Solutions to Final Exam

1. $F(x,y,u,p,q) = q - y p^2$

where $p = u_x$, $q = u_y$

Initial Data: $x_0(\xi) = \xi$, $y_0(\xi) = 0$, $u_0(\xi) = \xi$

so $F(x_0, y_0, u_0, p_0, q_0) = 0 \Rightarrow q_0 - y_0(\xi)\, p_0^2 = 0 \Rightarrow q_0 = 0$

$u_0' = p_0 x_0' + q_0 y_0' \Rightarrow 1 = p_0 \cdot 1 + 0 \cdot 0 \Rightarrow p_0 = 1$

Charpit's Equation:

$$\frac{dx}{ds} = F_p = -2yp, \quad x(0) = \xi \qquad \text{— (1)}$$

$$\frac{dy}{ds} = F_q = 1, \quad y(0) = 0 \qquad \text{— (2)}$$

$$\frac{dp}{ds} = -F_x - pF_u = 0, \quad p(0) = 1 \qquad \text{— (3)}$$

$$\frac{dq}{ds} = -F_y - qF_u = +p^2, \quad q(0) = 0 \qquad \text{— (4)}$$

$$\frac{du}{ds} = pF_p + qF_q = p(-2yp) + q \cdot 1 = -2yp^2 + q, \quad u(0) = \xi \qquad \text{— (5)}$$

Solving (2) $\Rightarrow y = s$

Solving (3) $\Rightarrow p = 1$

Solving (4) $\Rightarrow q = +s$

Solving (1) $\Rightarrow \frac{dx}{ds} = -2s \Rightarrow x = \xi - s^2$

Solving (5) $\Rightarrow \frac{du}{ds} = -2s + s = \cancel{}s$, $\Rightarrow u = \xi - \frac{1}{2}s^2$

$\Rightarrow \xi = x + s^2 = x + y^2$, $s = y$

$u = \xi - \frac{1}{2}s^2 = x + y^2 - \frac{1}{2}y^2 = x + \frac{1}{2}y^2$.

2. $\partial_t \,(2\partial_t + \partial_x) = x.$

$$\left.\begin{aligned} \partial_\xi &= \partial_t \\ \partial_\eta &= 2\partial_t + \partial_x \end{aligned}\right\} \Rightarrow \quad \begin{aligned} \partial_t &= \partial_\xi \\ \partial_x &= \partial_\eta - 2\partial_\xi \end{aligned}$$

$$\begin{pmatrix} \partial_t \\ \partial_x \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} \partial_\xi \\ \partial_\eta \end{pmatrix}$$

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix}$$

$$\xi = t - 2x, \quad \eta = x$$

$$\partial_\xi \partial_\eta u = \eta \qquad\qquad \partial_\eta (\partial_\xi u) = \eta$$

$$\Rightarrow \partial_\xi u = \frac{\eta^2}{2} + c(\xi)$$

$$u = \frac{\eta^2}{2}\xi + f(\xi) + g(\eta)$$

$$u = \frac{x^2}{2}(t - 2x) + f(t - 2x) + g(x)$$

3. $u(x,t) = \sum u_n(t) \sin\left(\frac{n\pi}{l}x\right)$

$$= \sum_{n=1}^{+\infty} u_n(t) \sin(nx)$$

$e^t \sin x = \sum f_n(t) \sin(nx)$. $\Rightarrow$ $f_1(t) = e^t$, $f_n(t) = 0,\ n \neq 1$

$u(x,0) = \sum \phi_n(0) \sin nx$

$h(t) = 1,\ j(t) = 0,$

$k = 1,\ \lambda_n = n^2$

So

$$\begin{cases} u_n' + n^2 u_n = \frac{2n\pi}{\pi^2}(1-0) + f_n(t) \\ u_n(0) = 0 \end{cases}$$

$n = 1$

$$\begin{cases} u_1' + u_1 = \frac{2}{\pi} + e^t \\ u_1(0) = 0 \end{cases} \Rightarrow (e^t u_1)' = \frac{2}{\pi} e^t + e^{2t}$$

$$\Rightarrow u_1 = e^{-t} \int_0^t \left(\frac{2}{\pi} e^t + e^{2t}\right) dt$$

$$= e^{-t}\left(\frac{2}{\pi}(e^t - 1) + \frac{1}{2}(e^{2t} - 1)\right)$$

$$= \frac{2}{\pi}(1 - e^{-t}) + \frac{1}{2}(e^t - e^{-t})$$

$n \geq 2$.

