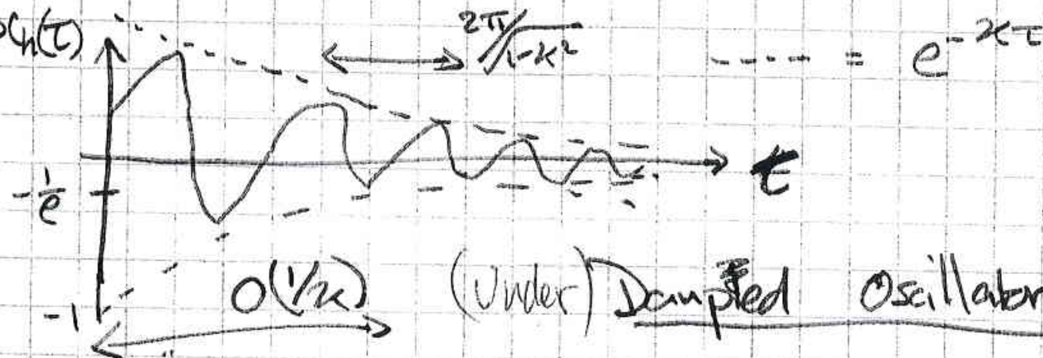
Case 1: $0 < \kappa < 1$.

In this case, $\lambda_\pm = -\kappa \pm \sqrt{\kappa^2 - 1}$ are complex.

Therefore, $x_h(\tau) = e^{-\kappa\tau}\left(A\sin(\sqrt{1-\kappa^2}\,\tau) + B\cos(\sqrt{1-\kappa^2}\,\tau)\right)$.

This oscillates with period $\frac{2\pi}{\sqrt{1-\kappa^2}}$ but has amplitude $e^{-\kappa\tau}$ that has e-folding time (half-life) $\frac{1}{\kappa}$:



(Under) Damped Oscillator

Case 2: $\kappa = 1$

In this case, $\lambda_\pm = -\kappa, -\kappa$, repeated roots!

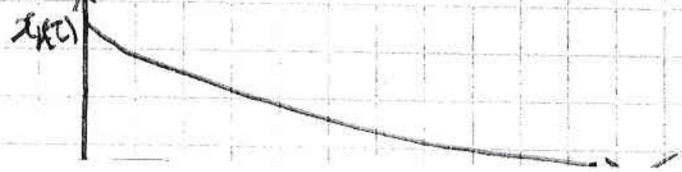
So $x_h(\tau) = (A + B\tau)e^{-\kappa\tau}$.

Critically Damped (no oscillations)

Case 3: $\kappa > 1$

In this case, $\lambda_\pm = -\kappa \pm \sqrt{\kappa^2 - 1}$ are real, distinct.

So, $x_h(\tau) = Ae^{\lambda_+\tau} + Be^{\lambda_-\tau}$.



Overdampted

In all cases, $x_h(\tau) \to 0$ as $\tau \to \infty$.

Now consider the forced case with $f(\tau) = \sin\tau$:

$$\frac{d^2x}{d\tau^2} + 2\kappa\frac{dx}{d\tau} + x = \sin\tau.$$

If $\kappa = 0$, we would have resonance. However, for $\kappa \neq 0$, we don't:

We have $x(\tau) = x_h(\tau) + x_p(\tau)$.

Let's try $x_p(\tau) = C\cos\tau + D\sin\tau$. Then,

$$-C\cos\tau - D\sin\tau - 2\kappa C\sin\tau + 2\kappa D\cos\tau + C\cos\tau + D\sin\tau = \sin\tau.$$

So, coeff. $\cos\tau$: $-C + 2\kappa D + C = 0 \Rightarrow D = 0$.

coeff. $\sin\tau$: $-D - 2\kappa C + D = 1 \Rightarrow C = -\frac{1}{2\kappa}$.

So $x_p(\tau) = -\frac{1}{2\kappa}\cos\tau$.

Then, $x(\tau) = x_h(\tau) - \frac{1}{2\kappa}\cos\tau$.

As $\tau \to \infty$, $x_h(\tau) \to 0$ and so $x(\tau) \to \frac{-1}{2\kappa}\cos\tau$.

No resonance. (No unbounded growth).

** § 3.5 Euler Equations

These are equations of the form:

$$ax^2y'' + bxy' + cy = F(x).$$

or $x^2y'' + bxy' + qy = F(x)$.

These eqns do not have constant coeffs but we can still solve them.

<u>Observation</u>

The differential operator $D = x\frac{d}{dx}$ has <u>eigenfunction</u> $y(x) = x^n$ with e-val n:

$$D[y] = x\frac{dy}{dx} = x\left(nx^{n-1}\right) = nx^n = ny.$$

So, to solve $x^2y'' + pxy' + qy = f(x)$,

write $y = y_h(x) + y_p(x)$ and try $y_h(x) = x^n$.

Then, $\frac{d^2y_h}{dx^2} = n(n-1)x^{n-2}$, so

$$x^2y_h'' + pxy_h' + qy_h = n(n-1)x^n + pnx^n + qx^n = 0.$$

To hold for all x, we need coeff. $x^n = 0$, i.e.

$$n(n-1) + pn + q = 0$$

$$\text{or } n^2 + (p-1)n + q = 0 \Rightarrow n = n_\pm = \frac{1-p \pm\sqrt{(p-1)^2 - 4q}}{2}$$

So $y_h(x) = Ax^{n_+} + Bx^{n_-}$.

We find $y_p(x)$ by similar "guessing" of an <u>ansatz</u>.

<u>Example</u> $x^2y'' + xy' - y = 0$. Here $p = 1$ and $q = -1$

$$\text{So } n_\pm = \frac{0 \pm \sqrt{0+4}}{2} = \pm 1$$

[illegible] is the general soln

§4. Systems of linear first-order ODEs

Consider two functions $y_1(t)$ and $y_2(t)$. The simplest system of linear first-order ODEs is given by

$$\dot{y}_1 = ay_1 + by_2 + f_1(t) \qquad \text{and} \qquad y_1(t_0) = y_{10}$$
$$\dot{y}_2 = cy_1 + dy_2 + f_2(t). \qquad \qquad y_2(t_0) = y_{20}. \quad \text{ICs.}$$

If we let $\underline{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, $\underline{f} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ and $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$,

then the system is the same as:

$$\dot{\underline{y}} = M\underline{y} + \underline{f}(t) \qquad \underline{\text{2-dimensional system}}$$

We can consider a system of n coupled ODEs:

$$\dot{y}_1 = m_{11}y_1 + m_{12}y_2 + \dots + m_{1n}y_n + f_1$$
$$\dot{y}_2 = m_{21}y_1 + m_{22}y_2 + \dots + m_{2n}y_n + f_2 \qquad \text{plus ICs}$$
$$\vdots \qquad \ddots \qquad \vdots$$
$$\dot{y}_n = m_{n1}y_1 + m_{n2}y_2 + \dots + m_{nn}y_n + f_n$$

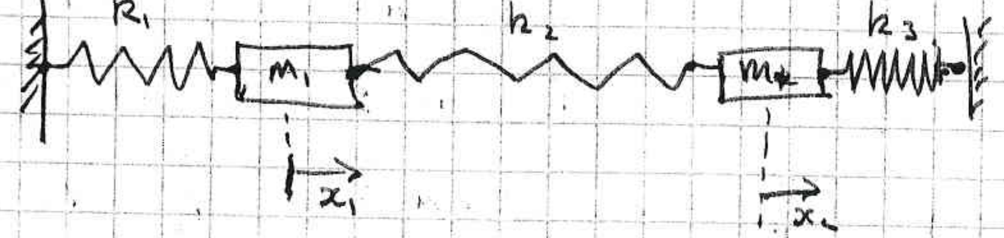
and we can write it as

$$\dot{\underline{y}} = M\underline{y} + \underline{f} \quad \text{where} \quad \underline{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}, \quad \underline{f} = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix}$$

and $M = \begin{pmatrix} m_{11} & m_{12} & \cdots & m_{1n} \\ m_{21} & m_{22} & \cdots & m_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ m_{n1} & m_{n2} & \cdots & m_{nn} \end{pmatrix}$. $\qquad \underline{\text{n-dimensional}}$ system

Example

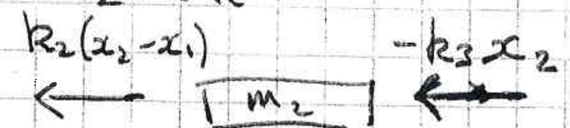
let's consider a coupled mass-spring system:



Where x_1, x_2 are displacements from resting positions of m_1, m_2.

The forces acting on m_1 are: (when $x_1, x_2 > 0$)

and on m_2 are:



So, Newton's 2nd Law gives:

- $m_1 \ddot{x}_1 = k_2(x_2 - x_1) - k_1 x_1$
- $m_2 \ddot{x}_2 = -k_3 x_2 - k_2(x_2 - x_1)$

or,

- $m_1 \ddot{x}_1 = -(k_1 + k_2) x_1 + k_2 x_2$
- $m_2 \ddot{x}_2 = k_2 x_1 - (k_2 + k_3) x_2$.

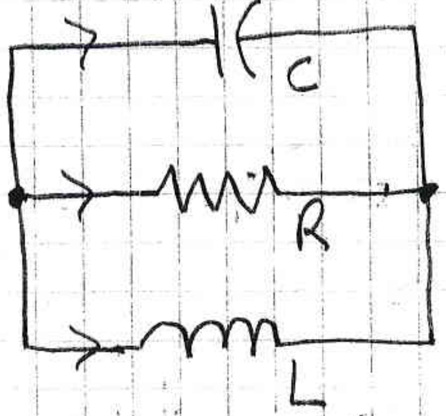
This is in fact a system of 2nd-order ODEs.

This isn't such a bad problem, since:

Example

Every 2nd-order ODE can be converted to a two-dimensional

Parallel LRC Circuit



Recall

Capacitor: $C\dfrac{dV}{dt} = I$; $C\dot{V}_c = I_c$

Resistor: $V = IR$; $V_R = RI_R$

Inductor: $L\dfrac{dI}{dt} = V$; $L\dot{I}_L = V_L$

Let I_C, I_R, I_L be currents, V_C, V_R, V_L be voltages.

