Be sure this exam has 17 pages including the cover.

The University of British Columbia

MATH 400, Section 101, 2019-2020

Final Exam

2.5 Hours

Name ______________________ Signature ______________________

Student Number ______________________ Section ______________________

This exam consists of **10** questions. No notes. No calculators are allowed. A list of formulas is provided on the last page. Write your answer in the blank page provided.

Problem	max score	score
1.	12	
2.	20	
3.	8	
4.	6	
5.	8	
6.	6	
7.	8	
8.	8	
9.	8	
10.	16	
total	100	

(12 points) 1. This question has three parts.

(4 points) (a) Find the general solutions to

$$u_x + 2xyu_y = u.$$

(6 points) (b) Solve the following problem and find the domain of existence

$$u_x + 2xyu_y = u,$$

$$u(x,1) = e^{2x}, -\infty < x < +\infty.$$

(2 points) (c) Find the function $h(x)$ so that the following problem admits a solution:

$$u_x + 2xyu_y = u,$$

$$u(x, e^{x^2}) = h(x), \quad -\infty < x < +\infty.$$

(a) Characteristic curve

$$\frac{dx}{1} = \frac{dy}{2xy} \Rightarrow 2x\,dx = \frac{1}{y}dy$$

$$\Rightarrow x^2 = \ln y + c \Rightarrow y = ce^{x^2}$$

$$\xi = ye^{-x^2}$$

Let $x' = x$, $y' = ye^{-x^2}$, $u = U$ $\Rightarrow U_{x'} = U \Rightarrow U = f(\xi)e^{x'}$

General sol'ns: $u = f(ye^{-x^2})e^{x}$

(b) $f(e^{-x^2})e^{x} = e^{2x} \Rightarrow f(e^{-x^2}) = e^{x}$

Let $X = e^{-x^2} \Rightarrow x = \pm\sqrt{-\ln X}$

$$f(X) = e^{\pm\sqrt{-\ln X}}$$

$$u(x,y) = e^{\pm\sqrt{-\ln(ye^{-x^2})}}\, e^{x}$$

First, $0 < ye^{-x^2} \leq 1 \Rightarrow 0 < y \leq e^{x^2}$

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Then $u(x,1) = e^{x} \Rightarrow$

$$e^{\pm\sqrt{x^2}} = e^{x}$$

So $u(x,y) = \begin{cases} e^{\sqrt{-\ln(ye^{-x^2})}}, & 0 < y \le e^{x^2},\ x > 0 \\ e^{-\sqrt{-\ln(ye^{-x^2})}}, & 0 < y \le e^{x^2},\ x < 0 \end{cases}$

Domain of existence $0 < y \le e^{x^2}$

(c) $u(x, e^{x^2}) = f(1)\, e^{x}$

So $h(x) = A e^{x}$

(20 points) 2. Solve the following quasi-linear partial differential equation

$$u_t + (4-2u)u_x = 0, \ -\infty < x < +\infty, t > 0,$$

$$u(x,0) = \begin{cases} 2, & x < 0; \\ 1, & 0 < x < 2; \\ 2, & 2 < x < +\infty. \end{cases}$$

(2 points) (a) Find the expansion fan solution.

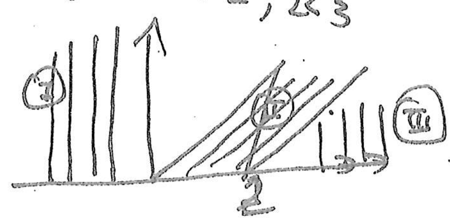
(13 points) (b) Find the solution before the characteristic curve hits the expansion fan.

(5 points) (c) Find the solution after the characteristic curve hits the expansion fan.

