$$\int_{-L}^{L} f(x)\cos\frac{m\pi x}{L}\,dx = \cancel{\frac{A_0}{2}\int_{-L}^{L}\cos\frac{m\pi x}{L}\,dx} \to 0$$

$$+\sum_{n=1}^{\infty} A_n\int_{-L}^{L}\cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx$$

$$+\,\cancel{B_n\int_{-L}^{L}\cos\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\,dx} \to 0.$$

So, $$\int_{-L}^{L} f(x)\cos\frac{m\pi x}{L}\,dx = A_m\int_{-L}^{L}\cos^2\frac{m\pi x}{L}\,dx$$

$$= L A_m$$

$$\Rightarrow\quad A_m = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{m\pi x}{L}\,dx \qquad m = 1, 2, \ldots$$

Also, $$B_m = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{m\pi x}{L}\,dx \qquad m = 1, 2, \ldots$$

What about A_0? Just integrate ⊛ without multiplying

$$\int_{-L}^{L} f(x)\,dx = \frac{A_0}{2}\int_{-L}^{L} dx \;+\; 0$$

$$\Rightarrow\quad A_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx.$$

(so $\frac{A_0}{2}$ is the ~~average~~ average of $f(x)$).

So $$A_m = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{m\pi x}{L}\,dx \qquad m = \underline{\underline{0}}, 1, 2, \ldots$$

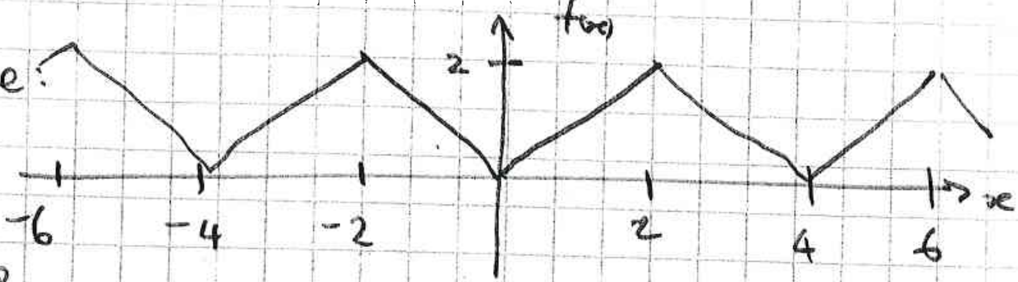
$$B_m = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{m\pi x}{L}\,dx, \qquad m = 1, 2, \ldots$$

Example

Let $f(x) = \begin{cases} -x & -2 \le x < 0 \\ x & 0 \le x < 2 \end{cases}$

with period 4 : $f(x+4) = f(x)$. So $L = 2$.

Triangular wave:



Then, $A_0 = \frac{1}{2}\int_{-2}^{2} f(x)\,dx = \frac{1}{2}\int_{-2}^{0}(-x)\,dx + \frac{1}{2}\int_0^2 x\,dx$

$= 1 + 1$

$= 2.$

For $m>0$, $A_m = \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{m\pi x}{2}\,dx$

$= \frac{1}{2}\int_{-2}^{0}(-x)\cos\frac{m\pi x}{2}\,dx + \frac{1}{2}\int_0^2 x\cos\frac{m\pi x}{2}\,dx$

$= \frac{4}{(m\pi)^2}\left(\cos m\pi - 1\right)$ (by parts)

$= \begin{cases} -8/(m\pi)^2 & m \text{ odd} \\ 0 & m \text{ even}. \end{cases}$

Next, $B_m = \frac{1}{2}\int_{-2}^{2} f(x)\sin\frac{m\pi x}{L}\,dx = 0$

since $f(x)\sin\frac{m\pi x}{L}$ is an odd function.

[$g(x)$ even $\Rightarrow g(-x) = g(x)$. $g(x)$ odd $\Rightarrow g(-x) = -g(x)$]

So, $f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{2} + B_n \sin\frac{n\pi x}{2}$

$$\Rightarrow \quad f(x) = 1 + \sum_{n \text{ odd}} \frac{-8}{(n\pi)^2} \cos\frac{n\pi x}{2}$$

$$= 1 - \frac{8}{\pi^2} \sum_{n \text{ odd}} \frac{\cos(n\pi x/2)}{n^2}$$

$$= 1 - \frac{8}{\pi^2} \sum_{n=1}^{\infty} \frac{\cos[(2n-1)\pi x/2]}{(2n-1)^2}.$$

Notice that in this example, $B_n = 0$ all n.

In general, if $f(x)$ is <u>even</u>, then $B_n = 0$ all n.

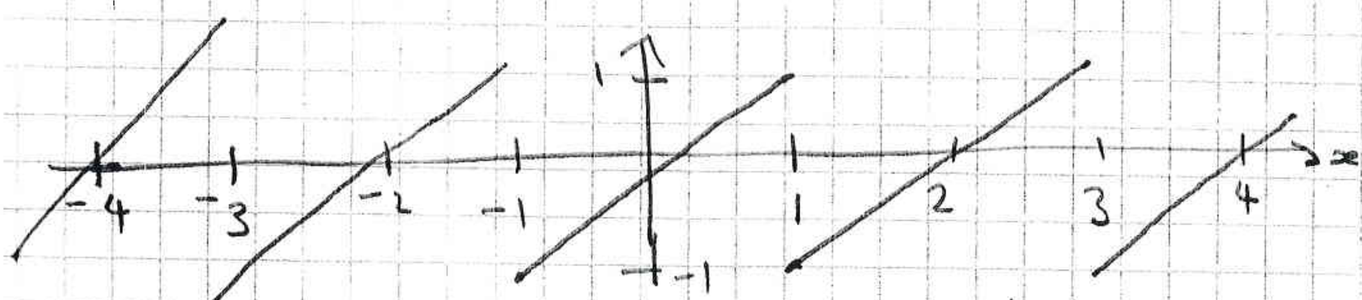
if $f(x)$ is <u>odd</u>, then $A_n = 0$ all n.

if $f(x)$ is <u>neither</u>, then, we <u>conclude nothing</u>.