Kirchoff's 2nd law to upper loop: $V_C - V_R = 0$

lower loop: $V_R - V_L = 0$

Kirchoff's 1st Law to either node: $I_C + I_R + I_L = 0$

So $V_C = V_R = RI_R = V_L = L\dot{I}_L$

and $C\dot{V}_C = I_C = -I_R - I_L = -\dfrac{V_R}{R} - I_L = -\dfrac{V_C}{R} - I_L$

So; $L\dot{I}_L = V_C$ and $C\dot{V}_C = -\dfrac{V_C}{R} - I_L$

or,
$$\begin{pmatrix} \dot{I}_L \\ \dot{V}_C \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{L} \\ -\frac{1}{C} & -\frac{1}{RC} \end{pmatrix} \begin{pmatrix} I_L \\ V_C \end{pmatrix}$$

Let $\ddot{y} + p\dot{y} + qy = f(t)$.

Let $y_1(t) = y(t)$ and $y_2(t) = \dot{y}(t)$.

Thus, $\dot{y}_2 = \ddot{y} = f(t) - p\dot{y} - qy$

$= f(t) - py_2 - qy_1$

also, $\dot{y}_1 = \dot{y} = y_2$.

So the system is:

$$\dot{y}_1 = y_2$$
$$\dot{y}_2 = -qy_1 - py_2 + f(t)$$

and so $\dot{\underline{y}} = M\underline{y} + \underline{f}$ where

$$\underline{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad \underline{f} = \begin{pmatrix} 0 \\ f(t) \end{pmatrix}, \quad M = \begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix}.$$

To solve the system of 1^{st}-order ODEs

$\dot{\underline{y}} = M\underline{y} + \underline{f}$, we again find the homogeneous solution and a particular solution, so that

$\underline{y} = \underline{y}_h + \underline{y}_p$, where

$\dot{\underline{y}}_h = M\underline{y}_h$ and $\dot{\underline{y}}_p = M\underline{y}_p + \underline{f}$.

We'll start by solving the homogeneous problem.

§4.1 Homogeneous systems of linear, first-order ODEs

- These are systems of the form $\dot{\underline{y}} = M\underline{y}$ with $\underline{y} = (y_1(t), y_2(t), \dots, y_n(t))^T$ and M an $n \times n$ matrix with elements M_{ij}.

Observation

The differential operator $\frac{d}{dt}$ has eigenfunction $\underline{y} = \underline{v}e^{\lambda t}$ for any constant vector $\underline{v}$; with e-val λ.

$$\frac{d}{dt}\underline{y} = \frac{d}{dt}(\underline{v}e^{\lambda t}) = \underline{v}\frac{de^{\lambda t}}{dt} = \lambda \underline{v}e^{\lambda t} = \lambda \underline{y}.$$

Idea

- Let's try looking for solutions of the form $\underline{y} = \underline{v}e^{\lambda t}$.

Then, $\dot{\underline{y}} = M\underline{y} \Rightarrow \lambda\underline{v}e^{\lambda t} = M\underline{v}e^{\lambda t}$.

Since $e^{\lambda t} \neq 0$, we have $M\underline{v} = \lambda\underline{v}$.

So, we solve the ODE system if $\underline{v}$ is an eigenvector of the matrix M with eigenvalue λ!

We find λ by solving

$$\det(M - \lambda I) = 0$$ where I is the $n \times n$ identity matrix.

Then, $\underline{v}$ is found by solving $M\underline{v} = \lambda\underline{v}$, given λ.

- We will only consider the case when $n = 2$, so M is a 2×2 matrix and $\underline{y} = (y_1(t), y_2(t))^T$

Example

Let $\dot{y} = \begin{pmatrix} -4 & 24 \\ 1 & -2 \end{pmatrix} y . \equiv My$

Try $y = \underline{v} e^{\lambda t}$. Then, $M\underline{v} = \lambda \underline{v}$. So,

$= \det(M - \lambda I) = \det \begin{pmatrix} -4-\lambda & 24 \\ 1 & -2-\lambda \end{pmatrix}$

$= (-4-\lambda)(-2-\lambda) - 24$

$= 8 + 6\lambda + \lambda^2 - 24 = \lambda^2 + 6\lambda - 16$

$= (\lambda+8)(\lambda-2)$

So $\lambda = 2$ or -8.

In order to find $\underline{v}$, we solve $M\underline{v} = \lambda\underline{v}$, or $(M - \lambda I)\underline{v} = 0$.

$\underline{\lambda = 2}$ let $\underline{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$. Then,

$(M - \lambda I)\underline{v} = \begin{pmatrix} -6 & 24 \\ 1 & -4 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0.$

So $v_1 = 4v_2 \Rightarrow \underline{v} = a \begin{pmatrix} 4 \\ 1 \end{pmatrix}$ any const. a.

$\underline{\lambda = -8}$ let $\underline{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$. Then,

$(M - \lambda I)\underline{v} = \begin{pmatrix} 4 & 24 \\ 1 & 6 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0.$

So $V_1 = -6v_2 \Rightarrow \underline{v} = b \begin{pmatrix} -6 \\ 1 \end{pmatrix}$ any const. b.

Then, the <u>general solution</u> is:

$$\underline{y} = a\begin{pmatrix}4\\1\end{pmatrix}e^{2t} + b\begin{pmatrix}-6\\1\end{pmatrix}e^{-8t}$$

where $a, b = \text{const}$.

Now let ICs be $y_1(0) = 0$ and $y_2(0) = 1$.

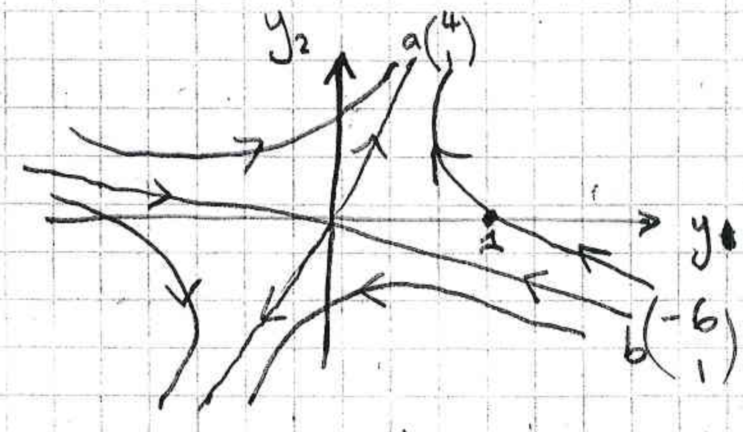
We have
$$y_1(t) = 4ae^{2t} - 6be^{-8t}$$
$$y_2(t) = ae^{2t} + be^{-8t}.$$

So $y_1(0) = 0 \Rightarrow 4a - 6b = 0 \Rightarrow a = \frac{3}{2}b$.

$y_2(0) = 1 \Rightarrow a + b = 1 \Rightarrow \frac{3}{2}b + b = 1 \Rightarrow b = \frac{2}{5}$.

and $a = \frac{3}{2}b = \frac{3}{2}\cdot\frac{2}{5} = \frac{3}{5}$.

Hence, $$\begin{pmatrix}y_1(t)\\y_2(t)\end{pmatrix} = \frac{3}{5}\begin{pmatrix}4\\1\end{pmatrix}e^{2t} + \frac{2}{5}\begin{pmatrix}-6\\1\end{pmatrix}e^{-8t}.$$



A <u>saddle point</u>!

For the general case $\dot{\underline{y}} = M\underline{y}$, we have two e-values λ_1 and λ_2. Then, provided $\lambda_1 \neq \lambda_2$ we have two e-vecs $\underline{v}_1$ and $\underline{v}_2$ and the general soln is:

nen $\lambda_{1,2} = \sigma \pm i\omega$, there is a complex conjugate r of e-vecs:

Example $M = \begin{pmatrix} -\frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{pmatrix}$. Then $0 = \det(M - \lambda I)$,

, $0 = \lambda^2 + \lambda + \frac{5}{4} \Rightarrow \lambda = -\frac{1}{2} + i, -\frac{1}{2} - i$.

$= -\frac{1}{2}+i$ $\quad (M - \lambda I)\underline{v} = \begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix} \underline{v} \Rightarrow v_2 = iv_1 \Rightarrow \underline{v} = \begin{pmatrix} 1 \\ i \end{pmatrix}$

$= -\frac{1}{2}-i$ $\quad (M - \lambda I)\underline{v} = \begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix} \underline{v} \Rightarrow v_2 = -iv_1 \Rightarrow \underline{v} = \begin{pmatrix} 1 \\ -i \end{pmatrix}$

e have two complex-valued solns

$$\underline{y}_a = e^{-\frac{1}{2}t + it} \begin{pmatrix} 1 \\ i \end{pmatrix} = e^{-\frac{1}{2}t} \begin{pmatrix} \cos t + i\sin t \\ i\cos t - \sin t \end{pmatrix}$$

$$\underline{y}_b = e^{-\frac{1}{2}t - it} \begin{pmatrix} 1 \\ -i \end{pmatrix} = e^{-\frac{1}{2}t} \begin{pmatrix} \cos t - i\sin t \\ -i\cos t - \sin t \end{pmatrix}.$$

ut, $\underline{y}_a + \underline{y}_b = e^{-\frac{1}{2}t} \begin{pmatrix} 2\cos t \\ -2\sin t \end{pmatrix}$

$$\underline{y}_a - \underline{y}_b = e^{-\frac{1}{2}t} \begin{pmatrix} 2i\sin t \\ -2i\cos t \end{pmatrix}$$

So let $\underline{v}_a = \frac{\underline{y}_a + \underline{y}_b}{2}$ and $\underline{v}_b = \frac{\underline{y}_a - \underline{y}_b}{2i}$ both real.

Then if $\dot{\underline{y}} = M\underline{y}$, we have gen. soln

$$\underline{y} = A\underline{v}_a + B\underline{v}_b = e^{-\frac{1}{2}t}\left[A\begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + B\begin{pmatrix} \sin t \\ \cos t \end{pmatrix}\right].$$

n general, if $\underline{v} = \underline{a} \pm i\underline{b}$, then for $\lambda = \sigma \pm i\omega$, then

$$\underline{y} = A\underline{v}_a + B\underline{v}_b \text{ where}$$

$$\underline{v}_a = e^{\sigma t}(\underline{a}\cos\omega t - \underline{b}\sin\omega t)$$

and $\underline{v}_b = e^{\sigma t}(\underline{a}\sin\omega t + \underline{b}\cos\omega t)$.

