Sol'ns to Problems 1-3 of HW6 (similar problems)

1 (a). $x^2 X'' - 5x X' + \lambda X = 0, \quad 1 < x < 2, \quad X(1) = X(2) = 0.$

Characteristic root equation: $X = x^r$

$$r(r-1) - 5r + \lambda = 0 \Rightarrow r^2 - 6r + \lambda = 0$$

$$r = \frac{6 \pm \sqrt{36 - 4\lambda}}{2} = 3 \pm \sqrt{9 - \lambda}$$

Case 1. $\lambda < 9$. $r_1 \neq r_2$

$$X = c_1 x^{r_1} + c_2 x^{r_2} \Rightarrow X(1) = 0 \Rightarrow c_1 + c_2 = 0$$

$$X(2) = 0 \Rightarrow c_1 2^{r_1} + c_2 2^{r_2} = 0$$

$$\begin{vmatrix} 1 & 1 \\ 2^{r_1} & 2^{r_2} \end{vmatrix} = 2^{r_2} - 2^{r_1} \neq 0 \Rightarrow c_1 = c_2 = 0$$

Case 2: $\lambda = 9$, $r_1 = r_2 = 3$

$$X = c_1 x^3 + c_2 x^3 \ln x \Rightarrow X(1) = 0 \Rightarrow c_1 = 0$$

$$X(2) \Rightarrow c_2 = 0$$

Case 3: $\lambda > 9$, $r = 3 \pm \sqrt{\lambda - 9}\, i$

$$X = c_1 x^3 \cos(\sqrt{\lambda - 9} \ln x) + c_2 x^3 \sin(\sqrt{\lambda - 9} \ln x)$$

$$X(1) = 0 \Rightarrow c_1 = 0$$

$$X(2) = 0 \Rightarrow c_2 \sin(\sqrt{\lambda - 9} \ln 2) = 0 \Rightarrow \sqrt{\lambda - 9} \ln 2 = n\pi$$

$$\lambda = 9 + \left(\frac{n\pi}{\ln 2}\right)^2, \quad n = 1, 2, \dots$$

$$X = x^3 \sin\left(\frac{n\pi \ln x}{\ln 2}\right)$$

Sturm-Liouville: $X'' - \frac{5}{x} X' + \frac{\lambda}{x^2} X = 0$, $\left.\begin{array}{l} p = \mu \\ p' = -\frac{5}{x}\mu \\ w = \frac{1}{x^2}\mu \end{array}\right\} \Rightarrow \begin{array}{l} p = x^{-5} \\ \mu = x^{-5} \\ \Rightarrow w = x^{-7} \end{array}$

$$(x^{-5} X')' + \lambda x^{-7} X = 0.$$

$$f(x) = \sum f_n X_n(x) = \sum f_n x^3 \sin\left(\frac{n\pi}{\ln 2} \ln x\right)$$

$$f_n = \frac{\int_1^2 x^{-7} f(x) x^3 \sin\left(\frac{n\pi}{\ln 2} \ln x\right) dx}{\int_1^2 x^{-7} \left(x^3 \sin\left(\frac{n\pi}{\ln 2} \ln x\right)\right)^2 dx}$$

(b) Characteristic root eqn is: Consider $\boxed{X''-4X'+\lambda X=0}$

$X(0)=X(1)=0$

$$r^2-4r+\lambda=0 \ . \quad r=2\pm\sqrt{4-\lambda}$$

Case1 $\lambda<2$, $r_1\neq r_2$

$$X=c_1e^{r_1x}+c_2e^{r_2x}, \quad X(0)=0,\ X(1)=0 \Rightarrow$$

$$c_1e^{r_1\cdot 0}+c_2e^{r_2\cdot 0}=0 \qquad \begin{vmatrix} e^{r_1 0} & e^{r_2 0} \\ e^{2r_1} & e^{2r_2} \end{vmatrix}\neq 0 \Rightarrow c_1=c_2=0$$

