Solution formula for inhomogeneous wave equation:

$$\begin{cases} u_{tt} = c^2 u_{xx} + f(x,t), & t>0 \\ u(x,0)=\phi(x),\ u_t(x,0)=\psi(x) \end{cases}$$

$$u(x,t) = \frac{1}{2}\left[\phi(x-ct)+\phi(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(y)\,dy + \frac{1}{2c}\int_0^t\left(\int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,dy\right)ds$$

Example: Solve

$$\begin{cases} u_{tt} = 4u_{xx} + \cos x \\ u(x,0)=0,\ u_t(x,0)=0 \end{cases}$$

Solution: $u(x,t) = \frac{1}{2}[0+0] + \frac{1}{2c}\int_{x-ct}^{x+ct} 0\,dy$

$$+\frac{1}{2c}\int_0^t\left(\int_{x-c(t-s)}^{x+c(t-s)}\cos y\,dy\right)ds$$

$$= \frac{1}{4}\int_0^t\left(\int_{x-2(t-s)}^{x+2(t-s)}\cos y\,dy\right)ds = \frac{1}{4}\int_0^t \sin y\Big|_{x-2(t-s)}^{x+2(t-s)}\,ds$$

$$= \frac{1}{4}\int_0^t \left(\sin(x+2(t-s)) - \sin(x-2(t-s))\right)ds$$

$$= \frac{1}{2}\int_0^t \cos x\sin 2(t-s)\,ds = \cos x\cdot\left(+\frac{1}{4}\cos 2(t-s)\right)\Big|_0^t$$

$$= \cos x\,\frac{1}{4}(1-\cos 2t) = +\frac{1}{2}\cos x\sin^2 t$$