s) - y_0) + c Y(s) = G(s)$$

$$Y(s) = \frac{as y_0 + b y_0 + a y_1}{as^2 + bs + c} + \frac{G(s)}{as^2 + bs + c}$$

2) Method of Partial Fractions:

$\dfrac{P_n(s)}{P_m(s)}$.

$P_m(s)$ contains	Partial Fractions
$s-a$	$\dfrac{A}{s-a}$
$(s-a)^2$	$\dfrac{A}{s-a} + \dfrac{B}{(s-a)^2}$
$(s-a)^2 + b^2$	$\dfrac{A(s-a)}{(s-a)^2 + b^2} + \dfrac{B}{(s-a)^2 + b^2}$

3). Useful Formulas

$f(t)$	$F(s)$
t^n	$\frac{n!}{s^n}$
$\cos bt$	$\frac{s}{s^2+b^2}$
$\sin bt$	$\frac{b}{s^2+b^2}$
$e^{at} f(t)$	$F(s-a)$
$t f(t)$	$-F'(s)$
$f'(t)$	$sF(s)-f(0)$
$u_c(t)$	$\frac{e^{-cs}}{s}$
$\delta_c(t)$	e^{-cs}
$u_c(t) f(t-c)$	$e^{-cs} F(s)$
$\delta_c(t) f(t)$	$f(c)\, e^{-cs}$
$f*g=\int_0^t f(t-\tau)g(\tau)\,d\tau$	$F(s)\,G(s)$