<u>Example</u>

Let $f(x) = x$ for $-1 < x < 1$

and let $f(x+2) = f(x)$. So $L = 1$.



<u>Sawtooth wave</u>

Then, $f(x)$ is <u>odd</u>, so $A_n = 0$, $n = 0, 1, 2, \ldots$

But, $B_n = \int_{-1}^{1} f(x) \sin n\pi x \, dx = \int_{-1}^{1} x \sin n\pi x \, dx$

$$= \frac{-1}{n\pi} x \cos n\pi x \Big|_{-1}^{1} + \underbrace{\int_{-1}^{1} \frac{1}{n\pi} \sin n\pi x \, dx}_{0}$$

$$= \frac{-2}{n\pi}\cos n\pi$$

$$= \frac{-2}{n\pi}(-1)^n$$

So $f(x) = \sum_{n=1}^{\infty} \frac{-2}{n\pi}(-1)^n \sin n\pi x.$

We can simplify A_n, B_n for even / odd functions:

Let $g(x)$ be <u>even</u> then

$$\int_{-L}^{L} g(x)\,dx = \int_{-L}^{0} g(x)\,dx + \int_{0}^{L} g(x)\,dx$$

$u = -x$

$$= \int_{L}^{0} g(-u)(-du) + \int_0^L g(x)\,dx$$

$$= \int_0^L g(u)\,du + \int_0^L g(x)\,dx$$

$$= 2\int_0^L g(x)\,dx.$$

Hence, if $f(x)$ is <u>even</u>, then $B_n = 0$ <u>and</u>

$$A_n = \frac{2}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx$$

$$= \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$$

Also, if $f(x)$ is <u>odd</u>, then $A_n = 0$ <u>and</u>

$$B_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$$

$$= \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx.$$

We can now build periodic functions:

Let $f(x)$ be given on $0 < x < L$.

We can build an <u>even</u> periodic function from $f(x)$

by letting $f_c(-x) = f(x)$ and $f_c(x+2L) = f_c(x)$

~~odd an odd~~

then $f_c(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L}$

with $A_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$

<u>Cosine series</u>.

We can build an <u>odd</u> periodic function from $f(x)$ by letting $f_s(-x) = -f(x)$ and $f_s(x+2L) = f_s(x)$. Then,

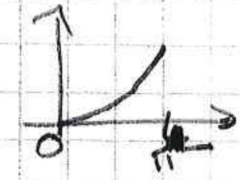
$f_s(x) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}$

with $B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$

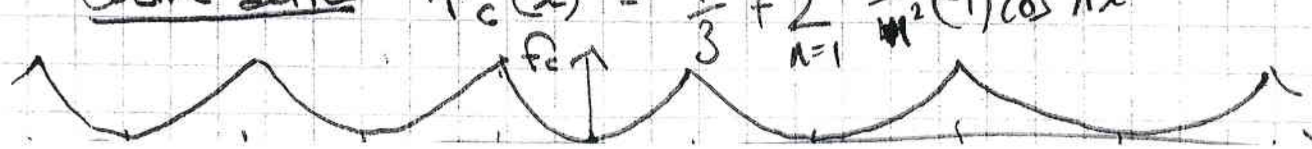
<u>sine series</u>.

<u>Example</u>

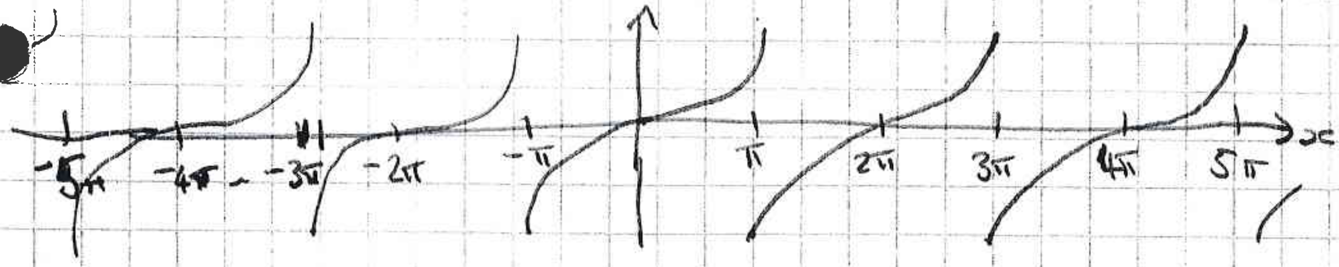
Let $f(x) = x^2$ for $0 < x < \pi$ (so $L = \pi$)



<u>Cosine series</u> $f_c(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^n \cos nx$



, Sine series $f_s(x) = \sum_{n=1}^{\infty} \left(\frac{2\pi}{n}(-1)^{n+1} + \frac{4}{\pi n^3}\left((-1)^n - 1\right) \right) \sin nx$



We will now show that Fourier series allow us to solve PDEs on simple domains.

§7. Separation of variables for PDEs

For a linear PDE ~~of the form~~ for the function $y(x,t)$ of the form

$$a_n \frac{\partial^{(n)} y}{\partial x^{(n)}} + a_{n-1} \frac{\partial^{(n-1)} y}{\partial x^{(n-1)}} + \dots + a_1 \frac{\partial y}{\partial x} + b_n \frac{\partial^{(n)} y}{\partial t^{(n)}} + \dots + b_1 \frac{\partial y}{\partial t} =$$

~~we can~~ with $a < x < b$ and $t > 0$

and IC $y(x,0) = y_0(x)$ and BCs $y(a,t) = y_a$, $y(b,t) = y_b$, (consts), we can find a soln by separation of variables.