$$c_1e^{2r_1}+c_2e^{2r_2}=0$$

Case2 $\lambda=2$, $r_1=r_2=2$

$$X=c_1e^{2x}+c_2xe^{2x}, \quad X(0)=0 \Rightarrow c_1=0$$

$$X(1)=0 \Rightarrow c_2=0$$

Case3. $\lambda>4$, $r=2\pm\sqrt{\lambda-4}\,i$

$$X=c_1e^{2x}\cos(\sqrt{\lambda-4}\,x)+c_2e^{2x}\sin(\sqrt{\lambda-4}\,x)$$

$$X(0)=0 \Rightarrow c_1=0$$

$$X(1)=0 \Rightarrow \sin(\sqrt{\lambda-4})=0 \Rightarrow \lambda=4+(n\pi)^2,\ n=1,2,\dots$$

$$X=e^{2x}\sin(n\pi x)$$

Sturm-Liouville: $\left.\begin{array}{l} p=\mu \\ p'=-4\mu \\ w=\mu \end{array}\right\} \Rightarrow p=e^{-4x} \Rightarrow \mu=e^{-4x}=w$

$$(e^{-4x}X')'+\lambda e^{-4x}X=0, \quad w=e^{-4x}$$

$$f=\sum f_n e^{2x}\sin(n\pi x)$$

$$f_n=\frac{\int_0^1 e^{-4x}f(x)e^{2x}\sin n\pi x\,dx}{\int_0^1 e^{-4x}(e^{2x}\sin(n\pi x))^2dx}$$

2. $\begin{cases} u_t = x^2 u_{xx} - 5x u_x \\ u(x,0) = 1 \\ u(1,t) = u(2,t) = 0 \end{cases}$

<u>step1</u>. $x^2 X'' - 5x X + \lambda X = 0, \quad T' + \lambda T = 0, \quad X(1) = X(2) = 0$

step2. $\lambda_n = 9 + \left(\frac{n\pi}{\ln 2}\right)^2, \quad X = x^3 \sin\left(\frac{n\pi}{\ln 2} \ln x\right)$

$T = e^{-\lambda_n t}$

step3. $u(x,t) = \sum a_n e^{-\lambda_n t} x^3 \sin\left(\frac{n\pi}{\ln 2} \ln x\right)$

$u(x,0) = 1 = \sum a_n x^3 \sin\left(\frac{n\pi}{\ln 2} \ln x\right)$

$$a_n = \frac{\int_1^2 x^{-7} \cdot 1 \, x^3 \sin\left(\frac{n\pi}{\ln 2} \ln x\right) dx}{\int_1^2 x^{-7} \left(x^3 \sin\left(\frac{n\pi}{\ln 2} \ln x\right)\right)^2 dx}$$

3. $\begin{cases} u_{tt} = u_{xx} - 4x u_x, \quad 0 < x < 1 \\ u(0,t) = u(1,t) = 0 \\ u(x,0) = x, \ u_t(x,0) = 0 \end{cases}$

<u>step1</u>. $X'' - 4X' + \lambda X = 0, \quad T'' + \lambda T = 0, \quad X(0) = X(1) = 0$

step2 $\lambda_n = 4 + (n\pi)^2, \quad X_n = e^{2x} \sin(n\pi x)$

$T = c_1 \cos \sqrt{\lambda_n}\, t + c_2 \sin(\sqrt{\lambda_n}\, t)$

step2, $u(x,t) = \sum e^{2x} \sin(n\pi x) \left(a_n \cos\sqrt{\lambda_n} t + b_n \sin(\sqrt{\lambda_n} t)\right)$

$u(x,0) = x = \sum a_n e^{2x} \sin(n\pi x)$

$$a_n = \frac{\int_0^1 e^{-4x} x \, e^{2x} \sin(n\pi x)}{\int_0^1 e^{-4x} (e^{2x} \sin n\pi x)^2}$$

$u_t(x,0) = 0 \Rightarrow b_n = 0$