Solutions to Final Exam

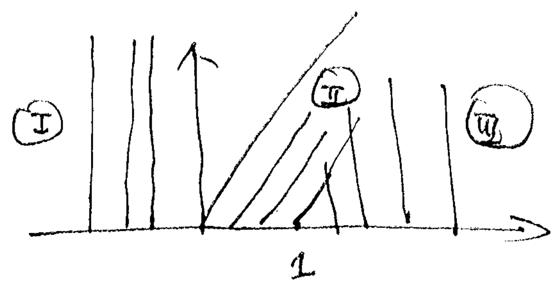
Problem (i): Method of Characteristics

$$\frac{dt}{1} = \frac{dx}{u^3} = \frac{du}{0}$$

$$\frac{dx}{dt} = u^3, \quad x(0) = \xi$$

$$\frac{du}{dt} = 0, \quad u = f_0(\xi) = \begin{cases} 1, & 0<\xi<1 \\ 0, & \text{otherwise} \end{cases}$$

So $x = (f_0(\xi))^3 t + \xi$



For $\xi<0$, $\quad x = \xi$

For $1<\xi$; $\quad x=\xi$

For $0<\xi<1$, $f_0(\xi)=1$, $x = t+\xi$, $0<\xi<1$

5 pts

Expansion Fan between (I) & (II): $\quad u = H(\lambda), \ \lambda = \frac{x}{t}$

$$u^3 = \lambda \Rightarrow H^3(\lambda) = \lambda \Rightarrow H = \lambda^{\frac{1}{3}} = \left(\frac{x}{t}\right)^{\frac{1}{3}}$$

3 pts

Shock between (II) and (III):

$$\begin{cases} \dfrac{ds}{dt} = \dfrac{[Q]}{[u]} = \dfrac{\frac{1}{4}u_+^4 - \frac{1}{4}u_-^4}{u_+ - u_-} = \dfrac{\frac{1}{4}\cdot 0^4 - \frac{1}{4}1^4}{0-1} = \dfrac{1}{4} \\ s(0) = 1 \end{cases}$$

5 pts

$$s = \frac{1}{4}t + 1$$

Prob

So the final solution is

$$u(x,t) = \begin{cases} 0, & x < 0 \\ (\frac{x}{t})^{\frac{1}{3}}, & 0 < x < t \\ 1, & t < x < \frac{t}{4} + 1 \\ 0 & 1 + \frac{t}{4} < x < +\infty \end{cases}$$ — (2pts)

(ii). The shock occurs when the expansion fan hits the shock:

$$\begin{cases} x = t \\ x = \frac{t}{4} + 1 \end{cases} \Rightarrow t_B = \frac{t_B}{4} + 1 \Rightarrow t_B = \frac{4}{3}, \ x_B = \frac{4}{3}$$

From $t > t_B$, we need a shock

$$\begin{cases} \frac{ds}{dt} = \frac{[Q]}{[u]} = \frac{\frac{1}{4}u_+^4 - \frac{1}{4}u_-^4}{u_+ - u_-} \\ \qquad = \frac{\frac{1}{4}\cdot 0 - \frac{1}{4}(\frac{x}{t})^{\frac{4}{3}}}{0 - (\frac{x}{t})^{\frac{1}{3}}} = \frac{1}{4}(\frac{x}{t}) \\ s(\frac{4}{3}) = \frac{4}{3} \end{cases}$$

$$\frac{ds}{s^{\frac{1}{3}}} = \frac{1}{4} t^{-\frac{1}{3}} dt \qquad \frac{3}{2} s^{\frac{2}{3}} = \frac{1}{4} \times \frac{3}{2} t^{\frac{2}{3}} + C$$

$\ln s = \frac{1}{4} \ln t$ $\qquad s^{\frac{2}{3}} = \frac{1}{4} t^{\frac{2}{3}} + C$

$s = C\, t^{\frac{1}{4}}$ $\qquad (\frac{4}{3})^{\frac{2}{3}} = \frac{3}{4}(\frac{4}{3})^{\frac{2}{3}} + C \Rightarrow C = \frac{3}{4}(\frac{4}{3})^{\frac{2}{3}} = (\frac{3}{4})^{\frac{1}{3}}$

