MATH 256

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- Handout with contact details / office hours / course outline etc.

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§0. Introduction

Differential equations allow us to express the <u>rate of change</u> of any given quantity. For example, recall Newton's 2nd Law applied to a point particle:

$$F = ma,$$

where a is the acceleration of the particle, m is its mass, and F is the force applied.

We know that a is the rate of change of the velocity, v, which itself is the rate of change of the position, x. Therefore,

$$a = \frac{dv}{dt} \quad \text{and} \quad v = \frac{dx}{dt} \quad \text{and} \quad a = \frac{d^2x}{dt^2}.$$

We often denote differentiation wrt time with a "dot":

$$a = \dot{v}, \quad v = \dot{x}, \quad a = \ddot{x}.$$

Therefore, $F = ma$ is the same as:

$$m\ddot{x} = m\dot{v} = ma = F \qquad (*)$$

We can imagine that F might be constant, or both $m\ddot{x} = F$ and $m\dot{v} = F$ are <u>differential eqns</u>:

<u>Example</u>

Let's consider motion in a constant gravitational field. Then, $F = -mg$, where g is the acceleration due to gravity. Then, X simplifies to:

$$\frac{d^2x}{dt^2} = \ddot{x} = -g.$$

We know how to integrate once:

$$v = \dot{x} = \frac{dx}{dt} = -gt + c$$

where c is a <u>constant of integration</u>.

We may integrate once more:

$$x(t) = -\tfrac{1}{2}gt^2 + ct + B$$

where B is another constant.

To find C and B we need <u>initial conditions</u>. These provide added context to the physical situation.

For example, suppose that the particle is at rest at $x=0$ when $t = 0$. This tells us that:

$$x(0) = 0 \quad \text{and} \quad \dot{x}(0) = v(0) = 0.$$

But, $v = -gt + c$. So $v(0) = 0 \Rightarrow c = 0$.

Then, $x = -\frac{1}{2}gt^2 + B$, so $x(0) = 0 \Rightarrow B = 0$.

Hence $\boxed{x = -\frac{1}{2}gt^2}$ GRAPH

Example

In reality, a particle experiences drag as it falls. A good model of this drag is to assume that there is an added force proportional to v^2. In this case, $\circledast$ becomes:

$$m\dot{v} = -mg + kv^2, \quad k = \text{const.} \quad (\dagger)$$

We don't have the tools to solve this yet, but we can investigate the steady state. This occurs when there is no rate of change of v, ie $\dot{v} = 0$. The velocity at which this occurs is called the terminal velocity, and is found by solving $(\dagger)$ when $\dot{v}$ ie:

$$m\dot{v} = -mg + kv_T^2 = 0$$

where $v_T \equiv$ terminal velocity. Rearranging, we find:

$$V_T = \pm\sqrt{\frac{mg}{k}}.$$ But g acts downwards, so $V_T = -\sqrt{\frac{mg}{k}}$.

§ 0.1 Terminology of DEs

Defn

A linear ordinary differential equation (linear ODE) for the function $y(x)$ is a relation of the form:

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \dots + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x)$$

- The order of the ODE is n.
- The ODE is linear because none of the terms

depend on powers of y or its derivatives (eg y^2, $\left(\frac{dy}{dx}\right)^3$, $\sin(y)$).

- If $f=0$, then the ODE is <u>homogeneous</u>.
- If the $a_i(x)$ are in fact constant (do not depend on x), then the ODE is <u>constant coefficient</u>.

§1 Linear, First Order ODEs

We will start with linear, first order ODEs. These provi[de] a good starting point, and highlight some key features of ODEs that will be important later when we consider nonlinear first order ODEs and ODEs of higher order.

§1.1 Homogeneous, constant coeff. first order linear OD[Es]

These are ODEs of the form:

$$a\frac{dy}{dx} + by = 0 \quad \text{where } a \neq 0 \text{ and } b \text{ are constants.}$$

For differentiation wrt space, x, we often write
$$y' \equiv \frac{dy}{dx}, \quad y'' \equiv \frac{d^2y}{dx^2} \quad \text{etc.}$$

Let's simplify and write:
$$y' + ky = 0, \qquad \text{where } k = b/a.$$

Observation

When $k=-1$, we obtain $y' = y$ (or $\frac{dy}{dx} = y$). Recall that this relation in fact defines the exponential function $\exp(x)$, or the exponent of the number e, i.e. e^x. Therefore, $y_p(x) = e^x$ solves $y_p' = y_p$, and is a particular solution.

However, it is not the general solution, since every function of the form $y(x) = Ae^x$, for some constant A, are solutions. "A" is in fact a constant of integration: see later. This is also the only type of solution ~~(see later)~~, and hence is the general soluti

To determine A, we need a boundary condition.

Eg. If $y(0) = y_0$ where $y_0 = $ const, then we require

$$y(0) = Ae^0 = A = y_0 \Rightarrow A = y_0$$

so that the sln is $y(x) = y_0 e^x$.

GRAPH $\rightarrow$ vary y_0

Remarks

- An n-th order linear ODE has a general solution with n constants of integration. (see later $n=1,2$
- To determine n unknown constants, we require

n initial or boundary conditions. (ICs or BCs)

- The solution $y(x) = y_0 e^x$ satisfies the ODE and the BC.

To demonstrate that Ae^x is the general solution, we need only show that $y(x) = y_0 e^x$ is <u>unique</u>.

What about general k? It is reasonable to hypothesise that the exp. function will again be important. We start with some definitions:

<u>Def^n</u> A <u>differential operator</u> is an operation that may be applied to a differentiable function $y(x)$ and is written:

$$D = a_n \frac{d^n}{dx^n} + a_{n-1} \frac{d^{n-1}}{dx^{n-1}} + \dots + a_1 \frac{d}{dx} + a_0,$$

and is understood through its action on $y(x)$:

$$D[y] := a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_1 \frac{dy}{dx} + a_0 y.$$

Then, our general linear ODE is written compactly as

$$D[y] = f, \quad \text{for some function } f(x).$$

Then, we have:

- The ODE is <u>homogeneous</u> if $f = 0$, and so $y = 0$ is a solution. (Though <u>not</u> the general solution).

• The ODE is <u>linear</u> if the differential operator D is linear in the sense that for all functions $y_1(x)$, $y_2(x)$ and all constants a and b, we have:

$$D[ay_1 + by_2] = aD[y_1] + bD[y_2]$$

and so we require that $a_i(x)$ depend on x <u>only</u> (and not y).

<u>Def$^{\underline{n}}$</u>

Let D be a <u>linear</u> differential operator. Then, the function $y(x)$ is an <u>eigenfunction</u> of D with <u>eigenvalue</u> λ if

$$D[y] = \lambda y.$$

[c.f. eigenvectors / eigenvalues of matrices]

<u>Observation</u>

The function $y(x) = e^{\lambda x}$ is an eigenfunction of the differential operator $D = \frac{d}{dx}$, since

$$D[y] = \frac{d}{dx}\left[e^{\lambda x}\right] = \lambda\left(e^{\lambda x}\right) = \lambda y.$$

<u>Idea</u>

Since $e^{\lambda x}$ is (relatively) unchanged by $\frac{d}{dx}$, let's try it as an <u>ansatz</u> (fancy German term that mathematicians use to mean "educated guess"): for the solution to our ODE.

Example

Solve $y' + ky = 0$ with BC $y(0) = y_0$.

Sol^n Let's try the ansatz $y(x) = e^{\lambda x}$. Then,

$$y' + ky = \lambda e^{\lambda x} + k e^{\lambda x} = (\lambda + k) e^{\lambda x} = 0.$$

So we have a soln if $\lambda + k = 0$ (since $e^{\lambda x} \neq 0$)

Hence, $\lambda = -k$.

Then general solution to the ODE is therefore

$y(x) = Ae^{-kx}$ some constant A.

The BC $y(0) = y_0 \Rightarrow A = y_0$. So the soln is:

$$y(x) = y_0 e^{-kx}.$$

Example

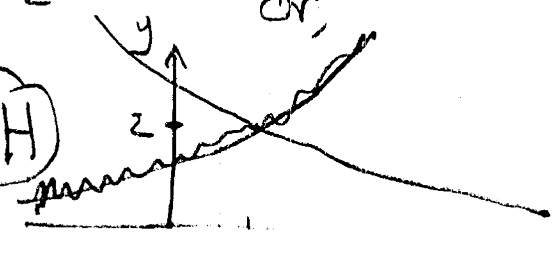
Solve $2y' + 3y = 0$ with BC $y(1) = 2$.

Sol^n Gen. sln is $y = Ae^{-\frac{3}{2}x}$ since $y' + \frac{3}{2}y = 0$ (so $k = \frac{3}{2}$). Then, $y(1) = Ae^{-3/2} = 2$. So

$A = 2e^{\frac{3}{2}}$ and $y(x) = 2e^{\frac{3}{2}} e^{-\frac{3}{2}x}$ or,

$y(x) = 2e^{-\frac{3}{2}(x-1)}$.

GRAPH



§ 1·2 Inhomogeneous, linear const. coeff. first order ODEs

These take the form:

$$y' + ky = f(x).$$

We will consider various $f(x)$. For compactness let's write $D[y] := y' + ky$, so that the ODE is $D[y] = f$.

Observation

Suppose that the function $y_p(x)$ satisfies the ODE, i.e. that $D[y_p] = f$. Then $y_p(x)$ is called a particular solution.

How do we find the general solution?

Note that unlike for the homogeneous case, we cannot multiply y_p by a constant, since:

$$D[Ay_p] = Ay_p' + Aky_p = A(y_p' + ky_p)$$

$$= Af \neq f \qquad (\text{unless } A = 1).$$

and so Ay_p does not solve the ODE.

However, if $y_h(x)$ is a solution to the homogeneous problem, $D[y_h] = 0$, then $y(x) = y_p(x) + y_h(x)$ solves the O[DE]

$$D[y_p + y_h] = D[y_p] + D[y_h] \quad \text{since } D \text{ linear}$$

$$= f + 0.$$

In this case, we know that $y_h(x) = Ae^{-kx}$, for some constant A.

Hence, the general s!n to $D[y] = f$ is:

$y(x) = y_p(x) + Ae^{-kx}$, where $y_p(x)$ is any solution.

Given the BC $y(0) = y_0$, we see that

$$y_p(0) + Ae^{-0} = y_p(0) + A = y_0$$

and so $A = y_0 - y_p(0)$, and the soln is:

$$y(x) = y_p(x) + (y_0 - y_p(0))e^{-kx}.$$

Idea

Provided that we can find any function $y_p(x)$ that solves $D[y_p] = f$, then the ODE can be solved by adding the homogeneous general soln $y_h(x)$ to $y_p(x)$.

