Lecture 17: Heat Conduction Problems with time-independent inhomogeneous boundary conditions

(*Compiled 8 November 2018*)

In this lecture we consider heat conduction problems with inhomogeneous boundary conditions. To determine a solution we exploit the linearity of the problem, which guarantees that linear combinations of solutions are again solutions. In particular, (excuse the pun) we first determine a *well chosen* particular solution, known as the *steady-state solution* that can be used to remove the inhomogeneous boundary conditions. This reduces the problem to one of solving the same boundary value problem but with *homogeneous* boundary conditions **and** an augmented initial condition. Although the steady state solution is a natural choice in this case, the choice of particular solution, *as always*, is by no means unique. To solve the homogeneous boundary value problems we demonstrate two distinct methods: Method I: comprises the more elementary *method of separation of variables*; while Method II introduces the more generally applicable *method of eigenfunction expansions*.

Key Concepts: Inhomogeneous Boundary Conditions, Particular Solutions, Steady state Solutions; Separation of variables, Eigenvalues and Eigenfunctions, Method of Eigenfunction Expansions.

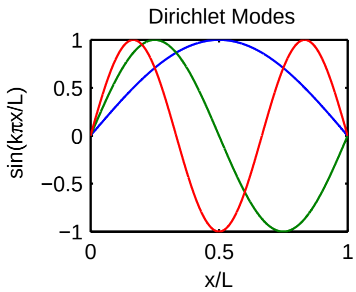
Reference Sections: Boyce and Di Prima Sections 10.5, 10.6, and 11.2

17 Heat Conduction Problems with inhomogeneous boundary conditions

17.1 A Summary of Eigenvalue Boundary Value Problems and their Eigenvalues and Eigenfunctions

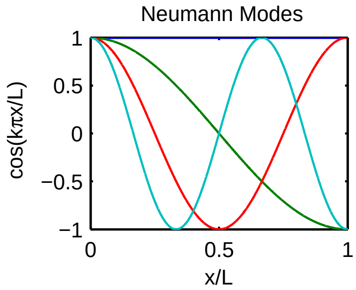
Thus far we have discussed five fundamental Eigenvalue problems: The Dirichlet Problem; The Neumann Problem; Periodic Boundary Conditions; and two types of Mixed Boundary Value Problems. Since these Eigenvalue problems will recur throughout the remainder of the course, for convenient reference we list the boundary value ODEs and the corresponding eigenvalues and eigenfunctions. We also plot the first few eigenfunctions, which can be seen to satisfy the prescribed boundary conditions. You will see that it is sometimes convenient to remember these so that we do not need to resort to the method of separation of variables in order to derive the appropriate eigenvalues and eigenfunctions for a particular problem. Recognizing the appropriate eigenvalues and eigenfunctions for a given problem gives rise to the so-called *method of eigenfunction expansions*, which we introduce in it simplest form in this lecture. There is, however, a word of warning. If the boundary value problem is new, in that it includes extra terms that cannot be grouped with the time variables in the separation process, it is necessary to consider the new eigenvalue problem in order to determine the appropriate eigenfunctions and eigenvalues.

I: The Dirichlet Problem: *Ice both sides*



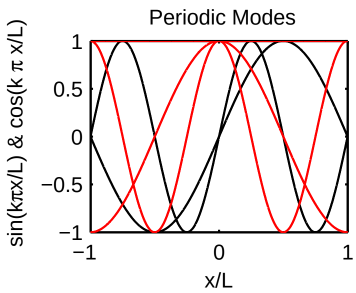
$$\left.\begin{array}{c} X'' + \lambda^2 X = 0 \\ X(0) = 0 = X(L) \end{array}\right\} \Longrightarrow \left\{\begin{array}{c} \lambda_n = \frac{n\pi}{L},\ n = 1, 2, \ldots \\ X_n(x) = \sin\left(\frac{n\pi x}{L}\right) \end{array}\right.$$

II: The Neumann Problem: *Insulation both sides*



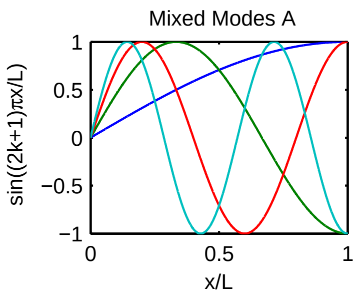
$$\left.\begin{array}{c} X'' + \lambda^2 X = 0 \\ X'(0) = 0 = X'(L) \end{array}\right\} \Longrightarrow \left\{\begin{array}{c} \lambda_n = \frac{n\pi}{L},\ n = 0, 1, 2, \ldots \\ X_n(x) = \cos\left(\frac{n\pi x}{L}\right) \end{array}\right.$$