$s(\frac{4}{3}) = \frac{4}{3} \Rightarrow \frac{4}{3} = C(\frac{4}{3})^{\frac{1}{4}} \Rightarrow C = (\frac{4}{3})^{\frac{3}{4}}$

$$s = \left(\frac{6}{8}\right)^{\frac{3}{6}} t^{\frac{1}{6}}$$

So
$$s = \left(\frac{1}{4}t^{\frac{2}{3}} + \left(\frac{3}{4}\right)^{\frac{1}{3}}\right)^{\frac{3}{2}}, \quad t > \frac{4}{3}$$
3 pts

The solution when $t > t_B$ is

$$u(x,t) = \begin{cases} 0, & x < 0 \\ \left(\frac{x}{t}\right)^{\frac{1}{3}}, & 0 < x < \left(\frac{1}{4}t^{\frac{2}{3}} + \left(\frac{3}{4}\right)^{\frac{1}{3}}\right)^{\frac{3}{2}} \\ 0, & \left(\frac{1}{4}t^{\frac{2}{3}} + \left(\frac{3}{4}\right)^{\frac{1}{3}}\right)^{\frac{3}{2}} < x < +\infty \end{cases}$$
1 pt

Problem 2: (i). Consider

$$\begin{cases} u_{tt} - 4u_{xx} = 0, & 0 < x < +\infty,\ t > 0 \\ u(x,0) = 0,\ u_t(x,0) = 0, & 0 < x < +\infty \\ u(0,t) = h(t), & t > 0 \end{cases}$$

The general solution of $u_{tt} - 4u_{xx} = 0$ is
$$u = f(x-ct) + g(x+ct).$$
3 pts

Now BC: $u(x,0) = 0,\ x > 0 \Rightarrow f(x) + g(x) = 0,\ x > 0$ ①

$u_t(x,0) = 0,\ x > 0 \Rightarrow -cf'(x) + cg'(x) = 0,\ x > 0$ ②

$u(0,t) = h(t),\ t > 0 \Rightarrow f(-ct) + g(ct) = h(t),\ t > 0.$

3 pts

①,② $\Rightarrow$: $\left.\begin{array}{l} f(x) + g(x) = 0 \\ f(x) - g(x) = A \end{array}\right\} \Rightarrow f(x) = \frac{A}{2},\ g(x) = -\frac{A}{2},\ x > 0$
1 pt

Prob

$$\sigma = ct$$

$$f(-\sigma) + g(\sigma) = h\left(\frac{\sigma}{c}\right), \quad \sigma > 0$$

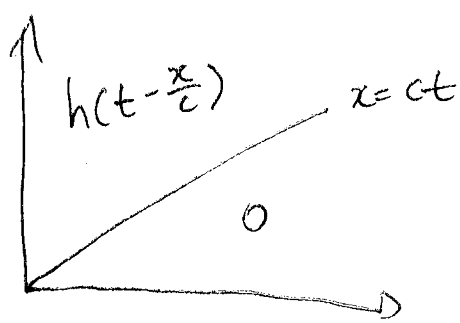
So
$$f(x) = \begin{cases} \frac{A}{2}, & x > 0 \\ h\left(\frac{-x}{c}\right) + \frac{A}{2}, & x < 0 \end{cases}$$

1 pts

So
$$u(x\cancel{=ct}) = f(x-ct) + g(x+ct)$$
$$= f(x-ct) + \frac{A}{2}$$
$$= \begin{cases} 0 & x - ct > 0 \\ h\left(\frac{ct-x}{c}\right), & x - ct < 0 \end{cases}$$

$$u(x) = \begin{cases} 0, & x > ct \\ h\left(t - \frac{x}{c}\right), & x < ct \end{cases}$$

2 pts



(ii) We decompose this problem into two problems

$$u_1 \quad \begin{cases} u_{tt} - 4u_{xx} = f(x,t), & 0 < x < +\infty,\ t > 0 \\ u(x,0) = \phi(x),\ u_t(x,0) = \psi(x), & 0 < x < +\infty \\ u(0,t) = 0,\ t > 0 \end{cases}$$

$$u_2 \quad \begin{cases} u_{tt} - 4u_{xx} = f(x,t) \\ u(x,0) = 0,\ u_t(x,0) = 0 \\ u(0,t) = h(t) \end{cases}$$

3pts

By (ii), $u_2(x,t) = \begin{cases} 0, & x \geq ct \\ h(t - \frac{x}{c}), & x < ct \end{cases}$

$$= \begin{cases} 0, & x > ct \\ e^{t - \frac{x}{c}}, & x < ct \end{cases}$$

Now we solve u_1 by the method of reflect ~~separation of varia~~

$$f_{ext}(x,t) = \begin{cases} f(x,t), & x > 0 \\ -f(-x,t), & x < 0 \end{cases} = xt$$

$$\phi_{ext}(x) = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$$

2pts

$$\psi_{ext}(x) = \begin{cases} \psi(x), & x > 0 \\ -\psi(-x), & x < 0 \end{cases} = \sin x$$