It is often correct to choose an ansatz for y_p that has the same functional form as $f(x)$. "Method of undetermined coeffs".

Example

Solve $y' + ky = B$ for some constant B and $y(0) = y_0$

Try $y_p(x) = C = \text{const}$.

Then $D[y_p] = y_p' + ky_p = 0 + kC = B$

So $C = B/k$ and $y_p(x) = B/k$.

Then, the full soln is:

$$y(x) = \frac{B}{k} + \left(y_0 - \frac{B}{k}\right)e^{-kx}.$$

<u>Example</u>

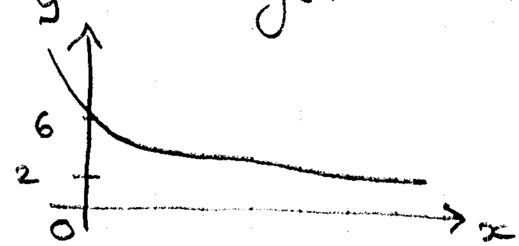
Solve $y' + 2y = 4$, $y(0) = 6$.

Then try $y_p = C$. So $y_p' + 2y_p = 0 + 2C = 4$.

So $C = 2$.

Then, $y(x) = 2 + Ae^{-2x}$ and $y(0) = 6 \Rightarrow A = 4$.

So $y(x) = 2 + 4e^{-2x}$.



<u>Example</u>

Solve $y' + ky = 2x$ and $y(0) = y_0$.

We try (incorrectly) $y_p = Bx$, $B = \text{const}$.

Then, $y_p' + ky_p = B + Bkx = 2x$.

<u>Recall</u>: Two polynomials are equal if there coeffs. match

So, OK if $B = 0$ <u>and</u> $Bk = 2$!!!

There is no choice of B that solves this <u>for all x</u>.

<u>Problem</u>: y_p' brings a constant term (B), which cannot be cancelled elsewhere in the equation.

<u>Sol^n</u> Instead, try $y_p(x) = Bx + C$, $B, C = \text{const}$.

Then, $y_p' + ky_p = B + k(Bx + C) = 2x$.

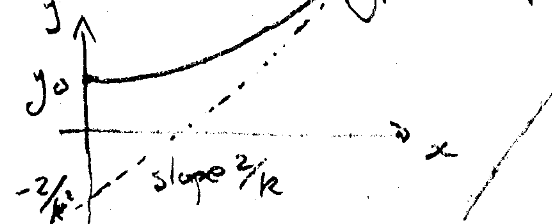
We can choose B and C for this to be valid for all x. Equating coefficients:

Coeff. x^1 equal: $kB = 2 \Rightarrow B = 2/k$

Coeff. x^0 (constants) equal: $B + kC = 0 \Rightarrow C = -\frac{B}{k} = -\frac{2}{k^2}$

Hence, $y_p(x) = \frac{2}{k}x - \frac{2}{k^2}$, and solution is

$$y(x) = \frac{2}{k}x - \frac{2}{k^2} + \left(y_0 + \frac{2}{k^2}\right)e^{-kx} \qquad \left(y_p(0) = -\frac{2}{k^2}\right)$$



Example

Solve $y' + ky = 3x^2$ and $y(0) = y_0$.

Try $y_p(x) = Bx^2 + Cx + D$.

Example

Solve $y' + ky = 3x^2 + 2x + 1$ and $y(0) = y_0$

Try $y_p(x) = Bx^2 + Cx + D$.

Example

Solve $y' + ky = \sin(3x)$ and $y(0) = y_0$.

If we try $y_p(x) = B\sin(3x)$, we run into trouble.

Should instead try $y_p(x) = B\sin(3x) + C\cos(3x)$.

Then, $y_p' + ky_p = 3B\cos(3x) - 3C\sin(3x) + k(B\sin(3x) + C\cos(3x))$

$= \sin(3x)$

For this to be valid for all x, we equate coeffs:

Coeff $\sin(3x)$: $-3C + kB = 1$ (1)

Coeff $\cos(3x)$: $3B + kC = 0$ (2).

Then, (2) $\Rightarrow$ $C = -3B/k$. so

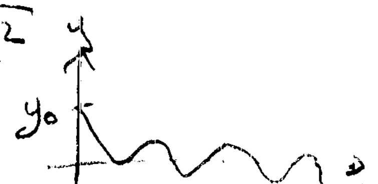
$$(1) \Rightarrow \quad -3\underbrace{\left(\frac{-3B}{k}\right)}_{=C} + kB = 1$$

or, $B\left(\frac{9}{k} + k\right) = 1 \Rightarrow B = \dfrac{1}{\frac{9}{k} + k} = \dfrac{k}{9+k^2}$

and $C = -\dfrac{3B}{k} = -\dfrac{3}{9+k^2}$

Hence, $y_p(x) = \dfrac{k\sin(3x)}{9+k^2} - \dfrac{3\cos(3x)}{9+k^2}$

So $y(x) = y_p + \left(y_0 - \dfrac{3}{9+k^2}\right)e^{-kx}$



Example

Solve $y' + ky = e^{-ax}$ and $y(0) = y_0$ where $a \neq k$.

Try $y_p(x) = Be^{-ax}$. Then $y_p' + ky_p = -aBe^{-ax} + kBe^{-ax}$
$= e^{-ax}$.

For this to hold for all x, we equate coeffs of e^{-ax}:

i.e. $-aB + kB = 1 \Rightarrow B = \dfrac{1}{k-a}$.

Observation

This doesn't work if $k = a$! This is because

Be^{-kx} is an <u>eigenfunction</u> of D with zero e-value: $D[Be^{-kx}] = \frac{d}{dx}(Be^{-kx}) + kBe^{-kx}$

$= -Bke^{-kx} + kBe^{-kx} = 0.$

[c.f. attempting to solve the matrix eqn $A\underline{x} = \underline{b}$ where $\underline{b}$ is an e-vector of A with zero e-value]

There is still a solution, but we have to do more work.

Physically, this is related to the problem of forcing a system at its <u>resonant frequency</u> (see later)

It seems reasonable that e^{-kx} is still relevant to $y' + ky = e^{-kx}$. We can try instead the general form: $y_p(x) = g(x)e^{-kx}$ and see what happens.

This is called the "method of variation of parameters".

Then, $y_p' + ky_p = g'e^{-kx} - kge^{-kx} + kge^{-kx}$.

$= g'(x)e^{-kx} = e^{-kx}.$

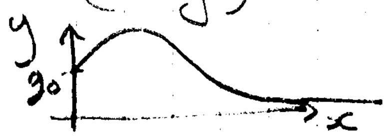
So, $g'(x) = 1$ and $g(x) = x + C$, and so $y_p(x) = (x+C)e^{-kx}$. Note that $Ce^{-kx} \propto y_h(x)$ so we can set $C=0$, and find $y_p(x) = xe^{-kx}$. (Since $y_h(x)$ already accounted for).

Thus, $y(x) = y_p(x) + y_h(x) = xe^{-kx} + Ae^{-kx}$.

The BC $y(0)=y_0$ gives

$$0 + A = y_0 \Rightarrow A = y_0 \text{ and hence}$$

$$y(x) = xe^{-kx} + y_0e^{-kx} = (x+y_0)\,e^{-kx}.$$



<u>Example</u>

Solve $y' + ky = xe^{-ax}$ with $y(0) = y_0$ and $k \neq a$.

Try $y_p = \cancel{Ax} \, (Bx + C)e^{-ax}$.

<u>Example</u>

Solve $y' + ky = xe^{-kx}$ with $y(0) = y_0$

Try $y_p = (Bx^2 + Cx + D)e^{-kx}$.

<u>Example</u>

Solve $y' + ky = (x^2 + 2x + 4)e^{-ax}$ with $y(0) = y_0$.

If $a \neq k$ try $y_p = (Bx^2 + Cx + D)e^{-ax}$.

If $a = k$ try $y_p = (Ex^3 + Bx^2 + Cx + D)e^{-kx}$.

<u>In general</u>: need to go one power of x higher if $a = k$.

<u>Example</u>

In a radioactive rock, isotope A decays to isotope B with rate $\propto$ a, # nuclei A. Isotope B decays to isotope C with different rate $\propto$ b, # nuclei B.

Initially, $a = a_0$ and $b = 0$. Find $b(t)$.

$$\frac{da}{dt} = -k_A a \quad \Rightarrow \quad a = a_0 e^{-k_A t}$$

$$\frac{db}{dt} = k_A a - k_B b \quad , \quad k_A \neq k_B.$$

$$\frac{db}{dt} + k_B b = k_A a_0 e^{-k_A t} \quad \cancel{k_B b}.$$

Hom. sln is $b_h(t) = Ae^{-k_B b}$.

Try particular sln $b_p(t) = Be^{-k_A t}$. Find

$$\frac{db_p}{dt} + k_B b = -k_A B e^{-k_A t} + B k_B e^{-k_A t} = k_A a_0 e^{-k_A t}.$$

So, $B(k_B - k_A) = k_A a_0$

$$\Rightarrow B = \frac{k_A a_0}{k_B - k_A}.$$

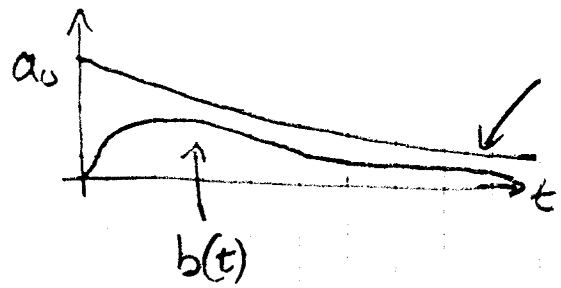
Hence, $b(t) = b_p(t) + b_h(t)$

$$\Rightarrow b(t) = \frac{k_A a_0}{k_B - k_A} e^{-k_A t} + Ae^{-k_B t}.$$

Then, $b(0) = 0 \Rightarrow 0 = \underbrace{\frac{k_A a_0}{k_B - k_A}}_{= B} + A$

so $A = -B$ and

$$b(t) = \frac{k_A a_0}{k_B - k_A}\left(e^{-k_A t} - e^{-k_B t}\right)$$



Suppose a geologist doesn't know a_0. We have

$$\frac{b(t)}{a(t)} = \frac{k_A}{k_B - k_A}\left(1 - e^{(k_A - k_B)t}\right).$$

So, the rock can be dated based on the ratio of the two isotopes A and B.