The key idea is to look for a soln of the form

$$y(x,t) = X(x)\,T(t).$$

We will do this for the heat wave and Laplace equations

§7.1 Heat equation for a conducting rod with homogeneous boundary conditions

Consider a heat-conducting rod of length L:

The temperature inside the rod, $y(x,t)$ satisfies the linear, homogeneous PDE:

$$\frac{\partial y}{\partial t} = \chi \frac{\partial^2 y}{\partial x^2}$$

where $\chi = \frac{k}{\rho s}$ is the thermal diffusivity, k is the thermal conductivity, ρ is the density and s is the specific heat.

χ has units $\text{length}^2/\text{time}$.

We will consider the evolution of an initial temp. dist. $y(x,0) = f(x)$ subject to fixed temperature boundary conditions $y(0,t) = a$, $y(L,t) = b$.

If $a, b \neq 0$, then the problem is inhomogeneous.

We can solve $a, b \neq 0$ via the soln for $a = b = 0$, which we do

We look for a soln of the form

$$y(x,t) = X(x)T(t).$$

This is called <u>separation of variables</u>.

Then, $\dfrac{\partial y}{\partial t} = XT'$ and $\dfrac{\partial^2 y}{\partial x^2} = X''T$.

So, $XT' = \kappa X''T$

or, $\dfrac{1}{\kappa}\dfrac{T'}{T} = \dfrac{X''}{X}$.

Now, the LHS is a <u>function of t only</u> and the RHS is a <u>function of x only</u>.

The only way this can be correct is if they are both constants, i.e.

$$\frac{1}{\kappa}\frac{T'}{T} = -\lambda \quad \underline{\text{and}} \quad \frac{X''}{X} = -\lambda$$

or, $T' = -\kappa\lambda T$ <u>and</u> $X'' = -\lambda X$.

Hence, we reduce the PDE with 2 variables to 2 ODEs.

The solution to $T' = -\kappa\lambda T$ is $T(t) = Ce^{-\kappa\lambda t}$

some C = const.

We also need to solve $X'' = -\lambda X$ with 2 BCs.

We have $y(0,t) = y(L,t) = 0$, so $X(0) = X(L) = 0$.

The general soln to $X'' = -\lambda X$ is one of:

$\underline{\lambda < 0}$ $\quad X = Ae^{\sqrt{-\lambda}x} + Be^{-\sqrt{-\lambda}x}$.

Then $X(0) = 0 \Rightarrow A + B = 0 \Rightarrow A = -B$

$X(L) = 0 \Rightarrow Ae^{\sqrt{-\lambda}L} + Be^{-\sqrt{-\lambda}L} = 0$

$\Rightarrow B\left(e^{-\sqrt{-\lambda}L} - e^{\sqrt{-\lambda}L}\right) = 0$

$\Rightarrow B = 0 \Rightarrow A = 0 \Rightarrow X = 0$

$\underline{\lambda = 0}$

$X'' = 0 \Rightarrow X = Ax + B$.

Then $X(0) = 0 \Rightarrow B = 0$

$X(L) = 0 \Rightarrow AL = 0 \Rightarrow A = 0 \Rightarrow X = 0$.

$\underline{\lambda > 0}$ $\quad X = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x$

Then $X(0) = 0 \Rightarrow A = 0$

$X(L) = 0 \Rightarrow B\sin(\sqrt{\lambda}L) = 0$

So, <u>either</u> $B = 0$ (so $X = 0$ ..) <u>or</u> $\sqrt{\lambda}L = n\pi \Rightarrow \lambda =$ some integer n.

So write solution $X_n(x) = B_n \sin\left(\frac{n\pi x}{L}\right)$.

Then, we have ~~$y_n(x,t)$~~ $T \propto e^{-\kappa\lambda t}$

So $T_n \propto e^{-\kappa n^2\pi^2 t/L^2}$

We therefore have solns $y_n(x,t)$ given by

$$y_n(x,t) = X_n T_n = B_n \sin\frac{n\pi x}{L} e^{-\varkappa n^2\pi^2 t/L^2}, \quad n = 1, 2, \ldots$$

<u>but</u> none of these y_n will satisfy $y(x,0) = f(x)$ (in general).

<u>Observation</u>

The PDE is <u>linear</u> and <u>homogeneous</u> and has <u>homogeneous BCs</u>. So if y_p and y_q both solve the eqn, then so do $ay_p + by_q$ with $a, b = \text{const}$:

$$\frac{\partial}{\partial t}(ay_p + by_q) = b\frac{\partial y_q}{\partial t} + a\frac{\partial y_p}{\partial t} = \varkappa a \frac{\partial^2 y_p}{\partial x^2} + \varkappa b \frac{\partial^2}{\partial}$$

and $$\varkappa \frac{\partial^2}{\partial x^2}(ay_p + by_q) = \varkappa a \frac{\partial^2 y_p}{\partial x^2} + \varkappa b \frac{\partial^2 y_q}{\partial x^2}$$

also, $(ay_p + by_q)(0 \text{ or } L, t)$

$$= a\, y_p(0 \text{ or } L, t) + b y_q(0 \text{ or } L, t)$$
$$= 0 + 0 = 0.$$

<u>Idea</u>

We have infinitely many solns y_n. If we add any number of y_n together, we still have a solution.

Let's add <u>all solns</u> together, and <u>then</u> try to satisfy $y(x,0) = f(x)$.

We write our soln as:

$$y(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}\, e^{-\kappa n^2\pi^2 t/L^2}.$$

Then we need $f(x) = y(x,0)$, or:

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} \cdot 1$$

i.e. the B_n are the coeffs in the sine series of $f(x)$!

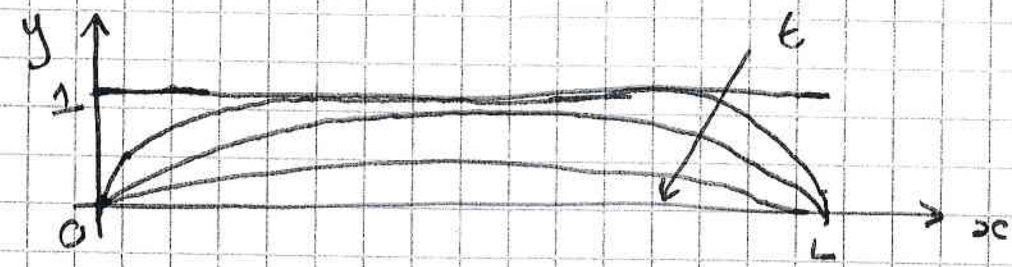
$$\text{So } B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx.$$

Example

Let $f(x) = 1$.