2d

By d'Alembert's Formula:

$$u_1(x,t) = \frac{1}{2}\left[\phi_{ext}(x-ct) + \phi_{ext}(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{ext}(y)\,dy$$

$$+ \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f_{ext}(y)\,dy\,ds$$

$$= \frac{1}{2}\left[\begin{cases} 1, & x > ct \\ -1, & x < ct \end{cases} + 1\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \sin y\,dy$$

$$+ \frac{1}{2c}\int_0^t \left(\int_{x-c(t-s)}^{x+c(t-s)} y\,dy\right) ds$$

$$= \frac{1}{2}\left[\begin{cases} 2, & x > ct \\ 0, & x < ct \end{cases}\right] + \frac{1}{2c}\cdot\left(\cos(x-ct) - \cos(x+ct)\right)$$

$$+ \frac{1}{2c}\int_0^t \frac{1}{2}\left[(x+c(t-s))^2 - (x-c(t-s))^2\right] ds$$

$$= \frac{1}{2}\left[\begin{cases} 2, & x > ct \\ 0, & x < ct \end{cases}\right] + \frac{1}{c}\sin x \sin ct$$

$$\underbrace{+ \frac{1}{2c}\int_0^t 2xc(t-s)\,ds} \qquad \frac{1}{2}x \cdot \frac{t^2}{8}$$

4 pts

Thus

$$
u(x,t)=\begin{cases} 1+\frac{1}{2}\sin x\sin ct+\frac{1}{6}xt^2, & x>ct \\ b\left(t-\frac{x}{c}\right)+\frac{1}{2}\sin x\sin ct+\frac{1}{6}xt^2, & x<ct \end{cases}
$$

— 1pt

(iii) Let $v=u_1-u_2$, where u_1, u_2 are two solutions.

Then $v_{tt}-4v_{xx}=0$

$v(x,0)=0,\ v_t(x,0)=0 \Rightarrow$

$v(0,t)=0,\ t>0 \Rightarrow v_t(0,t)=0$

1 pt

Define energy functional:

$$E(t)=\frac{1}{2}\int_0^{+\infty} v_t^2+\frac{c^2}{2}\int_0^{+\infty} v_x^2$$

2pts

Then

$$
\begin{aligned}
\frac{dE}{dt} &= \int v_t v_{tt}+c^2 v_x v_{tx} \\
&= c^2\int v_t v_{xx}+v_x v_{tx} \\
&= c^2\int_0 (v_t v_x)_x = +c^2 v_t v_x \Big|_0^{+\infty} \\
&= 0-c^2 v_t(0,t)v_x(0,t)=0.
\end{aligned}
$$

2pts

~~Prob~~

Now $E(t) = E(0) = \frac{1}{2}\int v_t(x,0)^2 + \frac{c^2}{2}\int v_x^2(x,0)$

$= 0$

So $E(t) = 0 \Rightarrow \int v_t^2 + c^2 \int v_x^2 = 0$

$\Rightarrow v_t = v_x = 0 \Rightarrow v = \text{constant}$

But $v(0,t) = 0 \Rightarrow v = 0$.

1pt

Problem 3

(i). Step 1 $u = X(x)T(t)$

$X'' + 2X' + X + \lambda X = 0, \quad 0 < x < 1$

$T' + \lambda T = 0.$

$X(0) = 0, \quad 2X'(1) - X(1) = 0$

2pts

Step 2. Solve EVP

$X'' + 2X' + X + \lambda X = 0$

$(e^{2x}X')' + e^{2x}X + \lambda e^{2x}X = 0.$

3pts

λ is real

Case 1 $\lambda = -\gamma^2 < 0. \quad \gamma > 0.$

$X = e^{rx} \qquad r^2 + 2r + 1 + \lambda = 0 \qquad (r+1)^2 = \gamma^2$

$r_1 = -1 + \gamma$, $r_2 = -1 - \gamma$

So $X(x) = e^{-x}(c_1 \cosh \gamma x + c_2 \sinh \gamma x)$.

$X(0) = 0 \Rightarrow c_1 = 0$.

