Lecture 5: Examples of Frobenius Series: Bessel's Equation and Bessel Functions

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In this lecture we will consider the Frobenius series solution of the Bessel equation, which arises during the process of separation of variables for problems with radial or cylindrical symmetry. Depending on the parameter ν in Bessel's equation, we obtain roots of the indicial equation that are: distinct and real, repeated, and which differ by an integer.

Key Concepts: Frobenius Series Solutions, Bessel's equation; Bessel Functions.

5 Bessel Functions

5.1 Bessel's Function of Order $\nu \notin \{\ldots, -2, -1, 0, 1, 2\ldots\}$

$$Ly = x^2 y'' + xy' + (x^2 - \nu^2)y = 0 \tag{5.1}$$

$x = 0$ is a regular Singular Point: therefore let $y = \sum_{n=0}^{\infty} a_n x^{n+r}$.

$$0 = \sum_{n=0}^{\infty} a_n \big[(n+r)(n+r-1) + (n+r) - \nu^2\big] x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r+2} \tag{5.2}$$

$$m = n+2 \quad n = m-2$$

$$n = 0 \Rightarrow m = 2$$

$$0 = \sum_{m=2}^{\infty} \left\{a_m \big[(m+r)^2 - \nu^2\big] + a_{m-2}\right\} x^{m+r} + a_0 \left\{r^2 - \nu^2\right\} x^r + a_1 \left\{(1+r)^2 - \nu^2\right\} x^{r+1} \tag{5.3}$$

$$\begin{array}{ll} x^r > & a_0 \neq 0 \Rightarrow r = \pm\nu \quad \text{Indicial Eq. Roots} \\ x^{r+1} > & a_1\left\{r^2 + 2r + 1 - \nu^2\right\} = 0, \quad a_1 = 0 \text{ provided } \nu \neq -\frac{1}{2} \text{ and if } \nu = -\frac{1}{2} \text{ then } a_1 \text{ is arbitrary.} \\ x^{m+r} > & a_m = -\dfrac{a_{m-2}}{(m+r)^2 - \nu^2} \quad m \geq 2 \end{array} \tag{5.4}$$

$r = \nu$:

$$a_m = -\frac{a_{m-2}}{(m+\nu)^2 - \nu^2} = -\frac{a_{m-2}}{m^2 + 2m\nu} = -\frac{a_{m-2}}{m(m+2\nu)}$$

$$a_2 = -\frac{a_0}{2(2+2\nu)} = -\frac{a_0}{2^2(1+\nu)} \quad a_4 = -\frac{a_2}{4(4+2\nu)} = \frac{(-1)^2 a_0}{2.2^4(2+\nu)(1+\nu)}$$

$$\dots a_{2m} = \frac{(-1)^m a_0}{m! 2^{2m}(1+\nu)\dots(m+\nu)} \tag{5.5}$$

$$y_1(x) = x^\nu \sum_{m=0}^{\infty} \frac{(-1)^m (x/2)^{2m}}{m!(1+\nu)(2+\nu)\dots(m+\nu)} \quad \overset{x\to 0}{\to} 0$$

$r = -\nu$:

$$a_m = -\frac{a_{m-2}}{m(m-2\nu)}$$

$$a_2 = -\frac{a_0}{2(2-2\nu)} = -\frac{a_0}{2^2(1-\nu)}, \quad a_4 = -\frac{a_2}{4(4-2\nu)} = \frac{(-1)^2 a_0}{22^4(1-\nu)(2-\nu)}$$

$$\dots a_{2m} = \frac{(-1)^m a_0}{m! 2^{2m}(1-\nu)\dots(m-\nu)} \tag{5.6}$$

$$y_2(x) = x^{-\nu} \sum_{m=0}^{\infty} \frac{(-1)^m (x/2)^{2m}}{m!(1-\nu)\dots(m-\nu)} \quad \overset{x\to 0}{\to} \infty$$

5.2 Bessel's Function of Order $\nu = 0$ - repeated roots:

In this case

$$Ly = x^2 y + xy' + x^2 y = 0$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$Ly = \sum_{n=0}^{\infty} a_n \big\{(n+r)(n+r-1) + (n+r)\big\} x^{n+r} + a_n x^{n+r+2} = 0$$

$$m = n+2 \quad n = m-2 \tag{5.7}$$

$$0 = \sum_{n=2}^{\infty} \big[a_n(n+r)^2 + a_{n-2}\big] x^{n+r} + a_0\big[r(r-1)+r\big]x^r + a_1\big[(r+1)r + r + 1\big]x^{r+1} = 0$$

The indicial equation is: $a_0 r^2 = 0 \quad r_{1,2} = 0, 0$ a double root.

$r_1 = 0 \Rightarrow a_1.1 = 0 \Rightarrow a_1 = 0$.