Then
$$B_n = \frac{2}{L}\int_0^L \sin\frac{n\pi x}{L}\,dx$$
$$= \frac{2}{L}\left(\frac{-L}{n\pi}\right)\cos\frac{n\pi x}{L}\Big|_0^L$$
$$= \frac{-2}{n\pi}\left(\cos n\pi - 1\right)$$
$$= \begin{cases} 0 & n \text{ even} \\ \frac{4}{n\pi} & n \text{ odd}. \end{cases}$$

$$\text{So } y(x,t) = \cancel{\frac{4}{\pi}\sum_{n=1}^{\infty}}\ \frac{4}{\pi}\sum_{n \text{ odd}} \frac{\sin(n\pi x/L)}{n}\, e^{-\kappa n^2\pi^2 t/L^2}.$$



As $t \to \infty$, each term in sum $\to 0$, so
$y(x,t) \to 0$ also.

This is true for any B_n:

$$y(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} e^{-\lambda n^2\pi^2 t/L^2}$$

$$\to 0 \quad \text{as } t \to \infty$$

for every $f(x)$ initial temp.

We'll now show how to solve the ~~equations~~ PDE when there are inhomogeneous BCs, i.e.

$$y(0,t) = a, \quad y(L,t) = b$$

not both zero.

§ 7.2 Heat equation for a conducting rod with inhomogeneous boundary conditions

Idea

As $t \to \infty$, we might imagine that a steady temp distribution is reached, i.e. $y(x,t) \to \bar{y}(x)$.

Yes:

As $t \to \infty$, soln becomes independent of time, so

$$\kappa \frac{\partial^2 \bar{y}}{\partial x^2} = \frac{\partial \bar{y}}{\partial t} = 0$$

or $$\frac{d^2 \bar{y}}{dx^2} = 0.$$

This gives $\bar{y} = Ax + B$.

Then, $\bar{y}(0) = a \Rightarrow B = a$.

and $\bar{y}(L) = b \Rightarrow AL + a = b \Rightarrow A = \frac{b-a}{L}$.

Hence, $$\bar{y} = \frac{b-a}{L} x + a.$$

Now, lets write the full, time-dependent soln as

$$y(x,t) = \bar{y}(x) + y_0(x,t).$$

where we expect $y_0(x,t) \to 0$ as $t \to \infty$.

First, ~~B~~ the BCs imply:

$$y(0,t) = \bar{y}(0) + y_0(0,t)$$

$$\Rightarrow \quad a = a + y_0(0,t)$$

$$\Rightarrow \quad y_0(0,t) = 0$$

Also, $$y_0(L,t) = 0$$

so $y_0(x,t)$ satisfies <u>homogeneous BCs</u>.

What PDE does $y_0(x,t)$ satisfy?

We have $\dfrac{\partial y}{\partial t} = \dfrac{\partial \bar{y}}{\partial t} + \dfrac{\partial y_0}{\partial t} = \dfrac{\partial y_0}{\partial t}$.

and $\kappa \dfrac{\partial^2 y}{\partial x^2} = \kappa \dfrac{\partial^2 \bar{y}}{\partial x^2} + \kappa \dfrac{\partial^2 y_0}{\partial x^2} = \kappa \dfrac{\partial^2 y_0}{\partial x^2}$.

Hence,

$$\frac{\partial y_0}{\partial t} = \kappa \frac{\partial^2 y_0}{\partial x^2}.$$

So $y_0(x,t)$ satisfies the same eqn and BCs as §7.1

$$\text{So} \quad y_0(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} e^{-\kappa n^2\pi^2 t/L^2}.$$

Hence $y(x,t) = \bar{y}(x) + y_0(x,t)$

$$\Rightarrow y(x,t) = \frac{b-a}{L}x + b + \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} e^{-\kappa n^2\pi^2 t/L^2}.$$

Finally, the IC is $y(x,0) = f(x)$, so

$$f(x) = \frac{b-a}{L}x + b + \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}$$

or, $g(x) \equiv f(x) - \frac{b-a}{L}x - b = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}$.

$$\text{So} \quad B_n = \frac{2}{L}\int_0^L g(x) \sin\frac{n\pi x}{L}\, dx.$$

NB/ Use $\underline{g(x)}$ not $f(x)$ because IC applies to y $\underline{\text{not}}$ y_0.

**

§7.3 Similarity solutions for the heat equation

Some PDEs admit a <u>similarity solution</u>, for which

$y(x,t) = y(s(x,t))$ so y depends on <u>one</u> special variable $s(x,y)$. ~~We~~

Consider $\dfrac{\partial y}{\partial t} = \kappa \dfrac{\partial^2 y}{\partial x^2}$.

Let $y = y(s)$ with $s = \dfrac{x}{2\sqrt{\kappa t}}$.

Then, the chain rule gives $\dfrac{\partial y}{\partial t} = \dfrac{\partial s}{\partial t}\dfrac{\partial y}{\partial s} = \dfrac{-s}{2t}\dfrac{\partial y}{\partial s}$

and $\dfrac{\partial^2 y}{\partial x^2} = \dfrac{1}{2\kappa t}\dfrac{\partial^2 y}{\partial s^2}$.

Hence, $-\dfrac{s}{2t}\dfrac{\partial y}{\partial s} = \kappa \cdot \dfrac{1}{\cancel{4}\kappa t}\dfrac{\partial^2 y}{\partial s^2}$

or, $\dfrac{\partial^2 y}{\partial s^2} = -2s\dfrac{\partial y}{\partial s}$. an <u>ODE</u>.