$$\begin{cases} u_n' + n^2 u_n = \frac{2n}{\pi} \\ u_n(0) = 0 \end{cases}$$

$$u_n = \frac{2}{\pi n} + A e^{-n^2 t} = \frac{2}{n\pi}(1 - e^{-n^2 t})$$

So

$$u(x,t) = \left(\frac{2}{\pi}(1 - e^{-t}) + \frac{1}{2}(e^t - e^{-t})\right)\sin x + \sum_{n=2}^{+\infty} \frac{2}{n\pi}(1 - e^{-n^2 t}) \sin nx$$

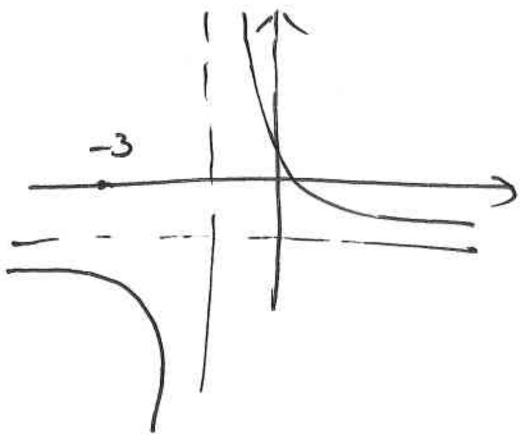
4. (i). S1). $u = X(t)\,T(t)$

$$X'' + \lambda X = 0, \quad T'' + \lambda T = 0.$$

S2) $\begin{cases} X'' + \lambda X = 0 \\ X'(0) + 3X(0) = 0 \\ X'(1) = 0 \end{cases}$ $\qquad T'' + \lambda T = 0$

S3) $a_0 = -3$, $a_\ell = 0$, ~~k~~l, Hyperbola graph

$(a_0+1)(a_\ell+1) = 1$



Region II: one negative eigenvalue

all positive

negative eigenvalue equation: $\lambda = -\gamma_{-1}^2 < 0$

$$\tanh \gamma_{-1} = \frac{(a_0 + a_\ell)\gamma}{-a_0 a_\ell - \gamma^2} = -\frac{-3}{\gamma} = \frac{3}{\gamma_{-1}}$$

$$X_{-1} = \cosh \gamma_{-1} x - \frac{3}{\gamma_{-1}} \sinh \gamma_{-1} x$$

positive eigenvalues

$$\lambda = \beta^2 > 0, \quad \tan(\beta) = \frac{-3\beta}{\beta^2 - 0} = -\frac{3}{\beta}$$

$$X_n(x) = \cos \beta x - \frac{3}{\beta} \sin \beta x$$

S4) $\gamma = -\gamma_1^2 < 0$

$$T'' - \gamma_1^2 T = 0$$

$$T = a \cosh \gamma_1 t + b \sinh \gamma_1 t$$

$\gamma = \beta^2$, $T = a \cos\beta t + b \sin\beta t$

S4) $u = (a_1 \cosh \gamma_1 t + b_1 \sin \gamma_1 t)(\cosh \gamma_1 x - \frac{3}{\gamma_1} \sin \gamma_1 x)$

$$+ \sum_{n=1}^{+\infty} (a_n \cos\beta t + b_n \sin\beta t)(\cos\beta x - \frac{3}{\beta} \sin\beta x)$$

$u(x,0) = \phi(x) \Rightarrow a_1 X_1(x) + \sum a_n X_n(x) = \phi(x)$

$u_t(x,0) = \psi(x) \Rightarrow -\gamma_1 b_1 X_1 + \sum \beta b_n X_n = \psi(x)$

$\psi = 0 \Rightarrow b_1 = b_n = 0$

$$a_1 = \frac{\int_0^1 \phi(x) X_1}{\int_0^1 X_1^2}, \quad a_n = \frac{\int_0^1 \phi(x) X_n}{\int_0^1 X_n^2}$$