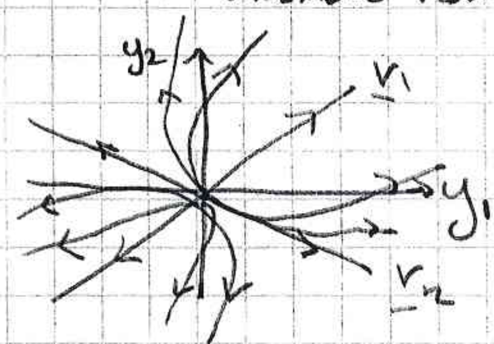
$$\underline{y}(t) = a\underline{v}_1 e^{\lambda_1 t} + b\underline{v}_2 e^{\lambda_2 t}.$$

- There are then a number of cases:

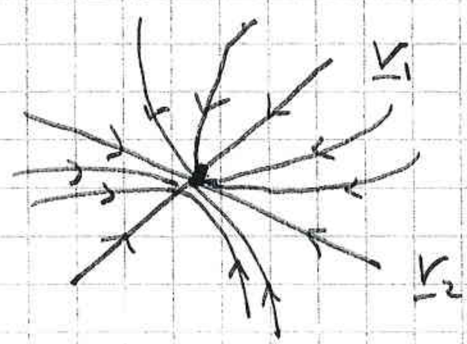
1) λ_1 and λ_2 both real, $\lambda_1 < 0$ and $\lambda_2 > 0$ (say).
$\rightarrow$ saddle point as above.

2) λ_1 and λ_2 both real, same sign. $|\lambda_1| > |\lambda_2|$ (say).

a) $\lambda_1, \lambda_2 > 0$
$\rightarrow$ unstable node



b) $\lambda_1, \lambda_2 < 0$
$\rightarrow$ stable node



3) λ_1 and λ_2 complex, $\lambda_1 = \overline{\lambda_2} = \sigma + i\omega$.

~~Then $\underline{y}(t) = a\underline{v}_1 e^{\sigma t}\cos\omega t + b\underline{v}_2 e^{\sigma t}\sin\omega t$.~~

a) $\sigma < 0$
$\rightarrow$ stable spiral

b) $\sigma > 0$
$\rightarrow$ unstable spiral

c) $\sigma = 0$
$\rightarrow$ center

But what about repeated roots, $\lambda_1 = \lambda_2 = \lambda$?

~~Example~~ If we still have two distinct e-vecs, then we're OK: $\underline{y}(t) = a\underline{v}_1 e^{\lambda t} + b\underline{v}_2 e^{\lambda t}$.
with $\underline{v}_1$ and $\underline{v}_2$ linearly independent.

were probably expecting to find $g = t$.
turns out that the right thing to do is try
$= t\begin{pmatrix}1\\-1\end{pmatrix}e^{2t} + \underline{a}e^{2t}$ for some constant vector
$\underline{a} \neq \begin{pmatrix}1\\-1\end{pmatrix}c$.

hen $\dot{\underline{y}} = \begin{pmatrix}1\\-1\end{pmatrix}e^{2t} + 2t\begin{pmatrix}1\\-1\end{pmatrix}e^{2t} + 2\underline{a}e^{2t}$

nd $M\underline{y} = 2t\begin{pmatrix}1\\-1\end{pmatrix}e^{2t} + M\underline{a}\,e^{2t}$.

hen for $\dot{\underline{y}} = M\underline{y}$, we need

$$\begin{pmatrix}1\\-1\end{pmatrix} + 2\underline{a} = M\underline{a}$$

$$\Rightarrow \quad (M - 2I)\underline{a} = \begin{pmatrix}1\\-1\end{pmatrix}.$$

$\underline{a}$ is called a generalised eigenvector of M

et $\underline{a} = \begin{pmatrix}a_1\\a_2\end{pmatrix}$. Then $(M-2I)\underline{a} = \begin{pmatrix}-1 & -1\\1 & 1\end{pmatrix}\underline{a} = \begin{pmatrix}1\\-1\end{pmatrix}$

$\Rightarrow \quad a_1 + a_2 = -1$ (twice).

let $a_1 = k$ then $a_2 = -k-1$ and

$$\underline{a} = \begin{pmatrix}0\\-1\end{pmatrix} + k\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}0\\-1\end{pmatrix} + k\underline{v}$$

o wlog $k=0$ and $\underline{y} = t\begin{pmatrix}1\\-1\end{pmatrix}e^{2t} + \begin{pmatrix}0\\-1\end{pmatrix}e^{2t}$

and gen. soln is

$$\underline{y} = A\begin{pmatrix}1\\-1\end{pmatrix}e^{2t} + B\left[t\begin{pmatrix}1\\-1\end{pmatrix}e^{2t} + \begin{pmatrix}0\\-1\end{pmatrix}e^{2t}\right].$$

So, $\dot{y} = My \Rightarrow \dot{g}\begin{pmatrix}1\\-1\end{pmatrix}e^{2t} + 2g\begin{pmatrix}1\\-1\end{pmatrix}e^{2t} = Mg\begin{pmatrix}1\\-1\end{pmatrix}e^{2t}$

$$= ge^{2t} M\begin{pmatrix}+1\\-1\end{pmatrix} = ge^{2t}\, 2\begin{pmatrix}1\\-1\end{pmatrix}$$

So $\dot{g}\begin{pmatrix}1\\-1\end{pmatrix}e^{2t} = 0$. So $\dot{g} = 0$ $\Rightarrow g = t$.

$g = c$

Hence, $y = t\begin{pmatrix}1\\-1\end{pmatrix}e^{2t}$ is a soln.

The general soln is therefore:

$$\underline{y}(t) = a\begin{pmatrix}1\\-1\end{pmatrix}e^{2t} + bt\begin{pmatrix}1\\-1\end{pmatrix}e^{2t}.$$

with $a, b = $ const.

In general, if $\lambda_1 = \lambda_2 = \lambda$ and we have only one e-vec $\underline{v}$

then $\underline{y} = a\underline{v}e^{\lambda t} + b(t\underline{v}e^{\lambda t} + \underline{w}e^{\lambda t})$

where $(M - \lambda I)\underline{w} = \underline{v}$.

§ 4.2 Inhomogeneous systems of linear, first-order ODEs

We'll now return to

$$\dot{\underline{y}} = M\underline{y} + \underline{f}.$$

We search for a solution of the form

$\underline{y} = \underline{y}_h + \underline{y}_p$ where $\dot{\underline{y}}_h = M\underline{y}_h$ (which we know)

and $\underline{y}_p$ is a particular solution.

As before, we "guess" an ansatz for $\underline{y}_p$ based on the

functional form of $f(t)$

Example

Solve $\dot{y} = \begin{pmatrix} -4 & 24 \\ 1 & -2 \end{pmatrix} y + \underline{f}_0$

with $\underline{f}_0$ = const. vector.

Try $y_p = \underline{v}$ with $\underline{v}$ = const. vector.

Then, $\dot{y}_p = 0$, so $0 = \begin{pmatrix} -4 & 24 \\ 1 & -2 \end{pmatrix} \underline{v} + \underline{f}_0 = M\underline{v} + \underline{f}_0.$

so $M\underline{v} = -\underline{f}_0 \Rightarrow \underline{v} = -M^{-1}\underline{f}_0$

Eg. If $\underline{f}_0 = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$, then $\underline{v} = -M^{-1}\begin{pmatrix} +4 \\ +1 \end{pmatrix}$

We have $M^{-1} = \frac{1}{\det M}\begin{pmatrix} -2 & -24 \\ -1 & -4 \end{pmatrix}$ and $\det M = 8 - 24 = -16$

so $M^{-1} = -\frac{1}{16}\begin{pmatrix} -2 & -24 \\ -1 & -4 \end{pmatrix}$

and $\underline{v} = -M^{-1}\underline{f}_0 = -\left(-\frac{1}{16}\right)\begin{pmatrix} -2 & -24 \\ -1 & -4 \end{pmatrix}\begin{pmatrix} 4 \\ 1 \end{pmatrix}$

$= \frac{1}{16}\begin{pmatrix} -8-24 \\ -4-4 \end{pmatrix} = \frac{1}{16}\begin{pmatrix} -32 \\ -8 \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} 4 \\ 1 \end{pmatrix}$

So, general soln is:

$y = y_h + -\frac{1}{2}\begin{pmatrix} 4 \\ 1 \end{pmatrix} = a\begin{pmatrix} 4 \\ 1 \end{pmatrix}e^{2t} + b\begin{pmatrix} -6 \\ 1 \end{pmatrix}e^{-8t} - \frac{1}{2}\begin{pmatrix} 4 \\ 1 \end{pmatrix}.$

Then, if $y_1(0) = 0$ and $y_1(1) = 1$, we have:

$y_1(0) = 0 \Rightarrow 4a - 6b - 2 = 0 \Rightarrow a = \frac{3b}{2} + \frac{1}{2}$

$y_2(0) = 1 \Rightarrow a + b - \frac{1}{2} = 1 \Rightarrow \frac{3b}{2} + \frac{1}{2} + b - \frac{1}{2} = 1$

$\Rightarrow b = \frac{2}{5}$ and $a = \frac{2}{5}\cdot\frac{3}{2} + \frac{1}{2} = \frac{3}{5} + \frac{1}{2} = \frac{6}{10} + \frac{5}{10} = \frac{11}{10}$

Hence, $y = \frac{11}{10}\begin{pmatrix}4\\1\end{pmatrix}e^{2t} + \frac{2}{5}\begin{pmatrix}-6\\1\end{pmatrix}e^{-8t} - \frac{1}{2}\begin{pmatrix}4\\1\end{pmatrix}$.

Example

Let $\dot{y} = \begin{pmatrix}-4 & 24\\1 & -2\end{pmatrix}y + f_0 e^t \equiv My + \underline{f}_0 e^t$.

with $\underline{f}_0$ = const. vector. Try $y_p = \underline{v}\, e^t$ with $\underline{v}$ = const. vector.

Then $\dot{y}_p = \underline{v} e^t$. Also $My_p = e^t M\underline{v}$.

Hence, $\underline{v} e^t = e^t M\underline{v} + \underline{f}_0 e^t$.

$$\Rightarrow \underline{v} = M\underline{v} + \underline{f}_0$$

$$\Rightarrow (M - I)\underline{v} = -\underline{f}_0$$

$$\Rightarrow \underline{v} = -(M-I)^{-1}\underline{f}_0.$$

Now, $M - I = \begin{pmatrix}-5 & 24\\1 & -3\end{pmatrix}$. So $\det(M-I) = 15 - 24 = -9$

$$\Rightarrow (M-I)^{-1} = \frac{-1}{9}\begin{pmatrix}-3 & -24\\-1 & -5\end{pmatrix}$$

Let $\underline{f}_0 = \begin{pmatrix}4\\1\end{pmatrix}$. Then $\underline{v} = -(M-I)^{-1}\underline{f}_0$

$$= -\left(\frac{-1}{9}\right)\begin{pmatrix}-3 & -24\\-1 & -5\end{pmatrix}\begin{pmatrix}4\\1\end{pmatrix} = \frac{1}{9}\begin{pmatrix}-12-24\\-4-5\end{pmatrix} = \frac{1}{9}\begin{pmatrix}-36\\-9\end{pmatrix}$$

$$= -\begin{pmatrix}4\\1\end{pmatrix}.$$

Hence, $y = y_h + y_p = a\begin{pmatrix}4\\1\end{pmatrix}e^{2t} + b\begin{pmatrix}-6\\1\end{pmatrix}e^{-8t} - \begin{pmatrix}4\\1\end{pmatrix}e^{t}$.