Let $g = \dfrac{\partial y}{\partial s}$. Then $\dfrac{\partial g}{\partial s} = -sg \Rightarrow g = Ae^{-s^2}$.

So $\dfrac{\partial y}{\partial s} = Ae^{-s^2} \Rightarrow y = A\int_0^s e^{-\tilde{s}^2}\, d\tilde{s} + B$.

This form is difficult to work with on $0 < x < L$

but we can consider $L \to \infty$ with this form, whereas §7.1 and §7.2 don't work if $L \to \infty$.

Example

Consider an infinitely long rod $x > 0$ with $y(x,0) = 0$ and $y(0,t) = 1$.

Then, $y = A\int_0^S e^{-\tilde{S}^2} d\tilde{S} + B$.

Then $y(0,t) = 1 \Rightarrow$ is at $x = 0$, so $S = \frac{x}{2\sqrt{\kappa t}} = 0$

$$\text{So } 1 = A\cancel{\int_0^0 e^{-\tilde{S}^2} d\tilde{S}}^{\,0} + B$$

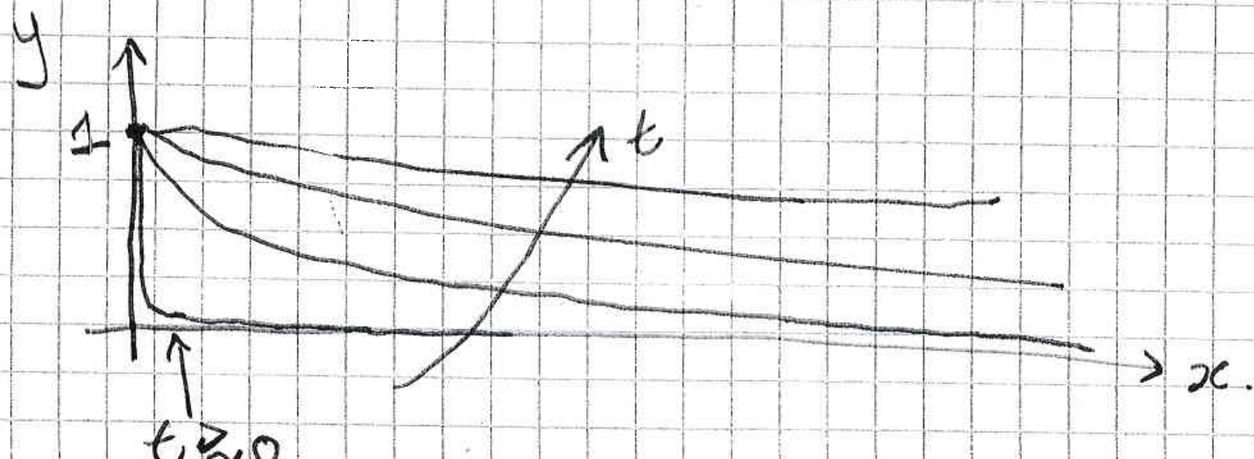
$$\Rightarrow B = 1.$$

Next, as $t \to 0_+$, we have $S \to \infty$, so

$$y(x,0) = 0 = A\underbrace{\int_0^\infty e^{-\tilde{S}^2} d\tilde{S}}_{\sqrt{\pi}/2} + 1$$

$$\Rightarrow A = -2/\sqrt{\pi}$$

$$\text{So } y(x,t) = \frac{-2}{\sqrt{\pi}}\int_0^{\frac{x}{2\sqrt{\kappa t}}} e^{-S^2} dS + 1.$$



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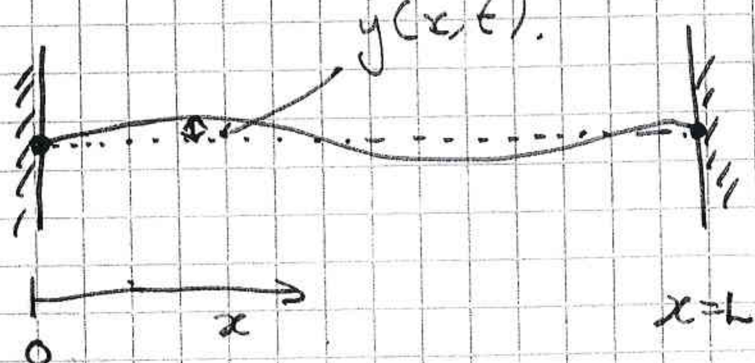
§7-4 The wave equation for an elastic string of length L

Consider an elastic string stretched ~~~~ tightly between two walls:



If $y(x,t)$ is the displacement of the string wrt the horizontal, then neglecting any damping, ~~the~~ y satisfies:

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2},$$

where $c = \sqrt{T/\rho}$ is the <u>wave speed</u> (units $\frac{\text{Length}}{\text{time}}$)

and T is the tension in the string, and ρ is its density.

The boundary conditions on the displacement are:

$$y(0,t) = y(L,t) = 0.$$

We [illegible] can consider two types of I.C:

Prescribed position

$$y(x,0) = f(x) \quad \text{and} \quad \frac{\partial y}{\partial t}(x,0) = 0$$

Prescribed velocity

$$y(x,0) = 0 \quad \text{and} \quad \frac{\partial y}{\partial t}(x,0) = g(x)$$

In either case, the PDE is solved via separation of variables.

Prescribed Position

Let $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$, $y(0,t) = y(L,t) = 0$

and $y(x,0) = f(x)$, $\frac{\partial y}{\partial t}(x,0) = 0$.

Let's write $y(x,t) = X(x)T(t)$.

Then, $XT'' = c^2 X'' T$, or

$$\frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T} = -\lambda .$$

So $X'' = -\lambda X$ and $T'' = -\lambda c^2 T$.

We have $X(0) = X(L) = 0$ and $T'(0) = 0$

But any non-homogeneous BCs cannot be applied yet.

