Lecture 6: Introduction to Partial Differential Equations

(*Compiled 3 March 2014*)

In this lecture we will introduce the three basic partial differential equations we consider in this course. After briefly discussing the classification of these equations we go through the modeling process to arrive at these three equations from a number of different physical situations. We deliberately explore the different paths to arrive at the same partial differential equations to emphasize way in which the models from disparate applications share the same features.

Key Concepts: Partial Differential Equations (PDEs); Elliptic, Parabolic, Hyperbolic PDEs; The heat Equation, The Wave Equation, and Laplace's Equation, Modeling and Derivation of PDEs.

6 Introduction to PDEs

6.1 Classification of PDEs

Ordinary Differential Equations (ODE) - Equations which define functions of a single independent variable by prescribing a relationship between the values of the function and its derivatives.

Example 1 A nonlinear second order ODE

$$y''(x) + e^{y(x)} = 0. \tag{6.1}$$

Partial Differential Equations (PDE) - Involve multivariable functions $u(x,t)$, $u(x,y)$ that are determined by prescribing a relationship between the function value and its partial derivatives.

Example 2 A Linear First Order PDE

$$a(x,y)\frac{\partial}{\partial x}u(x,y) + b(x,y)\frac{\partial}{\partial y}u(x,y) = c(x,y) \tag{6.2}$$

Example 3 A Nonlinier First Order PDE

$$a(x,y,u)\frac{\partial}{\partial x}u(x,y) + b(x,y,u)\frac{\partial}{\partial y}u(x,y) = c(x,y,u) \tag{6.3}$$

Example 4 Some Classic Linear Second Order PDEs:

Quadric	Classification	Eq.	Name
$T = X^2$	Parabolic	$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2}$	The Heat Equation or Diffusion Equation
$X^2 + Y^2 = k$	Elliptic	$\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = f(x,y)$	Poisson's Equation $f \not\equiv 0$ Laplace's Equation $f = 0$
$T^2 - c^2 X^2 = k$	Hyperbolic	$\frac{\partial^2 u(x,t)}{\partial t^2} - c^2 \frac{\partial^2 u(x,t)}{\partial x^2} = 0$	The Wave Equation

By analogy with quadric surfaces $aX^2 + 2bXY + c^2Y^2 + \cdots = k$ that can be reduced to a standard form by coordinate rotation, the most general linear 2nd order PDE

$$au_{xx} + 2bu_{xy} + cu_{yy} + \cdots \tag{6.4}$$

can be reduced by a transformation of coordinates to one of the Heat, Laplace or the Wave Equations.

6.2 Modeling and the derivation of PDE

Mathematical modeling is the process of writing down an equation or a system of equations that describe a particular physical, chemical, biological, or economic system that we wish to understand at a more fundamental level and whose behavior we would like to predict and perhaps even control. Mathematical modeling is an art-form in which a loose set of tools are applied to arrive at a self-consistent model, which can give a faithful representation of the behavior of the target system - sometimes with startling results. Tools that are typically used to build mathematical models include: conservation principles and balance laws that must obviously govern the behavior of the system we are trying to describe, e.g. conservation of mass, fluid, chemicals, fruit flies or balance of linear momentum, Newton's Second and Third Laws of Mechanics. In this process it is very important to be very mindful of the dimensions of the variables that we define so that we do not commit the cardinal sin of "adding apples to oranges". Dimensional analysis, rather than being a mere check for consistency, has evolved to an extremely useful sub-field of ODE and PDE that can be used to derive properties of certain solutions and even to identify special solutions that could not be obtained by other techniques. Other reality checks in the modeling process are obtained by ensuring that the model does not violate some very fundamental physical or economic principle - such as the second law of thermodynamics or the postulate of a liquid market.