§ 1.3 Non-constant coeff. linear first-order ODEs

These are eqns of the form

$$a(x)y' + b(x)y = c(x).$$

Let's divide through by $a(x)$ and consider:

$$y' + p(x)y = f(x) \quad , \quad p(x) = \frac{b(x)}{a(x)}, \quad f(x) = \frac{c(x)}{a(x)}.$$

Observation

In § 1.2, when $p(x) = k = \text{const}$, we had $y' + ky = f(x)$. The LHS can be written as:

$$y' + ky = (e^{kx}y)'e^{-kx}, \quad \text{since:}$$

$$e^{-kx}\frac{d}{dx}(e^{kx}y) = e^{-kx}\left[y\frac{d}{dx}e^{kx} + e^{kx}\frac{dy}{dx}\right]$$

$$= ky + \frac{dy}{}$$

Therefore, $e^{-kx}(e^{kx}y)' = f(x)$

$$\Rightarrow (e^{kx}y)' = e^{kx}f(x). \quad (*)$$

Hence $e^{kx}y = \int e^{kx}f(x)\,dx.$

$$\Rightarrow y(x) = e^{-kx}\int e^{kx}f(x)\,dx.$$

Is another way to get the solution.

Example

When $f(x) = 1$, we have $y(x) = e^{-kx}\int e^{kx}\,dx$.

$$= e^{-kx}\left(\frac{1}{k}e^{kx} + C\right) = \frac{1}{k} + C e^{-kx}$$

where C is the ~~integrating fact~~ constant of integration.

This is the same as for §1.2.

Note that we had to multiply through by e^{kx}.

Idea

For general $p(x)$, can we find a function $q(x)$ such that, after multiplying the ODE by $q(x)$, we reduce it to $\frac{d}{dx}[q(x)y(x)] = q(x)f(x)$ as in $(*)$?

Def^n If we can find $q(x)$ such that

$$y' + p(x)y = f(x) \Leftrightarrow (q(x)y(x))' = q(x)f(x),$$ then

we call $q(x)$ an integrating factor.

If we find such a function $q(x)$, then the sln to the ODE can be found:

$$(qy)' = qf \Rightarrow qy = \int q(x) f(x) dx$$

and so $y(x) = \frac{1}{q(x)} \int q(x) f(x) dx$.

How do we find $q(x)$? Multiply through:

$$y' + p(x) y = f(x)$$

$$\Rightarrow \quad q(x) y' + q(x) p(x) y = q(x) y(x)$$

Want $q(x) y' + q(x) p(x) y = (q(x) y)'$ (†)

But, from chain rule, $(q(x)y)' = q'y + qy'$

Putting into (†), we get

$$\frac{dq}{dx} = q(x) p(x).$$

or $\frac{1}{q} \frac{dq}{dx} = p \Rightarrow \int \frac{1}{q} \frac{dq}{dx} dx = \int p \, dx$

But, change of variables in ~~log~~ integral with $s = q(x)$ gives $ds = \frac{dq}{dx} dx$.

so $\int \frac{1}{q} \frac{dq}{dx} dx = \int \frac{1}{s} ds = \log s = \log(q)$

Hence, $\log(q) = \int p \, dx \Rightarrow q = \exp\left(\int p \, dx\right)$.

Example

Let $xy' + (1-x)y = 1$.

Then, $y' + \left(\frac{1}{x} - 1\right) y = \frac{1}{x}$.

So, $p(x) = \frac{1}{x} - 1$, and so $\int p(x)\,dx = \int \frac{1}{x} - 1 \, dx$

$= \log x - x$.

Hence, $q(x) = e^{\log x - x} = e^{\log x} e^{-x} = xe^{-x}$ (no const.)

Then, $(qy)' = qf$, i.e. $(xe^{-x}y)' = xe^{-x} \cdot \frac{1}{x} = e^{-x}$.

Hence, $xe^{-x}y = C - e^{-x}$

$$\Rightarrow y = \frac{Ce^{x} - 1}{x}.$$

Now suppose that y is finite at $x = 0$. Then $C = 1$ since

$$\lim_{x \to 0} \frac{e^{x} - 1}{x} = 1$$

$$\Rightarrow y = \frac{e^{x} - 1}{x}.$$

§2. Nonlinear, first order ODEs

Nonlinear first order ODEs are of the form:

$$Q(x,y)\frac{dy}{dx} + P(x,y) = 0.$$

Examples $\frac{dy}{dx} = y^2$, $y\frac{dy}{dx} = 1$, $\frac{dy}{dx} = \sin(y)$,

$\frac{dy}{dx} = y^{\frac{1}{2}}$. $(y^2 + x)\frac{dy}{dx} = y^3(x + \sin(y))$ etc.

§2.1 Seperable ODEs

A first-order ODE is **seperable** if it can be written as:

$$q(y)\frac{dy}{dx} = p(x).$$

In this case, we may integrate both sides:

$$\int q\frac{dy}{dx}\,dx = \int p\,dx.$$

or $$\int q\,dy = \int p\,dx.$$

and this leads to the solution.

Example

$$\frac{dy}{dx} = \frac{y}{x}. \Rightarrow \frac{1}{y}dy = \frac{1}{x}dx$$

Then, $$\int \frac{1}{y}dy = \int \frac{1}{x}dx.$$

$$\Rightarrow \log y = \log x + c. \qquad c = \text{const.}$$

$$\Rightarrow y = Ax, \qquad A = e^c.$$

(This eqn is in fact linear).

Example

$$\frac{dy}{dx} = xy^2 \Rightarrow \frac{1}{y^2}dy = x\,dx.$$

$$\Rightarrow \int \frac{1}{y^2}dy = \int x\,dx \Rightarrow -\frac{1}{y} = \frac{1}{2}x^2 - c.$$

$$\Rightarrow \frac{1}{y} = c - \frac{1}{2}x^2. \Rightarrow y = \frac{1}{c - \frac{1}{2}x^2}.$$

Example

$$\frac{dy}{dx} = \frac{x^2}{1-y^2} \Rightarrow (1-y^2)\,dy = x^2\,dx.$$

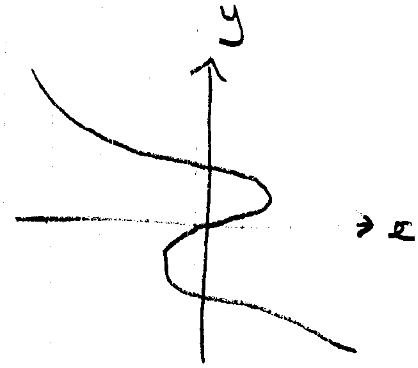
$$\Rightarrow \int (1-y^2)\,dy = \int x^2\,dx \Rightarrow y - \tfrac{1}{3}y^3 = \tfrac{1}{3}x^3 + C.$$

$$\Rightarrow y^3 - 3y + x^3 = A, \qquad A = -3C.$$

implicit equation for y.

Eg Let $y(0) = 0$. Then $A = 0$ and

$$y^3 - 3y + x^3 = 0$$



Example

$$\frac{dy}{dx} = -\frac{x}{y} \quad \text{with} \quad y(0) = 1.$$

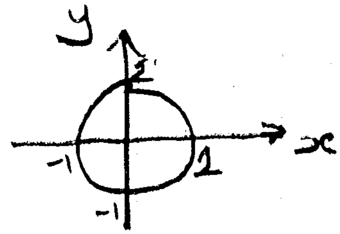
$$\Rightarrow y\,dy = -x\,dx. \Rightarrow \int y\,dy = \int -x\,dx$$

$$\Rightarrow \tfrac{1}{2}y^2 = -\tfrac{1}{2}x^2 + C.$$

$$\Rightarrow y^2 + x^2 = 2C$$

Then $y(0)=1 \Rightarrow 1^2 + 0^2 = 2C \Rightarrow C = \frac{1}{2}$ and

$$x^2 + y^2 = 1$$



Example

Let's return to motion in a constant gravitational field, with drag.
Recall that the velocity v ~~satisfies~~

$$m\dot{v} = -mg + kv^2, \qquad k = \text{const}.$$

$$\Rightarrow \frac{m}{kv^2 - mg}\, dv = dt$$

$$\Rightarrow \int \frac{m}{kv^2 - mg}\, dv = \int dt = t + c.$$

Now, $\displaystyle \frac{m}{kv^2 - mg} = \frac{m}{k} \cdot \frac{1}{v^2 - \frac{mg}{k}} = \frac{m}{k} \cdot \frac{1}{v^2 - v_T^2}$

where $v_T \equiv$ terminal velocity $(\dot{v} = 0)$; $-mg + kv_T^2 = 0 \Rightarrow v_T = \sqrt{\frac{mg}{k}}$

$$= \frac{-m}{2v_T k} \cdot \left(\frac{1}{v + v_T} - \frac{1}{v - v_T} \right).$$

So $\displaystyle \int \frac{+m}{kv^2 - mg}\, dv = \frac{-m}{2v_T k} \int \left(\frac{1}{v + v_T} - \frac{1}{v - v_T} \right) dv$

$$= \frac{-m}{2v_T k} \left(\log|v + v_T| - \log|v - v_T| \right)$$

$$= \frac{+m}{2v_T k} \log \left| \frac{v - v_T}{v + v_T} \right| = t + c.$$

So $\displaystyle \left| \frac{v - v_T}{v + v_T} \right| = \exp\left(\frac{+2v_T k}{m} (t + c) \right) = A e^{+\frac{2v_T k t}{m}}$. $\rightarrow$ etc.

$$\Rightarrow |v - v_T| = |v + v_T|\, A\, \cancel{e^{+(mt/(2v_T k))}}$$

~~$\Rightarrow v(1 \mp A e^{-mt/(2v_T k)}) = -v_T(1 \pm A e^{-mt/(2v_T k)})$.~~

$$\cancel{v = \frac{-v_T\left(1+Ae^{-mt/2v_Tk}\right)}{1-Ae^{-mt/2v_Tk}}}$$

~~Suppose that v(0) = 0. Then~~

~~$|v_T| = |v_T| A \Rightarrow A = 1$~~

~~So $|v+v_T| = |v-v_T| e^{-mt/2v_Tk}$.~~

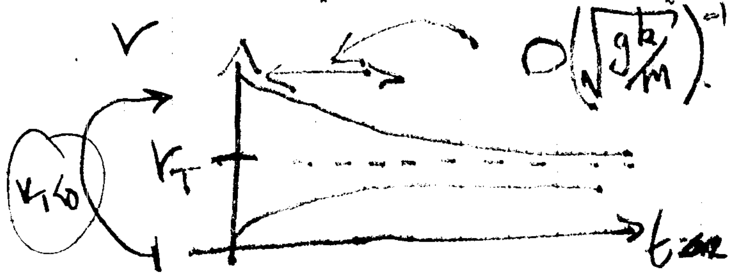
$$\cancel{|v-v_T| = |v+v_T|Ae^{-mt/2v_Tk} = |v+v_T|e^{-\frac{mt}{2\sqrt{\frac{mg}{k}}k}}A}$$

so as $t \to \infty$, RHS $\to 0$, so $v \to v_T$

for all A, ie <u>regardless of IC</u>.

