Be sure this exam has 10 pages including the cover

The University of British Columbia

MATH 256, Sections 102 and 103

Final Exam – December 4, 2014

Name ____________________________ **Signature** ____________________________

Student Number ____________________

This exam consists of **8** questions. No notes. Simple numerics calculators are allowed. List of Laplace Transform is provided. Write your answer in the blank page provided.

Problem	max score	score
1.	15	
2.	10	
3.	10	
4.	10	
5.	10	
6.	20	
7.	10	
8.	15	
total	100	

(15 points) 1. Solve the following ordinary differential equation

$$\frac{dy}{dx} = \frac{y^2 + 2xy}{x^2}, \qquad y(1) = 1$$

Sol'n: This is either a Bernouli or a homogeneous

$$\frac{dy}{dx} - \frac{2}{x} y = \frac{1}{x^2} y^2$$

$$v = y^{-1} \quad \Rightarrow \quad v' + \frac{2}{x} v = -\frac{1}{x^2}$$

$$\mu = e^{\int \frac{2}{x}} = x^2$$

$$\int \mu g = -\int x^2 \frac{1}{x^2} dx = -x$$

$$v = \frac{1}{x^2}(c - x)$$

$$y = \frac{x^2}{c-x}, \quad y(1) = 1 \Rightarrow \quad 1 = \frac{1}{c-1} \Rightarrow c = 2$$

$$y = \frac{x^2}{2-x}$$

Interval of existence, $0 < x < 2$

(10 points) 2. Find the critical points of the following population model

$$\frac{dy}{dx} = (y^2 - 1)\,(e^y - 1)$$

and classify the stability/instability of these critical points.

Sol'n: $f(y)=0 \Rightarrow y^2-1=0,\ e^y-1=0$

$\Rightarrow y=-1, 0, 1$

At $y=-1$, $f'(y) = -2(e^{-1}-1) > 0 \Rightarrow$ unstable

At $y=0$, $f'(y) = -1 \Rightarrow$ stable

At $y=1$, $f'(y) = 2(e-1) > 0 \Rightarrow$ unstable

(10 points) 3. Use whatever method to solve the following second order differential equation

$$y'' - 3y' + 2y = e^t, \quad y(0) = 1, y'(0) = 0$$

Sol'n: Use method of undetermined coefficients

$y_h = c_1 e^t + c_2 e^{2t}$

$y_p = Ate^t$

$\Rightarrow A(2-3) = 1 \Rightarrow A = -1$

$y = c_1 e^t + c_2 e^{2t} - te^t$

$y(0)=1 \Rightarrow c_1 + c_2 = 1$

$y'(0)=1 \Rightarrow c_1 + 2c_2 - 1 = 1$

$\Rightarrow \begin{cases} c_1 + c_2 = 1 \\ c_1 + 2c_2 = 2 \end{cases} \Rightarrow$

$c_1 = 0$

$c_2 = 1$

$y = e^{2t} - te^t$

(10 points) 4. Consider the following second order equation

$$ty'' - y' + 4t^3 y = 0, \quad t > 0$$

Check that $y_1 = \cos(t^2)$ is a solution. Use the reduction of order to find the second solution of the homogeneous problem.

Hint: You may use the integration formula $\int \frac{1}{cos^2(x)} dx = \tan(x)$.

Sol'n: $y_2 = v y_1$. Then

$$v' = \frac{W}{y_1^2}$$

$$W = e^{+\int \frac{1}{t}} = t$$

$$v' = \frac{t}{\cos^2(t^2)}$$

$$v = \int \frac{t}{\cos^2(t^2)} dt \overset{u=t^2}{=} \frac{1}{2} \int \frac{1}{\cos^2 u} du = \frac{1}{2} \tan(t^2)$$

$$y_2 = \frac{1}{2} \tan t^2 \cos t^2 = \frac{1}{2} \sin t^2$$

(15 points) 5. Use whatever method to obtain the general solutions of

$$\mathbf{x}' = \begin{pmatrix} 1 & -3 \\ -2 & 2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} t-2 \\ 2t \end{pmatrix}$$

Sol'n: $A = \begin{pmatrix} 1 & -3 \\ -2 & 2 \end{pmatrix}$, $\det(A - \lambda I) = 0 \Rightarrow \lambda^2 - 3\lambda + 2 - 6 = 0$

$\lambda^2 - 3\lambda - 4 = 0 \Rightarrow \lambda_1 = -1,\ \lambda_2 = 4$

$\xi^{(1)} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$, $\xi^{(2)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$x_p = at + b$

$Aa + \begin{pmatrix} 1 \\ 2 \end{pmatrix} = a \Rightarrow a = \begin{pmatrix} -\frac{7}{6} \\ -\frac{1}{3} \end{pmatrix}$

$Ab + \begin{pmatrix} -2 \\ 0 \end{pmatrix} = 0 \Rightarrow b = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$