(a). $4 - 2U = \frac{x}{t} \Rightarrow U = 2 - \frac{x}{2t}$

(b) General sol'ns are

$x - \xi = (4 - 2f(\xi))t$, $\quad u = f(\xi) = \begin{cases} 2, & \xi < 0 \\ 1, & 0 < \xi < 2; \\ 2, & 2 < \xi \end{cases}$



For $\xi < 0 \Rightarrow \quad x - \xi = 0 \quad \Rightarrow x = \xi < 0$

$0 < \xi < 2 \Rightarrow \quad x - \xi = 2t \Rightarrow x - 2t = \xi$

$\xi > 2 \Rightarrow \quad x - \xi = 0 \Rightarrow x = \xi > 2$

Between I & II, insert an expansion fan

Between II and III, insert a shock curve

$Q(u) = 4u - u^2$, $\quad \frac{ds}{dt} = \frac{Q(u^+) - Q(u^-)}{u^+ - u^-} = 4 - (u^+ + u^-) = 4 - (2+1)$
$= 1$

$S(0) = 2$

$S = t + 2$

Shock curve hits expansion fan at $\begin{cases} x = 2t \\ x = t + 2 \end{cases} \Rightarrow t = 2, x = 4$

For $t < 2$

$$u(x,t) = \begin{cases} 2, & x < 0 \\ 2 - \frac{x}{2t} & 0 < x < 2t \\ 1, & 2t < x < t+2 \\ 2, & x > t+2 \end{cases}$$

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(c) For $t>2$, insert a shock curve.

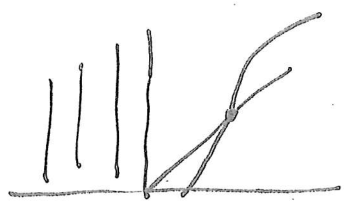
$$u^- = 2 - \frac{x}{2t} = 2 - \frac{s}{2t}$$

$$u^+ = 2.$$

$$\frac{ds}{dt} = 4 - (u^+ + u^-) = 4 - \left(2 + 2 - \frac{s}{2t}\right) = \frac{s}{2t}$$

$$s(2) = 4$$

$$s = 2\sqrt{2t}$$



For $t>2$

$$u(x,t) = \begin{cases} 2, & x<0 \\ 2 - \frac{x}{2t}, & 0 < x < 2\sqrt{2t} \\ 2, & 2\sqrt{2t} < x \end{cases}$$

(8 points) 3. This problem has two parts.

(2 points) (a) Determine the type of the following second order PDE

$$u_{xx} - 2u_{xy} + u_{yy} - u_x - u_y = 0$$

(6 points) (b) Find a linear transformation so that the PDE becomes one of the following standard form:

$$u_{\xi\xi} + u_{\eta\eta} = 0; \; u_{\xi\xi} - u_{\eta\eta} = 0; \; u_{\xi\xi} - u_\eta = 0$$

(a) $\partial_x^2 - 2\partial_x\partial_y + \partial_y^2 - \partial_x - \partial_y$

$= (\partial_x - \partial_y)^2 - (\partial_x + \partial_y)$

parabolic

(b) Let $\partial_\xi = \partial_x - \partial_y$ $\Rightarrow$ $\partial_x = \frac{1}{2}\partial_\xi + \frac{1}{2}\partial_\eta$

$\partial_\eta = \partial_x + \partial_y$ $\quad$ $\partial_y = -\frac{1}{2}\partial_\xi + \frac{1}{2}\partial_\eta$

$$\begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \partial_\xi \\ \partial_\eta \end{pmatrix}$$

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Under the linear transformation

$$\xi = \frac{x}{2} - \frac{y}{2}, \quad \eta = \frac{x}{2} + \frac{y}{2}$$

the PDE becomes

$$u_{\xi\xi} - u_\eta = 0$$

(6 points) 4. Solve the following heat equation

$$u_t = \frac{1}{4}u_{xx}, -\infty < x < +\infty, t > 0,$$

$$u(x,0) = \begin{cases} 1, & x < 0; \\ 2, & 0 < x < 1; \\ 0, & x > 1 \end{cases}$$

