List of Formulas for Midterm 2 of MATH400-101-2019

Part I: Second order PDEs: general Formula

1. Wave Equation on the whole line:

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x,t), -\infty < x < +\infty, \\ u(x,0) = \phi(x),\ u_t(x,0) = \psi(x), -\infty < x < +\infty \end{cases} \tag{1}$$

D'Alembert's formula

$$u(x,t) = \frac{1}{2}(\phi(x-ct) + \phi(x+ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)dy + \frac{1}{2c}\int_0^t (\int_{x-c(t-s)}^{x+c(t-s)} f(y,s)dy)ds$$

2. Diffusion Equation on the whole line:

$$\begin{cases} u_t - ku_{xx} = f(x,t), -\infty < x < +\infty, t > 0 \\ u(x,0) = \phi(x), -\infty < x < +\infty \end{cases} \tag{2}$$

Solution formula

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)dy + \int_0^t \int_{-\infty}^{+\infty} S(x-y,t-s)f(y,s)dyds$$

where

$$S(x,t) = \frac{1}{\sqrt{4k\pi t}}e^{-\frac{x^2}{4kt}}$$

3. Wave Equation on the half line:

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x,t), 0 < x < +\infty, t > 0 \\ u(x,0) = \phi,\ u_t(x,0) = \psi(x), 0 < x < +\infty \\ u(0,t) = 0 \end{cases} \tag{3}$$

Method of Reflection: extend f, ϕ, ψ oddly to $(-\infty, +\infty)$:

$$\phi_{ext} = \begin{cases} \phi(x),\ x > 0; \\ -\phi(-x),\ x < 0 \end{cases} \quad \psi_{ext} = \begin{cases} \psi(x),\ x > 0; \\ -\psi(-x),\ x < 0 \end{cases} \quad f_{ext} = \begin{cases} f(x,t),\ x > 0; \\ -f(-x,t),\ x < 0 \end{cases}$$

$$u(x,t) = \frac{1}{2}(\phi_{ext}(x-ct) + \phi_{ext}(x+ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{ext}(y)dy + \frac{1}{2c}\int_0^t (\int_{x-c(t-s)}^{x+c(t-s)} f_{ext}(y,s)dy)ds$$

There is a similar formula for Neumann boundary condition.
Inhomogeneous BC: $u(0,t) = h(t)$. Use $V(x,t) = u(x,t) - xh(t)$.

4. Heat Equation on the half line:

$$\begin{cases} u_t - c^2 u_{xx} = f(x,t), 0 < x < +\infty, t > 0 \\ u(x,0) = \phi, 0 < x < +\infty \\ u(0,t) = 0 \end{cases} \tag{4}$$

Method of Reflection: extend f, ϕ oddly to $(-\infty, +\infty)$:

$$\phi_{ext} = \begin{cases} \phi(x),\ x > 0; \\ -\phi(-x),\ x < 0 \end{cases} \quad f_{ext} = \begin{cases} f(x,t),\ x > 0; \\ -f(-x,t),\ x < 0 \end{cases}$$

$$u(x,t) = \int S(x-y,t)\phi_{ext}(y)dy + \int_0^t \int S(x-y,t-s) f_{ext}(y,s)dyds$$

There is a similar formula for Neumann boundary condition.

Inhomogeneous BC: $u(0,t) = h(t)$. Use $V(x,t) = u(x,t) - xh(t)$.

5. Wave equation in bounded interval

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0, 0 < x < l, t > 0 \\ u(x,0) = \phi,\ u_t(x,0) = \psi(x), 0 < x < l \\ u(0,t) = u(l,t) \end{cases} \tag{5}$$

Method of Extension: extend f, ϕ, ψ periodically to $(-\infty, +\infty)$:

$$\phi_{ext} = \begin{cases} \phi(x),\ 0 < x < l; \\ -\phi(-x), l < x < 0; \\ \phi(x \pm 2l), \end{cases} \quad \psi_{ext} = \begin{cases} \psi(x),\ 0 < x < l; \\ -\psi(-x), l < x < 0; \\ \psi(x \pm 2l), \end{cases}$$

$$u(x,t) = \frac{1}{2}(\phi_{ext}(x-ct) + \phi_{ext}(x+ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{ext}(y)dy$$

Part III: Boundary Value Problems and Method of Separation of Variables

1. Method of Separation of Variations

Step 1: Find the right separated functions. Plug into PDE and obtain one Eigenvalue Problem (EVP) and one ODE.

Step 2: Solve (EVP) and (ODE)

Step 3: Sum-up. Plug in the inhomogeneous BC and find the coefficients.

