MATH 256 Written Assignment 4

In Questions 3 through 7 you may use any standard results of the Laplace transform provided that they are explicitly stated. Otherwise, you must show all your working.

1. Calculate the Laplace transform of the following functions explicitly (i.e. using the integral formula and showing all your steps):
   (a) $f(t) = t^2$.
   (b) $f(t) = \cos(2t)$.
   (c) $f(t) = te^t$.

2. Calculate the Laplace transform of the following functions explicitly (i.e. using the integral formula and showing all your steps):
   (a)
$$f(t) = \begin{cases} 0 & \text{for } t \leq 1 \\ t & \text{for } 1 < t \leq 2 \\ 0 & \text{for } t > 2 \end{cases}.$$
   (b)
$$f(t) = \begin{cases} 0 & \text{for } t \leq \pi \\ -\sin(t) & \text{for } \pi < t \leq 2\pi \\ 0 & \text{for } t > 2\pi \end{cases}.$$

3. For the following expressions, list all the terms which must appear in their partial fraction expansion (you do not have to find the coefficients of each term):
   (a)
$$\frac{s}{s^2 - 1}$$
   (b)
$$\frac{1}{(s^2 - 1)(s^2 + 1)}$$
   (c)
$$\frac{s^2(s - 1)}{s^2(s - 4)(s - 2)^2}$$
   (d)
$$\frac{s^3 - 3}{(s^2 + 5)^2(s + 10)^3}.$$

4. Write the following functions $F(s)$ in terms of particial fractions and hence find the functions $f(t)$ for which $F(s) = \mathcal{L}[f(t)]$. **Hint:** $\mathcal{L}[e^{at}\sin\omega t] = \frac{\omega}{(s-a)^2+\omega^2}$ and $\mathcal{L}[e^{at}\cos\omega t] = \frac{s-a}{(s-a)^2+\omega^2}$.
   (a)
$$F(s) = \frac{2s^2 + 5s + 1}{s^2(s^2 + 4)},$$
   (b)
$$F(s) = \frac{8s - 22}{s^2 - 6s + 10}.$$

5. Use Laplace transforms to solve the following ODEs: **Hint:** $\mathcal{L}[e^{at}\sin\omega t] = \frac{\omega}{(s-a)^2+\omega^2}$ and $\mathcal{L}[e^{at}\cos\omega t] = \frac{s-a}{(s-a)^2+\omega^2}$.

   (a) $y'' - y' - 6y = 0$ with $y(0) = 1$ and $y'(0) = -1$.

   (b) $4y'' + 3y' = 4$ with $y(0) = -2$ and $y'(0) = -3$.

   (c) $y'' - 2y' + 2y = \cos(t)$ with $y(0) = 1$ and $y'(0) = 0$.

6. For the following functions $f(t)$, sketch the functions for $t \geq 0$, write them in terms of the heaviside step function $H(t)$, and hence find their Laplace transforms: **Note:** the heavside step funciton evaluated at $t - c$, i.e. $H(t - c)$ is sometimes written $u_c(t)$.

   (a)
$$f(t) = \begin{cases} 0 & \text{for } t < 2 \\ (t-2)^2 & \text{for } t \geq 2 \end{cases}.$$

   (b)
$$f(t) = \begin{cases} 0 & \text{for } t < 1 \\ t^2 - 2t + 2 & \text{for } t \geq 1 \end{cases}.$$

   (c)
$$f(t) = \begin{cases} 0 & \text{for } t < 1 \\ 1 & \text{for } 1 \leq t < 2 \\ 2 & \text{for } t \geq 2 \end{cases}.$$

   (d)
$$f(t) = \begin{cases} t & \text{for } t < 1 \\ 1 & \text{for } t \geq 1 \end{cases}.$$

7. Find the functions $f(t)$ whose Laplace transforms are given by the following functions $F(s)$.

   (a)
$$F(s) = \frac{2}{(s-1)^3}$$

   (b)
$$F(s) = \frac{e^{-s}}{s^2 + s - 2}$$

   (c)
$$F(s) = \frac{2e^{-2s}}{s^2 - 4}$$

   (d)
$$F(s) = \frac{e^{-s} + e^{-2s} - e^{-3s} - e^{-4s}}{s}$$