Write your solution in terms of the error function $\mathrm{Erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-p^2}\,dp$. Hint: the source function is given by

$$S(x,t) = \frac{1}{\sqrt{4k\pi t}}e^{-\frac{x^2}{4kt}}$$

Sol'ns:

$$u(x,t) = \frac{1}{\sqrt{\pi t}}\int_{-\infty}^{+\infty} e^{-\frac{(x-y)^2}{t}}\phi(y)\,dy$$

$$= \frac{1}{\sqrt{\pi t}}\left(\int_{-\infty}^{0} e^{-\frac{(x-y)^2}{t}}dy + 2\int_0^1 e^{-\frac{(x-y)^2}{t}}dy\right)$$

$y = x + \sqrt{t}\,p$

$$= \frac{1}{\sqrt{\pi t}}\left(\int_{-\infty}^{-\frac{x}{\sqrt{t}}} e^{-p^2}\sqrt{t}\,dp + 2\int_{-\frac{x}{\sqrt{t}}}^{\frac{1-x}{\sqrt{t}}} e^{-p^2}\sqrt{t}\,dp\right)$$

$$= \frac{1}{\sqrt{\pi}}\left(\int_{-\infty}^{0} e^{-p^2}dp + \int_0^{-\frac{x}{\sqrt{t}}} e^{-p^2}dp + 2\left(\int_0^{\frac{1-x}{\sqrt{t}}} - \int_0^{-\frac{x}{\sqrt{t}}} e^{-p^2}dp\right.\right.$$

$$= \frac{1}{\sqrt{\pi}}\left(\frac{\sqrt{\pi}}{2} \quad \frac{\sqrt{\pi}}{2}\mathrm{Erf}\left(-\frac{x}{\sqrt{t}}\right) + 2\frac{\sqrt{\pi}}{2}\mathrm{Erf}\left(\frac{1-x}{\sqrt{t}}\right)\right)$$

$$= \frac{1}{2} - \frac{1}{2}\mathrm{Erf}\left(-\frac{x}{\sqrt{t}}\right) + \mathrm{Erf}\left(\frac{1-x}{\sqrt{t}}\right)$$

$$= \frac{1}{2} + \frac{1}{2}\mathrm{Erf}\left(\frac{x}{\sqrt{t}}\right) - \mathrm{Erf}\left(\frac{x-1}{\sqrt{t}}\right)$$

(8 points) 5. This problem has two parts.

(2 points) (a) Find the general solutions to

$$u_{tt} = 4u_{xx}$$

(6 points) (b) Solve the following wave equation for $u(x,t)$

$$u_{tt} = 4u_{xx}, 0 < x < +\infty, t > 0,$$

$$u(0,t) = t^2, 0 < t < +\infty$$

$$u(x,0) = 0, u_t(x,0) = 0, 0 < x < +\infty.$$

(a) $u = f(x+2t) + g(x-2t)$

(b) $u(0,t) = t^2,\ t>0 \Rightarrow f(2t) + g(-2t) = t^2,\ t>0$

$u(x,0)=0 \Rightarrow f(x) + g(x) = 0,\ x>0$

$u_t(x,0)=0 \Rightarrow 2f'(x) - 2g'(x) = 0,\ x>0$

So $f(x) = g(x) = 0,\ x>0$

$g(-2t) = t^2 - f(2t) = t^2$

$X = -2t < 0 \Rightarrow g(X) = \left(-\frac{X}{2}\right)^2 = \frac{1}{4}X^2$

$f(x) = 0,\ x>0$

$g(x) = \begin{cases} 0, & x>0 \\ \frac{1}{4}x^2, & x<0 \end{cases}$

$u(x,t) = f(x+2t) + g(x-2t) = g(x-2t) = \begin{cases} 0, & x>2t \\ \frac{1}{4}(x-2t)^2, & x<2t \end{cases}$

(6 points) 6. This problem contains two parts.

(2 points) (a) Write the following eigenvalue problem into a standard Sturm-Liouville eigenvalue problem

$$X'' + \frac{1}{x}X' - 2xX' + \lambda X = 0.$$

(4 points) (b) Consider the following eigenvalue problem

$$X'' + \lambda X = 0, 0 < x < 2,$$

$$X'(0) + 4X(0) = 0, X'(2) + X(2) = 0.$$

Determine the number of negative eigenvalues and write down the algebraic equation for the negative eigenvalues.