$\underline{f} = \underline{f}_0 \cdot$ polynomial degree n $\Rightarrow$ $y_p = \underline{v} \cdot$ polynomial degree n.

$\underline{f} = \underline{f}_0 \cdot \cos\omega t$ or . $\Rightarrow$ $y_p = \underline{v}_1 \cos\omega t + \underline{v}_2 \sin\omega t$.

Example

Let $\underline{\dot{y}} = \begin{pmatrix} -4 & 24 \\ 1 & -2 \end{pmatrix} \underline{y} + \begin{pmatrix} 4 \\ 1 \end{pmatrix} e^{2t} \equiv M\underline{y} + \underline{f}_0 e^{2t}$

Now, $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$ is e-vec of M with e-val 2, so

$\begin{pmatrix} 4 \\ 1 \end{pmatrix} e^{2t}$ is contained in $\underline{y}_h$.

We can't try $\underline{y}_p = \underline{v} e^{2t}$. Instead, let's try

$\underline{y}_p = \underline{v}\, g(t)\, e^{2t}$.

Then, $\underline{\dot{y}}_p = \underline{v} \dot{g} e^{2t} + 2\underline{v} g e^{2t}$.

So, $\underline{v} \dot{g} e^{2t} + 2\underline{v} g e^{2t} = M\underline{y}_p + \begin{pmatrix} 4 \\ 1 \end{pmatrix} e^{2t}$

$= g e^{2t} M\underline{v} + \begin{pmatrix} 4 \\ 1 \end{pmatrix} e^{2t}$.

So, $\dot{g}\underline{v} + 2g\underline{v} = gM\underline{v} + \begin{pmatrix} 4 \\ 1 \end{pmatrix}$.

* If we now choose $\underline{v} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$, we have $M\underline{v} = 2\underline{v}$,

so $\dot{g}\begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} \Rightarrow \dot{g} = 1 \Rightarrow g = t$.

So $\underline{y}_p = t\begin{pmatrix} 4 \\ 1 \end{pmatrix} e^{2t}$.

**

We won't consider $\underline{f} \propto \underline{y}_h$ in this course.

~~§5. Laplace Transforms~~

**

§ 4.3 The Lorenz equations

The Lorenz equns are possibly the most famous system of ODEs. They are:

$$\dot{x} = 10(y-x)$$
$$\dot{y} = x(28-z)-y$$
$$\dot{z} = xy - \frac{8}{3}z.$$

This is a <u>nonlinear</u> system of equations.

We can find <u>fixed points</u> or <u>steady states</u> by solving

$\dot{x} = \dot{y} = \dot{z} = 0$, i.e.

$$0 = 10(y-x) \quad (1)$$
$$0 = 28x - xz - y \quad (2)$$
$$0 = xy - \frac{8}{3}z \quad (3).$$

Now, (1) $\Rightarrow x = y$. So,

$$(2) \Rightarrow 28x - xz - x = 0 \quad (4)$$
$$(3) \Rightarrow x^2 - \frac{8}{3}z = 0 \quad (5)$$

Then (4) $\Rightarrow x = 0$ or $27 = z$.

Then (5) $\Rightarrow z = 0$ or $x^2 = 72 = y^2$

$\therefore$ fixed points are $(0,0,0)$, $(\sqrt{72}, \sqrt{72}, 27)$, $(-\sqrt{72}, -\sqrt{72}, 27)$

We can find the stability of a fixed point (x_0, y_0, z_0) by setting $x = x_0 + \varepsilon x_1$, $y = y_0 + \varepsilon y_1$, $z = z_0 + \varepsilon z_1$, where ε is small, and Taylor expanding, as in §2.2

It turns out that all three fixed points are unstable. So what happens? Chaos!

www.paulbourke.net/fractals/lorenz

www.youtube.com/watch?v=5xu-9D4ahVU →

§5 Laplace Transforms

Many practical problems involve forcing that is either discontinuous or impulsive (over a very short time interval). The techniques covered so far can be used for these problems but it becomes awkward to do so. A much more powerful method is to use Laplace transforms. We'll start by defining the Laplace transform, then show how to solve (linear) ODEs with it, then consider discontinuous and impulsive forcing.

§5.1 Properties of the Laplace transform

Let $f(t)$ be any function defined for $t \geq 0$. Then, the Laplace transform of $f(t)$ is another function $F(s)$ given by $F(s) = \mathcal{L}[f(t)] \equiv \int^{\infty} e^{-st} f(t)\, dt$.

provided that the integral converges.

<u>Example</u>

Let $f(t) = 1$. Then the Laplace transform of $f(t)$ is the function

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt = \int_0^\infty e^{-st}\, dt$$

$$= -\frac{e^{-st}}{s}\Big|_0^\infty$$

$$= \frac{1}{s} \quad \text{provided } s > 0.$$

<u>Example</u>

Let $f(t) = t^{\cancel{n}}$ ~~with n an integer~~. Then,

$$F(s) = \int_0^\infty e^{-st} t^{\cancel{n}}\, dt = -\frac{te^{-st}}{s}\Big|_0^\infty - \int_0^\infty -\frac{e^{-st}}{s}\, dt$$

$$= 0 + \int_0^\infty \frac{e^{-st}}{s}\, dt \quad \text{provided } s > 0$$

$$= -\frac{e^{-st}}{s^2}\Big|_0^\infty = \frac{1}{s^2} \quad \text{provided } s > 0.$$

<u>Example</u>

Let $f(t) = t^n$ with n an integer. Then, after integrating by parts n times, we find

$$F(s) = \frac{n!}{s^{n+1}} \quad \text{provided } s > 0.$$

Example

Let $f(t) = e^{at}$. Then, we have

$$F(s) = \int_0^\infty e^{-st} e^{at} = \int_0^\infty e^{(a-s)t} dt$$

$$= \left.\frac{e^{(a-s)t}}{a-s}\right|_0^\infty = \frac{1}{s-a} \quad \text{provided } s>a.$$

Example

Let $f(t) = \sin \omega t$. Then, we have

$$F(s) = \int_0^\infty e^{-st} \sin \omega t \, dt$$

$$= \left.\frac{e^{-st}\sin \omega t}{-s}\right|_0^\infty + \int_0^\infty \frac{\omega e^{-st}}{s} \cos \omega t \, dt$$

$$= 0 + \left. -\frac{\omega e^{-st}}{s^2} \cos \omega t \right|_0^\infty - \int_0^\infty \frac{\omega^2}{s^2} e^{-st} \sin \omega t \, dt$$

$$= \frac{\omega}{s^2} - \frac{\omega^2}{s^2} F(s), \quad \text{provided } s>0.$$

So, $F(s)\left(1 + \frac{\omega^2}{s^2}\right) = F(s)\left(\frac{s^2+\omega^2}{s^2}\right) = \frac{\omega}{s^2}$

$$\Rightarrow F(s) = \frac{\omega}{s^2+\omega^2}. \quad \text{provided } s>0.$$

Example

Let $f(t) = \cos \omega t$. Then $F(s) = \frac{s}{s^2+\omega^2}$ provided $s>$

Let $f(t) = \sinh at$. Then $F(s) = \frac{a}{s^2-\omega^2}$ provided $s>|a|$.

Let $f(t) = \cosh at$. Then $F(s) = \frac{s}{s^2-\omega^2}$ provided $s>|a|$.

Example

Let $f(t) = t\sin\omega t$. Then,

$$F(s) = \int_0^\infty e^{-st}\, t\sin\omega t\, dt$$

$$= \frac{e^{-st}\, t\sin\omega t}{-s}\Bigg|_0^\infty + \int_0^\infty \frac{e^{-st}}{s}\left[\sin\omega t + \omega t\cos\omega t\right]dt$$

$$= 0 + \frac{1}{s}\mathcal{L}[\sin\omega t] + \int_0^\infty \frac{e^{-st}}{s}\,\omega t\cos\omega t\, dt$$

provided $s>0$

$$= \frac{\omega}{s(s^2+\omega^2)} + \frac{e^{-st}}{-s^2}\omega t\cos\omega t\Bigg|_0^\infty + \int_0^\infty \frac{\omega e^{-st}}{s^2}\left[\cos\omega t - \omega t\sin\omega t\right] dt$$

$$= \frac{\omega}{s(s^2+\omega^2)} + 0 + \frac{\omega}{s^2}\mathcal{L}[\cos\omega t] - \frac{\omega^2}{s^2}F(s).$$

So, $$\left(1+\frac{\omega^2}{s^2}\right)F(s) = \left(\frac{s^2+\omega^2}{s^2}\right)F(s)$$

$$= \frac{\omega}{s(s^2+\omega^2)} + \frac{\omega}{s^2}\cdot\frac{s}{s^2+\omega^2} = \frac{2\omega}{s(s^2+\omega^2)}$$

$$\Rightarrow F(s) = \frac{2\omega s}{(s^2+\omega^2)^2}.$$ provided $s>0$

Example

Let $f(t) = t\cos\omega t$. Then $F(s) = \dfrac{s^2-a^2}{(s^2+\omega^2)^2}$ provided $s>0$

We can also calculate a number of general properties of the Laplace transform.

Example

Let $f(t) = ag(t) + bh(t)$. Then

$$F(s) = \int_0^\infty e^{-st}\left(ag(t) + b\,h(t)\right)dt$$

$$= a\int_0^\infty e^{-st} g(t)\,dt + b\int_0^\infty e^{-st} h(t)\,dt$$

$$= aG(s) + bH(s).$$

So, the Laplace transform is linear in the sense that for all functions $g(t)$ and $h(t)$ and all constants a,

$$\mathcal{L}[af + bg] = a\mathcal{L}[f] + b\mathcal{L}[g].$$

Example

Let $f(t) = \dot{y}(t)$ where $\dot{y}(t) = \frac{dy}{dt}$.