III: The Periodic Boundary Value Problem: *The closed ring*



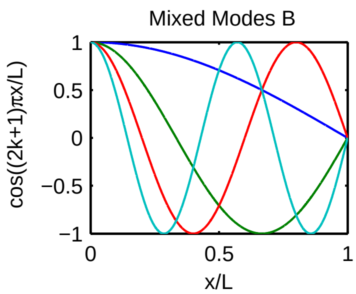
$$\left.\begin{array}{c} X'' + \lambda^2 X = 0 \\ X(-L) = X(L) \\ X'(-L) = X'(L) \end{array}\right\} \Longrightarrow \left\{\begin{array}{c} \lambda_n = \frac{n\pi}{L},\ n = 0, 1, 2, \ldots \\ X_n(x) \in \left\{1, \cos\left(\frac{n\pi x}{L}\right), \sin\left(\frac{n\pi x}{L}\right)\right\} \end{array}\right.$$

IV: Mixed Boundary Value Problem A: *Ice Left and Insulation Right*



$$\left.\begin{array}{c} X'' + \lambda^2 X = 0 \\ X(0) = 0 = X'(L) \end{array}\right\} \Longrightarrow \left\{\begin{array}{c} \lambda_k = \frac{(2k+1)\pi}{2L},\ k = 0, 1, 2, \ldots \\ X_n(x) = \sin\left(\frac{(2k+1)\pi}{2L}x\right) \end{array}\right.$$

V: Mixed Boundary Value Problem B: *Insulation Left and Ice Right*



$$\left.\begin{array}{c} X'' + \lambda^2 X = 0 \\ X'(0) = 0 = X(L) \end{array}\right\} \Longrightarrow \left\{\begin{array}{c} \lambda_k = \frac{(2k+1)\pi}{2L},\ k = 0, 1, 2, \ldots \\ X_n(x) = \cos\left(\frac{(2k+1)\pi}{2L}x\right) \end{array}\right.$$

17.2 Specified Constant Temperatures

Example 17.1

$$u_t = \alpha^2 u_{xx} \qquad 0 < x < L, \quad t > 0 \tag{17.1}$$

$$\text{BC: } u(0,t) = u_0 \qquad u(L,t) = u_1 \quad u_0, u_1 \text{ constants} \tag{17.2}$$

$$u(x,0) = g(x). \tag{17.3}$$

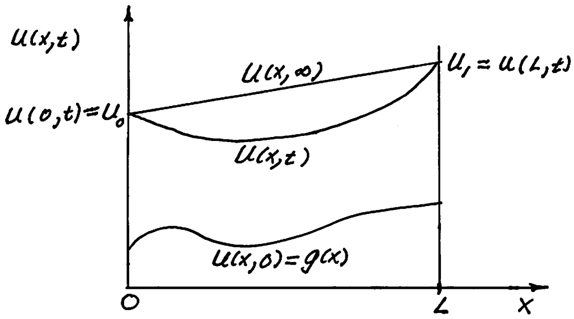


FIGURE 1. Initial, transient, and steady solutions to the heat conduction problem (17.3) with inhomogeneous Dirichlet BC

Subtracting out a particular solution:

Firstly consider the steady-state solution (i.e., when $u_t = 0$) which we denote by $u_\infty(x)$. In this case (17.1) becomes

$$\alpha^2 u_\infty''(x) = 0 \Rightarrow u_\infty(x) = A_0 x + B_0$$

$$u_\infty(0) = B_0 = u_0 \quad u_\infty(L) = A_0 L + u_0 = u_1 \Rightarrow \quad \begin{array}{c} u_\infty(x) = \left(\frac{u_1 - u_0}{L}\right) x + u_0 \\ \text{steady state solution} \end{array}.$$

Let $u(x,t) = u_\infty(x) + v(x,t)$. Substitute into (17.1)

$$u_t = \big(u_\infty(x) + v(x,t)\big)_t = \alpha^2 \big(u_\infty(x) + v(x,t)\big)_{xx} \Rightarrow v_t = \alpha^2 v_{xx} \tag{17.4}$$

since $\big(u_\infty(x)\big)_{xx} = 0$. Substitute into (17.2)

$$\begin{array}{lllllllll} u(0,t) & = & u_0 & = & u_\infty(0) + v(0,t) & = & u_0 + v(0,t) & \Rightarrow & v(0,t) = 0 \\ u(L,t) & = & u_1 & = & u_\infty(L) + v(L,t) & = & u_1 + v(L,t) & \Rightarrow & v(L,t) = 0. \end{array}$$

Substitute into (17.3)

$$u(x,0) = g(x) = u_\infty(x) + v(x,0) \Rightarrow v(x,0) = g(x) - u_\infty(x).$$