Modeling is an extremely broad topic, which arguably includes all of physics, physical chemistry, mathematical biology, and about which many books have been written. Therefore we will not have time to explore this topic in much detail in this course. We will, however, explore a few examples to illustrate the modeling process. One aspect of the remainder of this lecture to which you should pay particular attention is the way in which we can arrive at precisely the same equation in spite of the fact that we are considering completely different physical systems with very different meanings attached to the dependent and independent variables. Thus modeling is a tremendously unifying process, which can highlight the fundamental similarities between the behavior of seemingly disparate physical systems. In fact, we will only be studying three equations in this course! However, the richness of the diverse applications of these few equations is what makes Applied Mathematics so interesting. Indeed, it is the reason that Applied Mathematicians are in such high demand in almost every field of industry from: geoengineering, e.g., mining, extracting petroleum, geophysical prospecting; to every branch of engineering, e.g. to design more efficient circuits, medical devices and imaging techniques, or to designing safer aircraft. The focus of this course will be on what comes after the model has been built, i.e., given a mathematical model how do we find a solution? Given this emphasis you may be tempted to forget about the modeling aspect of the PDE you will find that you can derive much insight about the behavior of the solutions by keeping in mind the physical meaning behind the variables. For example, a simple "thought experiment", with a mental picture of one of the physical systems to which a given PDE applies, can be used to check a solution that you have derived to see if it makes sense.

6.3 A one dimensional Conservation Law

6.3.1 *Traffic flow on a highway*

Consider the traffic flow on a highway and let $u(x,t)$ be the *density* of cars at x at time t.

$$[u] = \# \text{ of cars/unit length.} \tag{6.5}$$

Let $q(x,t)$ be the *flux* of cars at x at time t.

$$[q] = \# \text{ of cars/unit time.} \tag{6.6}$$

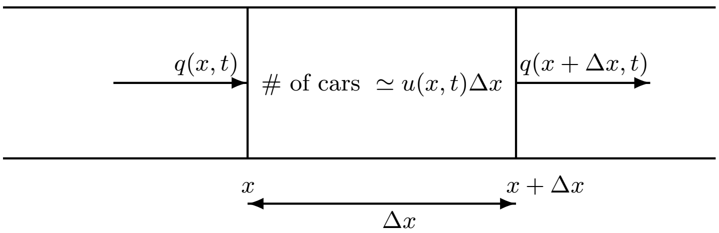

FIGURE 1. Traffic flow along the x axis with density $u(x,t)$ (cars/unit length) and flux $q(x,t)$ (cars/second) at x & instant t.

Now the change in the number of cars within the interval $[x, x+\Delta x]$ is approximated by

$$\Delta u \Delta x = \{u(x,t+\Delta t) - u(x,t)\}\Delta x \simeq \{q(x,t) - q(x+\Delta x,t)\}\Delta t \tag{6.7}$$

Now divide (6.7) by $\Delta t \Delta x$ and let $\Delta x \to 0$ and $\Delta t \to 0$ and we obtain the following conservation law PDE:

$$\frac{\partial u}{\partial t} + \frac{\partial q}{\partial x} = 0 \tag{6.8}$$

This limiting process is frequently referred to as "taking the continuum limit".

Observations

This partial differential equation represents the conservation of a quantity $u(x,y)$ that is subject to a flux $q(x,t)$, which is why it is called a conservation law. Depending on the context and the definitions of u and q, the conservation law can be used to represent the following physical situations (among many) in which quantities are conserved:

- conservation of cars
- conservation of heat
- conservation of chemicals.
- conservation of fluid.

We observe that the conservation law relates the gradients of two distinct quantities u and q. In order to have enough information to solve for one of the variables, we need to provide another equation. This is sometimes provided by what is known as *an equation of state* in thermodynamics or *a constitutive relation* in continuum mechanics. For example, how does q change with u or its derivatives, i.e., $q = q(u)$, $q = q(x, t, u)$, or $q = q(x, t, u, \frac{\partial u}{\partial x})$?