Then, $F(s) = \int_0^\infty e^{-st}\dot{y}\,dt = e^{-st}y\Big|_0^\infty + \int_0^\infty se^{-st}y\,dt$

$$= -y(0) + sY(s)$$

where $Y(s) = \mathcal{L}[y]$ is the Laplace transform of $y(t)$

Example

Let $f(t) = \ddot{y}(t)$ where $\ddot{y} = \frac{d^2y}{dt^2}$. Then,

$$F(s) = \int_0^\infty e^{-st}\ddot{y}\,dt = e^{-st}\dot{y}\Big|_0^\infty + s\int_0^\infty e^{-st}\dot{y}\,dt$$

$$= -\dot{y}(0) + s\mathcal{L}[\dot{y}] = -\dot{y}(0) + s^2Y(s) - sy(0).$$

These last ~~~~ examples allow us to solve linear 2nd-order ODEs with ICs at $t=0$!

§5.2 Solving linear ODEs with the Laplace transform

We will now use the Laplace transform to solve ~~equation~~ ODEs of the form
$$\ddot{y} + p\dot{y} + qy = f(t) \quad \text{with} \quad y(0)=a,\ \dot{y}(0)=b.$$
with $p, q = \text{const}$ and $f(t)$ one of the previous functions.

To do this, we take the Laplace transform of both sides of the ODE:
$$\mathcal{L}[\ddot{y} + p\dot{y} + qy] = \mathcal{L}[f] = F(s).$$

But, $\mathcal{L}$ is linear and so

$$\mathcal{L}[\ddot{y} + p\dot{y} + qy] = \mathcal{L}[\ddot{y}] + p\,\mathcal{L}[\dot{y}] + q\,\mathcal{L}[y]$$
$$= s^2 Y(s) - \dot{y}(0) - s\,y(0) + p\left(sY(s) - y(0)\right) + qY(s)$$
$$= (s^2 + ps + q)Y(s) - \dot{y}(0) - y(0)(s+p)$$
$$= (s^2 + ps + q)Y(s) - b - a(s+p). = F(s)$$

So,
$$Y(s) = \frac{F(s) + a(s+p) + b}{s^2 + ps + q}.$$

So if $\frac{F(s) + a(s+p) + b}{s^2+ps+q}$ is the transform of a function we

$y(t)$ is this function.

<u>Example</u>

Let $\ddot{y} - 3\dot{y} + 2y = e^{3t}$, $y(0) = 1$, $\dot{y}(0) = 0$.

We know from §3.3 that the soln is:

$$y(t) = +\tfrac{5}{2}e^{t} - 2e^{2t} + \tfrac{1}{2}e^{3t}.$$

Let's obtain the same result using Laplace transforms.

We have $\mathcal{L}[\ddot{y}] - 3\mathcal{L}[\dot{y}] + 2\mathcal{L}[y] = \mathcal{L}[e^{3t}]$

or, $s^2 Y - \dot{y}(0) - s\,y(0) - 3(sY - y(0)) + 2Y = \dfrac{1}{s-3}$

$$\Rightarrow (s^2 - 3s + 2)Y(s) - 0 - s + 3 = \frac{1}{s-3}$$

$$\Rightarrow (s^2 - 3s + 2)Y(s) = \frac{1}{s-3} + s - 3.$$

$$\Rightarrow Y(s) = \frac{1}{(s-3)(s^2-3s+2)} + \frac{s-3}{(s^2-3s+2)}$$

$$= \frac{1}{(s-3)(s-2)(s-1)} + \frac{s-3}{(s-2)(s-1)}$$

Now, $\dfrac{1}{(s-3)(s-2)(s-1)} = \dfrac{A}{s-3} + \dfrac{B}{s-2} + \dfrac{C}{s-1}$

$$= \frac{A(s-2)(s-1) + B(s-3)(s-1) + C(s-3)(s-2)}{(s-3)(s-2)(s-1)}$$

So $1 = s^2(A+B+C) - s(3A+4B+5C) + 2A+3B+6C$

Equating coeffs. s^n:

Coeff s^2: $A+B+C=0$ ①

Coeff s^1: $3A+4B+5C=0$ ②

Coeff. s^0: $2A+3B+6C=1$ ③.

① $\Rightarrow A=-B-C$ so ② $\Rightarrow -3B-3C+4B+5C=0$

$\Rightarrow B+2C=0 \Rightarrow B=-2C.$

So ③ $\Rightarrow -2B-2C+3B+6C=1$

$\Rightarrow 4C-2C-6C+6C=1$

$\Rightarrow C=\frac{1}{2} \Rightarrow B=-1 \Rightarrow A=\frac{1}{2}.$

So $\frac{1}{(s-3)(s-2)(s-1)} = \frac{1}{2}\cdot\frac{1}{s-3} - \frac{1}{s-2} + \frac{1}{2}\cdot\frac{1}{s-1}.$

Next, $\frac{s-3}{(s-2)(s-1)} = \frac{D}{s-2} + \frac{E}{s-1}$

$= \frac{D(s-1)+E(s-2)}{(s-2)(s-1)}$

So $s-3 = s(D+E) - D - 2E.$

So Coeff: s^1: $D+E=1$ ④

Coeff: s^0: $-D-2E=-3$ ⑤.

Then ④ $\Rightarrow D=1-E$, so ⑤ $\Rightarrow -1+E-2E=-3$

$\Rightarrow E=2 \Rightarrow D=-1$

So $\frac{s-3}{(s-2)(s-1)} = -\frac{1}{s-2} + \frac{2}{s-1}.$

Hence,

$$Y(s) = \frac{1}{(s-3)(s-2)(s-1)} + \frac{s-3}{(s-2)(s-1)}$$

$$= \frac{1}{2}\cdot\frac{1}{s-3} - 2\cdot\frac{1}{s-2} + \frac{5}{2}\cdot\frac{1}{s-1}.$$

$$= \frac{1}{2}\mathcal{L}[e^{3t}] - 2\,\mathcal{L}[e^{2t}] + \frac{5}{2}\mathcal{L}[e^{t}]$$

$$= \mathcal{L}\left[\frac{1}{2}e^{3t} - 2e^{2t} + \frac{5}{2}e^{t}\right]$$

$$\Rightarrow y(t) = \frac{1}{2}e^{3t} - 2e^{2t} + \frac{5}{2}e^{t}.$$ The same answer!

This is quite long and messy, but it illustrates the method. When we use step-functions and impulses, this method is far easier than those of previous sections.

We also have two "shifting theorems" that can help with solving ODEs:

<u>Example</u> (First shifting theorem).

Let $f(t) = e^{at}g(t)$ and let $G(s) = \mathcal{L}[g(t)]$.

Then, $F(s) = \mathcal{L}[f(t)] = \int_0^\infty e^{-st}\, e^{at}\, g(t)\,dt$

$$= \int_0^\infty e^{-(s-a)t}\, g(t)\,dt$$

$$= G(s-a).$$

//

Example

Let $f(t) = e^{at} \sin \omega t$. Then

$F(s) = \mathcal{L}[f(t)] = G(s-a)$ where $G(s) = \mathcal{L}[\sin \omega t]$.

But, $\mathcal{L}[\sin \omega t] = \dfrac{\omega}{s^2+\omega^2}$.

So, $F(s) = \dfrac{\omega}{(s-a)^2+\omega^2}$.

Example

Solve $\ddot{y} + 2\dot{y} + 5y = 0$ with $y(0) = 1$, $\dot{y}(0) = -2$.

Take Laplace transform of both sides:

$$s^2 Y(s) - \dot{y}(0) - s\,y(0) + 2sY(s) - 2y(0) + 5Y(s) = 0$$

$$\Rightarrow (s^2+2s+5)\,Y(s) = +\dot{y}(0) + s\,y(0) - 2y(0) = s$$

$$\Rightarrow Y(s) = \frac{s}{s^2+2s+5} = \frac{s}{(s+1)^2+4} = \frac{s+1}{(s+1)^2+4} - \frac{1}{(s+1)^2+4}$$

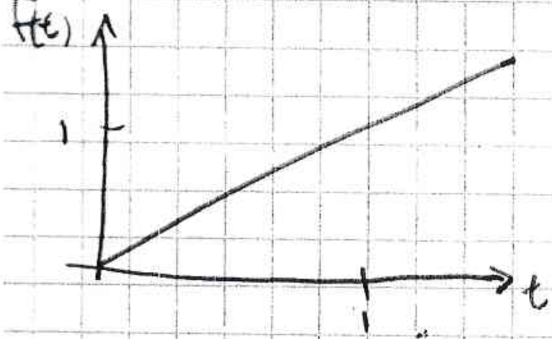
$$= \mathcal{L}[e^{-t}\cos 2t] - \tfrac{1}{2}\mathcal{L}[e^{-t}\sin 2t]$$

$$\Rightarrow y(t) = e^{-t}\cos 2t - \tfrac{1}{2}e^{-t}\sin 2t.$$

Before we show the second shifting theorem, we need to introduce step-functions and discontinuous forcing.

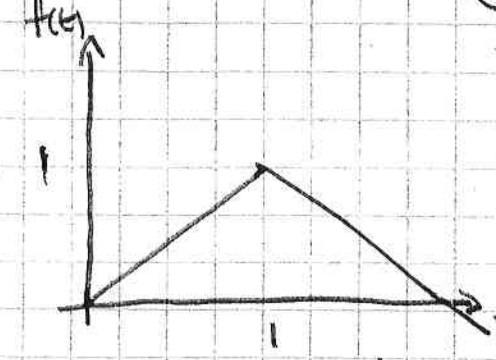
§ 5.3 Step functions and discontinuous forcing.