Thus we have to solve a new problem for v which has homogenous BC:

$$\begin{array}{lll} v_t & = & \alpha^2 v_{xx} \\ v(0,t) & = & 0 = v(L,t) \\ v(x,0) & = & g(x) - u_\infty(x). \end{array} \tag{17.5}$$

Method I: Solving the homogeneous problem using Separation of Variables

Separate variables: $v(x,t) = X(x)T(t)$.

$$\frac{\dot{T}(t)}{\alpha^2 T(t)} = \frac{X''(x)}{X(x)} = -\lambda^2 = \text{ const}$$

$$T(t) = c e^{-\lambda^2 \alpha^2 t}$$

$$X'' + \lambda^2 X = 0 \quad X(0) = 0 = X(L) \Rightarrow X_n(x) = \sin\left(\frac{n\pi x}{L}\right),\ \lambda_n = \frac{n\pi}{L},\ n = 1, \ldots$$

$$\left.\begin{array}{r} X'' + \lambda^2 X = 0 \\ X(0) = 0 = X(L) \end{array}\right\} \Longrightarrow \left\{\begin{array}{c} \lambda_n = \frac{n\pi}{L}, n = 1, 2, \ldots \\ X_n(x) = \sin\left(\frac{n\pi x}{L}\right) \end{array}\right.$$

Thus

$$v(x,t) = \sum_{n=1}^{\infty} b_n \mathrm{e}^{-\alpha^2\left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi x}{L}\right) \tag{17.6}$$

$$v(x,0) = g(x) - u_\infty(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \Rightarrow b_n = \frac{2}{L}\int_0^L \{g(x) - u_\infty(x)\} \sin\left(\frac{n\pi x}{L}\right)\, dx$$

Thus the solution to the inhomogeneous problem is:

$$\begin{aligned} u(x,t) &= u_\infty(x) + v(x,t) && (17.7) \\ &= u_0 + \left(\frac{u_1 - u_0}{L}\right) x + \sum_{n=1}^{\infty} b_n \mathrm{e}^{-\alpha^2\left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi x}{L}\right) && (17.8) \end{aligned}$$

where

$$b_n = \frac{2}{L}\int_0^L \{g(x) - u_\infty(x)\} \sin\left(\frac{n\pi x}{L}\right)\, dx. \tag{17.9}$$

Method II: Solving the homogeneous problem using Eigenfunction Expansions

In order to solve the boundary value problem (17.1)-(17.3) we could recognize that $\left\{\sin\left(\frac{n\pi x}{L}\right)\right\}_{n=1}^{\infty}$ are eigenfunctions of the spatial operator:

$$-\frac{\partial^2}{\partial x^2} \tag{17.10}$$

along with the homogeneous Dirichlet BC $v(0,t) = 0 = v(L,t)$. We therefore assume an eigenfunction expansion of the form:

$$\begin{aligned} v(x,t) &= \sum_{n=1}^{\infty} \hat{v}_n(t) \sin\left(\frac{n\pi x}{L}\right) && (17.11) \\ \frac{\partial v}{\partial t} &= \sum_{n=1}^{\infty} \dot{\hat{v}}_n(t) \sin\left(\frac{n\pi x}{L}\right) \quad \text{and} && (17.12) \\ \frac{\partial^2 v}{\partial x^2} &= -\sum_{n-1}^{\infty} \hat{v}_n(t) \left(\frac{n\pi}{L}\right)^2 \sin\left(\frac{n\pi x}{L}\right) && (17.13) \\ v_t = \alpha^2 v_{xx} &\Rightarrow \sum_{n=1}^{\infty} \left\{\dot{\hat{v}}_n(t) + \alpha^2 \left(\frac{n\pi}{L}\right)^2 \hat{v}_n(t)\right\} \sin\left(\frac{n\pi x}{L}\right) = 0. && (17.14) \end{aligned}$$

Therefore

$$\begin{aligned} \dot{\hat{v}}_n(t) &= -\alpha^2 \left(\frac{n\pi}{L}\right)^2 \hat{v}_n(t) \quad \text{A simple ODE for } \hat{v}_n(t)\text{:} && (17.15) \\ \Rightarrow \hat{v}_n(t) &= \hat{v}_n(0) \mathrm{e}^{-\alpha^2\left(\frac{n\pi}{L}\right)^2 t}. && (17.16) \end{aligned}$$

Therefore

$$v(x,t) = \sum_{n=1}^{\infty} \hat{v}_n(0) \mathrm{e}^{-\alpha^2 \left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi x}{L}\right) \tag{17.17}$$

$$v(x,0) = \sum_{n=1}^{\infty} \hat{v}_n(0) \sin\left(\frac{n\pi x}{L}\right) = g(x) - u_\infty(x) \tag{17.18}$$

$$\hat{v}_n(0) = \frac{2}{L} \int_0^L \{g(x) - u_\infty(x)\} \sin\left(\frac{n\pi x}{L}\right)\, dx \tag{17.19}$$

which is the same solution as that in (17.8) above.