Hint: You can use the formula sheet at the end of the exam paper.

(a) $\mu X'' + (\frac{1}{x} - 2x)\mu X' + \lambda \mu X = 0$

$\mu = p, \quad p' = (\frac{1}{x} - 2x)\mu, \quad w = \mu$

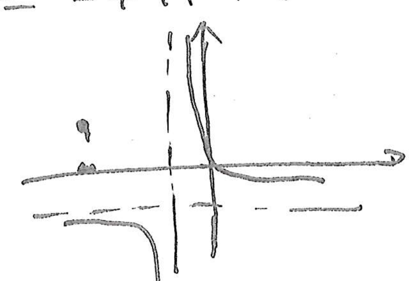
$\frac{p'}{p} = \frac{1}{x} - 2x \Rightarrow \ln p = \ln x - x^2$

$p = x e^{-x^2} = \mu = w$

$(xe^{-x^2} X')' + \lambda (x e^{-x^2} X) = 0$

(b) $l = 2, \quad a_0 = -4, \quad a_l = 1$

$a_0 + a_l + a_0 \cdot a_l \cdot l = -4 + 1 + (-4) \times 1 \cdot 2 = -5 < 0$



Region III. One negative eigenvalues

$\lambda = -\gamma^2, \quad \tanh(\gamma l) = -\frac{(a_0 + a_l)\gamma}{\gamma^2 + a_0 a_l}$

$\tanh(2\gamma) = \frac{3\gamma}{\gamma^2 - 4}$

(8 points) 7. Consider the following wave equation

$$u_{tt} = 4u_{xx}, 0 < x < 2, t > 0,$$

$$u(0,t) = 0,\ u(2,t) = 0,$$

$$u(x,0) = 0,\ u_t(x,0) = 1, 0 < x < 2.$$

(4 points) (a) Use the method of reflection to find $u(1,1)$.

(4 points) (b) Use the method of separation of variables to find $u(1,1)$.

(a) $\phi = 0 \Rightarrow \phi_{ext} = 0$

$\psi = 1 \Rightarrow \psi_{odd} = \begin{cases} 1, & 0<x<2 \\ -1, & -2<x<0 \end{cases}$, $\psi_{ext}(x+4) = \psi_{ext}(x)$

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{ext}(y)\,dy$$

$$u(1,1) = \frac{1}{4}\int_{-1}^{3} \psi_{ext}(y)\,dy = \frac{1}{4}\left(\int_{-1}^{1}\psi_{ext}(y)dy + \int_{1}^{3}\psi_{ext}(y)\,dy\right)$$

$$= \frac{1}{4}\left(0 + \int_1^2 \psi_{ext}(y)dy + \int_2^3 \psi_{ext}(y)\,dy\right)$$

$$= \frac{1}{4}\left(0 + \int_1^2 1\cdot dy + \int_2^3 (-1)\,dy\right) = 0$$

(b) $l = 2$, $c = 2$.

$$u(x,t) = \sum \left(a_n \cos\left(\frac{n\pi c}{l}t\right) + b_n \sin\left(\frac{n\pi c}{l}t\right)\right)\sin\left(\frac{n\pi}{l}x\right)$$

$\cancel{u(1,1) = \sum (a_n \cos(n\pi) + b_n \sin(n\pi))\sin(\frac{n\pi}{2})}$

$u(x,0) = 0 \Rightarrow a_n = 0$

$$u_t(x,0) = 1 = \sum b_n \frac{n\pi c}{l}\sin\left(\frac{n\pi}{l}x\right)$$

$$b_n \frac{n\pi c}{l} = \frac{2}{l}\int_0^l 1 \sin\left(\frac{n\pi}{l}x\right)dx$$

$$b_n = \frac{2}{n\pi c}\cdot\frac{l}{n\pi}(1-(-1)^n) = \frac{2}{(n\pi)^2}(1-(-1)^n)$$

$$\boxed{\begin{aligned} &u(1,1) \\ &= \sum_n b_n \sin n\pi \sin\left(\frac{n\pi}{2}\right) \\ &= \sum_n \frac{2}{(n\pi)^2}(1-(-1)^n)\sin n\pi \sin\left(\frac{n\pi}{2}\right)\end{aligned}}$$