As before, $X(0) = X(L) = 0 \Rightarrow \lambda > 0$ for $X \neq 0$

and $X_n = B_n \sin \frac{n\pi x}{L}$ and $\lambda = \frac{n^2\pi^2}{L^2}$

with $n = 1, 2, 3, \ldots$

So, $T_n'' = -\frac{n^2\pi^2 c^2}{L^2} T_n$

$$\Rightarrow T_n = A \cos \frac{n\pi ct}{L} + B \sin \frac{n\pi ct}{L}.$$

But $T'(0) = 0 \Rightarrow B = 0$

So we have infinitely many solns:

$$y_n(x,t) = X_n T_n = B_n \sin\frac{n\pi x}{L}\cos\frac{n\pi c t}{L}$$

and hence

$$y(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}\cos\frac{n\pi c t}{L}.$$

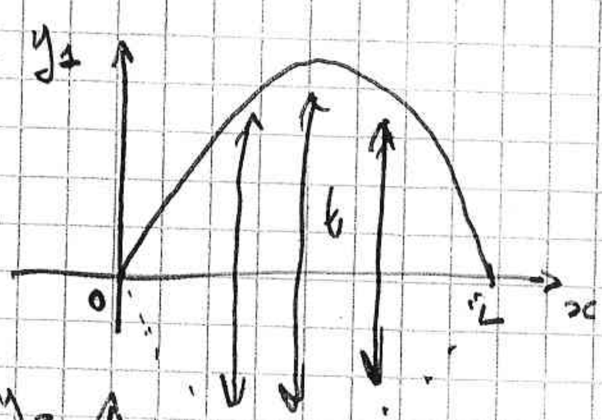
Then, B_n found by setting $y(x,0) = f(x)$, or

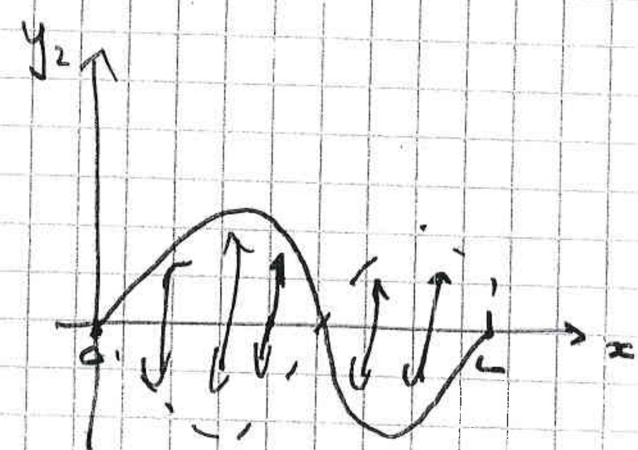
$$f(x) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}\cdot 1$$

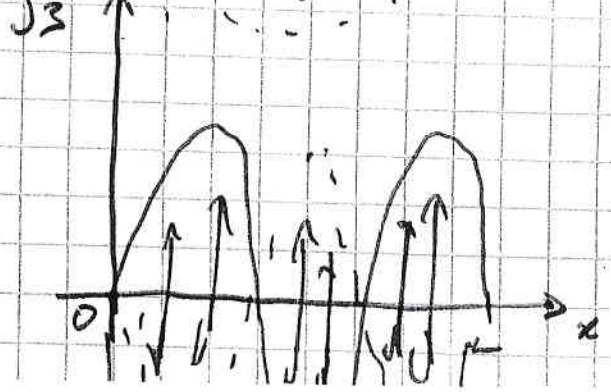
so $B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$

sine series.

Let's plot some y_n s:







The normal modes $\sin\frac{n\pi x}{L}$ are shapes oscillating due to $\cos\frac{n\pi c t}{L}$,

~~Example~~

Prescribed velocity

Now consider $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$, $y(0,t) = y(L,t) = 0$

and ICs $y(x,0) = 0$, $\frac{\partial y}{\partial t}(x,0) = g(x)$.

Let $y = X(x)T(t)$. Then, as before,
$X(0) = X(L) = 0$, $T(0) = 0$

$$X'' = -\lambda X \text{ and } T'' = -\lambda c^2 T, \quad \lambda > 0.$$

So $X_n = B_n \sin\frac{n\pi x}{L}$ and $\lambda = \frac{n^2\pi^2}{L^2}$.

Then $T_n'' = -\frac{n^2\pi^2c^2}{L^2} T_n$

So $T_n = A\cos\frac{n\pi ct}{L} + B\sin\frac{n\pi ct}{L}$

Now, $T(0) = 0 \Rightarrow A = 0$.

So $y_n = X_nT_n = B_n \sin\frac{n\pi x}{L} \sin\frac{n\pi ct}{L}$

and $y(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} \sin\frac{n\pi ct}{L}$.

We need $\frac{\partial y}{\partial t}(x,0) = g(x)$, or

$$g(x) = \sum_{n=1}^{\infty} \frac{n\pi c}{L} B_n \sin\frac{n\pi x}{L} \cdot 1.$$

So $B_n = \frac{2}{n\pi c}\int_0^L g(x) \sin\frac{n\pi x}{L}\,dx$.

Example

Let $\frac{\partial y}{\partial t}(x,0) = V\delta(x - L/2)$, δ-function.

Impulsively striked string at $x = L/2$.

So $g(x) = V\delta(x - L/2)$

and $B_n = \frac{2}{n\pi c}\int_0^L g(x) \sin\frac{n\pi x}{L}\,dx$

$$= \frac{2}{n\pi c}\int_0^L V\delta(x - L/2) \sin\frac{n\pi x}{L}\,dx$$

$$= \frac{2V}{n\pi c}\sin\left(\frac{n\pi}{2}\right)$$

$$= \begin{cases} 0 & n \text{ even} \\ \frac{2V}{n\pi c} & n = 4k+1 \quad k = 0,1,2,\ldots \\ -\frac{2V}{n\pi c} & n = 4k+3 \quad k = 0,1,2,\ldots \end{cases}$$

Ex* §7.5 Propagation of waves on an infinite elastic spring.