Example 17.2

$$u_t = \alpha^2 u_{xx} \qquad 0 < x < L, \quad t > 0 \tag{17.20}$$

$$\textit{BC: } u(0,t) = u_0 \quad u_x(L,t) = 0 \tag{17.21}$$

$$\textit{IC: } u(x,0) = g(x). \tag{17.22}$$

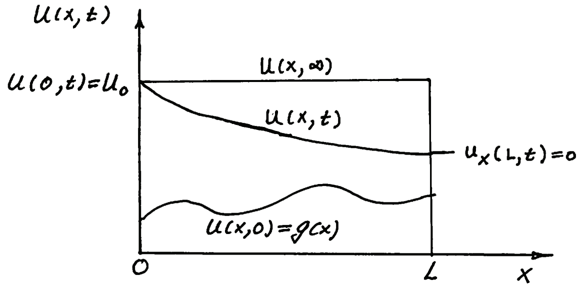


FIGURE 2. Initial, transient, and steady solutions to the heat conduction problem (17.3) with inhomogeneous Mixed BC

Look for a steady solution: $u''_\infty(x) = 0$.

$$u_\infty(x) = Ax + B \quad u_\infty(0) = B = u_0 \quad u'_\infty(x) = A = 0 \tag{17.23}$$

Therefore

$$u_\infty(x) = u_0. \tag{17.24}$$

Let $u(x,t) = u_\infty(x) + v(x,t) = u_0 + v(x,t)$.

$$u_t = \alpha^2 u_{xx} \Rightarrow v_t = \alpha^2 v_{xx} \tag{17.25}$$

$$u(0,t) = u_0 \Rightarrow u_0 = u_0 + v(0,t) \Rightarrow v(0,t) = 0 \tag{17.26}$$

$$u_x(L,t) = 0 \Rightarrow 0 = v_x(L,t) \Rightarrow v_x(L,t) = 0 \tag{17.27}$$

$$u(x,0) = u_0 + v(x,0) = g(x) \Rightarrow v(x,0) = g(x) - u_0 \tag{17.28}$$

Thus $v(x,t)$ satisfies

$$v_t = \alpha^2 v_{xx} \tag{17.29}$$
$$v(0,t) = 0 = v_x(L,t) \tag{17.30}$$
$$v(x,0) = g(x) - u_0. \tag{17.31}$$

$u(x,0) = u_0 + v(x,0) = g(x) \Rightarrow v(x,0) = g(x) - u_0$.

We now need a solution $v(x,t) = X(x)T(t)$ to (17.29):

$$\frac{\dot{T}(t)}{\alpha^2 T(t)} = \frac{X''(x)}{X(x)} = -\lambda^2$$
$$\dot{T}(t) = -\lambda^2\alpha^2 T(t) \Rightarrow T(t) = c\mathrm{e}^{-\lambda^2\alpha^2 t} \tag{17.32}$$
$$X'' + \lambda^2 X = 0; \quad X(0) = 0 = X'(L) \tag{17.33}$$
$$\Rightarrow X(x) = A\cos(\lambda x) + B\sin(\lambda x) \tag{17.34}$$
$$X'(x) = -A\lambda\sin(\lambda x) + B\lambda\cos(dx)$$
$$X(0) = A = 0 \tag{17.35}$$
$$\text{Therefore } X'(L) = B\lambda\cos(\lambda L) = 0 \Rightarrow \lambda_k = (2k-1)\frac{\pi}{2L},\ k = 1,2,3,\ldots \tag{17.36}$$
$$\text{or } \lambda = 0 \quad \text{which yields the trivial solution.} \tag{17.37}$$

Therefore

$$v(x,t) = \sum_{k=1}^{\infty} b_k \mathrm{e}^{-\lambda_k^2\alpha^2 t}\sin\left(\frac{(2k-1)}{2L}\pi x\right) \tag{17.38}$$
$$v(x,0) = \sum_{k=1}^{\infty} b_k \sin\left(\frac{(2k-1)}{2L}\pi x\right) = g(x) - u_0 \tag{17.39}$$
$$\Rightarrow b_n = \frac{2}{L}\int_0^L \{g(x) - u_0\}\sin\left(\frac{(2k-1)}{2L}\pi x\right)\,dx. \tag{17.40}$$

Returning to $u(x,t) = u_0 + v(x,t)$:

$$u(x,t) = u_0 + \sum_{k=1}^{\infty} b_k \mathrm{e}^{-\alpha^2\lambda_k^2 t}\sin\left(\frac{(2k-1)}{2L}\pi x\right). \tag{17.41}$$