2. Standard Eigenvalue Problems

$$X'' + \lambda X = 0, 0 < x < l$$

2.1) Dirichlet BC: $X(0) = X(l) = 0$

$$\lambda_n = \frac{(n\pi)^2}{l^2}, X_n = \sin(\frac{n\pi}{l}x), n = 1, 2, ...$$

2.2) Neumann BC: $X'(0) = X'(l) = 0$

$$\lambda_n = \frac{(n\pi)^2}{l^2}, X_n = \cos(\frac{n\pi}{l}x), n = 0, 1, 2, ...$$

2.3) Periodic BC: $X(0) = X(l), X'(0) = X'(l)$

$$\lambda_n = \frac{(2n\pi)^2}{l^2}, X_n = a\cos(\frac{2n\pi}{l}x) + b\sin(\frac{2n\pi}{l}x), n = 0, 1, 2, ...$$

2.4) Summary of Robin boundary condition eigenvalue problems

$$\begin{cases} X'' + \lambda X = 0, 0 < x < l, \\ X'(0) - a_0 X(0), X'(l) + a_l X(l) = 0 \end{cases}$$

Hyperbola:

$$a_0 + a_l + a_0 a_l l = 0 \equiv (a_0 + \frac{1}{l})(a_l + \frac{1}{l}) - \frac{1}{l^2} = 0$$

divides the parameter space (a_0, a_l) into Five Regions. Depending on the regions, the number of negative or zero eigenvalues can be determined.

Equation for negative eigenvalues:

$$\lambda = -\gamma^2,\ \tanh(\gamma l) = -\frac{(a_0 + a_l)\gamma}{\gamma^2 + a_0 a_l},\ X = \cosh(\gamma x) + \frac{a_0}{\gamma}\sinh(\gamma x)$$

Equation for zero eigenvalue

$$a_0 + a_l + a_0 a_l l = 0, \lambda = 0, X = 1 - a_0 x$$

Equation for positive eigenvalue

$$\lambda = \beta^2,\ \tan(\beta l) = \frac{(a_0 + a_l)\beta}{\beta^2 - a_0 a_l},\ \ X = \cos(\beta x) + \frac{a_0}{\beta}\sin(\beta x)$$

3. Sturm-Liouville Eigenvalue Problem

$$(p(x)X)^{'} + \lambda w(x)X = 0, 0 < x < l,$$

$$X^{'}(0) - h_0 X(0) = 0, X^{'}(l) + h_1 X(l) = 0$$

Lagrange's identity:

$$\int_0^l [f(pg^{'})^{'} - g(pf^{'})^{'}] = (pfg^{'} - pf^{'}g)|_0^l$$

1) all eigenvalues are real
2) $\lambda_1 > 0$ if $h_0 > 0, h_1 > 0$
3) Different eigenfunctions are orthogonal with respect to the weight function w:

$$\int_0^l w(x)X_n X_m dx = 0$$

4) eigenvalues are discrete and approach to infinity

$$\lambda_1 < \lambda_2 < ... < \lambda_n < ..., \lambda_n \to +\infty$$

5) Expansion with respect to the eigenfunctions:

$$f(x) = \sum_n A_n X_n(x)$$

$$A_n = \frac{\int_0^l X_n f w(x) dx}{\int_0^l X_n^2 w(x) dx}$$

4. Typical Sturm-Liouville Eigenvalue Problems
4.1. $aX^{''} + bX^{'} + \lambda X = 0$
$X(x) \sim e^{rx}$
4.2. $ax^2 X^{''} + bxX^{'} + \lambda X$=0
$X(x) \sim x^r$
4.3. $X^{''} + \frac{1}{x}X^{'} + \lambda X = 0$
$X(x) = J_0(\sqrt{\lambda}x)$
3. Method of Separation of Variables for heat equation/wave equation
1) Diffusion equation without source:

$$u_t = k u_{xx}, 0 < x < l$$

$$u(x, 0) = \phi$$

$$u(0, t) = 0, u(l, t) = 0$$

$$u(x, t) = \sum_{n=1}^{\infty} a_n e^{-k\lambda_n t} \sin(\frac{n\pi}{l} x)$$

where

$$a_n = \frac{2}{l}\int_0^l \phi(x)\sin(\frac{n\pi}{l}x)dx$$

Similar formula for wave equation.

2) Diffusion equation with source:

$$u_t = ku_{xx} + f(x,t),$$

$$u(x,0) = \phi$$

$$u(0,t) = g(t), u(l,t) = h(t)$$

Expansion:

$$u = \sum_{n=1}^{\infty} u_n(t)\sin(\frac{n\pi}{l}x)$$

$$f(x,t) = \sum_{n=1}^{\infty} f_n(t)\sin(\frac{n\pi}{l}x)$$

$$\phi(x) = \sum_{n=1}^{\infty} \phi_n \sin(\frac{n\pi}{l}x)$$

Then we need to solve

$$u_n^{'} + k\lambda_n u_n = \frac{2nk\pi}{l^2}(g(t) - (-1)^n h(t)) + f_n(t)$$

$$u_n(0) = \phi_n$$

Part III: Properties of Second Order PDEs

1. Well-posedness of PDE problems: (a) existence (b) uniqueness (c) stability

1.1. Well-posedness of wave equations via d'Alembert's formula

1.2. Well-posedness of heat equation via heat equation formula.

2. For wave equation

$$u_{tt} = c^2 u_{xx}$$

Domain of dependence, domain of influence

3. The energy of wave equation

$$E(t) = \frac{1}{2}\int_{-\infty}^{\infty} u_t^2(x,t)dt + \frac{c^2}{2}\int_{-\infty}^{+\infty} u_x^2(x,t)dx$$

$$\frac{dE}{dt} = 0$$

The energy of diffusion equation

$$E(t) = \frac{1}{2}\int_{-\infty}^{+\infty} u(x,t)dx$$

$$\frac{dE}{dt} \leq 0$$

4. Uniqueness of wave and diffusion equations by energy method.