$= 0$

(8 points) 8. Consider the following Laplace equation

$$u_{rr} + \frac{1}{r}u_r + \frac{u_{\theta\theta}}{r^2} = 0, 0 \le r < 1, 0 \le \theta < 2\pi$$

$$u_r(1,\theta) + u(1,\theta) = \cos(2\theta)$$

(4 points) (a) Use the method of separation of variables to find a solution.

(4 points) (b) Show that the solution is unique. Hint: you may use the following divergence theorem

$$\int\int_\Omega w\Delta w = \int_{\partial\Omega} w\frac{\partial w}{\partial n} - \int\int_\Omega |\nabla w|^2$$

(a). $u = \frac{a_0}{2} + \sum_{n=1}^{+\infty} r^n (a_n \cos n\theta + b_n \sin n\theta)$

$u_r(1,\theta) = \sum_{n=1}^{+\infty} n r^{n-1} (a_n \cos n\theta + b_n \sin n\theta)$

$= \sum_{n=1}^{+\infty} n (a_n \cos n\theta + b_n \sin n\theta)$

$u_r(1,\theta) + u(1,\theta) = \frac{a_0}{2} + \sum_{n=1}^{+\infty} (n+1)(a_n \cos n\theta + b_n \sin n\theta)$

$= \cos 2\theta$

$\Rightarrow a_0, a_n = 0, n \neq 2, b_n = 0$

$(2+1) a_2 = 1 \Rightarrow a_2 = \frac{1}{3}$

$u(r,\theta) = r^2 \cdot \frac{1}{3} \cos 2\theta$

(b). Let u_1, u_2 be two sol'ns. $w = u_1 - u_2$. Then

$$\begin{cases} \Delta w = w_{rr} + \frac{1}{r} w_r + \frac{w_{\theta\theta}}{r^2} = 0 & \text{in } \Omega = B_1(0) \\ \frac{\partial w}{\partial n} + w = w_r(1,\theta) + w(1,\theta) = 0, & \text{on } \partial\Omega = B_1(0) \end{cases}$$

$0 = \iint_{B_1} w\Delta w = \int_{\partial B_1} w \frac{\partial w}{\partial n} - \iint_{B_1} |\nabla w|^2 = -\int_{\partial B_1} w^2 - \iint_{B_1} |\nabla w|^2$

$\iint_{B_1} |\nabla w|^2 + \int_{\partial B_1} w^2 = 0 \Rightarrow w \equiv C$, $w = 0$ on $\partial B \Rightarrow w \equiv 0$ in B_1

So $u_1 \equiv u_2$ in B_1

(8 points) 9. Use the method of separation of variables to solve the following Laplace equation

$$\Delta u = u_{xx} + u_{yy} = 0, 0 < x < +\infty, 0 < y < \pi$$

$$u(x,0) = u_y(x,\pi) = 0, 0 < x < +\infty$$

$$u(0,y) = y, 0 < y < \pi$$

$$u(x,y) \text{ is bounded}$$

Hint: $\int y \sin(ay)dy = -\frac{y}{a}\cos(ay) + \frac{1}{a^2}\sin(ay) + C$.