The $v = v_T$ <u>steady state</u> is thus <u>stable</u>.

Go back: $|v-v_T| = |v+v_T|Ae^{-\left(\frac{mt}{2\sqrt{\frac{mg}{k}}k}\right)} = |v+v_T|Ae^{-\left(\frac{1}{2}\sqrt{\frac{m}{gk}}\right)^{-1}t}$



[c.f. half-life]

§ 2.2 Autonomous Equations and Stability

An <u>autonomous</u> ODE is one that does not depend on the differentiation variable.

$$\frac{dy}{dt} = f(y).$$

A fixed point is a solution to $f(y)=0$, ie $\frac{dy}{dt}=0$.

Example

Again $m\dot{v}=-mg+kv^2 \Rightarrow$ fixed point $v=\pm v_T$.

Def^n

Let $y=a$ be a fixed point of $\frac{dy}{dt}=f(y)$, i.e. $f(a)=0$. The point a is stable if, when you slightly perturb away from a, you return to a.

Mathematically, given IC $y(0)=a+\varepsilon_0$, we say

a is stable if $y\to a$ as $t\to\infty$

unstable " $y\not\to a$ as $t\to\infty$,

when ε_0 is "small".

To see if a is stable or not, let

$y(t)=a+\varepsilon(t)$. and $\varepsilon(0)=\varepsilon_0$.

Then, $\frac{dy}{dt}=f(y) \Rightarrow \frac{d\varepsilon}{dt}=f(a+\varepsilon(t))$

But, ε small, so $f(a+\varepsilon(t))=f(a)+\varepsilon f'(a)+O(\varepsilon^2)$.

But $f(a)=0$, so $f(a+\varepsilon(t))=\varepsilon f'(a)+O(\varepsilon^2)$.

Then, $\frac{d\varepsilon}{dt}\simeq\varepsilon f'(a).\equiv b\varepsilon$ $\quad b=f'(a)$.

Then. $\varepsilon=\varepsilon_0 e^{bt}\to\{$ "∞" if $b>0$

So, the soln $y=a$ is $\begin{matrix}\text{stable} \\ \text{unstable}\end{matrix}$ if $f'(a) \begin{matrix} <0 \\ >0 \end{matrix}$.

Example

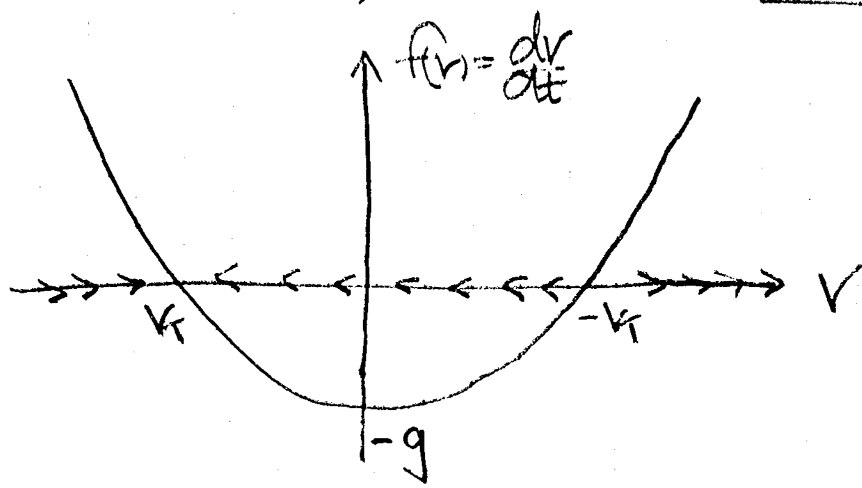
Again, $m\dot{v} = -mg + kv^2$; so $f(v) = -g + \frac{k}{m}v^2$.

and $f(\pm v_T) = 0$ where $v_T = -\sqrt{\frac{mg}{k}}$.

Then, $f'(v) = \frac{2kv}{m}$. so $f'(\pm v_T) = \mp \frac{2k}{m}\sqrt{\frac{mg}{k}}$

so $f'(v_T) \stackrel{<0}{\cancel{>0}}$ and $f'(-v_T) > 0$.

So, $v = v_T$ is stable, $v = -v_T$ is unstable.



Example

Population of sheep, $y = \#$ sheep. Birth rate αy, death rate βy.

$$\text{So, } \frac{dy}{dt} = (\alpha - \beta) y. \Rightarrow y = y_0 e^{(\alpha-\beta)t}$$

So exp. growth or decay if $\alpha \gtrless \beta$.

However, as pop. grows, there is competition. Probability of food being found $\propto y$. Prob. two individuals finding same food $\propto$

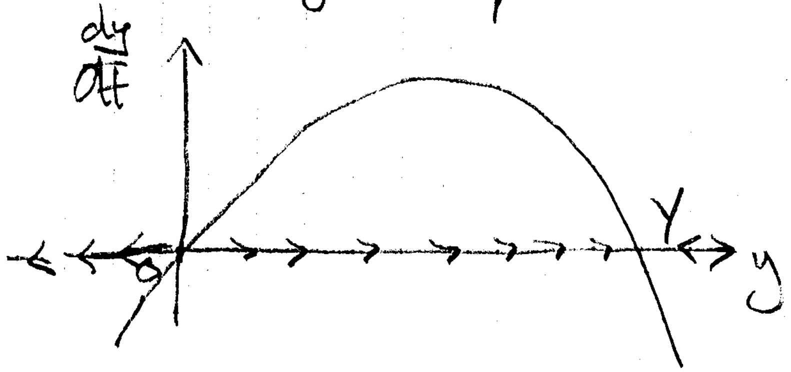
So, if food is scarce, extra death rate $\propto y^2$.

$$\Rightarrow \frac{dy}{dt} = (\alpha - \beta) y - \gamma y^2.$$

Let $r = \alpha - \beta$, $Y = r/\gamma$. Then,

$$\frac{dy}{dt} = ry\left(1 - \frac{y}{Y}\right). \equiv f(y)$$

This is the logistic eqn. (differential)



Fixed points $y = 0$, $y = Y$.

$f'(y) = r - \frac{2ry}{Y}$.

So $f'(0) = r > 0 \Rightarrow$ u.s.

$f'(Y) = -r < 0 \Rightarrow$ stable.

** § 2.3 Discrete Eqns.

In reality, ~~population isn't contin.~~ birth and death occur discretely (births in spring, deaths in winter). A better model might be $x_{n+1} = \lambda x_n(1 - x_n)$

For the population x at time n.

Discrete maps are $x_{n+1} = f(x_n)$

They have fixed points: ~~$x_n = f(x_n)$~~

$$x_{n+1} = x_n \Rightarrow x_n = f(x_n)$$

In this case, $x_n = \lambda x_n (1 - x_n)$

$$\Rightarrow x_n = 0 \text{ or } x_n = 1 - \frac{1}{\lambda}.$$

We can look at stability: let $x_n = X$ be a fixed point.

but $f(x+\varepsilon_n) \simeq f(x) + \varepsilon_n f'(x) + O(\varepsilon_n^2)$.

So $\varepsilon_{n+1} \simeq \varepsilon_n f'(x) = \varepsilon_{n-1}\left(f'(x)\right)^2 = \varepsilon_{n-2}\left(f'(x)\right)^3$

etc.

So $\varepsilon_n = \left[f'(x)\right]^n \varepsilon_0$.

and $|\varepsilon_n| \to \begin{cases} \text{"}\infty\text{"} \\ 0 \end{cases}$ if $|f'(x)| \begin{matrix} >1 \\ <1 \end{matrix}$

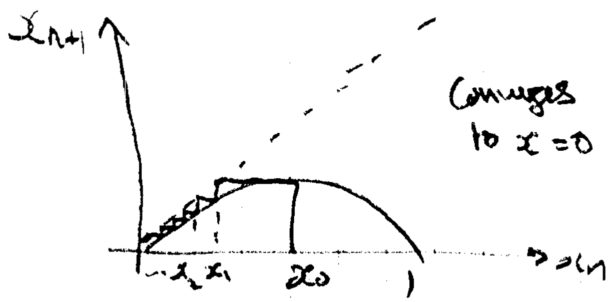
For logistic map, $f(x) = \lambda x(1-x)$, so $f'(x) = \lambda - 2\lambda x$.

Then, $f'(0) = \lambda$ so $x = 0$ is stable if $|\lambda| < 1$
unstable $|\lambda| > 1$

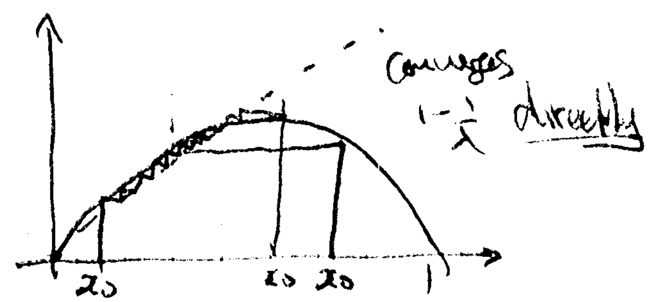
$f'(1-\frac{1}{\lambda}) = 2-\lambda$

so $x = 1-\frac{1}{\lambda}$ is stable / unstable if $1 < \lambda < 3$.