So $X(x) = c_2 e^{-x} \sinh \gamma x$

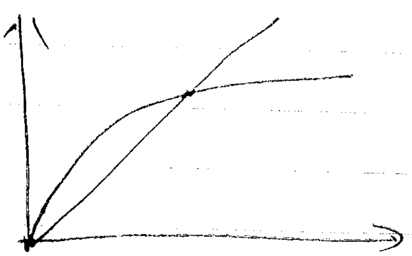
$X'(x) = -c_2 e^{-x} \sinh \gamma x + \gamma c_2 e^{-x} \cosh \gamma x$

$2X'(1) - X(1) = 0$

$\Rightarrow 2(-c_2 e^{-1} \sinh \gamma + \gamma c_2 e^{-1} \cosh \gamma) - c_2 e^{-1} \sinh \gamma = 0$

$\tanh \gamma = \frac{2}{3} \gamma$

5pts



So $\exists 1$ solution γ_0. $\Rightarrow X_0 = e^{-x} \sinh \gamma_0 x$

$T = e^{\cdot \gamma_0^2 t}$

Case 2 $\lambda = 0$.

$r^2 + 2r + 1 = 0 \Rightarrow r = -1$

$X(x) = e^{-x}(c_1 + c_2 x)$ $X(0) = 0 \Rightarrow c_1 = 0$

$2X'(1) - X(1) = 0 \Rightarrow$ impossible

2pt

Case 3. $\lambda=\beta^2,\ \beta>0$

$$(r+1)^2+\beta^2=0$$

$$r=-1\pm\beta i$$

$$X(x)=e^{-x}(c_1\cos\beta x+c_2\sin\beta x)$$

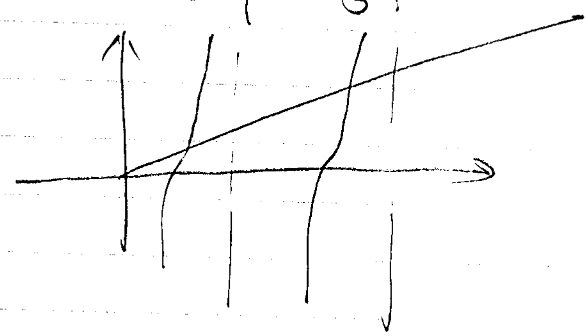
$X(0)=0\Rightarrow c_1=0 \qquad X(x)=e^{-x}\sin\beta x$

15 pts

$$X'=-e^{-x}\sin\beta x+\beta e^{-x}\cos\beta x$$

$$-2\sin\beta+2\beta\cos\beta-\sin\beta=0$$

$$\tan\beta=\frac{2}{3}\beta$$



$$\beta_n\in((n-1)\pi,\ n\pi),\quad n=1,2,\dots$$

$$X_n=e^{-x}\sin\beta_n x.$$

$$T=e^{-\beta_n^2 t}$$

step 3: Sum up

$$u(x,t)=a_0e^{\gamma_0^2 t}X_0(x)+\sum_{n=1}^{+\infty}a_n e^{-\beta_n^2 t}X_n(x)$$ — 1 pt

$$\phi(x) = a_0 X_0(x) + \sum_{n=1}^{+\infty} a_n X_n(x).$$

Now the ~~a~~ weight function is $e^{2x} = \omega(x)$

so

$$a_n = \frac{\int \phi X_n \omega}{\int X_n^2 \omega} \qquad \text{— 2pts.}$$

$$= \frac{\int_0^1 \phi X_n e^{2x} dx}{\int_0^1 X_n^2 e^{2x} dx} \qquad n = 0, 1, 2, \cdots$$

(ii). As $t \to +\infty$, $e^{-\beta_n^2 t} \longrightarrow 0$.

So the part $\sum_{n=1}^{+\infty} a_n e^{-\beta_n^2 t} X_n(x) \longrightarrow 0$ | 2pts