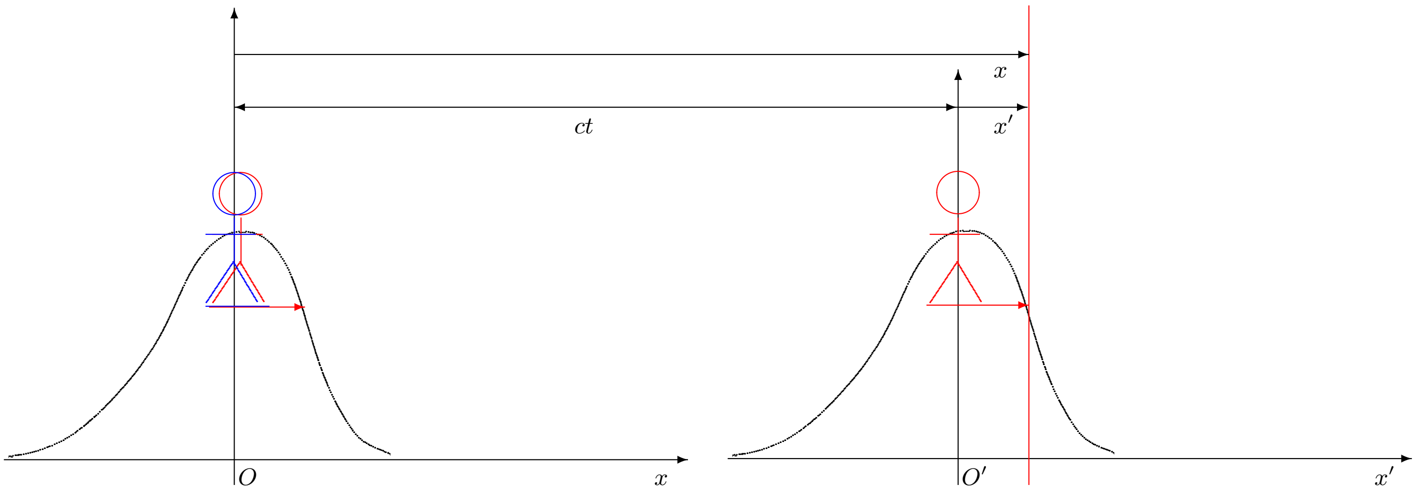

FIGURE 2. The Galilean transformation of coordinates from x to $x' = x - ct$

6.3.2 *Application: convection and the first order Wave Equation*

Assume that the flux of cars q increases linearly with the density of cars u, i.e., $q = cu,\ c > 0$, then it follows that

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0 \tag{6.9}$$

But this is just a wave equation. To see this consider two coordinate systems Ox and $O'x'$. Assume that at time $t = 0$ the coordinate systems Ox and $O'x'$ are coincident and that $O'x'$ moves at a speed c relative to coordinate system Ox and directed toward increasing x. To make the situation more realistic assume that the moving coordinate system is attached to a wave whose shape is shown in figure 2 and that the red "surfer" is riding with the wave. We assume that there is a blue "observer" attached to the fixed coordinate system Ox. At time $t = 0$, since the two coordinate systems Ox and $O'x'$ were coincident, the blue and red observers were at the same place. According to the surfer the functional form of the wave represented by the function $f(x')$ stays the same throughout the motion. We observe that the distance between the O' and the vertical red line remains x' throughout the motion, while the distance x from the centre O of the stationary coordinate system is related to x' by the so-called Galilean Transformation:

$$x' = x - ct \tag{6.10}$$

Thus according to the stationary observer, the functional form of the wave varies in space-time according to:

$$f(x') = f(x - ct) \tag{6.11}$$

Motivated by this property of a wave/signal moving to the right at a constant speed c, we are led to consider the following guess for a solution to (6.9):

$$\text{We guess that} \quad u(x,t) = f(x-ct) \quad \text{solves} \quad u_t + cu_x = 0 \tag{6.12}$$

$$\text{Take derivatives} \quad u_t = -cf' \quad u_x = f'$$

$$\text{Therefore} \quad u_t + cu_x = -cf' + cf' = 0, \quad \text{which implies} \quad u(x,t) = f(x-ct) \quad \text{solves (6.9).}$$

Thus $u_t + cu_x = 0$ has solutions of the form $u(x,t) = f(x - ct)$ for any sufficiently differentiable f, each of which represents a right moving wave of a given shape.

