MATH400-101, 2018-2019 Homework Assignment 6 (Due Date: by 6pm of November 20 or by 9am Nov. 21, 2018)

1. (30pts) Put the following two problems in standard Sturm-Liouville form, identify the weight function $w(x)$, and calculate the eigenvalues and eigenfunctions. (You can use Bessel function of order zero $J_0(r)$: $J_0^{''}+\frac{1}{r}J_0^{'}+J_0=0, J(0)=1$.) Write down the general formula for the expansion of a general function $f(x)$ in terms of the eigenfunctions.

$$(a)\quad x^2X^{''}+3xX^{'}+\lambda X=0, 1<x<2,\quad X(1)=0,\quad X(2)=0$$

$$(b)\quad X^{''}-2X^{'}+\lambda X=0, 0<x<1;\quad X(0)=0,\quad X(1)=0;\qquad (c)\quad X^{''}+\frac{1}{x}X^{'}+\lambda X=0, 0<x<1, X^{'}(1)+X(1)=0$$

2. (10pts) Solve the following diffusion equation

$$\begin{cases} u_t=x^2u_{xx}+3xu_x,\ 1<x<2 \\ u(x,0)=1, \\ u(1,t)=u(2,t)=0 \end{cases}$$

You are allowed to use results in Problem 1.

3. (10pts) Solve the following wave equation

$$\begin{cases} u_{tt}=u_{xx}-2u_x,\ 0<x<1 \\ u(x,0)=x, u_t(x,0)=0 \\ u(0,t)=u(1,t)=0 \end{cases}$$

You are allowed to use results in Problem 1.

4. (10pts) Solve the following wave equation

$$\begin{cases} u_t=u_{rr}+\frac{1}{r}u_r\ 0<r<1 \\ u(r,0)=r^2 \\ u_r(0,t)=0, u_r(1,t)+u(1,t)=0 \end{cases}$$

You are allowed to use results in Problem 1. Write your solution in terms of Bessel function of order zero J_0.

5.(20pts) Use the method of separation of variables to solve

$$\begin{cases} u_t=u_{xx}+e^t\sin(2x),\ 0<x<\pi \\ u(x,0)=\sin(3x) \\ u(0,t)=t,\ u(\pi,t)=0 \end{cases}$$

6.(20pts) Use the method of separation of variables to solve

$$\begin{cases} u_{tt}=u_{xx}+e^{-t}\sin(3x),\ 0<x<\pi \\ u(x,0)=\sin(x), u_t(x,0)=0 \\ u(0,t)=1,\ u(\pi,t)=0 \end{cases}$$

7.(20pts) (a) Use the method of separation of variables to solve

$$\begin{cases} u_{xx}+u_{yy}=0, 0<x<\pi,\ 0<y<\pi, \\ u(0,y)=u_x(\pi,y)=u(x,0)=0, \\ u(x,\pi)=\sin\frac{x}{2}-2\sin\frac{3x}{2} \end{cases}$$

(b) Use divergence theorem to show the solutions to (a) are unique.

8. (10pts) Solve

$$u_{xx}+u_{yy}+u_{zz}=1$$

in the spherical shell $1<r<2$ with the boundary condition $\frac{\partial u}{\partial r}=1$ on $r=1$ and $u=1$ on $r=2$. Use the divergence theorem to verify your solution.

Hint: First solve it in radial coordinate $u_{xx}+u_{yy}+u_{zz}=u_{rr}+\frac{2}{r}u_r$. Then show the uniqueness.