Review of 1st and Second Order Equations

1. First order equations
1.1) The solutions to

$$y' + p(t)y = 0, \qquad \text{is} \qquad y = Ce^{-\int p(t)dt}$$

1.2) The steps to solve

$$y' + p(t)y = g(t)$$

are

$$\text{compute } \mu(t) = e^{\int p(t)dt}, \quad \int \mu(t)g(t)dt =$$

$$y(t) = \frac{1}{\mu(t)}(C + \int \mu(t)g(t)dt)$$

1.3) Separable equation

$$\frac{dy}{dt} = h(t)k(y), \qquad \int \frac{dy}{k(y)} = \int h(t)dt$$

1.4) Bernoulli equation

$$y' + p(t)y = q(t)y^n, \quad \text{let } v = y^{1-n}$$

$$v' + (1-n)p(t)v = (1-n)q(t)$$

1.5) Homogeneous equation

$$\frac{dy}{dx} = \frac{ax+by}{cx+dy}$$

$$\text{let } v = \frac{y}{x},\ x\frac{dv}{dx} = \frac{a+bv}{c+dv} - v$$

1.6) Interval of Existence: three factors a) The solution, b) The Equation, c) the Initial Condition

1.7) Difference between linear and nonlinear: for linear equation, existence is global and uniqueness is guaranteed; for nonlinear equation, existence and uniqueness are nonlocal; for nonlinear nonsmooth $f(t,y)$ nonuniqueness

2. Linear Second Order Equations

$$y'' + p(t)y' + q(t)y = g(t)$$

2.1. Homogeneous case

$$y'' + p(t)y' + q(t)y = 0$$

2.1.1. Wronskian $W[y_1, y_2](t) = y_1 y_2' - y_1' y_2$.

$$\text{Abel's equation } W' + pW = 0$$

$$W(t) = W(t_0)e^{-\int_{t_0}^{t} p(s)ds}$$

2.1.2. Set of Fundamental Solutions y_1, y_2. All solutions are given by $y = c_1 y_1 + c_2 y_2$
2.1.3. Constant Coefficients:

$$ay'' + by' + cy = 0$$

$$\text{Characteristic equation } \; ar^2 + br + c = 0$$

- $b^2 - 4ac > 0$, two unequal real roots $r_1 \neq r_2$.

$$y_1 = e^{r_1 t}, y_2 = e^{r_2 t}$$

- $b^2 - 4ac < 0$, two complex roots: $r_1 = \lambda + i\mu, r_2 = \lambda - i\mu$

$$y_1 = e^{\lambda t}\cos(\mu t), \quad y_2 = e^{\lambda t}\sin(\mu t)$$

- $b^2 - 4ac = 0$, two equal roots: $r_1 = r_2 = r$

$$y_1 = e^{rt}, \; y_2 = te^{rt}$$

2.1.4. Euler's type equation

$$at^2 y'' + bty' + cy = 0$$

$$\text{Characteristic equation } \; ar(r-1) + br + c = 0, \quad ar^2 + (b-a)r + c = 0$$

- $(b-a)^2 - 4ac > 0$, two unequal real roots $r_1 \neq r_2$.

$$y_1 = t^{r_1}, y_2 = t^{r_2}$$

- $(b-a)^2 - 4ac = 0$, two equal roots: $r_1 = r_2 = r$

$$y_1 = t^r, \ y_2 = t^r \log t$$

- $(b-a)^2 - 4ac < 0$, two complex roots: $r_1 = \lambda + i\mu, r_2 = \lambda - i\mu$

$$y_1 = t^\lambda \cos(\mu \log t), \quad y_2 = t^\lambda \sin(\mu \log t)$$

2.1.5. Reduction of Order

$$y^{''} + p(t)y^{'} + q(t)y = 0$$

If y_1 is known, we can get y_2 by letting $y_2 = v(t)y_1$. Then v satisfies

$$v^{''} + (\frac{2y_1^{'}}{y_1} + p)v^{'} = 0$$

and

$$v^{'} = \frac{W}{y_1^2}$$

where $W = e^{-\int p(t)dt}$ is the Wronskian.

2.2 Inhomogeneous equations

$$y^{''} + py^{'} + qy = h(t)$$

$$y = y_p(t) + c_1 y_1 + c_2 y_2$$

where y_p is a particular solution and y_1, y_2–set of fundamental solutions of homogeneous problem.

2.2.1 Method One: Method of Undetermined Coefficients. Works only for

$$ay^{''} + by^{'} + cy = h(t)$$

- $h(t) = a_0 + a_1 t + ... + a_n t^n$

$$y_p = t^s(A_0 + A_1 t + ... + A_n t^n)$$

- $h(t) = e^{\alpha t}(a_0 + a_1 t + ... + a_n t^n)$

$$y_p = t^s e^{\alpha t}(A_0 + A_1 t + ... + A_n t^n)$$

- $h(t) = e^{\alpha t}(a_0 + a_1 t + ... + a_n t^n)\cos(\beta t)$ or $g(t) = e^{\alpha t}(a_0 + a_1 t + ... + a_n t^n)\sin(\beta t)$

$$y_p = t^s e^{\alpha t}[(A_0 + A_1 t + ... + A_n t^n)\cos(\beta t) + (B_0 + B_1 t + ... + B_n t^n)\sin(\beta t)]$$

- s equals either 0, or 1, or 2, is the least integer such that there are no solutions of the homogeneous problem in y_p

- $h(t) = h_1 + ... + h_m$

$$y_p = y_{p,1} + ...y_{p,m}$$

2.2.2. Method of Variation of Parameters

$$y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$$

where

$$\begin{cases} u_1^{'} y_1 + u_2^{'} y_2 = 0, \\ u_1^{'} y_1^{'} + u_2^{'} y_2 = h(t) \end{cases}$$

Formula:

$$u_1 = -\int \frac{y_2 g(t)}{W} dt, \quad u_2 = \int \frac{y_1 g(t)}{W} dt$$

$$y_p = -y_1(t)\int \frac{y_2 g(t)}{W} dt + y_2(t)\int \frac{y_1 g(t)}{W} dt$$