Observations and extensions:

- *A judicious guess:* Because (6.9) comprises a linear combination of a time derivative $\frac{\partial}{\partial t}$ and a spatial derivative $\frac{\partial}{\partial x}$, we might expect to find a solution of the form of an exponential of a linear function of x and t, since either derivative of such a function is in the form of a constant times the exponential. We therefore consider the trial solution of the form

$$u(x,t) = \mathrm{e}^{\mathrm{i}kx+\sigma t} \tag{6.13}$$

  Substituting (6.13) into (6.9) we obtain

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\mathrm{e}^{\mathrm{i}kx+\sigma t} = \{\sigma + \mathrm{i}kc\}\mathrm{e}^{\mathrm{i}kx+\sigma t}, \tag{6.14}$$

  which is a solution of provided σ and k satisfy the following "dispersion relation"

$$\sigma = -\mathrm{i}kc \tag{6.15}$$

  Thus we have obtained a special case of the solution derived in (6.12)

$$u(x,t) = g(x-ct) = \mathrm{e}^{\mathrm{i}k(x-ct)} \tag{6.16}$$

- *Shocking - a nonlinear wave equation:* What happens if q, instead of increasing linearly with u, can behave in a nonlinear way, i.e.,

$$q(u) = h(u), \text{ where } h \text{ is some given function of u} \tag{6.17}$$

  Combining this with the conservation law (6.8)can be written in the form

$$\frac{\partial u}{\partial t} + h'(u)\frac{\partial u}{\partial x} = 0 \tag{6.18}$$

  We note that since the wave speed $c\,h'(u)$ can vary in space, it is possible for certain initial conditions and functions h to have the waves that initiate for more negative values of x to crash into waves that initiate for more positive values of x. This phenomenon is resolved in wave mechanics by the formation of a *shock wave*, which represents a special solution to this over-specified situation in which there are potentially multiple values of the solution at certain points in the domain. This is the same phenomenon that occurs with formation of supersonic shock waves by aircraft or by the cracking of a whip.

- *A left moving wave:* What happens if $q = -cu$? In this case

$$\frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x} = 0 \tag{6.19}$$

We leave it as an exercise to show, in a similar way to the procedure used for the right moving wave, that (6.19) has a solution that represents a left moving wave.

- *The second order wave equation:* Note that if we apply the left $\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}$ and right $\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}$ moving wave operators in succession, we obtain

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)u(x,t) = \frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = 0, \tag{6.20}$$

which is the second order wave equation that has both left and right moving wave solutions (we will return to this later in the context of acoustic waves in a solid bar).

6.3.3 *Application: the convection-diffusion equation*

Consider the traffic flowing down the highway as shown in figure 1 and assume that the flux q increases linearly with the car density u. Now assume some agency from the driver in which she responds to an increase in the density of traffic by decreasing her speed, which results in a decrease in the flux of cars locally. This situation can be represented by a flux function of the form

$$q = cu - D\frac{\partial u}{\partial x} \tag{6.21}$$

Combining (6.21) with (6.8) we obtain the convection-diffusion equation

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = D\frac{\partial^2 u}{\partial x^2} \tag{6.22}$$

Observations:

- *A second order parabolic PDE:* Considering the highest derivatives that appear in each of the independent variables x and t we observe (from the chart at the beginning of this lecture) that the convection-diffusion is classified as a parabolic PDE.
- *A moving coordinate system:* Introduce the transformation $u(x,t) = U(\xi,t)$, where $\xi = x - ct$ and show that $U(\xi,t)$ satisfies the diffusion equation $U_t = DU_{\xi\xi}$. Can you interpret the removal of the convection term physically?
- *The dispersion relation and stability:* Consider solutions of (6.22) of the form $u(x,t) = \mathrm{e}^{\mathrm{i}kx+\sigma t}$. Determine the associated dispersion relation $\sigma = \sigma(k)$. Using the dispersion relation determine if the solution stable when $D > 0$ or when $D < 0$?