In the ODE $\ddot{y} + p\dot{y} + qy = f(t)$, often $f(t)$ is discontinuous. Eg if $y = Q$, the charge in a circuit and $f = V$, the applied voltage. We could have many different $f(t)$:



$f(t) = t$

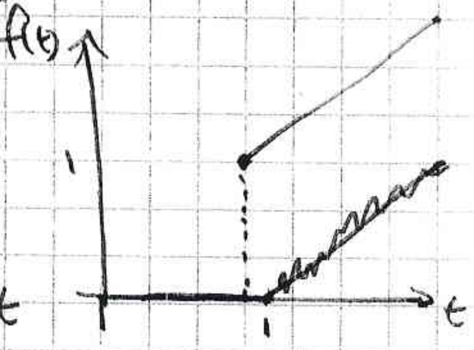
f continuous



$$f(t) = \begin{cases} t & t < 1 \\ 2-t & t > 1 \end{cases}$$

f continuous

$\dot{f}$ discontinuous

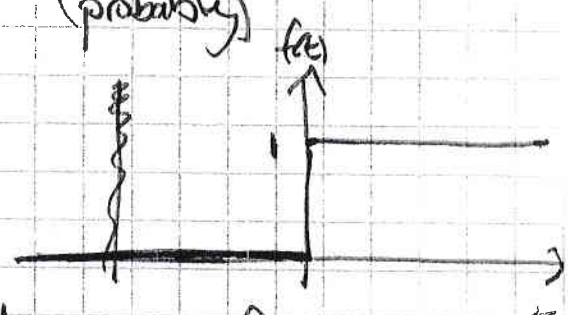


$$f(t) = \begin{cases} 0 & t < 1 \\ t & t > 1 \end{cases}$$

f discontinuous

The simplest discontinuous function is (probably)

$$f(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$$

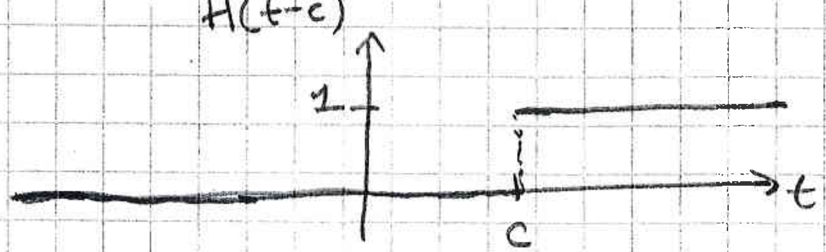


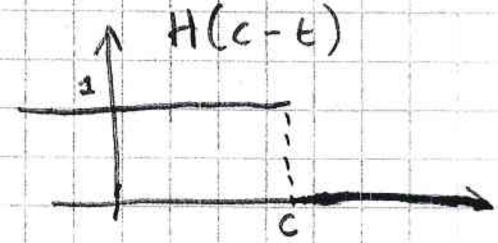
We call this function the Heaviside step function and give it a special symbol:

$$H(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0. \end{cases}$$

We can also write down a unit jump at $t=c$:

$$H(t-c) = \begin{cases} 0 & t-c<0 \\ 1 & t-c>0 \end{cases} = \begin{cases} 0 & t<c \\ 1 & t>c \end{cases}$$

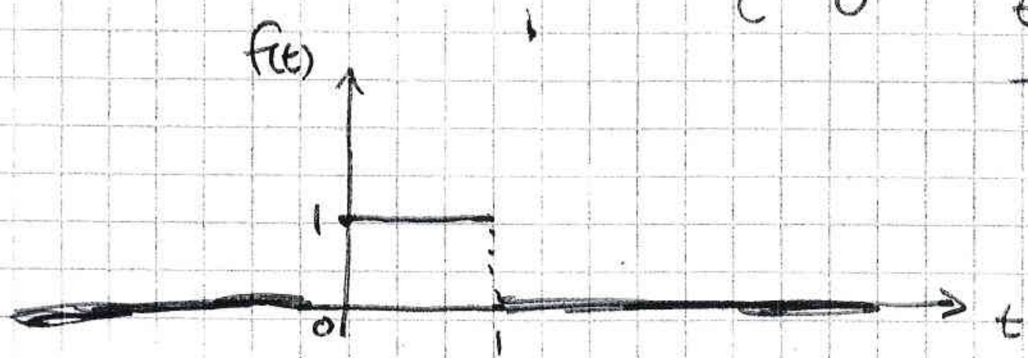




We can add step functions together:

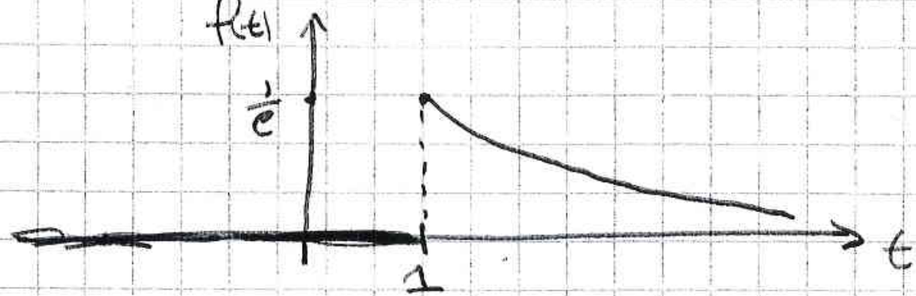
$$f(t) = H(t) + H(t-1) = \begin{cases} 0+0 & t<0 \\ 1+0 & 0<t<1 \\ 1+1 & t>1 \end{cases} = \begin{cases} 0 & t \\ 1 & 0< \\ 2 & t \end{cases}$$

$$f(t) = H(t) - H(t-1) = \begin{cases} 0 & t<0 \\ 1 & 0<t<1 \\ 0 & t>1 \end{cases}$$



We can turn functions on/off using $H(t)$.

$$f(t) = H(t-1)e^{-t} = \begin{cases} 0\times e^{-t} & t<1 \\ 1\times e^{-t} & t>1 \end{cases} = \begin{cases} 0 \\ e^{-t} \end{cases}$$



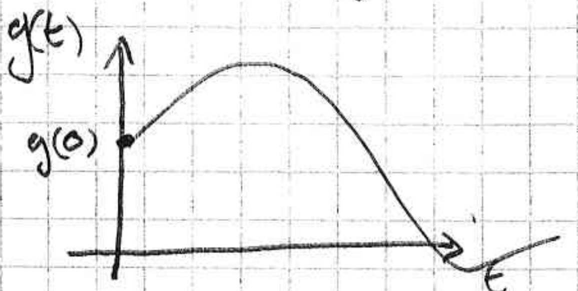
Example

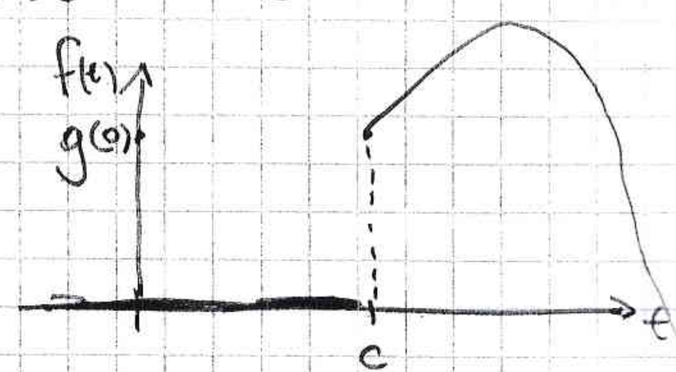
At the begin of this chapter we wrote down:

$$f(t) = \begin{cases} 0 & t<1 \\ t & t>1 \end{cases} = H(t-1)\,t.$$

We can also shift functions with $H(t)$:

$$f(t) = \begin{cases} 0 & t<c \\ g(t-c) & t>c \end{cases} = g(t-c)H(t-c)$$





Example

$$f(t) = \begin{cases} t & t<1 \\ 2-t & t>1 \end{cases} = \begin{cases} t & t<1 \\ 0 & t>1 \end{cases} + \begin{cases} 0 & t \\ 2-t & t \end{cases}$$

$$= t\,H(1-t) + (2-t)H(t-1).$$

We can take the Laplace transform of $H(t-c)$

$$\mathcal{L}[H(t-c)] = \int_0^\infty H(t-c) e^{-st}\, dt$$

$$= \int_c^\infty e^{-st}\, dt$$

$$= \left. \frac{-e^{-st}}{s} \right|_c^\infty$$

$$= \frac{e^{-cs}}{s} \quad \text{provided } s > 0.$$

We also have the <u>second shifting theorem</u>.

<u>Example</u>

Let $f(t) = H(t-c)\, g(t-c)$.

Then $F(s) = \int_0^\infty H(t-c)\, g(t-c)\, e^{-st}\, dt$

$$= \int_c^\infty g(t-c)\, e^{-st}\, dt \qquad u = t-c$$

$$= \int_0^\infty g(u)\, e^{-s(u+c)}\, du$$

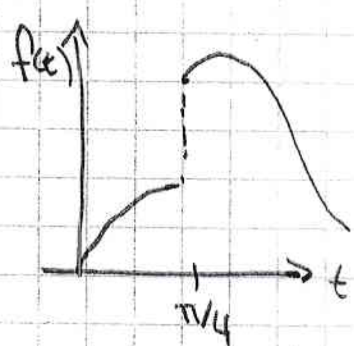
$$= e^{-sc} \int_0^\infty g(u) e^{-su}\, du$$

$$= e^{-sc}\, G(s)$$

where $G(s) = \mathcal{L}[g]$.

Example

Let $f(t) = \begin{cases} \sin t & t < \pi/4 \\ \sin t + \cos(t - \pi/4) & t > \pi/4 \end{cases}$



$$= \sin t + H(t - \pi/4)\cos(t - \pi/4)$$

So, $F(s) = \mathcal{L}[f] = \mathcal{L}[\sin t] + \mathcal{L}[H(t-\pi/4)\cos(t-\pi/4)]$

$$= \frac{1}{s^2+1} + e^{-\pi/4 s}\mathcal{L}[\cos t]$$

$$= \frac{1}{s^2+1} + e^{-\pi s/4}\frac{s}{s^2+1}$$

$$= \frac{1 + se^{-\frac{\pi s}{4}}}{s^2+1}.$$

Example

Let $F(s) = \mathcal{L}[f] = \dfrac{1 - e^{-2s}}{s^2}$.

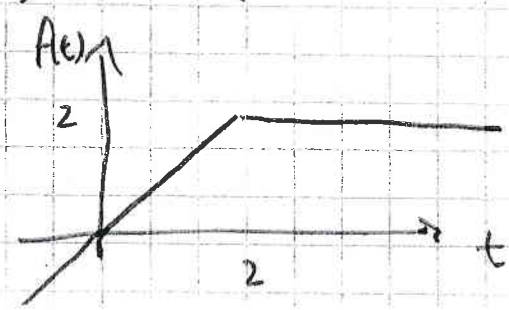
$$= \frac{1}{s^2} - e^{-2s}\cdot\frac{1}{s^2}$$