Consider $\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2}$ for $-\infty < x < \infty$.

This eqn admits characteristic variables:

Let $p = x + ct$ and $q = x - ct$.

Then, $\frac{\partial y}{\partial t} = \frac{\partial p}{\partial t}\frac{\partial y}{\partial p} + \frac{\partial q}{\partial t}\frac{\partial y}{\partial q} = c\frac{\partial y}{\partial p} - c\frac{\partial y}{\partial q}$.

Then, $$\frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial t}\left(c\frac{\partial y}{\partial q} - c\frac{\partial y}{\partial p}\right)$$

$$= \left(c\frac{\partial}{\partial q} - c\frac{\partial}{\partial p}\right)\left(c\frac{\partial y}{\partial q} - c\frac{\partial y}{\partial p}\right)$$

$$= c^2\frac{\partial^2 y}{\partial q^2} - 2c^2\frac{\partial^2 y}{\partial p\,\partial q} + c^2\frac{\partial^2 y}{\partial p^2}$$

Also, $$\frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial q^2} + 2\frac{\partial^2 y}{\partial p\,\partial q} + \frac{\partial^2 y}{\partial p^2}$$

So, $$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2}$$

$$\Rightarrow c^2\left(\frac{\partial^2 y}{\partial q^2} - 2\frac{\partial^2 y}{\partial p\,\partial q} + \frac{\partial^2 y}{\partial p^2}\right)$$

$$= c^2\left(\frac{\partial^2 y}{\partial q^2} + 2\frac{\partial^2 y}{\partial p\,\partial q} + \frac{\partial^2 y}{\partial p^2}\right)$$

$$\Rightarrow 4c^2\frac{\partial^2 y}{\partial p\,\partial q} = 0$$

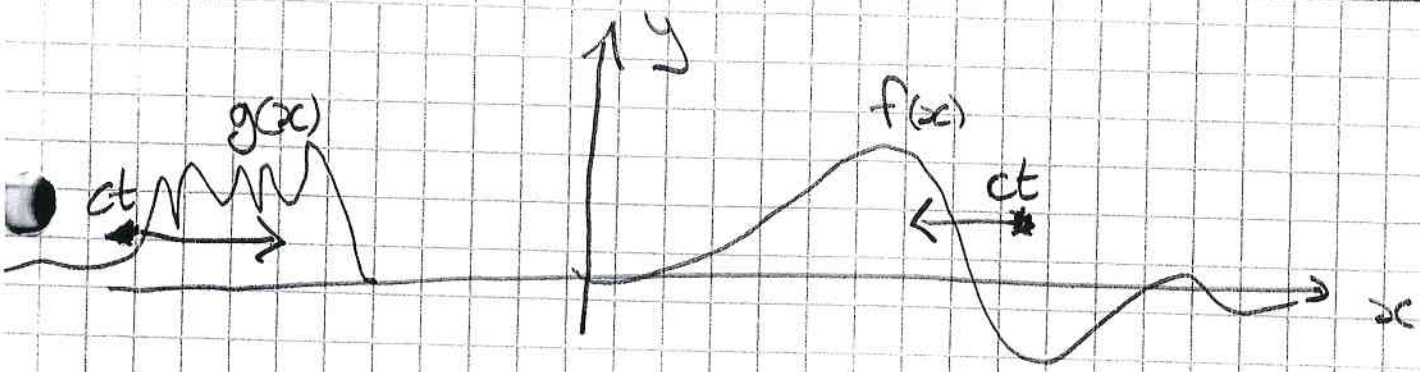
or $$\frac{\partial^2 y}{\partial p\,\partial q} = 0$$