So u is bounded if and only if

$$a_0 = 0$$

$$\Leftrightarrow \int_0^1 \phi(x) X_0(x) e^{2x} dx = 0 \qquad \text{3pts}$$

$$\Leftrightarrow \int_0^1 \phi(x) e^{+x} \sinh \gamma_0 x \, dx = 0.$$

Problem 4 : u satisfies

$$\Delta u = 0 \text{ in } D$$

$$u = 1 + x^2 + 3xy$$

so $u(a,\theta) = 1 + a^2\cos^2\theta + 3a^2\cos\theta\sin\theta$

$= 1 + a^2 \frac{1+\cos 2\theta}{2} + \frac{3a^2}{2}\sin 2\theta$

$= 1 + 2a^2 + \frac{a^2}{2}(\cos 2\theta + 3\sin 2\theta)$

4 pts

So $\max_{\partial D} u(a,\theta) = 1 + 2a^2 + \frac{\sqrt{1+3^2}}{2} a^2$

$\min_{\partial D} u(a,\theta) = 1 + 2a^2 - \frac{\sqrt{10}}{2} a^2$

By mean-value-property

$u(0) = \frac{1}{2\pi}\int_0^{2\pi} u(a,\theta)\, d\theta = 1 + 2a^2$ | 2 pts

By Maximum Principle

$\max_D u = \max_{\partial D} u = 1 + 2a^2 + \frac{\sqrt{10}}{2} a^2$ | 2 pts

$\min_D u = \min_{\partial D} u = 1 + 2a^2 - \frac{\sqrt{10}}{2} a^2$ | 2 pts

Problem 5: First we write it in polar coordinate

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0 \qquad r > 2, \quad 0 < \theta < \frac{\pi}{2}$$

$$u_\theta(r, 0) = 0, \qquad u(r, \tfrac{\pi}{2}) = 0.$$

$$u(2, \theta) = a^2 \sin^2\theta = 4\sin^2\theta = 4 \cdot \frac{1-\cos 2\theta}{2} = 2(1-\cos 2\theta)$$

u bdd.

5pts

Method of Separation of Variables

Step 1: $u = R(r)\,\Theta(\theta)$

$$R'' + \frac{1}{r} R' - \frac{\lambda}{r^2} R = 0, \quad R \text{ bdd}$$

$$\Theta'' + \lambda\Theta = 0$$

$$\Theta'(0) = 0, \quad \Theta(\tfrac{\pi}{2}) = 0.$$

2pts

Step 2 Solve EVP

$$\lambda = \left(\frac{(n-\frac{1}{2})\pi}{\frac{\pi}{2}}\right)^2 = (2n-1)^2, \quad n = 1, 2, \cdots$$

$$\Theta_n = \cos((2n-1)\theta)$$

$$R = c_1 r^{2n-1} + c_2 r^{-(2n-1)} = c_2 r^{-(2n-1)}, \quad \text{bdd}.$$

5pts

step 3

$$u = \sum_{n=1}^{+\infty} a_n r^{-(2n-1)} \cos((2n-1)\theta)$$

-3pt

$$u(2,\theta) = \sum_{n=1}^{+\infty} a_n\, 2^{-(2n-1)} \cos((2n-1)\theta)$$

$$= 2(1 - \cos 2\theta)$$

$$2^{-(2n-1)} a_n = \frac{\int_0^{\frac{\pi}{2}} 2(1-\cos 2\theta)\cos((2n-1)\theta)\, d\theta}{\int_0^{\frac{\pi}{2}} \cos^2 (2n-1)\theta \, d\theta}$$

$$= \frac{8}{\pi} \cdot \int_0^{\frac{\pi}{2}} \Big[\cos(2n-1)\theta - \overbrace{\cos 2\theta \cos(2n-1)\theta}^{\frac{1}{2}\left[\cos(2n+1)\theta + \cos(2n-3)\theta\right]}\Big]$$

$$= \frac{8}{\pi} \left\{ \frac{1}{2n-1} \sin(2n-1)\cdot\frac{\pi}{2} - \frac{\sin(2n+1)\frac{\pi}{2}}{2(2n+1)} - \frac{\sin(2n-3)\frac{\pi}{2}}{2(2n-3)} \right\}$$ 5pts

$$= \frac{8}{\pi} \left\{ \frac{(-1)^{n-1}}{2n-1} - \frac{(-1)^n}{2(2n+1)} - \frac{(-1)^n}{2(2n-3)} \right\}$$

$$= \frac{8}{\pi}(-1)^{n-1} \left\{ \frac{1}{2n-1} + \frac{1}{2(2n+1)} + \frac{1}{2(2n-3)} \right\}$$

$$= \frac{8}{\pi}(-1)^{n-1} \left\{ \frac{1}{2n-1} - \frac{1}{2n+1} \right\}$$

$$= \frac{16}{\pi}(-1)^{n-1} \cdot \frac{1}{4n^2-1}$$

So $$u(r,\theta) = \sum_{n=1}^{+\infty} \frac{16}{\pi}(-1)^{n-1}\frac{1}{4n^2-1}\left(\frac{2}{r}\right)^{2n-1} \cos(2n-1)\theta$$