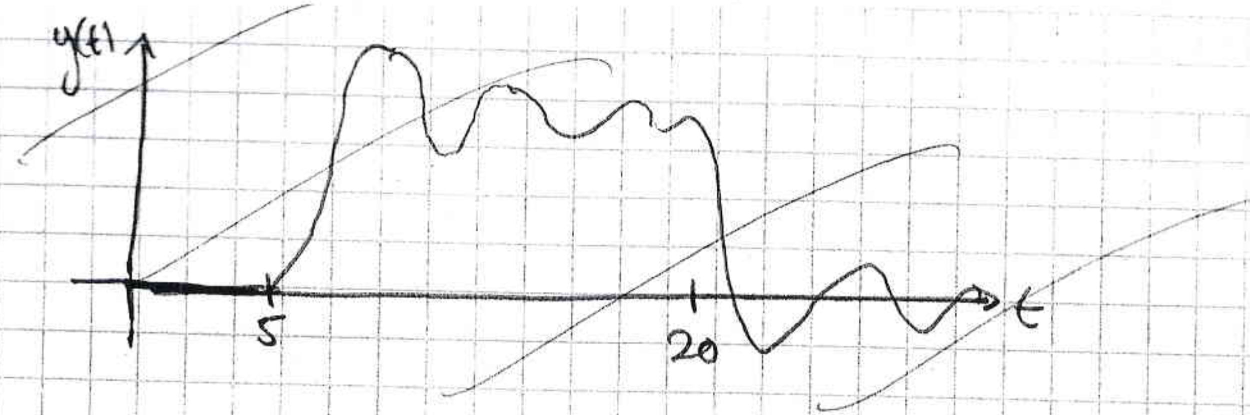
$$= \mathcal{L}[t] - e^{-2s}\mathcal{L}[t]$$

$$= \mathcal{L}[t] - \mathcal{L}[H(t-2)(t-2)]$$

So $f(t) = t - H(t-2)(t-2)$

$$= \begin{cases} t & t < 2 \\ 2 & t > 2 \end{cases}$$

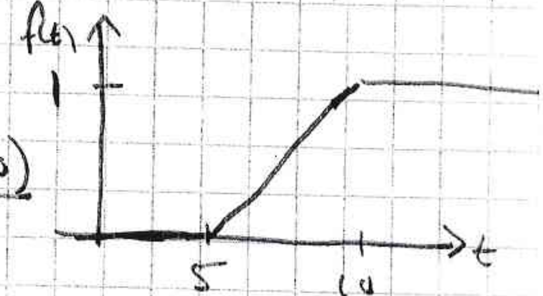




Example

Solve $\ddot{y} + 4y = f(t) = \begin{cases} 0 & 0 \le t < 5 \\ (t-5)/5 & 5 \le t < 10 \\ 1 & t > 10 \end{cases}$ Ramp f

with $y(0) = 0$, $\dot{y}(0) = 0$.



Then $f(t) = \dfrac{H(t-5)(t-5) - H(t-10)(t-10)}{5}$.

So $F(s) = \mathcal{L}[f] = \dfrac{e^{-5s}}{5s^2} - \dfrac{e^{-10s}}{5s^2}$

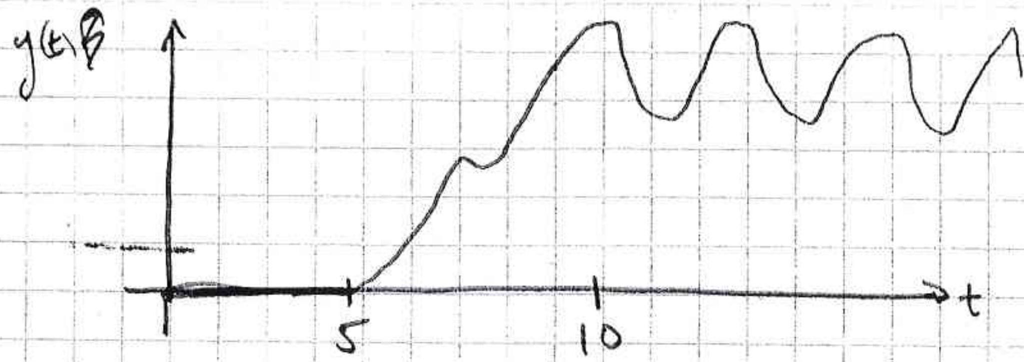
So $(s^2+4)Y(s) = \dfrac{(e^{-5s} - e^{-10s})}{5s^2}$

$$\Rightarrow Y(s) = \frac{e^{-5s} - e^{-10s}}{5s^2(s^2+4)}$$

$$= (e^{-5s} - e^{-10s})\left[\frac{1}{4}\cdot\frac{1}{s^2} - \frac{1}{4}\cdot\frac{1}{s^2+4}\right]$$

$$= (e^{-5s} - e^{-10s})\left(\frac{1}{4}\mathcal{L}[t] - \frac{1}{8}\mathcal{L}[\sin 2t]\right)$$

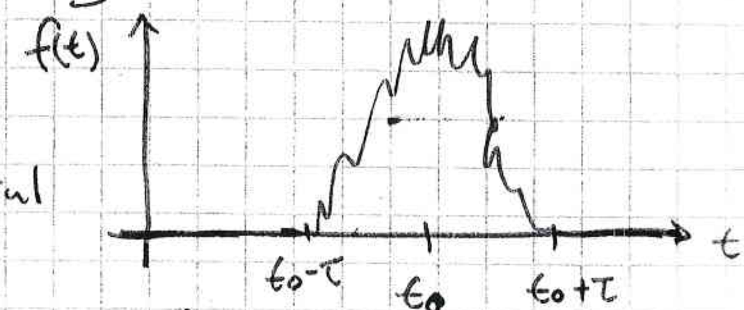
$$\Rightarrow y(t) = \frac{[H(t-5)(t-5) - H(t-10)]}{4} - \frac{[H(t-5) - H(t-10)]\sin 2(t-10)}{8}$$



Settles into steady oscillation.

§ 5.4 Impulses

Often, forcing occurs briefly over a short time interval and are "impulsive".



The impulse over the interval

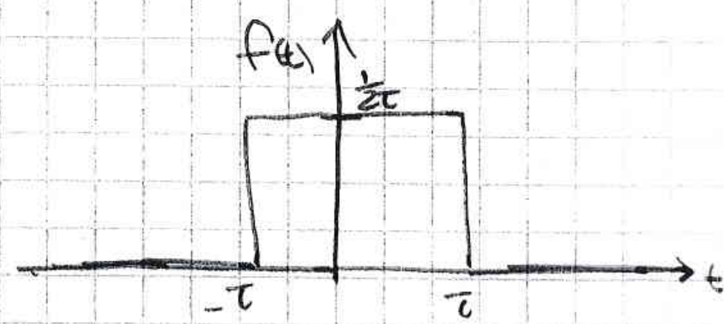
$$t_0 - \tau < t < t_0 + \tau$$

of a force $f(t)$ is $I = \int_{t_0-\tau}^{t_0+\tau} f(t)\,dt = \int_{-\infty}^{\infty} f(t)$

since $f = 0$ outside interval

Let's consider

$$f(t) = \begin{cases} \frac{1}{2\tau} & -\tau < t < \tau \\ 0 & t < -\tau \text{ or } t > \tau \end{cases}$$

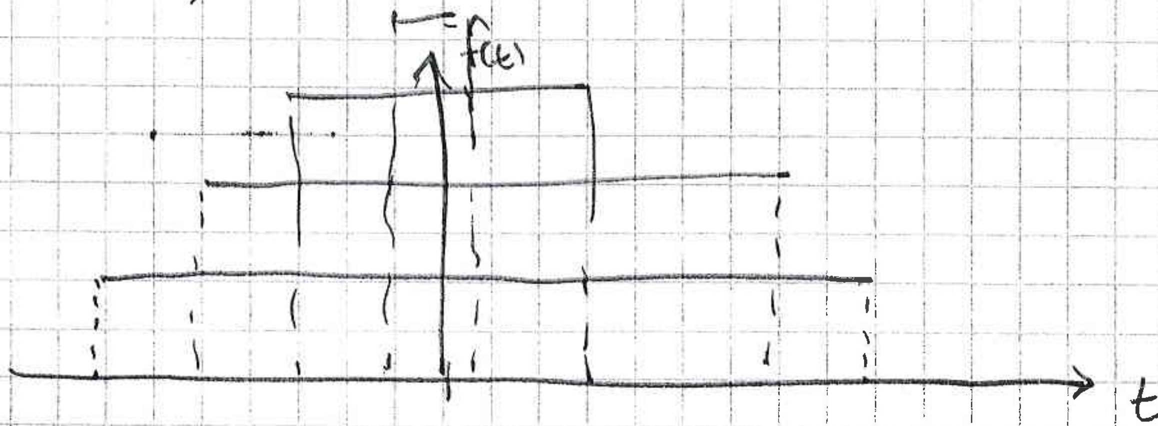


Then $I = \int_{-\infty}^{\infty} f(t)\,dt$

$$= \int_{-\tau}^{\tau} \frac{1}{2\tau}\,dt$$

$$= \left.\frac{t}{2\tau}\right|_{-\tau}^{\tau} = 1.$$

Now suppose time interval 2τ gets smaller and smaller, so $\tau \to 0$. Then $f(t)$ looks like:



As $\tau \to 0$, $f(t) \to 0$ everywhere <u>except</u> $t = 0$.

<u>But</u> $I = \int_{-\infty}^{\infty} f(t)\,dt = 1 \to 1$ as $\tau \to 0$.

It looks like $f(t) \to \begin{cases} 0 & t \neq 0 \\ \text{"}\infty\text{"} & t = 0 \end{cases}$

and $\int_{-\infty}^{\infty} f(t)\,dt = 1$.

This is the <u>idealized impulse</u> and we call this function the <u>Dirac delta function</u>, and give it a <u>special symbol</u> δ

so $\delta(t) = 0 \qquad t \neq 0$

$$\int_{-\infty}^{\infty} \delta(t)\,dt = 1.$$

The Dirac delta function has several properties:

<u>Example</u>

$$\int_{-\infty}^{\infty} \delta(t-c)\,dt \overset{u = t-c}{=} \int_{-\infty}^{\infty} \delta(u)\,du = 1.$$

Example

$$\int_1^\infty \delta(t)\,dt = \int_1^\infty 0\,dt = 0.$$

Example

$$\int_{-\infty}^{\infty} f(t)\,\delta(t)\,dt = \int_{-\infty}^{\infty} f(0)\,\delta(t)\,dt = f(0)\cdot 1$$
$$= f(0)$$

Example

$$\int_{-\infty}^{\infty} f(t)\,\delta(t-c)\,dt = \int_{-\infty}^{\infty} f(c)\,\delta(t-c)\,dt = f($$

The δ-function "picks out" ~~the~~ values of $f(t)$.

We can take the Laplace transform of the δ-function

Example Let $c>0$

$$\mathcal{L}[\delta(t-c)] = \int_0^\infty \delta(t-c)\,e^{-st}\,dt$$
$$= \int_0^\infty \delta(t-c)\,e^{-sc}\,dt$$
$$= e^{-sc}.$$

Example

$$\mathcal{L}[\delta(t)] = \lim_{c\to 0} \mathcal{L}[\delta(t-c)] = \lim_{c\to 0} e^{-sc} = \underline{1}.$$

We can solve ODES with δ- function forcing:

Example

Solve $\ddot{y} - 3\dot{y} + 2y = \delta(t-5)$ with $y(0) = \dot{y}(0) = 0$.