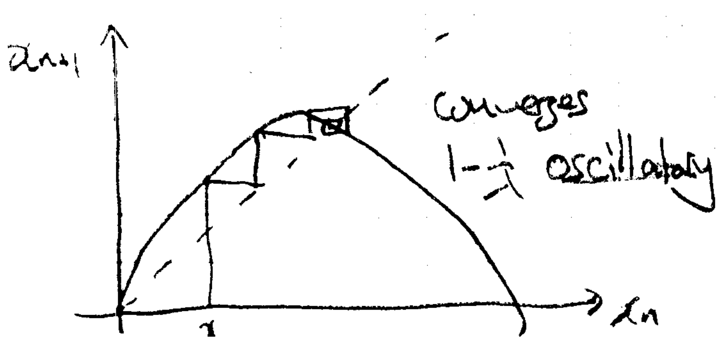
$\lambda < 1$



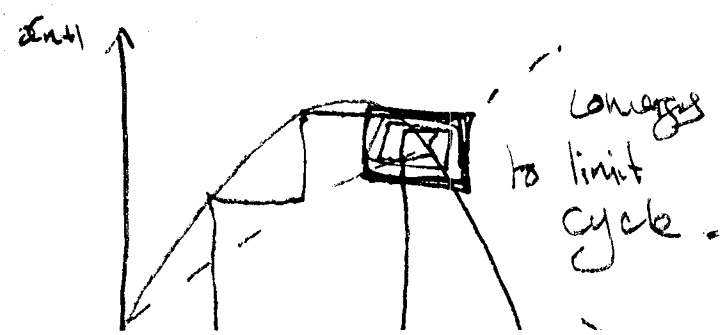
$1 < \lambda < 2$



$2 < \lambda < 3$

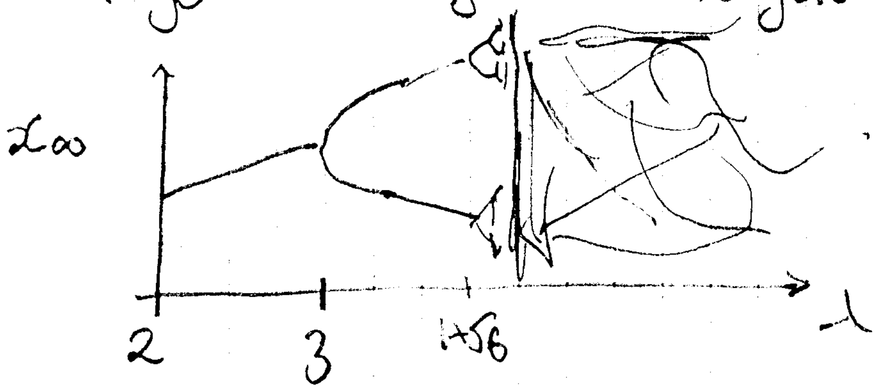


$3 < \lambda < 3.2$ ish



2 cycle $\rightarrow$ 4 cycle at $\lambda = 1+\sqrt{6}$

4 cycle $\rightarrow$ 8 cycle $\rightarrow$ 16 cycle $\rightarrow$ etc.



etc.

**

** § 2.4 Existence and Uniqueness

Every linear ODE ~~with~~ order n, with n ICs/BCs has a unique solution.

Some nonlinear equations also have unique solutions, but many do not:

Example (non-uniqueness)

$y' = y^{1/3}$ and $y(0) = 0$.

Then, $\frac{1}{y^{1/3}} dy = dx \Rightarrow \int \frac{1}{y^{1/3}} dy = \int dx$

$$\Rightarrow \frac{3}{2} y^{2/3} = x + c$$

$$\Rightarrow y = \left[\frac{2}{3}(x+c)\right]^{3/2}.$$

Then $y(0)=0 \Rightarrow c = 0$ and $y = \left(\frac{2}{3}x\right)^{3/2}$.

However, $y(x) = 0$ for all x is also a soln.

Also, $y(x) = \begin{cases} 0 & 0 \le x < x_0 \\ \ldots & \end{cases}$ are solns for all x_0 !

Example (non-existence)

$\frac{dy}{d\cancel{x}t} = y^2$ and $y(0) = 1$.

Then $\frac{1}{y^2} dy = dt \Rightarrow \int \frac{1}{y^2}\, dy = \int dt$

$\Rightarrow -\frac{1}{y} = t + c$

$\Rightarrow y = \frac{-1}{t + c}$.

Then $y(0) = 1 \Rightarrow c = -1$ so $y = \frac{1}{1-t}$.

as $t \to 1$, $y \to \infty$, so there is no solution for $t > 1$.

**

§3. Linear 2nd-order, ~~constant coeff.~~ ODEs

These are equations of the form:

$$a(x) y'' + b(x) y' + c(x) y = g(x) \quad , \quad \cancel{a, b, c \in \text{const.}}$$

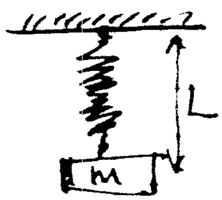
or, dividing through by a,

$$y'' + p(x) y' + q(x) y = f(x) \;, \quad p = \frac{b}{a}, \; q = \frac{c}{a}, \; f(x) = \frac{g(x)}{a}$$

Such equations represent oscillations, vibrations, and can help explain resonance.

Example

Consider a mass m attached to a spring when the system is at rest and the spring has length L.



Then, the forces acting on the mass are

gravity: $\quad -mg$

Hooke's law: $\quad kL$

where k is the spring constant. Then, since the spring is at rest, there is no net force, i.e. $\quad 0 = kL - mg$.

So the resting length is $L = \dfrac{mg}{k}$.

Now suppose that we extend the spring by a length x in addition to L, and let go. What happens?

The forces acting are now:

Gravity: $\quad -mg$

Hooke's law: $\quad k(x+L)$.

Since the spring will be in motion, we apply Newton's 2nd Law:

$$F = ma, \quad \text{or} \quad m\ddot{x} = +mg - k(x+L)$$
$$= +mg - kL - kx$$
$$= -kx.$$

i.e. $\quad m\ddot{x} + kx = 0$.

Then we apply initial conditions $x(0) = x_0$, $\dot{x}(0) = 0$.

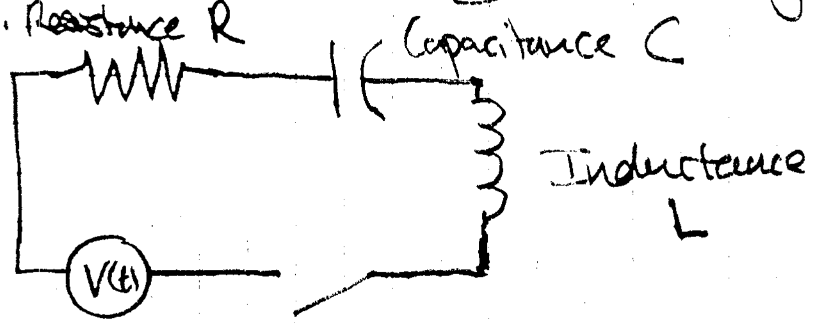
We will show that this has solution

$$x(t) = x_0 \cos\left(\sqrt{\frac{k}{m}}\, t\right) \rightarrow \text{oscillations!}$$

Oscillations don't decay $\rightarrow$ damping needed.

Example

Consider the following circuit diagram:



Let $L > \frac{1}{4}CR^2$.

The total ~~charge~~ current $I(t)$ is given by the rate of change of the ~~current~~ charge $Q(t)$, $I = \frac{dQ}{dt}$.

Then, Kirchoff's 2nd Law $\Rightarrow$ sum of voltage drops equals impressed voltage.

Drop across resistor : IR

Drop across capacitor : Q/C

Drop across inductor : $L\frac{dI}{dt}$.

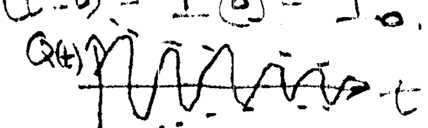
So,

$$L\frac{dI}{dt} + RI + \frac{1}{C}Q = V(t).$$

But, $I = \frac{dQ}{dt}$ and so $\frac{dI}{dt} = \frac{d^2Q}{dt^2}$. Hence,

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = V(t).$$

We then have ICs: $Q(0) = Q_0$ and $\frac{dQ}{dt}(t=0) = I(0) = I_0$.



When $V(t)=0$, the solution is

$$Q(t) = e^{-\sigma t}(A\sin\omega t + B\cos\omega t) \rightarrow \text{decaying oscillations!}$$

with $\sigma = +\frac{R}{2L}$, $\omega = \sqrt{R^2 + 4L/C}$, $A = I_0 + \sigma Q_0$, $B = Q_0$.)

We can solve the ~~gen~~ original ODE in two steps.

$$y'' + p(x)y' + q(x)y = f(x).$$

Step 1. Solve homogeneous problem, $f(x) = 0$,

$$y'' + py' + qy = 0$$

to find general homogeneous soln $y_h(x)$.

Step 2 Find particular solution $y_p(x)$ to the full eqns.

Then, general soln is given by:

$$y(x) = y_h(x) + y_p(x),$$

just as for first-order ODEs.

§ 3.1 Homogeneous, linear, second-order, ~~const/coeff~~, ODEs

First, we will show how to solve the homogeneous problem

$$y'' + py' + qy = 0 \quad ; \quad p, q = \text{const}.$$

Recall

$y = e^{\lambda x}$ is an eigenfunction of $D = \frac{d}{dx}$:

$$\frac{dy}{dx} = \lambda e^{\lambda x} = \lambda y, \quad \text{with e-val } \lambda$$

Observation

$y = e^{\lambda x}$ is also an eigenfunction of $D = \frac{d^2}{dx^2}$:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(\lambda e^{\lambda x}) = \lambda^2 e^{\lambda x} = \lambda^2 e^{\lambda x} \text{ with e-val } \lambda^2.$$

Idea

- Solutions of the form $e^{\lambda x}$ are likely still important here.

To solve $y''+py'+q=0$, let's try the ansatz: $y=e^{\lambda x}$.

Then, $y''+py'+qy = \lambda^2 e^{\lambda x} + p\lambda e^{\lambda x} + q e^{\lambda x} = 0$

Then, $e^{\lambda x} \neq 0$, so $\lambda^2 + p\lambda + q = 0$. $(*)$

This is the characteristic equation for λ of the ODE.

$(*)$ has two slns:

$$\lambda_\pm = \frac{-p \pm \sqrt{p^2-4q}}{2}.$$

~~Therefore~~ now

We therefore know that

$$y_1 = Ae^{\lambda_+ x} \quad \text{and} \quad y_2 = Be^{\lambda_- x}$$ solve the ODE

for any constants A, B, since the ODE is linear.

Also, since the ODE is linear, we also have the solution

$$y(x) = Ae^{\lambda_+ x} + Be^{\lambda_- x}.$$

However, we don't yet know if this is the general solution.

- There are three cases:

1) Both λ_+, λ_- real, $\lambda_+ \neq \lambda_-$

3) Both λ_+, λ_- real and $\lambda_+ = \lambda_- \equiv \lambda$.

We will consider each in turn:

Distinct, real roots

In case ①, we have λ_+, λ_- real, and $\lambda_+ \neq \lambda_-$

Therefore, $y = Ae^{\lambda_+ x} + Be^{\lambda_- x}$ is the general solution since $e^{\lambda_+ x}$ and $e^{\lambda_- x}$ are linearly independent, i.e. $e^{\lambda_+ x}$ cannot be written ~~as~~ ~~a~~ in terms of $e^{\lambda_- x}$ for all x: $e^{\lambda_+ x} \neq Ce^{\lambda_- x}$ for all x, for any C.