Recursion: $a_n = -\dfrac{a_{n-2}}{(n+r)^2} \quad n \geq 2$.

$$a_2 = -\frac{a_0}{2^2}; \quad a_4 = -\frac{a_2}{4^2} = \frac{a_0}{2^2 4^2}; \quad a_6 = -\frac{a_4}{6^2} = -\frac{a_0}{2^2 4^2 6^2}; \quad a_8 = \frac{a_0}{2^2 4^2 6^2 8^2} \tag{5.8}$$

$$a_{2m} = \frac{(-1)^m}{2^{2m}(m!)^2} a_0 \tag{5.9}$$

$$y_1(x) = \left\{1 + \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m}(m!)^2}\right\} = J_0(x)$$

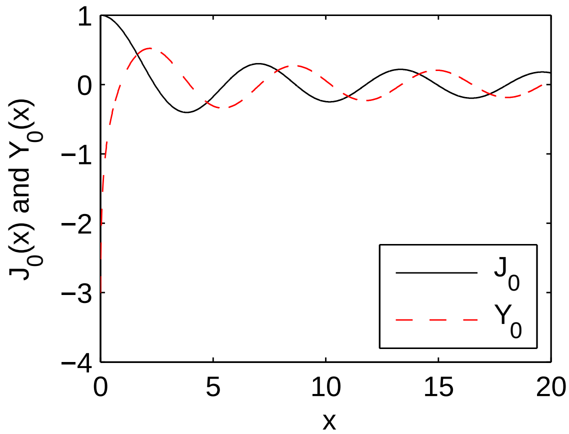

FIGURE 1. Zeroth order bessel functions $j_0(x)$ and $Y_0(x)$

To get a second solution

$$y(x,r) = a_0 x^r \left\{ 1 - \frac{x^2}{(2+r)^2} + \frac{x^4}{(2+r)^2(4+r)^2} + \cdots + \frac{(-1)^m x^{2m}}{(2+r)^2(4+r)^2\dots(2m+r)^2} + \quad \cdots \right\} \tag{5.10}$$

$$\left.\frac{\partial y}{\partial r}(x,r)\right|_{r=r_1} = a_0 \log x y_1(x) + a_0 x^r \sum_{m=1}^{\infty} (-1)^m x^{2m} \frac{\partial}{\partial r} \left\{ \frac{1}{(2+r)^2 \dots (2m+r)^2} \right\}.$$

Let

$$a_{2m}(r) = \{ \ \} \Rightarrow \ln a_{2m}(r) = -2\ln(2+r) - \dots - 2\ln(2m+r) \tag{5.11}$$

$$a'_{2m}(0) = \left.\left(-\frac{2}{2+r} - \frac{2}{4+r} \cdots - \frac{2}{(2m+r)} \right)\right|_{r=0} a_{2m}(0)$$

$$= \left(-1 - \frac{1}{2} - \dots - \frac{1}{m} \right) a_{2m}(0) = -H_m a_{2m}(0).$$

Let $H_m = 1 + \frac{1}{2} + \cdots + \frac{1}{m}$. Therefore

$$y_2(x) = J_0(x) \ln x + \sum_{m=1}^{\infty} \frac{(-1)^{m+1} H_m}{2^{2m}(m!)^2} x^{2m} \quad x > 0. \tag{5.12}$$

It is conventional to define

$$Y_0(x) = \frac{2}{\pi} \left[y_2(x) + (\gamma - \log 2) J_0(x) \right] \tag{5.13}$$

where

$$\gamma = \lim_{n \to \infty} (H_n - \log n) = 0.5772 \quad \text{Euler's Constant}$$

$$y(x) = c_1 J_0(x) + c_2 Y_0(x). \tag{5.14}$$

5.3 Bessel's Function of Order $\nu = \frac{1}{2}$:

Consider the case $\nu = 1/2$: $Ly = x^2y'' + xy' + \left(x^2 - \dfrac{1}{4}\right) y = 0$.
Let

$$y = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{5.15}$$

$$Ly = \sum_{n=0}^{\infty} a_n \left\{(n+r)^2 - \frac{1}{4}\right\} x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r+2} = 0 \quad \begin{aligned} m &= n+2 \\ n &= m-2 \\ n &= 0 \Rightarrow m = 2 \end{aligned} \tag{5.16}$$

$$Ly = a_0 \left\{r^2 - \frac{1}{4}\right\} x^r + a_1 \left\{(r+1)^2 - \frac{1}{4}\right\} x^{r+1} + \sum_{n=2}^{\infty} \left[a_n \left\{(n+r)^2 - \frac{1}{4}\right\} + a_{n-2}\right] x^{n+r} = 0.$$

Indicial Equation: $r^2 - \dfrac{1}{4} = 0, \quad r = \pm\dfrac{1}{2}$ Roots differ by an integer.
Recurrence: $a_n = -\dfrac{a_{n-2}}{(n+r)^2 - \frac{1}{4}} \quad n \geq 2.$
$r_1 = +1/2$:

$$\begin{aligned} a_n &= -\frac{a_{n-2}}{(n+\frac{1}{2})^2 - \frac{1}{4}} = -\frac{a_{n-2}}{(n+1)n} \quad n \geq 2; \left(\frac{9}{4} - \frac{1}{4}\right) a_1 = 0 \Rightarrow a_1 = 0 \\ a_2 &= -\frac{a_0}{3.2} \quad a_4 = \frac{(-1)^2 a_0}{5.4.3.2} \ldots a_{2n} = \frac{(-1)^n a_0}{(2n+1)!} \\ y_1(x) &= x^{\frac{1}{2}} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!} = x^{-\frac{1}{2}} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x^{-\frac{1}{2}} \sin x \end{aligned} \tag{5.17}$$