So, $$\frac{\partial y}{\partial q} = f(p)$$

and $$y = f(p) + g(q)$$

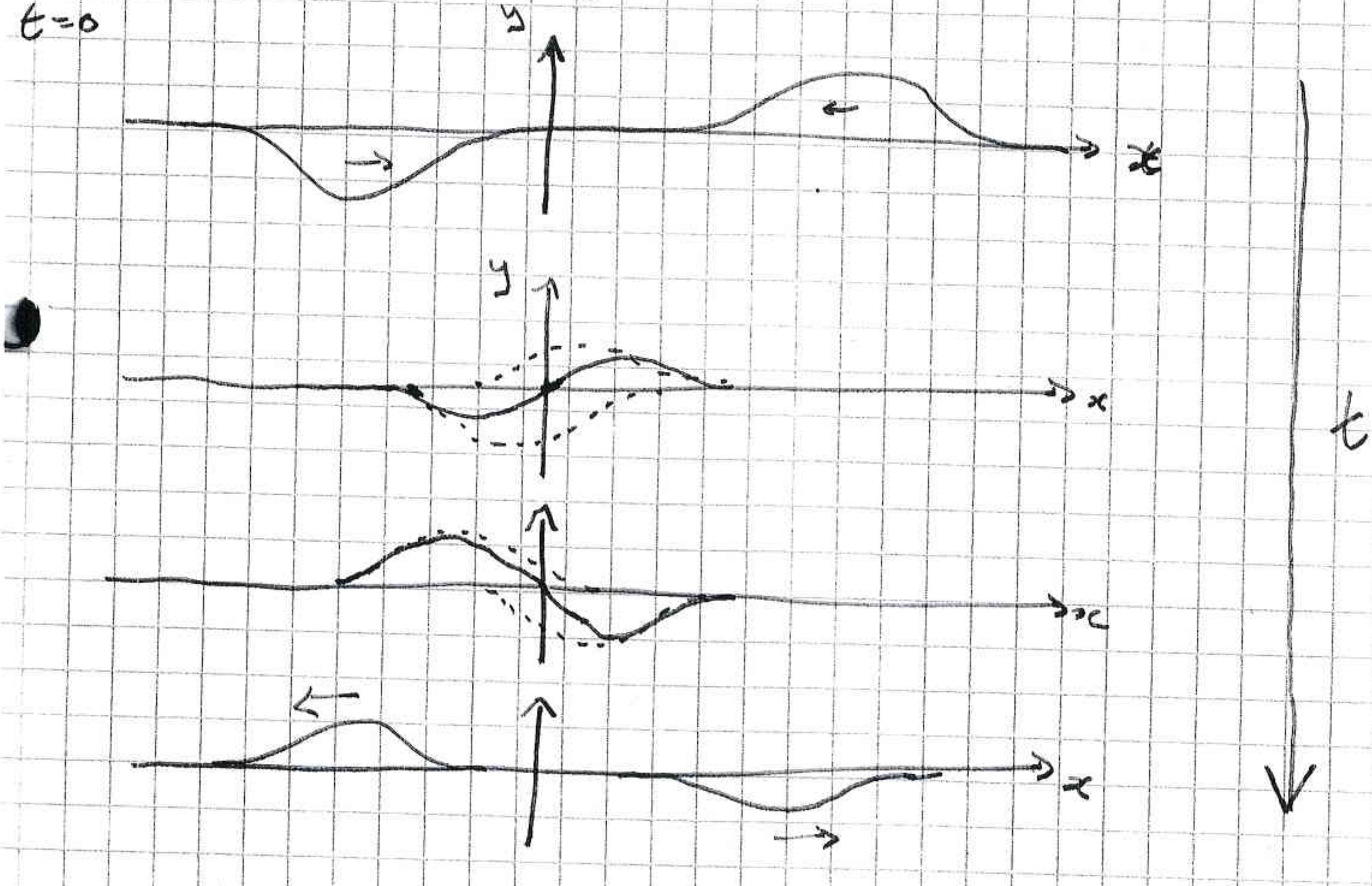
or, $$y(x,t) = f(x+ct) + g(x-ct)$$

The superposition of a ~~right~~ leftward propagating and rightward propagating disturbance.



Example

Let ~~...~~ $g(x) = -f(x)$.



Note that $y(0,t) = 0$. So this soln also solves wave eqn on $x>0$ only (if we ignore LHS) with the correct BC.

→ represents reflection at a boundary.

§7.6 Laplace equation

. The Laplace equation is $\nabla^2 u = 0$ for $u(\underline{x})$. In 2D this is:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{for } u(x,y).$$

<u>Example</u>

In more than 1D, the heat eqn is $\frac{\partial u}{\partial t} = \kappa \nabla^2 u$.
In steady state, $\frac{\partial u}{\partial t} = 0$, so we solve $\nabla^2 u = 0$.

<u>Example</u>

In inviscid irrotational, incompressible fluid mechanics, the velocity field is a vector field $\underline{u}(x,y)$ and it can be shown that $\underline{u} = \nabla\phi$ with $\nabla^2\phi = 0$.

Will will consider 2D domains that are <u>bounded</u>.
Eg Rectangle $0 < x < a$, $0 < y < b$.

We must apply BCs that come in two forms:

Dirichlet : u given on boundary

Neuman : $\underline{n}\cdot\nabla u$ given on boundary with normal $\underline{n}$.

In this course we will only consider the <u>Dirichlet proble</u>

We will now solve

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad , \quad 0<x<a \ , \ 0<y<b .$$

with $u(x,0) = 0 \ , \ u(x,b) = f(x)$ for $0<x<a$

and $u(0,y) = 0 \ , \ u(a,y) = 0$ for $0<y<b$

for some given $f(x)$.

We do this via separation of variables.

Let $u(x,y) = X(x)Y(y)$.

Then, $X''Y + XY'' = 0$

or, $\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda$.

So $X'' = -\lambda X$ and $Y'' = +\lambda Y$.

We have $X(0) = X(a) = 0$, so $\lambda > 0$

~~and $Y_n = B_n \sin \frac{n\pi y}{b}$ and $\lambda = \frac{n^2\pi^2}{b^2}$, $n=1,2,3,\ldots$~~

and $X_n = B_n \sin \frac{n\pi x}{a}$ and $\lambda = n^2\pi^2/a^2$.

Then, $Y_n'' = \frac{n^2\pi^2}{a^2} Y_n$, so

$$Y_n = A e^{\frac{n\pi y}{a}} + B e^{-n\pi y/a}.$$

Then, $Y_n(0) = 0 \Rightarrow A + B = 0 \Rightarrow B = -A$.

Then $Y_n = A\left(e^{n\pi y/a} - e^{-\frac{n\pi y}{a}}\right)$

$= 2A \sinh \frac{n\pi y}{a}$.

We have infinitely many solns of the form:

$$u_n(x,y) = X_n Y_n = B_n \sin \frac{n\pi x}{a} \sinh \frac{n\pi y}{a}$$

So, we write

$$u(x,y) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{a} \sinh \frac{n\pi y}{a}.$$

Finally, we need $u(\cancel{a,}\, x, b) = f(x)$, or

$$f(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{a} \cdot \underbrace{\sinh \frac{n\pi b}{a}}_{\text{const.}}$$

So $B_n = \dfrac{2}{a \sinh\left(\frac{n\pi b}{a}\right)} \displaystyle\int_0^L f(x) \sin \frac{n\pi x}{a}\, dx$

- $u(0,t)=5,\ u(2,t)=1,\ u(x,0)=4-2x$?

What is b_n?

show that spiral if $-1<\alpha<1$

and stable $\alpha>0$

US $\alpha<0$

$$\begin{pmatrix} 3 & -3 \\ \alpha & 1+\alpha \end{pmatrix}$$

$$\lambda^2-(4+\alpha)\lambda+3=0$$

$$\lambda=\frac{4+\alpha}{2}\pm\frac{1}{2}\sqrt{16+8\alpha+\alpha^2-12}$$

$$=\frac{4+\alpha}{2}\pm\frac{1}{2}\sqrt{\alpha^2+8\alpha+4}$$

When is it a spiral