(ii). Since $\cosh \gamma_1 t \to +\infty$ as $t \to +\infty$, we need

$$a_1 = 0 \Leftrightarrow \int_0^1 \phi(x) X_1(x)\, dx = 0$$

to ensure the boundededness of solutions

5. Use polar

$$
\begin{cases}
u_{rr} + \frac{1}{r} u_r + \frac{u_{\theta\theta}}{r^2} = 0, & 1 < r < 2, \quad 0 < \theta < \frac{\pi}{2} \\
u(1, \theta) = 2x^2 = 2\cos^2\theta = 1 + \cos 2\theta, & \\
u(2, \theta) = 0, & \\
u_\theta(r, 0) = u_\theta(r, \frac{\pi}{2}) = 0. &
\end{cases}
$$

Separation of variables

$$\Theta'' + \lambda \Theta = 0, \quad 0 < \theta < \frac{\pi}{2}$$

$$\Theta'(0) = \Theta'(\tfrac{\pi}{2}) = 0$$

$$\Theta = \cos 2n\theta, \quad \lambda = (2n)^2$$

$$n = 0, \quad R'' + \frac{1}{r} R' = 0 \Rightarrow R = a_0 \ln r + b_0$$

$$n \geq 1, \quad R'' + \frac{1}{r} R' - \frac{(2n)^2}{r^2} R \Rightarrow R = a r^{2n} + b r^{-2n}$$

$$u = (a_0 \ln r + b_0) + \sum_{n=1}^{+\infty} (a_n r^{2n} + b_n r^{-2n}) \cos 2n\theta$$

$$u(1, \theta) = 1 + \cos 2\theta \Rightarrow \quad a_0 \cdot \ln 1 + b_0 = 1 \Rightarrow b_0 = 1$$

$$a_2 + b_2 = 1 \quad =$$

$$u(2, \theta) = 0 \Rightarrow \quad a_0 \ln 2 + b_0 = 0 \quad \Rightarrow a_0 = -\frac{1}{\ln 2}$$

$$a_2 2^4 + b_2 2^{-4} = 0 \Rightarrow b_2 = 2^8 a_2$$

$$a_2 = \frac{1}{1 + 2^4}, \quad b_2 = \frac{2^4}{1 + 2^4}.$$

$$u = -\frac{1}{\ln 2} \ln r + 1 + \left(\frac{1}{1+2^4} r^2 + \frac{2^4}{1+2^4} r^{-2} \right) \cos 2\theta$$

6. Step 0: Solve steady-state first

$$\begin{cases} u_{0,rr}+\frac{1}{r}u_{0,r}-4=0 \\ u_{0,r}(1)+u_0(1)=4 \end{cases}$$

$\Rightarrow u_0 = r^2 + C$

$u_{0,r}(1)+u_0(1)=4 \Rightarrow C=1$

Now let $u = r^2+1+v$

$$\begin{cases} v_{tt} = v_{rr}+\frac{1}{r}v_r \\ v_r(1,t)+v(1,t)=0. \\ v(r,0)=\phi(r),\ v_t(r,0)=\psi(r) \end{cases}$$

The eigenvalue problem is

$R''+\frac{1}{r}R+\lambda R=0$ $\qquad R=J_0(\sqrt{\lambda}\,r)$

$R'(1)+R(1)=0$

$R'(1)=\sqrt{\lambda}J_0'(\sqrt{\lambda})$

$\Rightarrow \sqrt{\lambda}J_0'(\sqrt{\lambda})+J_0(\sqrt{\lambda})=0 \Rightarrow \sqrt{\lambda}=\hat{z}_n,$ ....

$u=\sum_{n=1}^{+\infty}(a_n\cos\hat{z}_n t+b_n\sin(\hat{z}_n t))J_0(\hat{z}_n t)$

$u(r,0)=\sum a_n J_0(\hat{z}_n t).=\phi(r) \Rightarrow a_n= \dfrac{\int_0^1 r\phi J_0(\hat{z}_n r)}{\int_0^1 rJ_0^2(\hat{z}_n r)}$

$u_t(r,0)=\sum \hat{z}_n b_n J_0(\hat{z}_n t)=\psi(r) \Rightarrow \hat{z}_n b_n=$ ______