$r_2 = -\dfrac{1}{2}$:

$$\begin{aligned} a_n &= -\frac{a_{n-2}}{(n-\frac{1}{2})^2 - \frac{1}{4}} = -\frac{a_{n-2}}{n(n-1)}, \quad n \geq 2, \\ n = 1 \Rightarrow a_1 &\left\{\left(-\frac{1}{2} + 1\right)^2 - \frac{1}{4}\right\} = a_1.0 = 0 \quad a_1 \text{ and } a_0 \text{ arbitrary.} \end{aligned} \tag{5.18}$$

a_0:

$$a_2 = -\frac{a_0}{2.1} \quad a_4 = \frac{(-1)^2 a_0}{4.3.2.1} \ldots \quad a_{2n} = \frac{(-1)^n a_0}{(2n)!} \tag{5.19}$$

a_1:

$$a_3 = -\frac{a_1}{3.2} \quad a_5 = \frac{(-1)^2 a_1}{5.4.3.2} \quad a_{2n+1} = \frac{(-1)^n a_1}{(2n+1)!} \tag{5.20}$$

$$\begin{aligned} y_2(x) &= a_0 x^{-\frac{1}{2}} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} + a_1 x^{-\frac{1}{2}} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \\ &= a_0 x^{-\frac{1}{2}} \cos x + a_1 x^{-\frac{1}{2}} \sin x \end{aligned} \tag{5.21}$$

$\nwarrow$ included in $y_1(x)$.

Note: In this case the recursion spawns another solution for the smaller root $r = -\frac{1}{2}$ so we get away without having to do anything special to get another solution. In the next subsection we give an example where this is not the case and we have to use our differentiation with respect to r trick. We could always use the method of reduction of order along with the first solution.

5.4 The roots differ by an integer - an example for enrichment

Let $Ly = xy'' - y = 0, \quad x = 0$ is a regular singular point.

$$\begin{aligned}
&y = \sum_{n=0}^{\infty} c_n x^{n+\alpha} \\
&\sum_{n=0}^{\infty} c_n (n+\alpha)(n+\alpha-1)x^{n+\alpha-1} - c_n x^{n+\alpha} = 0 \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \uparrow \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad p-1=n \\
&\sum_{n=1}^{\infty} \{c_n(x+\alpha)(n+\alpha-1) - c_{n-1}\} x^{n+\alpha-1} + c_0(\alpha-1)\alpha x^{\alpha-1} = 0
\end{aligned} \tag{5.22}$$

Indicial Equation: $(\alpha-1)\alpha = 0 \;\Rightarrow\; \alpha = 0, 1$ differ by integer.
Recurrence Rel: $c_n = \dfrac{c_{n-1}}{(n+\alpha)(n+\alpha-1)} \quad n \geq 1.$
Note: When $\alpha = 0, \quad c_1$ blows up!
Let $\alpha = 1 \Rightarrow \quad c_1 = \dfrac{c_0}{2}, \quad c_2 = \dfrac{c_0}{12}, \ldots.$

$$y_1(x) = c_0 x \left(1 + \frac{x}{2} + \frac{x^2}{12} + \cdots\right) = c_0 u_1(x). \tag{5.23}$$

Second Solution:

$$\begin{aligned}
\bar{y}(x,\alpha) = \alpha y(x,\alpha) &= c_0 x^{\alpha} \left\{\alpha + \frac{x}{1+\alpha} + \frac{x^2}{(1+\alpha)(2+\alpha)(1+\alpha)} + \cdots\right\} \\
\frac{\partial \bar{y}}{\partial \alpha} &= c_0 x^{\alpha} \ln x \left\{\alpha + \frac{x}{1+\alpha} + \cdots\right\} \\
&\quad + c_0 x^{\alpha} \left\{1 - \frac{x}{(1+\alpha)^2} - \frac{x^2}{(1+\alpha)^2(2+\alpha)}\left[\frac{2}{(1+\alpha)} + \frac{1}{(2+\alpha)}\right] + \cdots\right\} \\
\left.\frac{\partial \bar{y}}{\partial \alpha}\right|_{\alpha=0} &= c_0 \left\{x + \frac{x^2}{2} + \frac{x^3}{12} + \cdots\right\} \ln x + c_0 \left\{1 - x - \frac{5}{4}x^2 - \cdots\right\} = c_0 u_2.
\end{aligned} \tag{5.24}$$

Therefore $y(x) = (A + B\ln x)\left(x + \dfrac{x^2}{2} + \dfrac{x^3}{12} + \cdots\right) + B\left(1 - x - \dfrac{5}{4}x^2 - \cdots\right).$