Be sure this exam has 7 pages including the cover

The University of British Columbia

MATH 400, Sections 101

Final Exam –December 16, 2016

Name ______________________ **Signature** ______________________

Student Number ______________

This exam consists of **6** questions. No notes. Write your answer in the blank page provided.

Problem	max score	score
1.	20	
2.	10	
3.	15	
4.	20	
5.	15	
6.	20	
total	100	

(20 points) 1. Solve the following first order fully nonlinear PDE:

$$u_y - y u_x^2 = 0$$

$$u(x,0) = x, -\infty < x < +\infty$$

You are allowed to use the Charpit's equations

$$\begin{cases} \frac{dx}{ds} = F_p, x(0) = x_0(\xi) \\ \frac{dy}{ds} = F_q,\ y(0) = y_0(\xi) \\ \frac{dp}{ds} = -F_x - pF_u, p(0) = p_0(\xi) \\ \frac{dq}{ds} = -F_y - qF_u,\ q(0) = q_0(\xi) \\ \frac{du}{ds} = pF_p + qF_q,\ u(0) = u_0(\xi) \end{cases}$$

where (p_0, q_0) is computed via

$$\begin{cases} F(x_0, y_0, p_0, q_0) = 0 \\ u_0^{'} = p_0 x_0^{'} + q_0 y_0^{'} \end{cases}$$

(10 points) 2. Find the general solutions to the following equation

$$2u_{tt} + u_{tx} = x$$

(15 points) 3. Solve the following diffusion equation

$$u_t = u_{xx} + e^t \sin x, 0 < x < \pi, t > 0$$

$$u(x, 0) = 0, 0 < x < \pi$$

$$u(0, t) = 1, u(\pi, t) = 0$$

Hint: you are allowed to use the following formula:

$$u(x, t) = \sum_{n=1}^{\infty} u_n(t) \sin(\frac{n\pi}{l} x)$$

$$u_n^{'} + k\lambda_n u_n = \frac{2n\pi}{l^2}(h(t) - (-1)^n j(t)) + f_n(t)$$

$$u_n(0) = \phi_n$$

where $u(0, t) = h(t), u(l, t) = j(t)$.

(20 points) 4. Consider the following wave equation

$$u_{tt} = u_{xx} \quad 0 < x < 1, \quad t > 0$$
$$u(x,0) = \phi(x), \ u_t(x,0) = 0, \quad 0 < x < 1$$
$$u_x(0,t) + 3u(0,t) = 0, \qquad u_x(1,t) = 0, \ t > 0$$

(i) (15) Use the method of separation of variables to find the general solution. (You should give the equations for negative, zero or positive eigenvalues, and general solution with coefficients in terms of the eigenfunctions and ϕ.)

(ii) (5) What can you say about the asymptotic behavior of u as $t \to +\infty$? Justify your answer.

(15 points) 5. Use the method of separation of variables to find u of the following Laplace equation in a sector between two annulus

$$u_{xx} + u_{yy} = 0 \ \text{ in } \{1 < x^2 + y^2 < 4, x > 0, y > 0\}$$

$$u(x, y) = 2x^2 \text{ for } \ x^2 + y^2 = 1, x > 0, y > 0$$

$$u(x, y) = 0 \text{ for } \ x^2 + y^2 = 4, x > 0, y > 0$$

$$u_y(x, 0) = u_x(0, y) = 0, \text{ for } \ 1 < x^2 + y^2 < 4, x \geq 0, y \geq 0$$

Hint: use polar coordinate and the fact that $\cos^2\theta = \frac{1+\cos(2\theta)}{2}$.

(20 points) 6. Solve the following wave equation in terms of the Bessel function of order 0

$$\begin{cases} u_{tt} = u_{rr} + \frac{1}{r}u_r - 4, \ 0 \leq r < 1, \ t > 0, \\ u_r(1,t) + u(1,t) = 4, \ u(r,t) \ \text{ is bounded,} \\ u(r,0) = \phi(r), u_t(r,0) = \psi(r), \ 0 \leq r < 1. \end{cases}$$

Hint: find the steady-state solution first.

Here the Bessel function of order 0, denoted by $J_0(z)$, is given as the solution to

$$J_0'' + \frac{1}{z}J_0' + J_0 = 0, z \geq 0, \quad J_0(0) = 1$$