We need two ICs/BCs to solve fully:

Example

$$y'' - 3y' + 2y = 0, \quad \text{with } y(0) = 1 \text{ and } y'(0) = 0.$$

Try $y = e^{\lambda x}$. Then, $\lambda^2 - 3\lambda + 2 = 0 \Rightarrow (\lambda - 2)(\lambda - 1) = 0$.

So, $\lambda = 1$ or 2. Then, the general sln is

$$y(x) = Ae^{x} + Be^{2x}, \quad \text{and} \quad y'(x) = Ae^{x} + 2Be^{2x}$$

The BCs give: $y(0) = 1 \Rightarrow A + B = 1$ ①

$y'(0) = 0 \Rightarrow A + 2B = 0$ ②

Then, ② $\Rightarrow A = -2B$, so ① $\Rightarrow -2B + B = 1$

$\Rightarrow B = -1$ and $A = -2B = 2$.

Hence $y(x) = 2e^{x} - e^{2x}$.

Example

$$y'' - 4y' + 13y = 0.$$

Then $\lambda^2 - 4\lambda + 13 = 0$

$\Rightarrow \lambda = 2 \pm 3i$

So $y(x) = e^{2x}(A\cos 3x + B\sin 3x)$

Example

$$y'' + 16y = 0$$

Then $\lambda^2 + 16 = 0$

$\Rightarrow \lambda = \pm 4i$

So $y(x) = A\cos 4x + B\sin 4x.$

Example

$$y'' - 3y' + 2y = 0, \qquad y(0) = 1, \quad y(1) = 0$$

We know $y(x) = Ae^{x} + Be^{2x}$. Then, BCs give:

$y(0) = 1 \Rightarrow A + B = 1$ and $y(1) = 0 \Rightarrow Ae + Be^2 = 0$.

So, $A = -Be$ and $-Be + B = 1 \Rightarrow B = \frac{1}{1-e}$

and $A = \frac{-e}{1-e}$. So, $y(x) = \frac{-e}{1-e}e^{x} + \frac{1}{1-e}e^{2x}$.

Distinct, complex roots

In case (2), we have λ_+, λ_- complex, and

$\lambda_+ = \overline{\lambda_-} = \sigma + i\omega$. Then, the general solution is:

$$y(x) = Ae^{(\sigma+i\omega)x} + Be^{(\sigma-i\omega)x} \quad \text{with } A,B \text{ \underline{complex}}.$$

But, we also have $e^{(\sigma+i\omega)x} = e^{\sigma x}\cdot e^{i\omega x} = e^{\sigma x}(\cos\omega x + i\sin\omega x)$.

So, $y(x) = Ae^{\sigma x}(\cos\omega x + i\sin\omega x) + Be^{\sigma x}(\cos\omega x - i\sin\omega x)$

$$= e^{\sigma x}\left[(A+B)\cos\omega x + (A-B)i\sin\omega x\right]$$

$$= e^{\sigma x}\left[C\cos\omega x + D\sin\omega x\right]$$

with $C = A+B$, $D = i(A-B)$ real (if BCs real)

This is OK because $\cos\omega x$ and $\sin\omega x$ are linearly independent

Example

Return to the mass-spring system,

$m\ddot{x} + kx = 0$ with $x(0) = x_0$ and $\dot{x}(0) = 0$.

Try $x = e^{\lambda t}$. Then, $m\lambda^2 + k = 0$, so $\lambda = \pm i\sqrt{\frac{k}{m}}$.

So $\sigma = 0$ and $\omega = \sqrt{\frac{k}{m}}$. Then, the general solution is:

$$x(t) = A\cos\omega t + B\sin\omega t \text{ and } \dot{x}(t) = -A\omega\sin\omega t + B\omega\cos\omega t$$

Then, $x(0) = x_0 \Rightarrow A + 0 = x_0 \Rightarrow A = x_0$

and $\dot{x}(0) = 0 \Rightarrow B\omega = 0 \Rightarrow B = 0.$

So, $x(t) = x_0 \cos\omega t.$

Example

Return to the RLC circuit (series), without forcing:

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = 0 \quad \text{and} \quad L > \frac{1}{4}CR^2.$$

with $Q(0) = Q_0$ and $\dot{Q}(0) = I_0$.

Try $Q = e^{\lambda t}$. Then, $L\lambda^2 + R\lambda + \frac{1}{C} = 0$

$$\text{So, } \lambda = \frac{-R \pm \sqrt{R^2 - 4L/C}}{2L} = -\frac{R}{2L} \pm i\sqrt{\frac{4L}{C} - R^2}$$

so $\sigma = \frac{-R}{2L}$ and $\omega = \sqrt{\frac{4L}{C} - R^2}$.

Then, the general soln is:

$$Q(t) = e^{\sigma t}\left[A\cos\omega t + B\sin\omega t\right]$$

and $Q(0) = Q_0 \Rightarrow A = Q_0$.

We have $\dot{Q}(0) = I_0 \Rightarrow -\frac{R}{2L}Q_0 + \omega B = I_0$ (exercise).

$$\text{So } B = \frac{I_0}{\omega} + \frac{RQ_0}{2L\omega}$$

$$\text{and } Q(t) = e^{-\frac{R}{2L}t}\left[Q_0\cos\omega t + \left(\frac{I_0}{\omega} + \frac{RQ_0}{2L\omega}\right)\sin\omega t\right]$$

Repeated Real Roots

In case (3), we have λ_+, λ_- real and $\lambda_+ = \lambda_- \equiv \lambda$, say.
Then, the "two" solutions we've found are:

$$y_1 = Ae^{\lambda x} \quad \text{and} \quad y_2 = Be^{\lambda x}, \qquad A, B = \text{const}.$$

But, these are <u>not</u> linearly independant! In fact,

$$y_1 = C y_2 \quad \text{where} \quad C = \frac{B}{A} = \text{const}.$$

In reality, we've only found <u>one</u> solution, $y = De^{\lambda x}$.

Recall

We struggled to solve $y' + ky = e^{-kx}$ since $y_h = Ae^{-kx}$. But instead we tried $y_p = g(x)e^{-kx}$ and found $g(x) \propto x$.

Idea

The ODE clearly "likes" $y_1 = e^{\lambda x}$ as a solution. So let's try to find a second solution $y_2(x) = g(x)\, y_1(x) = g(x) e^{\lambda x}$.

Example

Find the general solution to $y'' + 4y' + 4 = 0$.

Try $y = e^{\lambda x}$. Then $\lambda^2 + 4\lambda + 4 = 0 \Rightarrow (\lambda + 2)^2 = 0$

so $\lambda = -2$ (twice).

So one solution is $y_1(x) = e^{-2x}$. Let's try $y_2(x) = g(x)e^{-2x}$.
Then, $y_2' = g'e^{-2x} - 2ge^{-2x}$ by chain rule. Also,

Method of reduction of order

For the general linear second-order homogeneous ODE:

$$y'' + p(x)y' + q(x)y = 0$$

How do we find y_1 and y_2 so that the general soln is $y = ay_1 + by_2$.

We need at least one soln $y_1(x)$: $y_1'' + py_1' + qy_1 = 0$.

Then let $y = g(x)y_1(x)$. So $y' = g'y_1 + gy_1'$ and $y'' = g''y_1 + 2g'y_1' + gy_1''$. Then, $y'' + py' + qy$

$$= g''y_1 + 2g'y_1' + \cancel{gy_1''} + pg'y_1 + \cancel{pgy_1'} + \cancel{qgy_1}$$
$$= 0$$

Let $h(x) = g'(x)$. Then $\frac{dh}{dx}y_1(x) + 2h(x)y_1'(x) + p(x)h(x)y_1(x) = 0$

or, $\frac{1}{h(x)}\frac{dh}{dx} = -2\frac{y_1'(x)}{y_1(x)} - p(x)$

$$\Rightarrow \ln(h) = -2\ln(y_1) - \int p\,dx$$

$$\Rightarrow h = \frac{e^{-\int p dx}}{y_1^2} \Rightarrow g = \int \frac{e^{-\int p dx}}{y_1^2}dx \Rightarrow y_2 = gy_1$$

Example Let $2x^2y'' + 3xy' - y = 0$. Given that $y_1(x) = \frac{1}{x}$ is a soln, find another: Let $y_2(x) = g(x)/x$.

We have $y'' + \frac{3}{2x}y' - \frac{1}{2x^2}y = 0$ so $p(x) = \frac{3}{2x}$ and $\int p\,dx = \frac{3}{2}\ln(x) = \ln(x^{3/2})$. So $e^{-\int p(x)dx} = x^{-3/2}$.

Then, $g = \int \frac{x^{-3/2}}{y_1^2}dx = \int x^2 x^{-3/2} dx = \int x^{+\frac{1}{2}} dx = \frac{2}{3}x^{\frac{3}{2}} + c$

So $y_2 = gy_1 = \left(\frac{2}{3}x^{\frac{3}{2}} + c\right)\cdot\frac{1}{x} = \frac{2}{3}x^{\frac{1}{2}} + \frac{c}{x}$ So $y_2 = x^{\frac{1}{2}}$.

Substituting into the ODE we find:

$$y_2'' + 4y_2' + 4y_2$$
$$= g''e^{-2x} - 4g'e^{-2x} + 4ge^{-2x} + 4g'e^{-2x} - 8ge^{-2x} + 4ge^{-2x}$$
$$= g''e^{-2x} = 0.$$

Then, $e^{-2x} \neq 0$, so $g'' = 0$.

Therefore, $g' = C$ and $g = Cx + B$, $C, B = \text{const}$.

So $y_2 = (Cx+B)e^{-2x} = Cxe^{-2x} + Be^{-2x}$.

Finally, $Be^{-2x} \propto y_1$, so we can set $B = 0$ since y_1 is already accounted for. Then, the general soln is:

$$y(x) = Ae^{-2x} + Cxe^{-2x} = (A + Cx)e^{-2x}.$$

In general!