Sol'n: Step 1. $u = X(x)\,Y(y)$

$\frac{X''}{X} + \frac{Y''}{Y} = 0$. $\Rightarrow \begin{cases} Y'' + \lambda Y = 0 \\ Y(0) = Y'(\pi) = 0 \end{cases}$

$Y(0) = Y'(\pi) = 0$

$X'' - \lambda X = 0$

Step 2. Solve (EVP) $\Rightarrow Y = \sin \beta y$, $Y'(\pi) = 0$

$\cos(\beta\pi) = 0 \Rightarrow \beta\pi = (n-\frac{1}{2})\pi \Rightarrow \beta = (n-\frac{1}{2})$

$\lambda = (n-\frac{1}{2})^2$, $n = 1, 2, \cdots$

$X = C_1 e^{\sqrt{\lambda}\,x} + C_2 e^{-\sqrt{\lambda}\,x}$

Since u is bounded, $X = C_2 e^{-\sqrt{\lambda}x} = C_2 e^{-(n-\frac{1}{2})x}$

Step 3. $u(x,y) = \sum_{n=1}^{+\infty} a_n e^{-(n-\frac{1}{2})x} \sin\left((n-\tfrac{1}{2})y\right)$

$u(0,y) = y \Rightarrow y = \sum_{n=1}^{+\infty} a_n \sin((n-\frac{1}{2})y)$

$$a_n = \frac{\int_0^\pi y \sin((n-\frac{1}{2})y)dy}{\int_0^\pi \sin^2((n-\frac{1}{2})y)dy} = \frac{2}{\pi}\int_0^\pi y \sin((n-\tfrac{1}{2})y)\,dy$$

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$$\int_0^{\pi} y \sin\left(\left(n-\tfrac{1}{2}\right) y\right)$$

$$= -\frac{y}{(n-\frac{1}{2})} \cos\left(\left(n-\tfrac{1}{2}\right) y\right) + \frac{1}{(n-\frac{1}{2})^2} \sin\left(\left(n-\tfrac{1}{2}\right) y\right)\Big|_0^{\pi}$$

$$= \frac{1}{(n-\frac{1}{2})^2} \sin\left(\left(n-\tfrac{1}{2}\right)\pi\right) = -\frac{(-1)^n}{(n-\frac{1}{2})^2}$$

$$u(x,y) = -\sum_{n=1}^{+\infty} \frac{2(-1)^n}{\pi (n-\frac{1}{2})^2} e^{-(n-\frac{1}{2})x} \sin\left(\left(n-\tfrac{1}{2}\right) y\right)$$

(16 points) 10. Use the method of separation of variables to solve the following diffusion equation

$$u_t = u_{rr} + \frac{1}{r}u_r + \frac{u_{\theta\theta}}{r^2} + u_{zz},\ 0 \le r < 1,\ 0 \le \theta < 2\pi, 0 < z < \pi$$

$$u(1, \theta, z, t) = 0,\ 0 \le \theta < 2\pi, 0 < z < \pi,$$

$$u(r, \theta, 0, t) = 0, u(r, \theta, \pi, t) = 0, 0 \le r < 1, 0 \le \theta < 2\pi,$$

$$u(r, \theta, z, 0) = r\cos\theta\sin(2z),\ 0 \le r < 1,\ 0 \le \theta < 2\pi, 0 < z < \pi.$$

(6 points) (a) Let $u(r, \theta, z, t) = R(r)\Theta(\theta)Z(z)T(t)$. Find the corresponding eigenvalue problems and ODE.

(7 points) (b) Solve the corresponding eigenvalue problems and ODE. You may use the Bessel function of order n:

$$J_n'' + \frac{1}{y}J_n' - \frac{n^2}{y^2}J_n + J_n = 0, J_n(y) \sim y^n \text{ as } y \to 0.$$

The zeroes of $J_n(y)$ are denoted by $z_{m,n}$, $m = 1, ..., +\infty$.

(3 points) (c) Sum-up and find the corresponding coefficients.