Then, $s^2 Y(s) - \dot{y}(0) - s y(0) - 3(sY(s) - y(0)) + 2Y = \mathcal{L}[\delta(t-5)]$

$$\Rightarrow (s^2 - 3s + 2) Y(s) = e^{-5s}$$

$$\Rightarrow Y(s) = \frac{e^{-5s}}{(s-1)(s-2)} = e^{-5s}\left[\frac{1}{s-2} - \frac{1}{s-1}\right]$$

$$= e^{-5s} \mathcal{L}\left[e^{2t} - e^{t}\right]$$

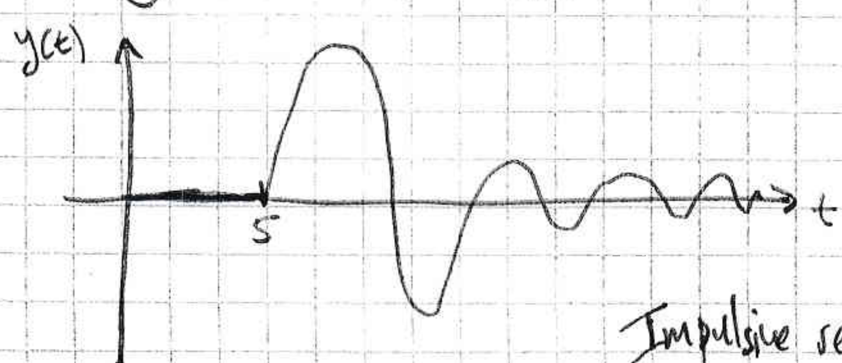
$$= \mathcal{L}\left[H(t-5)\left(e^{2(t-5)} - e^{(t-5)}\right)\right]$$

So $y(t) = H(t-5)\left(e^{2(t-5)} - e^{(t-5)}\right)$

Example

Solve $2\ddot{y} + \dot{y} + 2y = \delta(t-5)$, $y(0) = \dot{y}(0) = 0$

Then $y(t) = H(t-5) \cdot \frac{2}{\sqrt{15}} e^{-(t-5)/4} \sin \frac{\sqrt{15}}{4}(t-5)$



Impulsive response followed by relaxation.

Before finishing Laplace transforms, we show ~~to~~ to solve the ODE for <u>any</u> forcing $f(t)$ (in principle).

§5.5 Convolutions

Consider $F(s) = \mathcal{L}[f(t)]$ and $G(s) = \mathcal{L}[g(t)]$.

Let $H(s) = F(s)G(s)$. Can we find $h(t)$ so that $H(s) = \mathcal{L}[h(t)]$? Yes!

~~We have~~

$$H(s) = \int_0^\infty f(t) e^{-st}\,dt \cdot \int_0^\infty g(p) e^{-sp}\,dp$$

$$= \int_0^\infty\int_0^\infty f(t) e^{-st} g(p) e^{-sp}\,dp\,dt$$

$$= \int_0^\infty \int_0^\infty f(t)\, g(p)\, e^{-s(t+p)}\,dp\,dt, \quad u = t+p$$

$$= \int_0^\infty f(t) \int_t^\infty g(\tau - t) e^{-s\tau}\,d\tau\,dt$$

$$= \int_0^\infty \int_0^t f(t)\, g(\tau - t)\, e^{-st}\,d\tau\,dt$$

$$= \int_0^\infty e^{-st}\left[\int_0^t f(t)\, g(\tau - t)\,d\tau\right] dt$$

$$= \mathcal{L}\left[\int_0^t f(t)\, g(\tau - t)\,d\tau\right]$$

$$= \mathcal{L}[$$

If $h(t) = \int_0^t f(t-\tau)g(\tau)\,d\tau = \int_0^t f(\tau)g(t-\tau)\,d\tau$

Then $\mathcal{L}[h(t)] = F(s)G(s)$.

These integrals are useful all over applied math, and so we call them a special name. The <u>convolution</u> of $f(t)$ and $g(t)$ is <u>another function</u> $h(t)$ written

$$h(t) = (f*g)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau.$$

<u>Example</u>

Let $f(t) = \sin\omega t$ and $g(t) = t$. Then,

$$(f*g)(t) = \int_0^t f(\tau)\, g(t-\tau)\,d\tau$$

$$= \int_0^t \sin\omega\tau \cdot (t-\tau)\,d\tau$$

$$= \frac{\omega t - \sin\omega t}{\omega^2}.$$

The convolution has some nice properties.

$$f*g = g*f$$

$$f*(g+h) = f*g + f*h$$

$$f*(g*h) = (f*g)*h$$

$$f*0 = 0*f = 0.$$

However: $f*1 \neq f$ and $f*f$ can be $\gtrless 0$.

§ 6. Fourier Series

Functions are often *periodic* (in space or time). This means that for all x, we have

$$f(x+T) = f(x)$$

where f is periodic with *period* T. If f has period T, it also has period $2T$, $3T$, etc. We usually call the *smallest such T* the *fundamental period of f*.

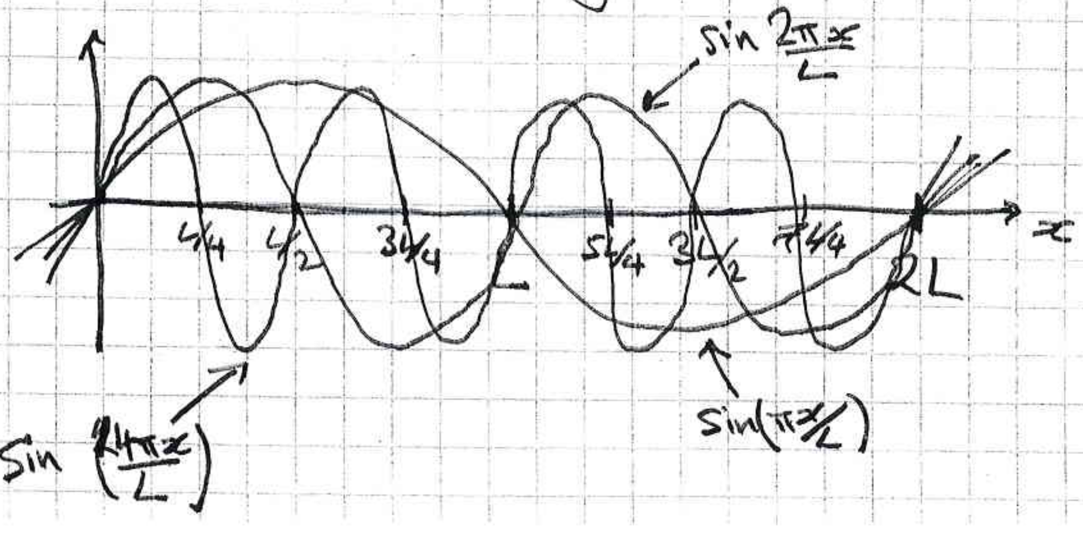
~~Example~~

§ 6.1 Properties of sine and cosine

~~$f(x) = \cos\left(\frac{n\pi x}{L}\right)$ with n an integer has fundamental period $T = \frac{2L}{n}$.~~

Example

Both $f(x) = \cos\left(\frac{n\pi x}{L}\right)$ and $f(x) = \sin\left(\frac{n\pi x}{L}\right)$ ~~have~~ where n is an integer (1, 2, 3, etc) have *fundament period* $\frac{2L}{n}$.
All of these functions (every n) have period $2L$.



The functions $\sin\frac{n\pi x}{L}$ and $\cos\frac{n\pi x}{L}$ are important for representing any periodic function of period $2L$. This is due to orthogonality.

Recall that two vectors $\underline{a}$, $\underline{b}$ are orthogonal if their inner-product $\langle \underline{a}, \underline{b}\rangle \equiv \underline{a}\cdot\underline{b} = 0$. We can define an inner-product for functions $f(x)$, $g(x)$, $-L<x<L$ by

$$\langle f,g\rangle = \int_{-L}^{L} f(x)g(x)\,dx .$$

Example

The functions $f(x) = \sin\frac{n\pi x}{L}$ and $g(x) = \sin\frac{m\pi x}{L}$ are orthogonal, with n, m integers and $m \neq n$.

$$\langle f,g\rangle = \int_{-L}^{L} \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\,dx$$

$\sin a \sin b = \frac{1}{2}\left[\cos(a-b) - \cos(a+b)\right]$

$$= \frac{1}{2}\int_{-L}^{L}\left(\cos\frac{(m-n)\pi x}{L} - \cos\frac{(m+n)\pi x}{L}\right)dx$$

$$= \frac{1}{2}\frac{L}{\pi}\left[\frac{\sin (m-n)\pi x/L}{m-n} - \frac{\sin (m+n)\pi x/L}{m+n}\right]_{-L}^{L}$$

$$= 0$$

Example.

$\cos 2a = \cos^2 a - \sin^2 a$

$\sin^2 a + \cos^2 a = 1$

Let $f(x) = \sin n\pi x/L$ with n an integer.

Then $\langle f,f\rangle = \int_{-L}^{L} \sin^2 n\pi x/L\,dx = \frac{1}{2}\int_{-L}^{L}\left(1 - \cos\frac{2n\pi x}{L}\right)dx$

$= \frac{1}{2}\left[x - \frac{L}{2n\pi}\sin\frac{2n\pi x}{L}\right]_{-L}^{L}$

In general, we have; for integers m and n,

$$\left\langle \cos\frac{n\pi x}{L}, \cos\frac{m\pi x}{L} \right\rangle = \begin{cases} 0 & n \neq m \\ L & m = n \end{cases}$$

$$\left\langle \cos\frac{n\pi x}{L}, \sin\frac{m\pi x}{L} \right\rangle = 0 \quad \text{all } m, n$$

$$\left\langle \sin\frac{n\pi x}{L}, \sin\frac{m\pi x}{L} \right\rangle = \begin{cases} 0 & n \neq m \\ L & m = n. \end{cases}$$

Just as orthogonal vectors $(0,1)^T$ and $(1,0)^T$ form a basis for 2D vectors, it turns out that $\left\{ \cos\frac{n\pi x}{L}, \sin\frac{m\pi x}{L}, \ n, m = 0, 1, 2, \dots \right\}$ form a "basis" for periodic functions with period $2L$.

§ 6.2 Writing periodic functions as a Fourier series

Let $f(x)$ have period $2L$. Then, the Fourier series of $f(x)$ is the following sum:

$$f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \left(A_n \cos\frac{n\pi x}{L} + B_n \sin\frac{n\pi x}{L} \right) \quad (*)$$

for some constants $\{A_n, B_n\}$.

How are the A_n, B_n determined?

By using orthogonality.

To show this, we mult. $(*)$ by $\cos\frac{m\pi x}{L}$ and integrate