$x = c_1 e^{-t} \begin{pmatrix} 3 \\ 2 \end{pmatrix} + c_2 e^{4t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} -\frac{7}{6} \\ -\frac{1}{3} \end{pmatrix} t + \begin{pmatrix} -1 \\ -1 \end{pmatrix}$

(20 points) 6. Use the method of Laplace transform to solve

$$y'' + 2y' + 2y = 2u_1(t) + e^t\delta(t-2), \quad y(0) = 1, y'(0) = 0$$

Sol'n: $s^2 Y(s) - s + 2(sY(s) - 1) + 2Y(s) = 2\dfrac{e^{-s}}{s} + e^2 \cdot e^{-2s}$

$$Y(s) = \frac{s+2}{s^2+2s+2} + \frac{2e^{-s}}{s(s^2+2s+2)} + \frac{e^2 e^{-2s}}{s^2+2s+2}$$

$$\frac{s+2}{s^2+2s+2} = \frac{s+1}{(s+1)^2+1} + \frac{1}{(s+1)^2+1}$$

$$\frac{2}{s(s^2+2s+2)} = \frac{1}{s} - \frac{s+1}{(s+1)^2+1} - \frac{1}{(s+1)^2+1}$$

Hence

$$y(t) = e^{-t}\cos t + e^{-t}\sin t$$

$$+ u_1(t)\left[1 - e^{-(t-1)}\cos(t-1) - e^{-(t-1)}\sin(t-1)\right]$$

$$+ e^2 u_2(t)\, e^{\cancel{-(t-2)}} \sin(t-2).$$

(10 points) 7. Consider the following function

$$f(x) = \begin{cases} 1, & -1 \le t < 0 \\ x, & 0 \le x < 1 \end{cases} \quad f(x+2) = f(x)$$

(7 points) (a) Compute the first three coefficients of full Fourier series expansion a_0, a_1, b_1.

(3 points) (b) Find out the values of the full Fourier series expansion at $x = -\frac{1}{2},\ 0,\ \frac{1}{2}$.

Sol'n . 7 (a). $a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi}{L}x\right)dx$

$b_n = \frac{1}{L}\int_{-1}^{1} f(x)\sin\left(\frac{n\pi}{L}x\right)dx$

$L=1$, $a_0 = \int_{-1}^{1} f(x)\,dx = \int_{-1}^{0} 1 + \int_{0}^{1} x = \frac{3}{2}$

$a_1 = \int_{-1}^{1} f(x)\cos(\pi x)\,dx = \int_{-1}^{0}\cos(\pi x)\,dx + \int_{0}^{1} x\cos(\pi x)\,dx$

$= \frac{1}{\pi}\sin(\pi x)\Big|_{-1}^{0} + \frac{x}{\pi}\sin(\pi x) + \frac{1}{\pi^2}\cos(\pi x)\Big|_{0}^{1}$

$= -\frac{2}{\pi^2}$

$b_1 = \int_{-1}^{1} f(x)\sin(\pi x)\,dx = \int_{-1}^{0}\sin(\pi x)\,dx + \int_{0}^{1} x\sin(\pi x)\,dx$

$= -\frac{1}{\pi}\cos\pi x\Big|_{-1}^{0} - \frac{1}{\pi}x\cos\pi x + \frac{1}{\pi^2}\sin\pi x\Big|_{0}^{1}$

$= -\frac{2}{\pi} + \frac{1}{\pi} = -\frac{1}{\pi}$

(b) $\frac{a_0}{2} + \sum\left(a_n\cos\left(\frac{n\pi x}{L}\right) + b_n\sin\left(\frac{n\pi x}{L}\right)\right) = \frac{1}{2}\left(f(x-) + f(x+)\right)$

so for $x = -\frac{1}{2}$ $= f(-\frac{1}{2}) = 1$

$x = 0$, $= \frac{1}{2}(1+0) = \frac{1}{2}$

$x = 1$, $f(\frac{1}{2}) = \frac{1}{2}$

(15 points) 8. Use the method of separation of variables to solve the following heat equation

$$u_t = u_{xx}, \quad 0 < x < \pi, \ t > 0;$$

$$u(x,0) = 2\cos^2(x), \quad 0 < x < \pi;$$

$$u_x(0,t) = 0, \quad u_x(\pi,t) = 0, \ t > 0.$$

Sol'n: $u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{+\infty} a_n e^{-\alpha^2 (\frac{n\pi}{L})^2 t} \cos(\frac{n\pi}{L} x)$

$= \frac{a_0}{2} + \sum_{n=1}^{+\infty} a_n e^{-n^2 t} \cos(nx)$

$u(x,0) = 2\cos^2 x = 1 + \cos 2x$

Hence $a_0 = 2$, $a_2 = 1$, $a_n = 0$ when $n \neq 0, 2$

$u(x,t) = 1 + e^{-4t} \cos(2x)$.