Sol'ns. (a). $\frac{T'}{T} = \frac{R'' + \frac{1}{r}R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} + \frac{Z''}{Z}$

$R(1) = 0$, $Z(0) = Z(\pi) = 0$

Θ is 2π-periodic

$(EVP)_1$ $Z'' + \lambda_1 Z = 0$, $Z(0) = Z(\pi) = 0$

$(EVP)_2$ $\Theta'' + \lambda_2\Theta = 0$, $\Theta(0) = \Theta(2\pi)$, $\Theta'(0) = \Theta'(2\pi)$

$$\frac{R'' + \frac{1}{r}R'}{R} - \frac{\lambda_2}{r^2} - \lambda_1 = -\lambda_3$$

$(EVP)_3$ $R'' + \frac{1}{r}R' - \frac{\lambda_2}{r^2}R + (\lambda_3 - \lambda_1)R = 0$, $R(1) = 0$

(ODE) $T' + \lambda_3 T = 0$

(b) $\lambda_1 = k^2$, $Z = \sin kz$

$\lambda_2 = -n^2$, $\Theta = a\cos n\theta + b\sin n\theta$

$R'' + \frac{1}{r}R' - \frac{n^2}{r^2}R + \mu R = 0$, $\mu = \lambda_3 - \lambda_1$

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$R = J_n(\sqrt{\mu}\, r)$, $\quad J_n(\sqrt{\mu}) = 0 \Rightarrow \sqrt{\mu} = z_{m,n}$

$\mu = z_{m,n}^2$

$\lambda_3 = \lambda_1 + \mu = k^2 + z_{m,n}^2$

$T = e^{-(k^2 + z_{m,n}^2)t}$

(c) $$u = \sum_{m=1}^{+\infty} \sum_{n=0}^{+\infty} \sum_{k=1}^{+\infty} \left(a_{m,n,k} \cos n\theta + b_{m,n,k} \sin n\theta\right) J_n(z_{m,n} r)\, e^{-(k^2}$$
$\sin kz$

$$u(r,\theta,z,0) = r\cos\theta \sin(2z)$$
$$= \sum_{m=1}^{+\infty} \sum_{n=0}^{+\infty} \sum_{k=0}^{+\infty} \left(a_{m,n,k} \cos n\theta + b_{m,n,k} \sin n\theta\right) J_n(z_{m,n} r) \sin$$

so $n=1$, $k=2$, $b_{m,n,k} = 0$

$$r = \sum_{m=1}^{+\infty} a_{m,1,2} J_1(z_{m,1} r)$$

$$a_{m,1,2} = \frac{\int_0^1 r J_1(z_{m,1} r)\, r\, dr}{\int_0^1 J_1^2(z_{m,1} r)^2\, r\, dr}$$

$$u(r,\theta,z,t) = \sum_{m=1}^{+\infty} \left(a_{m,1,2} J_1(z_{m,1} r)\right) \cos\theta \sin 2z\, e^{-(4+z_{m,1}^2}$$

Some formulas

1. Algebraic equations for negative and positive eigenvalues for Robin boundary conditions: $X'(0) - a_0 X(0) = 0, X'(l) + a_l X(l) = 0$

Equation for negative eigenvalues:

$$\lambda = -\gamma^2 < 0, \ \tanh(\gamma l) = -\frac{(a_0 + a_l)\gamma}{\gamma^2 + a_0 a_l}, \ X = \cosh(\gamma x) + \frac{a_0}{\gamma}\sinh(\gamma x)$$

Equation for zero eigenvalue

$$a_0 + a_l + a_0 a_l l = 0, \lambda = 0, X = 1 + a_0 x$$

Equation for positive eigenvalues

$$\lambda = \beta^2 > 0, \ \tan(\beta l) = \frac{(a_0 + a_l)\beta}{\beta^2 - a_0 a_l}, \quad X = \cos(\beta x) + \frac{a_0}{\beta}\sin(\beta x)$$

2. d'Alembert's formula for wave equation

$$u(x,t) = \frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)dy + \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y, ds)dyds$$

3. Eigenvalue problems for periodic boundary conditions:

$$\Theta'' + \lambda\Theta = 0, 0 < \theta < 2\pi, \Theta(0) = \Theta(2\pi), \Theta'(0) = \Theta'(2\pi).$$

$$\lambda_n = n^2, n = 0, 1, 2, ..., \Theta(\theta) = a\cos(n\theta) + b\sin(n\theta)$$