If $\lambda_+ = \lambda_- = \lambda$, the general sln is:

$$y(x) = (A + Bx)e^{-\lambda x}, \quad A, B = \text{const}.$$

** §3.2 Linear-Independence and the Wronskian

Fact

Two functions $y_1(x)$ and $y_2(x)$ are linearly independent if the Wronskian $W(y_1, y_2) \neq 0$, where

$$W(y_1, y_2) = \det \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} = y_1 y_2' - y_1' y_2.$$

If $W \neq 0$ for some x, the $W \neq 0$ for all x. (Abel's Thm)

Example

$$y_1(x) = e^{\lambda_1 x}, \quad y_2(x) = e^{\lambda_2 x} \quad \text{and } \lambda_1 \neq \lambda_2.$$

Then, $W(y_1, y_2) = y_1 y_2' - y_1' y_2 = \lambda_2 e^{(\lambda_1+\lambda_2)x} - \lambda_1 e^{(\lambda_1+\lambda_2)x}$

$$= (\lambda_2 - \lambda_1) e^{(\lambda_1+\lambda_2)x}$$

$$\neq 0 \quad \text{since } \lambda_1 \neq \lambda_2.$$

Example

$$y_1(x) = \sin \omega x, \quad y_2(x) = \cos \omega x:$$

Then, $W(y_1, y_2) = y_1 y_2' - y_1' y_2 = -\omega \sin^2 \omega x - \omega \cos^2 \omega x$

$$= -\omega \neq 0 \quad (\text{if } \omega \neq 0).$$

Example

$$y_1(x) = e^{\lambda x}, \quad y_2(x) = x e^{\lambda x}:$$

Then, $W(y_1, y_2) = y_1 y_2' - y_1' y_2 = e^{\lambda x}(\lambda x e^{\lambda x} + e^{\lambda x}) - \lambda e^{\lambda x}(x e^{\lambda x})$

$$= e^{2\lambda x} \neq 0.$$

**.

§3.3 Inhomogeneous linear, second-order, const. coeff. ODEs

We will now return to the ODEs of the form:

$$y'' + py' + qy = f(x), \quad p, q = \text{const}; \text{ or } D[y] = f(x)$$

where $D = \frac{d^2}{dx^2} + p\frac{d}{dx} + q$.

We will search for solutions of the form

$$y(x) = y_h(x) + y_p(x) \quad \text{where } y_h \text{ solves } D[y_h] = 0$$

and y_p is any solution to $D[y_p] = f$

Just as for first-order ODEs, we guess and ansatz for $y_p(x)$ that is of the same functional form as $f(x)$:

$f(x)$	$y_p(x)$	
e^{mx}	Ae^{mx}	$m \neq \lambda_+$ or λ_-
$\sin kx$ or $\cos kx$	$A\sin kx + B\cos kx$	$ik \neq \lambda_+$ or λ_-
$\sin kx\, e^{mx}$ or $\cos kx\, e^{mx}$	$(A\sin kx + B\cos kx)e^{mx}$	$m \pm ik \neq \lambda_+$ or λ_-
polynomial degree n	$(Ax^n + Bx^{n-1} + \dots + Fx + G)$	
(polynomial degree n)e^{mx}	$(Ax^n + Bx^{n-1} + \dots + Fx + G)e^{mx}$	
$y_h(x)$	$x\, y_h(x)$.	

Let's consider $y'' - 3y' + 2y = f(x)$.

The homogeneous solution is $y_h'' - 3y_h' + 2y_h = 0$.

Trying $y_h = e^{\lambda x}$ gives $\lambda^2 - 3\lambda + 2 = 0$

So $\lambda = 1$ or 2, and $y_h(x) = Ae^{x} + Be^{2x}$.

Now let's consider various $f(x)$:

Example

Solve $y'' - 3y' + 2y = e^{3x}$ with $y(0) = 1$, $y'(0) = 0$.

We have $y_h = Ae^{x} + Be^{2x}$, so $f(x) = e^{3x}$ is not part of the homogeneous soln. Therefore, we try $y_p(x) = Ce^{3x}$. Then, $y_p'' - 3y_p' + 2y_p = 9Ce^{3x} - 9Ce^{3x} + 2Ce^{3x}$

so $2Ce^{3x} = e^{3x} \Rightarrow C = \frac{1}{2}$. Hence $y_p(x) = \frac{1}{2}e^{3x}$

Then, the general soln is:

$$y(x) = y_h(x) + y_p(x) = Ae^{x} + Be^{2x} + \frac{1}{2}e^{3x}$$

with $A, B = \text{const}$.

We now apply BCs to $y(x)$ *** (and <u>not</u> to $y_h(x)$ or $y_p(x)$) ***

So, $y(0) = 1 \Rightarrow A + B + \frac{1}{2} = 1$ ①

$y'(0) = 0 \Rightarrow A + 2B + \frac{3}{2} = 0$ ②

Now, ① $\Rightarrow A = \frac{1}{2} - B$, so

② $\Rightarrow \frac{1}{2} - B + 2B + \frac{3}{2} = 0 \Rightarrow B = -2 \Rightarrow A = +\frac{5}{2}$

So $y(x) = +\frac{5}{2}e^{x} - 2e^{2x} + \frac{1}{2}e^{3x}$.

<u>Example</u>

Solve $y'' - 3y' + 2y = e^{2x}$, with $y(0) = 1$, $y'(0) = 0$.

Here, $f(x)$ is contained in $y_h(x)$, so $y_p(x) = Ce^{2x}$ <u>will</u> not work.

Therefore, we try $y_p(x) = Cxe^{2x}$. Then, $y_p' = 2Cxe^{2x} + Ce^{2x}$

$y_p'' = 4Cxe^{2x} + 4Ce^{2x}$

$$y_p'' - 3y_p' + 2y_p$$

$$= 4Cxe^{2x} + 4Ce^{2x} - 6Cxe^{2x} - 3Ce^{2x} + 2Cxe^{2x}$$

$$= Ce^{2x} = e^{2x}$$

So $C = 1$ and $y(x) = y_h(x) + y_p(x)$

$$= Ae^{x} + Be^{2x} + xe^{x}.$$

Then, $y(0) = 1 \Rightarrow A + B = 1$ ①

$y'(0) = 0 \Rightarrow A + 2B + 1 = 0$ ②

Variation of parameters

For the general linear second-order inhomogeneous ODE:

$$y'' + p(x)y' + q(x)y = f(x),$$

how do we find $y_p(x)$? Let $y_1(x)$, $y_2(x)$ be linearly indep. solutions for $f = 0$. So $W(y_1, y_2) \equiv y_1y_2' - y_2y_1' \neq 0$.

Try $y_p(x) = u(x)y_1(x) + v(x)y_2(x)$.

we choose $u'y_1 + v'y_2 = 0$. (*)

Then, $y_p' = \cancel{u'y_1} + uy_1' + \cancel{v'y_2} + vy_2'$

and $y_p'' = u'y_1' + uy_1'' + v'y_2' + vy_2''$.

Then $y_p'' + py_p' + qy_p = u'y_1' + \cancel{uy_1''} + v'y_2' + \cancel{vy_2''} + \cancel{puy_1'} + \cancel{pvy_2'} + \cancel{quy_1} + \cancel{qvy_2}$

So need $u'y_1' + v'y_2' = f(x)$

Now (*) $\Rightarrow$ $u' = -\dfrac{v'y_2}{y_1}$ $\Rightarrow$ $-\dfrac{v'y_2y_1'}{y_1} + v'y_2' = f$

$\Rightarrow v'(y_1y_2' - y_1'y_2) = y_1 f \Rightarrow v' = \dfrac{y_1 f}{W} \Rightarrow u' = -\dfrac{y_2 f}{W}$

Example

Solve $y'' - 2y' + y = e^x$. For y_1 and y_2 try $y \propto e^{\lambda x}$.

Then $\lambda^2 - 2\lambda + 1 = 0 \Rightarrow \lambda = 1$ (twice). So $y_1 = e^x$ and $y_2 = xe^x$.

Guess $y_p \propto x^2e^x$). We have $W(y_1, y_2) = e^x(e^x + xe^x) - xe^{2x} = e^{2x}$

so $v' = y_1 f/W = 1 \Rightarrow v = x + C$. Then $u' = -x \Rightarrow u = -\frac{1}{2}x^2 + B$.

Then $y_p(x) = (-\frac{1}{2}x^2 + B)e^x + (x + C)xe^x$.

$$= \frac{1}{2}x^2e^x + \underbrace{Be^x}_{y_1} + \underbrace{Cxe^x}_{y_2} \Rightarrow B = C = 0$$

so $y_p(x) = \frac{1}{2}x^2e^x$

So, ① $\Rightarrow$ $A = \cancel{3}\ 1-B$

so ② $\Rightarrow$ $1-B+2B+1=0 \Rightarrow B=-2 \Rightarrow A=3$

So $y(x) = 3e^{x} - 2e^{2x} + xe^{x}$.

__Example__ $f(x) = \cancel{C} = \text{const}$. Try $y_p(x) = D = \text{const}$.

__Example__ $f(x) = \sin 2x$. Try $y_p(x) = C\cos 2x + D\sin 2x$.

Then, $y_p'' - 3y_p' + 2y_p = -4C\cos 2x - 4D\sin 2x + 6C\sin 2x - 6D\cos 2x$
$+2C\cos 2x + 2D\sin 2x$

$= (-2C - 6D)\cos 2x + (6C - 2D)\sin 2x. = \sin 2x$

So, $-2C - 6D = 0$ and $6C - 2D = 1$

so $C = -3D$ and $6(-3D) - 2D = 1 \Rightarrow D = -\frac{1}{20}$

and $C = \frac{3}{20}$

So, $y(x) = y_h(x) + y_p(x) = Ae^{x} + Be^{2x} + \frac{3}{20}\cos 2x - \frac{1}{20}\sin 2x$

__Now__, we can apply BCs, if given.

__Example__ $f(x) = x^2$. Try $y_p(x) = Cx^2 + Dx + E$.

§3.4 Beating, ~~and~~ Resonance and Damping

We've seen that linear second-order ODEs can have oscillatory solns, when λ_+, λ_- are complex. When these solutions are forced, they can give rise to beating ~~and~~ or resonance. Also, resonance can be eliminated through damping.

To show these results, we'll start with the equation for free oscillations. This is also called a simple harmonic oscillator:

$$\ddot{y} + \omega_0^2 y = 0.$$

The mass-spring system is an example: $m\ddot{x} + kx = 0$.
The general solution to this homogeneous equation is

$$y(t) = A\cos\omega_0 t + B\sin\omega_0 t.$$

Now let's force this system, with another oscillator:

$$\ddot{y} + \omega_0^2 y = \sin\omega t.$$

When $\omega \neq \omega_0$ but $\omega \approx \omega_0$, we get beating.
When $\omega = \omega_0$ we get resonance.

