MATH400-101 Homework Assignment 7, 2018-2019 (Due Date: by 6pm, December 4, 2018)

Last homework. Good luck on finals

Please either hand in to my office or send it by email by 6pm of December 4th, 2018. The solutions will be put on my website on December 4th.

1. (10pts) Use the method of separation of variables to find solutions to

$$u_t = k(u_{xx} + u_{yy}), 0 < x < \pi, 0 < y < \pi$$

$$u(x, y, 0) = \sin(x), 0 < x < \pi, 0 < y < \pi$$

$$u(0, y, t) = u(\pi, y, t) = u(x, 0, t) = u(x, \pi, t) = 0$$

2. (10pts) Consider the following Laplace equation in an annulus

$$u_{xx} + u_{yy} = 0 \ \text{ in } \{1 < x^2 + y^2 < 4\}$$

$$u(x, y) = x^2 - x, \ \text{for} \ \ x^2 + y^2 = 1,$$

$$u(x, y) = y, \ \text{for} \ \ x^2 + y^2 = 4.$$

(i) Find out the maximum and minimum values of u in the annulus

(ii) Use the method of separation of variables to find u.

3. (10pts) Solve the following exterior domain problem

$$\begin{cases} u_{xx} + u_{yy} = 0, & \text{for } x^2 + y^2 > 1 \\ u(x, y) = 2y^2 - 3x & \text{for } x^2 + y^2 = 1 \\ u(x, y) \text{ is bounded} \end{cases}$$

4. (20pts) Use the method of separation of variables to solve the Laplace equation in the quarter plane

$$u_{xx} + u_{yy} = 0 \ \text{ in } \{x^2 + y^2 > 1, x > 0, y > 0\}$$

$$u(x, 0) = 0, \ \text{for} \ \ x^2 + y^2 > 1, x > 0,$$

$$u(0, y) = 0, \ \text{for} \ \ x^2 + y^2 > 1, y > 0,$$

$$\frac{\partial u}{\partial r} = 1 \ \text{ on } x^2 + y^2 = 1, x > 0, y > 0$$

$$u \text{ is bounded}$$

(ii) Suppose that u is bounded and that $\lim_{r \to +\infty} r|\nabla u(x, y)| = 0$, where $r = \sqrt{x^2 + y^2}$. Show that the solution to (i) is unique.

5.(20pts) Use the method of separation of variables to solve the following PDE:

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 \ \text{ in } D = \{(r, \theta) \mid 1 < r < 2, 0 < \theta < \frac{\pi}{4}\}$$

$$u(1, \theta) = 2\sin^2(2\theta), \ u(2, \theta) = 1$$

$$u_\theta(r, 0) = 0, u_\theta(r, \frac{\pi}{4}) = 0$$

6. (20pts) Solve the wave equation using the Bessel functions of order n

$$\begin{cases} u_{tt} = c^2(u_{rr} + \frac{1}{r}u_r + \frac{u_{\theta\theta}}{r^2}), \ 0 \le r < 1, 0 \le \theta < 2\pi, t > 0 \\ u(1, \theta, t) = t^2, \ t > 0, 0 \le \theta < 2\pi \\ u(r, \theta, 0) = 1, \ \ u_t(r, \theta, 0) = \sin\theta \end{cases}$$

Here the Bessel function of order n of first kind, denoted by $J_n(z)$, is given as the solution to

$$J'' + \frac{1}{z}J' + J - \frac{n^2}{z^2}J = 0, z > 0, \quad J(z) \sim z^n, \text{ as } z \to 0$$

7. (20pts) Solve the diffusion equation using the Bessel functions

$$\begin{cases} u_t = k(u_{rr} + \frac{1}{r}u_r + \frac{u_{\theta\theta}}{r^2}), \ 0 \le r < a, 0 \le \theta < 2\pi, t > 0 \\ u(a, \theta, t) = 0, t \ge 0, 0 \le \theta < 2\pi \\ u(r, \theta, 0) = 1 - 2\sin\theta \end{cases}$$

8. (20pts) Find an infinite series representation (in terms of the Bessel function) for the diffusion equation problem

$$\begin{cases} u_t = k(u_{rr} + \frac{1}{r}u_r + u_{zz}), \ 0 \le r < a, 0 < z < b, \ t > 0 \\ u_r(a, z, t) = 0, u(r, 0, t) = 0, \ u(r, b, t) = 0 \\ u(r, z, 0) = \phi(r, z) \end{cases}$$