Beating

Let $\omega \neq \omega_0$. Then we try the particular solution

$y_p(t) = C\cos\omega t + D\sin\omega t.$

Then $\dot{y}_p = -\omega C\sin\omega t + \omega D\cos\omega t$

and $\ddot{y}_p = -\omega^2 C\cos\omega t - \omega^2 D\sin\omega t.$

$$\text{So, } \ddot{y}_p + \omega_0^2 y_p = -\omega^2 C\cos\omega t - \omega^2 D\sin\omega t + \omega_0^2 C\cos\omega t + \omega_0^2 D\sin\omega t = \sin\omega t$$

So, coeff. $\cos\omega t$: $-\omega^2 C + \omega_0^2 C = 0 \Rightarrow C = 0$

Hence, $y_p(t) = \frac{1}{\omega_0^2-\omega^2}\sin\omega t$,

and $y(t) = y_h(t) + y_p(t) = A\cos\omega_0 t + B\sin\omega_0 t + \frac{\sin\omega t}{\omega_0^2-\omega^2}$.

Let's choose ICs $y(0) = 0$ and $\dot{y}(0) = \frac{-1}{\omega_0+\omega}$

Then $y(0)=0 \Rightarrow A=0$, and $\dot{y}(0)=0 \Rightarrow B = \frac{-1}{\omega_0^2-\omega^2}$

So $$y(t) = \frac{\sin\omega t - \sin\omega_0 t}{\omega_0^2-\omega^2}.$$

We see that there are significant problems if $\omega = \omega_0$!

<u>Fact</u> $\sin a - \sin b = 2\cos\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)$.

Therefore, $$y(t) = \frac{+2}{\omega_0^2-\omega^2}\cos\left(\frac{\omega+\omega_0}{2}t\right)\sin\left(\underbrace{\frac{\omega-\omega_0}{2}}_{-\Delta\omega/2}t\right).$$

Let $\Delta\omega = \omega_0 - \omega$. Then $\omega_0^2-\omega^2 = (\omega_0+\omega)(\omega_0-\omega) = \Delta\omega(\omega_0+\omega)$

Also, $\omega+\omega_0 = \omega+\omega_0+\omega_0-\omega_0-(\omega_0-\omega)$

$= 2\omega_0 - \Delta\omega$.

So, $$y(t) = \frac{-2}{\Delta\omega(\omega_0+\omega)}\cos\left(\left(\omega_0-\frac{\Delta\omega}{2}\right)t\right)\sin\left(\frac{\Delta\omega t}{2}\right).$$

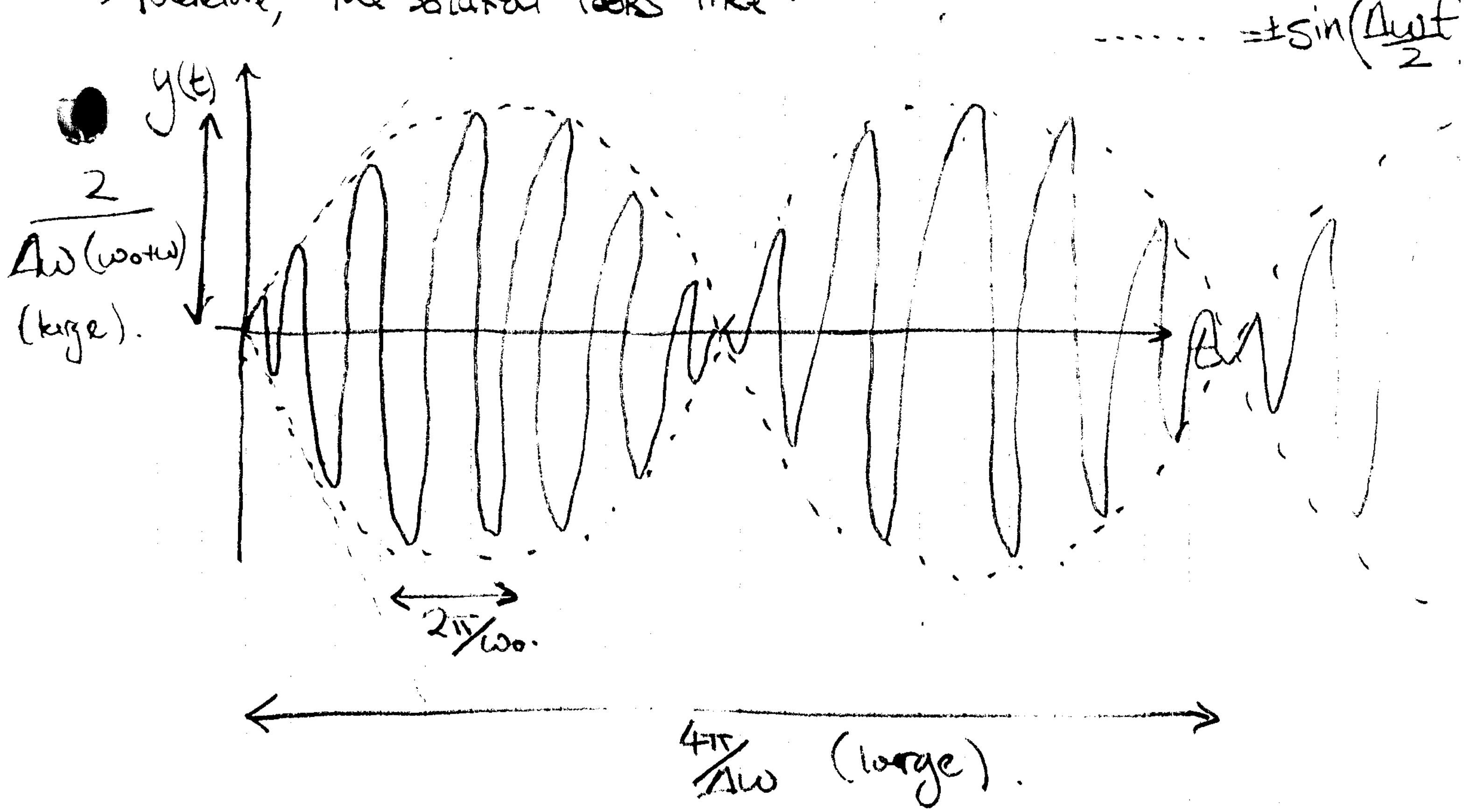
If $\omega \approx \omega_0$, then $\Delta\omega$ is very small and $\frac{1}{\Delta\omega}$ is very large

Also, $\cos\left(\left(\omega_0-\frac{\Delta\omega}{2}\right)t\right) \approx \cos\omega_0 t$.

The ~~wavelength~~ period of $\cos\omega_0 t$ is $\frac{2\pi}{\omega_0}$.

The ~~wavelength~~ period of ~~cos~~ sin $\Delta\omega t$ is $4\pi/\Delta\omega$

Therefore, the solution looks like:



This is beating!

Resonance

As $\Delta\omega \to 0$, i.e. as $\omega \to \omega_0$, we get resonance.

We have $y(t) = \frac{\mp 2}{\omega+\omega_0} \cdot \cos\left(\left(\omega - \frac{\Delta\omega}{2}\right)t\right) \frac{\sin\left(\frac{\Delta\omega t}{2}\right)}{\Delta\omega}$.

As $\Delta\omega \to 0$, $\cos\left(\left(\omega_0 - \frac{\Delta\omega}{2}\right)t\right) \to \cos\omega_0 t$

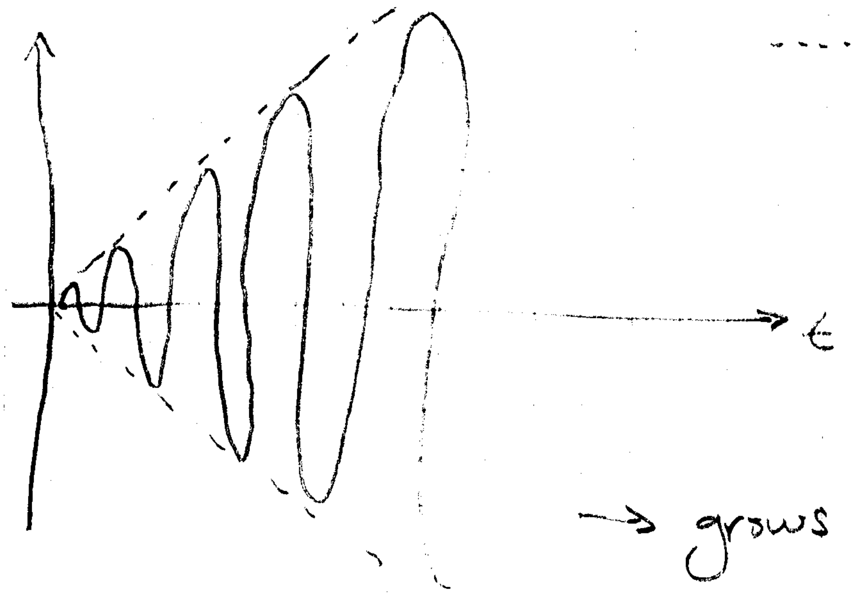
$$\frac{\mp 2}{\omega+\omega_0} \to \frac{\mp 1}{\omega_0}$$

$$\frac{\sin\left(\frac{\Delta\omega t}{2}\right)}{\Delta\omega} \to t/2.$$

So $y(t) \to \mp\frac{1}{2\omega_0} t \cos\omega_0 t$.

We could also find this by trying $y_p(t) = C\,t\cos\omega_0 t + D\,t\sin\omega_0 t$

- Therefore the solution looks like:



$\cdots\cdots = \pm \frac{1}{2\omega_0} t$

$\rightarrow$ grows without bound as $t \to \infty$!

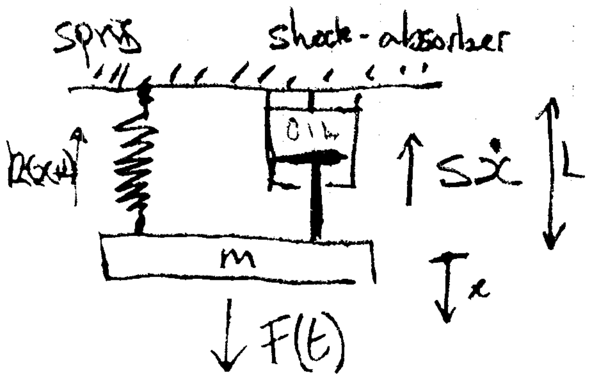
this is resonance!

Damping

Often, mechanical / electrical systems have some form of damping, that reduces or eliminates the effect of beating and resonance. We'll look at one example:

Example

Let's return to the mass-spring system, but add some damping:



Newton's 2nd Law gives:

$$m\ddot{x} = +mg - k(x+L) - S\dot{x} + F(t)$$

$$= \underbrace{mg - kL}_{=0} - kx - S\dot{x} + F(t)$$

$$\ddot{x} + \frac{S}{m}\dot{x} + \frac{k}{m}x = \frac{F(t)}{m}$$